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**ON COMPLETE MONOTONICITY OF LINEAR  
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## ON COMPLETE MONOTONICITY OF LINEAR COMBINATION OF FINITE PSI FUNCTIONS

BAI-NI GUO AND FENG QI

ABSTRACT. In the paper, the authors supply complete monotonicity of linear combination of finite psi functions and extend some known results.

### 1. Preliminaries

It is well-known [1, 12, 14] that the classical gamma function  $\Gamma(z)$  can be defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

or by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

and that the logarithmic derivative

$$\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$$

is called the psi or digamma function.

From [11, Chapter XIII], [30, Chapter 1], and [31, Chapter IV], we recall that an infinitely differentiable and nonnegative function  $f(x)$  is said to be completely monotonic on an interval  $I$  if and only if

$$(-1)^{m-1} f^{(m-1)}(x) \geq 0$$

for all  $m \in \mathbb{N}$  and  $x \in I$ . The Bernstein–Widder theorem [31, p. 161, Theorem 12b] states that a necessary and sufficient condition for  $f(x)$  to be completely monotonic on  $(0, \infty)$  is that

$$f(x) = \int_0^\infty e^{-xt} d\mu(t)$$

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for  $x \in (0, \infty)$ , where  $\mu$  is non-decreasing and the above integral converges. In other words or simply speaking, a function is completely monotonic on  $(0, \infty)$  if and only if it is a Laplace transform.

## 2. Motivations

In [2, Theorem 4.1], it was obtained that, if  $a_k$  and  $b_k$  for  $1 \leq k \leq m$  satisfy  $a_1 \geq a_2 \geq \cdots \geq a_m$  and  $b_1 \geq b_2 \geq \cdots \geq b_m > 0$ , then

$$\phi_0(x) = \sum_{k=1}^m a_k \psi(b_k x)$$

is completely monotonic on  $(0, \infty)$  if and only if

$$\sum_{k=1}^m a_k = 0 \quad \text{and} \quad \sum_{k=1}^m a_k \ln b_k \geq 0.$$

In [8, Lemma 2.1], the function  $\phi_0(x)$  was extended as

$$\phi_\delta(x) = \sum_{k=1}^m a_k \psi(b_k x + \delta)$$

for  $x > 0$  and  $\delta \geq 0$  and it was acquired that, if  $a_1 \geq a_2 \geq \cdots \geq a_m$  and  $b_1 \geq b_2 \geq \cdots \geq b_m \geq 0$  such that  $\sum_{k=1}^m a_k \geq 0$ , then, when  $\delta \geq \frac{1}{2}$ , the first derivative  $\phi'_\delta(x)$  is completely monotonic and, consequently, the function  $\phi_\delta(x)$  is increasing and concave, on  $(0, \infty)$ . These results were applied in [8] to discuss a problem arising in the context of statistical density estimation based on Bernstein polynomials.

In the proof of [29, Theorem 2.2], it was obtained that the function

$$\left( \sum_{i=1}^m a_i \right)^2 \psi' \left( 1 + x \sum_{i=1}^m a_i \right) - \sum_{i=1}^m a_i^2 \psi'(1 + a_i x), \quad a_i > 0$$

is completely monotonic and, consequently,

$$(1) \quad \left( \sum_{i=1}^m a_i \right) \psi \left( 1 + x \sum_{i=1}^m a_i \right) - \sum_{i=1}^m a_i \psi(1 + a_i x), \quad a_i > 0$$

is positive, increasing, and concave, with respect to  $x \in (0, \infty)$ . See also the proof of [13, Theorem 2.1]. In other words, the function defined by (1) is a Bernstein function of  $x \in (0, \infty)$ . For detailed information on the Bernstein functions, please refer to the monograph [30], the papers [21–23, 25, 26] and closely related references.

In this paper, we supply an alternative proof for the above complete monotonicity of the functions  $\phi_0(x)$  and  $\phi_\delta(x)$  for  $\delta \geq \frac{1}{2}$  on  $(0, \infty)$  and obtain slightly extended conclusions.

### 3. Main results and their proofs

We now state and prove our main results alternatively.

**Theorem 1.** *If  $\delta \geq \frac{1}{2}$ ,  $(a_i - a_j)(b_i - b_j) \gtrless 0$  for all  $1 \leq i, j \leq m$ , and  $\sum_{k=1}^m a_k \gtrless 0$ , then the first derivative  $\pm\phi'_\delta(x)$  is completely monotonic and, consequently, the function  $\pm\phi_\delta(x)$  is increasing and concave, on  $(0, \infty)$ .*

*If  $(a_i - a_j)(b_i - b_j) \gtrless 0$  for all  $1 \leq i, j \leq m$ , then the function  $\pm\phi_0(x)$  is completely monotonic on  $(0, \infty)$  if and only if  $\sum_{k=1}^m a_k = 0$  and  $\sum_{k=1}^m a_k \ln b_k \gtrless 0$ .*

*Proof.* Using the formula

$$\psi'(z) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-zt} dt, \quad \Re(z) > 0$$

in [1, p. 260, 6.4.1] gives

$$\begin{aligned} \psi'(\tau x + \delta) &= \int_0^\infty \frac{t}{1 - e^{-t}} e^{-(\tau x + \delta)t} dt \\ &= \int_0^\infty \frac{te^{-\delta t}}{1 - e^{-t}} e^{-\tau x t} dt \\ &= \frac{1}{\tau} \int_0^\infty h\left(\frac{v}{\tau}\right) e^{-vx} dv, \end{aligned}$$

where  $\tau > 0$  and  $h_\delta(t) = \frac{te^{-\delta t}}{1 - e^{-t}}$ . Hence

$$\phi'_\delta(x) = \sum_{k=1}^m a_k b_k \psi'(b_k x + \delta) = \int_0^\infty \left[ \sum_{k=1}^m a_k h_\delta\left(\frac{v}{b_k}\right) \right] e^{-vx} dv.$$

In [9, 32], it was established that the positive function

$$h_\delta(t) = \begin{cases} \frac{t}{e^{\delta t} - e^{(\delta-1)t}}, & t \neq 0, \\ 1, & t = 0, \end{cases}$$

is decreasing on  $\mathbb{R}$  if  $\delta \geq 1$ , increasing on  $\mathbb{R}$  if  $\delta \leq 0$ , increasing in  $(-\infty, 0)$  if  $\delta \leq \frac{1}{2}$ , and decreasing in  $(0, \infty)$  if  $\delta \geq \frac{1}{2}$ . For more information on properties and applications of  $h_\delta(t)$ , please refer to the papers and review articles [3, 4, 6, 15–19, 24, 28] and closely related references therein. Therefore, by virtue of the Čebyšev inequality in [10, p. 36, Section 2.5, Theorem 1], we acquire

$$\frac{1}{m} \sum_{k=1}^m a_k h_\delta\left(\frac{v}{b_k}\right) \gtrless \left( \frac{1}{m} \sum_{k=1}^m a_k \right) \left[ \frac{1}{m} \sum_{k=1}^m h_\delta\left(\frac{v}{b_k}\right) \right] \gtrless 0.$$

This means that the first derivative  $\pm\phi'_\delta(x)$  is completely monotonic on  $(0, \infty)$ . Consequently, the function  $\pm\phi_\delta(x)$  is increasing and concave on  $(0, \infty)$ .

When  $\delta = 0$ , the function  $h_0(t)$  is positive and increasing on  $[0, \infty)$  and

$$\frac{1}{m} \sum_{k=1}^m a_k h_0\left(\frac{v}{b_k}\right) \leq \left(\frac{1}{m} \sum_{k=1}^m a_k\right) \left[\frac{1}{m} \sum_{k=1}^m h_0\left(\frac{v}{b_k}\right)\right].$$

Accordingly, when  $\sum_{k=1}^m a_k \leq 0$ , the first derivative  $\mp \phi'_0(x)$  is completely monotonic and the function  $\mp \phi_0(x)$  is increasing on  $(0, \infty)$ . Furthermore, utilizing

$$\lim_{x \rightarrow \infty} [\ln x - \psi(x)] = 0$$

in [5, Theorem 1] and [7, 27] yields that the limit

$$\phi_0(x) = \sum_{k=1}^m a_k [\psi(b_k x) - \ln(b_k x)] + \sum_{k=1}^m a_k \ln b_k + (\ln x) \sum_{k=1}^m a_k \rightarrow \sum_{k=1}^m a_k \ln b_k$$

as  $x \rightarrow \infty$  is valid and

$$\mp \phi_0(x) \leq \mp \sum_{k=1}^m a_k \ln b_k$$

if and only if  $\sum_{k=1}^m a_k = 0$ . When and only when  $\sum_{k=1}^m a_k \ln b_k \geq 0$ , the function  $\pm \phi_0(x)$  is positive and, consequently, completely monotonic, on  $(0, \infty)$ . The proof of Theorem 1 is complete.  $\square$

*Remark 1.* This paper is a slight revision of the preprint [20].

## References

- [1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, National Bureau of Standards Applied Mathematics Series, **55**, For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, DC, 1964.
- [2] H. Alzer and C. Berg, *Some classes of completely monotonic functions. II*, Ramanujan J. **11** (2006), no. 2, 225–248. <https://doi.org/10.1007/s11139-006-6510-5>
- [3] W.-S. Cheung and F. Qi, *Logarithmic convexity of the one-parameter mean values*, Taiwanese J. Math. **11** (2007), no. 1, 231–237. <https://doi.org/10.11650/twjmath/1500404648>
- [4] B.-N. Guo and F. Qi, *A simple proof of logarithmic convexity of extended mean values*, Numer. Algorithms **52** (2009), no. 1, 89–92. <https://doi.org/10.1007/s11075-008-9259-7>
- [5] ———, *Two new proofs of the complete monotonicity of a function involving the PSI function*, Bull. Korean Math. Soc. **47** (2010), no. 1, 103–111. <https://doi.org/10.4134/BKMS.2010.47.1.103>
- [6] ———, *The function  $(b^x - a^x)/x$ : logarithmic convexity and applications to extended mean values*, Filomat **25** (2011), no. 4, 63–73. <https://doi.org/10.2298/FIL1104063G>
- [7] B.-N. Guo, F. Qi, J.-L. Zhao, and Q.-M. Luo, *Sharp inequalities for polygamma functions*, Math. Slovaca **65** (2015), no. 1, 103–120. <https://doi.org/10.1515/ms-2015-0010>
- [8] A. Leblanc and B. C. Johnson, *On a uniformly integrable family of polynomials defined on the unit interval*, JIPAM. J. Inequal. Pure Appl. Math. **8** (2007), no. 3, Article 67, 5 pp.

- [9] A.-Q. Liu, G.-F. Li, B.-N. Guo, and F. Qi, *Monotonicity and logarithmic concavity of two functions involving exponential function*, Internat. J. Math. Ed. Sci. Tech. **39** (2008), no. 5, 686–691. <https://doi.org/10.1080/00207390801986841>
- [10] D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, New York, 1970.
- [11] D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Mathematics and its Applications (East European Series), **61**, Kluwer Academic Publishers Group, Dordrecht, 1993. <https://doi.org/10.1007/978-94-017-1043-5>
- [12] F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010; Available online at <http://dlmf.nist.gov/>.
- [13] F. Ouimet, *Complete monotonicity of multinomial probabilities and its application to Bernstein estimators on the simplex*, J. Math. Anal. Appl. **466** (2018), no. 2, 1609–1617. <https://doi.org/10.1016/j.jmaa.2018.06.049>
- [14] F. Qi, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, Filomat **27** (2013), no. 4, 601–604. <https://doi.org/10.2298/FIL1304601Q>
- [15] F. Qi, P. Cerone, S. S. Dragomir, and H. M. Srivastava, *Alternative proofs for monotonic and logarithmically convex properties of one-parameter mean values*, Appl. Math. Comput. **208** (2009), no. 1, 129–133. <https://doi.org/10.1016/j.amc.2008.11.023>
- [16] F. Qi and J.-X. Cheng, *Some new Steffensen pairs*, Anal. Math. **29** (2003), no. 3, 219–226. <https://doi.org/10.1023/A:1025467221664>
- [17] F. Qi, J.-X. Cheng, and G. Wang, *New Steffensen pairs*, in *Inequality theory and applications*. Vol. I, 273–279, Nova Sci. Publ., Huntington, NY, 2001.
- [18] F. Qi and B.-N. Guo, *On Steffensen pairs*, J. Math. Anal. Appl. **271** (2002), no. 2, 534–541. [https://doi.org/10.1016/S0022-247X\(02\)00120-8](https://doi.org/10.1016/S0022-247X(02)00120-8)
- [19] ———, *Some properties of extended remainder of Binet’s first formula for logarithm of gamma function*, Math. Slovaca **60** (2010), no. 4, 461–470. <https://doi.org/10.2478/s12175-010-0025-7>
- [20] ———, *An alternative proof for complete monotonicity of linear combinations of many psi functions*, HAL archives (2018), available online at <https://hal.archives-ouvertes.fr/hal-01773131>.
- [21] ———, *The reciprocal of the weighted geometric mean is a Stieltjes function*, Bol. Soc. Mat. Mex. (3) **24** (2018), no. 1, 181–202. <https://doi.org/10.1007/s40590-016-0151-5>
- [22] ———, *The reciprocal of the weighted geometric mean of many positive numbers is a Stieltjes function*, Quaest. Math. **41** (2018), no. 5, 653–664. <https://doi.org/10.2989/16073606.2017.1396508>
- [23] ———, *Lévy-Khintchine representation of Toader-Qi mean*, Math. Inequal. Appl. **21** (2018), no. 2, 421–431. <https://doi.org/10.7153/mia-2018-21-29>
- [24] F. Qi, B.-N. Guo, and C.-P. Chen, *The best bounds in Gautschi-Kershaw inequalities*, Math. Inequal. Appl. **9** (2006), no. 3, 427–436. <https://doi.org/10.7153/mia-09-41>
- [25] F. Qi and W.-H. Li, *Integral representations and properties of some functions involving the logarithmic function*, Filomat **30** (2016), no. 7, 1659–1674. <https://doi.org/10.2298/FIL1607659Q>
- [26] F. Qi and D. Lim, *Integral representations of bivariate complex geometric mean and their applications*, J. Comput. Appl. Math. **330** (2018), 41–58. <https://doi.org/10.1016/j.cam.2017.08.005>
- [27] F. Qi and A.-Q. Liu, *Completely monotonic degrees for a difference between the logarithmic and psi functions*, J. Comput. Appl. Math. **361** (2019), 366–371. <https://doi.org/10.1016/j.cam.2019.05.001>

- [28] F. Qi, Q.-M. Luo, and B.-N. Guo, *The function  $(b^x - a^x)/x$  : ratio's properties*, in Analytic number theory, approximation theory, and special functions, 485–494, Springer, New York, 2014. [https://doi.org/10.1007/978-1-4939-0258-3\\_16](https://doi.org/10.1007/978-1-4939-0258-3_16)
- [29] F. Qi, D.-W. Niu, D. Lim, and B.-N. Guo, *Some logarithmically completely monotonic functions and inequalities for multinomial coefficients and multivariate beta functions*, HAL archives (2018), available online at <https://hal.archives-ouvertes.fr/hal-01769288>.
- [30] R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions*, second edition, De Gruyter Studies in Mathematics, **37**, Walter de Gruyter & Co., Berlin, 2012. <https://doi.org/10.1515/9783110269338>
- [31] D. V. Widder, *The Laplace Transform*, Princeton Mathematical Series, **6**, Princeton University Press, Princeton, NJ, 1941.
- [32] S.-Q. Zhang, B.-N. Guo, and F. Qi, *A concise proof for properties of three functions involving the exponential function*, Appl. Math. E-Notes **9** (2009), 177–183.

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