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**ON COMPLETE MONOTONICITY OF LINEAR
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ON COMPLETE MONOTONICITY OF LINEAR COMBINATION OF FINITE PSI FUNCTIONS

BAI-NI GUO AND FENG QI

ABSTRACT. In the paper, the authors supply complete monotonicity of linear combination of finite psi functions and extend some known results.

1. Preliminaries

It is well-known [1, 12, 14] that the classical gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

or by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

and that the logarithmic derivative

$$\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$$

is called the psi or digamma function.

From [11, Chapter XIII], [30, Chapter 1], and [31, Chapter IV], we recall that an infinitely differentiable and nonnegative function $f(x)$ is said to be completely monotonic on an interval I if and only if

$$(-1)^{m-1} f^{(m-1)}(x) \geq 0$$

for all $m \in \mathbb{N}$ and $x \in I$. The Bernstein–Widder theorem [31, p. 161, Theorem 12b] states that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0, \infty)$ is that

$$f(x) = \int_0^{\infty} e^{-xt} d\mu(t)$$

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for $x \in (0, \infty)$, where μ is non-decreasing and the above integral converges. In other words or simply speaking, a function is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform.

2. Motivations

In [2, Theorem 4.1], it was obtained that, if a_k and b_k for $1 \leq k \leq m$ satisfy $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_m > 0$, then

$$\phi_0(x) = \sum_{k=1}^m a_k \psi(b_k x)$$

is completely monotonic on $(0, \infty)$ if and only if

$$\sum_{k=1}^m a_k = 0 \quad \text{and} \quad \sum_{k=1}^m a_k \ln b_k \geq 0.$$

In [8, Lemma 2.1], the function $\phi_0(x)$ was extended as

$$\phi_\delta(x) = \sum_{k=1}^m a_k \psi(b_k x + \delta)$$

for $x > 0$ and $\delta \geq 0$ and it was acquired that, if $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_m \geq 0$ such that $\sum_{k=1}^m a_k \geq 0$, then, when $\delta \geq \frac{1}{2}$, the first derivative $\phi'_\delta(x)$ is completely monotonic and, consequently, the function $\phi_\delta(x)$ is increasing and concave, on $(0, \infty)$. These results were applied in [8] to discuss a problem arising in the context of statistical density estimation based on Bernstein polynomials.

In the proof of [29, Theorem 2.2], it was obtained that the function

$$\left(\sum_{i=1}^m a_i \right)^2 \psi' \left(1 + x \sum_{i=1}^m a_i \right) - \sum_{i=1}^m a_i^2 \psi'(1 + a_i x), \quad a_i > 0$$

is completely monotonic and, consequently,

$$(1) \quad \left(\sum_{i=1}^m a_i \right) \psi \left(1 + x \sum_{i=1}^m a_i \right) - \sum_{i=1}^m a_i \psi(1 + a_i x), \quad a_i > 0$$

is positive, increasing, and concave, with respect to $x \in (0, \infty)$. See also the proof of [13, Theorem 2.1]. In other words, the function defined by (1) is a Bernstein function of $x \in (0, \infty)$. For detailed information on the Bernstein functions, please refer to the monograph [30], the papers [21–23, 25, 26] and closely related references.

In this paper, we supply an alternative proof for the above complete monotonicity of the functions $\phi_0(x)$ and $\phi_\delta(x)$ for $\delta \geq \frac{1}{2}$ on $(0, \infty)$ and obtain slightly extended conclusions.

3. Main results and their proofs

We now state and prove our main results alternatively.

Theorem 1. *If $\delta \geq \frac{1}{2}$, $(a_i - a_j)(b_i - b_j) \geq 0$ for all $1 \leq i, j \leq m$, and $\sum_{k=1}^m a_k \geq 0$, then the first derivative $\pm\phi'_\delta(x)$ is completely monotonic and, consequently, the function $\pm\phi_\delta(x)$ is increasing and concave, on $(0, \infty)$.*

If $(a_i - a_j)(b_i - b_j) \geq 0$ for all $1 \leq i, j \leq m$, then the function $\pm\phi_0(x)$ is completely monotonic on $(0, \infty)$ if and only if $\sum_{k=1}^m a_k = 0$ and $\sum_{k=1}^m a_k \ln b_k \geq 0$.

Proof. Using the formula

$$\psi'(z) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-zt} dt, \quad \Re(z) > 0$$

in [1, p. 260, 6.4.1] gives

$$\begin{aligned} \psi'(\tau x + \delta) &= \int_0^\infty \frac{t}{1 - e^{-t}} e^{-(\tau x + \delta)t} dt \\ &= \int_0^\infty \frac{te^{-\delta t}}{1 - e^{-t}} e^{-\tau x t} dt \\ &= \frac{1}{\tau} \int_0^\infty h\left(\frac{v}{\tau}\right) e^{-vx} dv, \end{aligned}$$

where $\tau > 0$ and $h_\delta(t) = \frac{te^{-\delta t}}{1 - e^{-t}}$. Hence

$$\phi'_\delta(x) = \sum_{k=1}^m a_k b_k \psi'(b_k x + \delta) = \int_0^\infty \left[\sum_{k=1}^m a_k h_\delta\left(\frac{v}{b_k}\right) \right] e^{-vx} dv.$$

In [9, 32], it was established that the positive function

$$h_\delta(t) = \begin{cases} \frac{t}{e^{\delta t} - e^{(\delta-1)t}}, & t \neq 0, \\ 1, & t = 0, \end{cases}$$

is decreasing on \mathbb{R} if $\delta \geq 1$, increasing on \mathbb{R} if $\delta \leq 0$, increasing in $(-\infty, 0)$ if $\delta \leq \frac{1}{2}$, and decreasing in $(0, \infty)$ if $\delta \geq \frac{1}{2}$. For more information on properties and applications of $h_\delta(t)$, please refer to the papers and review articles [3, 4, 6, 15–19, 24, 28] and closely related references therein. Therefore, by virtue of the Čebyšev inequality in [10, p. 36, Section 2.5, Theorem 1], we acquire

$$\frac{1}{m} \sum_{k=1}^m a_k h_\delta\left(\frac{v}{b_k}\right) \geq \left(\frac{1}{m} \sum_{k=1}^m a_k\right) \left[\frac{1}{m} \sum_{k=1}^m h_\delta\left(\frac{v}{b_k}\right)\right] \geq 0.$$

This means that the first derivative $\pm\phi'_\delta(x)$ is completely monotonic on $(0, \infty)$. Consequently, the function $\pm\phi_\delta(x)$ is increasing and concave on $(0, \infty)$.

When $\delta = 0$, the function $h_0(t)$ is positive and increasing on $[0, \infty)$ and

$$\frac{1}{m} \sum_{k=1}^m a_k h_0\left(\frac{v}{b_k}\right) \geq \left(\frac{1}{m} \sum_{k=1}^m a_k\right) \left[\frac{1}{m} \sum_{k=1}^m h_0\left(\frac{v}{b_k}\right)\right].$$

Accordingly, when $\sum_{k=1}^m a_k \leq 0$, the first derivative $\mp\phi'_0(x)$ is completely monotonic and the function $\mp\phi_0(x)$ is increasing on $(0, \infty)$. Furthermore, utilizing

$$\lim_{x \rightarrow \infty} [\ln x - \psi(x)] = 0$$

in [5, Theorem 1] and [7, 27] yields that the limit

$$\phi_0(x) = \sum_{k=1}^m a_k [\psi(b_k x) - \ln(b_k x)] + \sum_{k=1}^m a_k \ln b_k + (\ln x) \sum_{k=1}^m a_k \rightarrow \sum_{k=1}^m a_k \ln b_k$$

as $x \rightarrow \infty$ is valid and

$$\mp\phi_0(x) \leq \mp \sum_{k=1}^m a_k \ln b_k$$

if and only if $\sum_{k=1}^m a_k = 0$. When and only when $\sum_{k=1}^m a_k \ln b_k \geq 0$, the function $\pm\phi_0(x)$ is positive and, consequently, completely monotonic, on $(0, \infty)$. The proof of Theorem 1 is complete. \square

Remark 1. This paper is a slight revision of the preprint [20].

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