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AN ALTERNATIVE PROOF FOR COMPLETE MONOTONICITY OF LINEAR COMBINATIONS OF MANY PSI FUNCTIONS

FENG QI AND BAI-NI GUO

ABSTRACT. In the paper, the authors supply an alternative proof for complete monotonicity of linear combinations of many psi functions and slightly extend some known results.

1. PRELIMINARIES

It is well-known [1, 12, 13] that the classical gamma function can be defined by

$$\Gamma(z) = \int_0^{\infty} t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

or by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$$

and that the logarithmic derivative $\psi(z) = [\ln \Gamma(z)]' = \frac{\Gamma'(z)}{\Gamma(z)}$ is called the psi or di-gamma function.

From [11, Chapter XIII], [28, Chapter 1], and [29, Chapter IV], we recall that an infinitely differentiable and nonnegative function $f(x)$ is said to be completely monotonic on an interval I if and only if $(-1)^{m-1} f^{(m-1)}(x) \geq 0$ for all $m \in \mathbb{N}$ and $x \in I$. The Bernstein–Widder theorem [29, p. 161, Theorem 12b] states that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0, \infty)$ is that $f(x) = \int_0^{\infty} e^{-xt} d\mu(t)$ for $x \in (0, \infty)$, where μ is a positive measure on $[0, \infty)$ such that the above integral converges. In other words or simply speaking, a function is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform.

2. MOTIVATIONS

In [2, Theorem 4.1], it was obtained that, if a_k and b_k for $1 \leq k \leq m$ satisfy $a_1 \geq a_2 \geq \dots \geq a_m$ and $b_1 \geq b_2 \geq \dots \geq b_m > 0$, then

$$\phi_0(x) = \sum_{k=1}^m a_k \psi(b_k x)$$

is completely monotonic on $(0, \infty)$ if and only if $\sum_{k=1}^m a_k = 0$ and $\sum_{k=1}^m a_k \ln b_k \geq 0$.

In [8, Lemma 2.1], the function $\phi_0(x)$ was extended as

$$\phi_{\delta}(x) = \sum_{k=1}^m a_k \psi(b_k x + \delta)$$

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for $x > 0$ and $\delta \geq 0$ and it was acquired that, if $a_1 \geq a_2 \geq \cdots \geq a_m$ and $b_1 \geq b_2 \geq \cdots \geq b_m \geq 0$ such that $\sum_{k=1}^m a_k \geq 0$, then, when $\delta \geq \frac{1}{2}$, the first derivative $\phi'_\delta(x)$ is completely monotonic and, consequently, the function $\phi_\delta(x)$ is increasing and concave, on $(0, \infty)$. These results were applied in [8] to discuss a problem arising in the context of statistical density estimation based on Bernstein polynomials.

In the proof of [27, Theorem 2.2], it was obtained that the function

$$\left(\sum_{i=1}^m \alpha_i \right)^2 \psi' \left(1 + x \sum_{i=1}^m \alpha_i \right) - \sum_{i=1}^m \alpha_i^2 \psi'(1 + \alpha_i x), \quad \alpha_i > 0$$

is completely monotonic and, consequently,

$$\left(\sum_{i=1}^m a_i \right) \psi \left(1 + x \sum_{i=1}^m a_i \right) - \sum_{i=1}^m a_i \psi(1 + a_i x), \quad \alpha_i > 0 \quad (1)$$

is positive, increasing, and concave, with respect to $x \in (0, \infty)$. In other words, the function defined by (1) is a Bernstein function of $x \in (0, \infty)$. For detailed information on the Bernstein functions, please refer to the monograph [28], the papers [17, 18, 19, 23, 24] and closely related references.

In this paper, we will supply an alternative proof for the above complete monotonicity of the functions $\phi_0(x)$ and $\phi_\delta(x)$ for $\delta \geq \frac{1}{2}$ on $(0, \infty)$ and obtain slightly extended conclusions.

3. MAIN RESULTS AND THEIR PROOF

We now state and prove our main results alternatively.

Theorem 1. *If $\delta \geq \frac{1}{2}$, $(a_i - a_j)(b_i - b_j) \geq 0$ for all $1 \leq i, j \leq m$, and $\sum_{k=1}^m a_k \geq 0$, then the first derivative $\pm \phi'_\delta(x)$ is completely monotonic and, consequently, the function $\pm \phi_\delta(x)$ is increasing and concave, on $(0, \infty)$.*

If $(a_i - a_j)(b_i - b_j) \geq 0$ for all $1 \leq i, j \leq m$, then the function $\pm \phi_0(x)$ is completely monotonic on $(0, \infty)$ if and only if $\sum_{k=1}^m a_k = 0$ and $\sum_{k=1}^m a_k \ln b_k \geq 0$.

Proof. Using the formula

$$\psi'(z) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-zt} dt, \quad \Re(z) > 0$$

in [1, p. 260, 6.4.1] gives

$$\begin{aligned} \psi'(\tau x + \delta) &= \int_0^\infty \frac{t}{1 - e^{-t}} e^{-(\tau x + \delta)t} dt \\ &= \int_0^\infty \frac{te^{-\delta t}}{1 - e^{-t}} e^{-\tau x t} dt = \frac{1}{\tau} \int_0^\infty h\left(\frac{v}{\tau}\right) e^{-vx} dv, \end{aligned}$$

where $\tau > 0$ and $h_\delta(t) = \frac{te^{-\delta t}}{1 - e^{-t}}$. Hence

$$\phi'_\delta(x) = \sum_{k=1}^m a_k b_k \psi'(b_k x + \delta) = \int_0^\infty \left[\sum_{k=1}^m a_k h_\delta\left(\frac{v}{b_k}\right) \right] e^{-vx} dv.$$

In [9, 30], it was established that the positive function

$$h_\delta(t) = \begin{cases} \frac{t}{e^{\delta t} - e^{(\delta-1)t}}, & t \neq 0 \\ 1, & t = 0 \end{cases}$$

is decreasing on \mathbb{R} if $\delta \geq 1$, increasing on \mathbb{R} if $\delta \leq 0$, increasing in $(-\infty, 0)$ if $\delta \leq \frac{1}{2}$, and decreasing in $(0, \infty)$ if $\delta \geq \frac{1}{2}$. For more information on properties and applications of $h_\delta(t)$, please refer to the papers and review articles [3, 4, 5, 14, 15, 16, 20, 21, 22, 26] and closely related references therein. Therefore, by virtue of the Čebyšev inequality in [10, p. 36, Section 2.5, Theorem 1], we acquire

$$\frac{1}{m} \sum_{k=1}^m a_k h_\delta\left(\frac{v}{b_k}\right) \begin{cases} \geq & \left(\frac{1}{m} \sum_{k=1}^m a_k\right) \left[\frac{1}{m} \sum_{k=1}^m h_\delta\left(\frac{v}{b_k}\right)\right] \\ \leq & \end{cases} \geq 0.$$

This means that the first derivative $\pm\phi'_\delta(x)$ is completely monotonic on $(0, \infty)$. Consequently, the function $\pm\phi_\delta(x)$ is increasing and concave on $(0, \infty)$.

When $\delta = 0$, the function $h_0(t)$ is positive and increasing on $[0, \infty)$ and

$$\frac{1}{m} \sum_{k=1}^m a_k h_0\left(\frac{v}{b_k}\right) \begin{cases} \leq & \left(\frac{1}{m} \sum_{k=1}^m a_k\right) \left[\frac{1}{m} \sum_{k=1}^m h_0\left(\frac{v}{b_k}\right)\right] \\ \geq & \end{cases}$$

Accordingly, when $\sum_{k=1}^m a_k \leq 0$, the first derivative $\mp\phi'_0(x)$ is completely monotonic and the function $\mp\phi_0(x)$ is increasing on $(0, \infty)$. Furthermore, utilizing

$$\lim_{x \rightarrow \infty} [\ln x - \psi(x)] = 0$$

in [6, Theorem 1] and [7, 25] yields that the limit

$$\phi_0(x) = \sum_{k=1}^m a_k [\psi(b_k x) - \ln(b_k x)] + \sum_{k=1}^m a_k \ln b_k + (\ln x) \sum_{k=1}^m a_k \rightarrow \sum_{k=1}^m a_k \ln b_k$$

as $x \rightarrow \infty$ is valid and

$$\mp\phi_0(x) \leq \mp \sum_{k=1}^m a_k \ln b_k$$

if and only if $\sum_{k=1}^m a_k = 0$. When and only when $\sum_{k=1}^m a_k \ln b_k \geq 0$, the function $\pm\phi_0(x)$ is positive and, consequently, completely monotonic, on $(0, \infty)$. The proof of Theorem 1 is complete. \square

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