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Non-abelian Radon transform and its applications

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Abstract: Considerations of the non-abelian Radon transform were started in [Manakov, Zakharov, 1981] in the framework of the theory of solitons in dimension 2+1. On the other hand, the problem of inversion of transforms of such a type arises in different tomographies, including emission tomographies, polarization tomographies, and vector field tomography. In this article we give a short review of old and recent results on this subject. This article is an extended version of the talk given at the conference "100 Years of the Radon Transform", Linz, 27-31 March 2017.

1. Introduction

We consider the transport equation

$$\theta \partial_x \psi + A(x, \theta) \psi = 0, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \quad (1)$$

where $\theta \partial_x = \sum_{j=1}^d \theta_j \partial / \partial x_j$ and A is a sufficiently regular function on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ with sufficient decay as $|x| \rightarrow \infty$.

We assume that A and ψ take values in $M(n, \mathbb{C})$ that is in $n \times n$ complex matrices.

For equation (1) we consider the "scattering" matrix S :

$$S(x, \theta) = \lim_{s \rightarrow +\infty} \psi^+(x + s\theta, \theta), \quad (x, \theta) \in T\mathbb{S}^{d-1}, \quad (2)$$

where

$$T\mathbb{S}^{d-1} = \{(x, \theta) \in \mathbb{R}^d \times \mathbb{S}^{d-1} : x\theta = 0\} \quad (3)$$

and $\psi^+(x, \theta)$ is the solution of (1) such that

$$\lim_{s \rightarrow -\infty} \psi^+(x + s\theta, \theta) = I, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \quad (4)$$

where I is the identity matrix.

We interpret $T\mathbb{S}^{d-1}$ as the set of all rays in \mathbb{R}^d . As a ray γ we understand a straight line with fixed orientation. If $\gamma = (x, \theta) \in T\mathbb{S}^{d-1}$, then $\gamma = \{y \in \mathbb{R}^d : y = x + t\theta, t \in \mathbb{R}\}$ (up to orientation) and θ gives the orientation of γ .

We say that S is the non-abelian Radon transform along oriented straight lines (or the non-abelian X-ray transform) of A .

We consider the following inverse problem for $d \geq 2$:

Problem 1. Given S , find A .

Note that S does not determine A uniquely, in general. One of the reasons is that S is a function on $T\mathbb{S}^{d-1}$, whereas A is a function on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ and

$$\dim \mathbb{R}^d \times \mathbb{S}^{d-1} = 2d - 1 > \dim T\mathbb{S}^{d-1} = 2d - 2.$$

In particular, for Problem 1 there is a gauge type non-uniqueness, that is S is invariant with respect to the gauge transforms

$$\begin{aligned} A &\rightarrow A', \\ A'(x, \theta) &= g^{-1}(x, \theta)A(x, \theta)g(x, \theta) + g^{-1}(x, \theta)\theta\partial_x g(x, \theta), \end{aligned} \tag{5}$$

where g is a sufficiently regular $GL(n, \mathbb{C})$ -valued function on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ and $g \rightarrow I$ sufficiently fast as $|x| \rightarrow \infty$.

In addition, in particular, for Problem 1 there are Boman type non-uniqueness (see [Bo], [GN]) and non-uniqueness related with solitons (see [N1]).

Equation (1), the "scattering" matrix S and Problem 1 arise, for example, in different tomographies (see Sections 2-6, 8), in differential geometry (see Section 7) and in the theory of the Yang-Mills fields (see Section 9). In Sections 2-9 we give a short review of old and recent results on this subject.

2. Classical X-ray transmission tomography

Problem 1 arises as a problem of the classical X-ray transmission tomography in the framework of the following reduction:

$$n = 1, \quad A(x, \theta) = a(x), \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \tag{6}$$

$$S(\gamma) = \exp[-Pa(\gamma)], \quad Pa(\gamma) = \int_{\mathbb{R}} a(x + s\theta)ds, \quad \gamma = (x, \theta) \in T\mathbb{S}^{d-1}, \tag{7}$$

where a is the X-ray attenuation coefficient of the medium, P is the classical Radon transform along straight lines (classical X-ray transform), $S(\gamma)$ describes the X-ray photograph along γ .

In this case, for $d \geq 2$,

$$S|_{TS^1(Y)} \text{ uniquely determines } a|_Y, \tag{8}$$

where Y is an arbitrary two-dimensional plane in \mathbb{R}^d , $TS^1(Y)$ is the set of all oriented straight lines in Y . In addition, this determination can be implemented via the Radon inversion formula for P in dimension $d = 2$; see [R]. In connection with this formula see also Remark 2 in Subsection 2.2.

For more information on the classical X-ray transmission tomography and on the classical X-ray transform, see, e.g., [GGG], [Na] and references therein.

3. Single-photon emission computed tomography (SPECT)

In SPECT one considers a body containing radioactive isotopes emitting photons. The emission data p in SPECT consist in the radiation measured outside the body by a family

of detectors during some fixed time (where expected p is described by $P_a f$ defined below). The basic problem of SPECT consists in finding the distribution f of these isotopes in the body from the emission data p and some a priori information concerning the body. Usually this a priori information consists in the photon attenuation coefficient a in the points of body, where this coefficient is found in advance by the methods of the classical X-ray transmission tomography (mentioned in Section 2).

Problem 1 arises as a problem of SPECT in the framework of the following reduction [N1]:

$$n = 2, \quad A_{11} = a(x), \quad A_{12} = f(x), \quad A_{21} = 0, \quad A_{22} = 0, \quad x \in \mathbb{R}^d, \quad (9)$$

$$S_{11} = \exp[-P_0 a], \quad S_{12} = -P_a f, \quad S_{21} = 0, \quad S_{22} = 1, \quad (10)$$

where a is the photon attenuation coefficient of the medium, f is the density of radioactive isotopes, $P_0 = P$ is defined in (7), P_a is the attenuated Radon transform along oriented straight lines (attenuated ray transform), $P_a f$ describes the expected emission data,

$$P_a f(\gamma) = \int_{\mathbb{R}} \exp[-Da(x + s\theta, \theta)] f(x + s\theta) ds, \quad \gamma = (x, \theta) \in TS^{d-1}, \quad (11)$$

$$Da(x, \theta) = \int_0^{+\infty} a(x + s\theta) ds, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \quad (12)$$

where D is the divergent beam transform.

In this case (as well as for the case of the classical X-ray transmission tomography), for $d \geq 2$,

$$S|_{TS^1(Y)} \text{ uniquely determines } a|_Y \text{ and } f|_Y, \quad (13)$$

where Y is an arbitrary two-dimensional plane in \mathbb{R}^d , $TS^1(Y)$ is the set of all oriented straight lines in Y . In addition, this determination can be implemented via the following inversion formula [N2]:

$$f = P_a^{-1} g, \quad \text{where } g = P_a f, \quad (14)$$

$$P_a^{-1} g(x) = \frac{1}{4\pi} \int_{\mathbb{S}} \theta^\perp \partial_x (\exp[-Da(x, -\theta)] \tilde{g}_\theta(\theta^\perp x)) d\theta, \quad (15)$$

$$\tilde{g}_\theta(s) = \exp(A_\theta(s)) \cos(B_\theta(s)) H(\exp(A_\theta) \cos(B_\theta) g_\theta)(s) + \exp(A_\theta(s)) \sin(B_\theta(s)) H(\exp(A_\theta) \sin(B_\theta) g_\theta)(s),$$

$$A_\theta(s) = (1/2) P_0 a(s\theta^\perp, \theta), \quad B_\theta(s) = H A_\theta(s), \quad g_\theta(s) = g(s\theta^\perp, \theta),$$

$$Hu(s) = \frac{1}{\pi} p.v. \int_{\mathbb{R}} \frac{u(t)}{s-t} dt, \quad (16)$$

$$x \in \mathbb{R}^2, \quad \theta^\perp = (-\theta_2, \theta_1) \text{ for } \theta = (\theta_1, \theta_2) \in \mathbb{S}^1, \quad s \in \mathbb{R}.$$

Remark 1. The assumptions on a and f in (13)-(15) can be specified as follows:

$$\begin{aligned} a, f & \text{ are real - valued, } a, f \in L^\infty(\mathbb{R}^2), \\ a, f & = O(|x|^{-\sigma}) \text{ as } |x| \rightarrow \infty \text{ for some } \sigma > 1, \end{aligned} \quad (17)$$

where Y is identified with \mathbb{R}^2 in (13).

Remark 2. For $a \equiv 0$, formulas (14), (15) are reduced to the classical Radon inversion formula for P defined in (7) for $d = 2$.

For more information on SPECT and for more results on the attenuated ray transform P_a we refer to [Na], [Ku1], [Ku2], [N2], [N3], [GuN1], [GuN2] and references therein.

4. Tomographies related with weighted Radon transforms

We consider the weighted Radon transforms P_W (along oriented straight lines) defined by the formula

$$P_W f(x, \theta) = \int_{\mathbb{R}} W(x + s\theta, \theta) f(x + s\theta) ds, \quad (x, \theta) \in T\mathbb{S}^{d-1}, \quad (18)$$

where $W = W(x, \theta)$ is the weight, $f = f(x)$ is a test function. The assumptions on W can be specified as follows:

$$\begin{aligned} W & \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1}), \\ W & = \bar{W}, \quad 0 < c_0 \leq W \leq c_1, \\ \lim_{s \rightarrow \pm\infty} W(x + s\theta, \theta) & = w_\pm(x, \theta), \quad (x, \theta) \in T\mathbb{S}^{d-1}. \end{aligned} \quad (19)$$

If $W = 1$, then P_W is reduced to the classical X-ray transform P defined in (7).

If

$$W(x, \theta) = \exp(-Da(x, \theta)), \quad (20)$$

where Da is defined by (12), then P_W is reduced to the classical attenuated ray transform P_a defined by (11), (12).

Transforms P_W with some other weights also arise in applications. For example, such transforms arise in positron emission tomography, optical tomography, fluorescence tomography; see [Na], [Ba], [MP].

The transforms $P_W f$ arise in the framework of the following reduction of the non-abelian Radon transform S :

$$n = 2, \quad A_{11} = \theta \partial_x \ln W(x, \theta), \quad A_{12} = f(x), \quad A_{21} = 0, \quad A_{22} = 0, \quad (21)$$

$$S_{11} = \frac{w_-}{w_+}, \quad S_{12} = -\frac{1}{w_+} P_W f, \quad S_{21} = 0, \quad S_{22} = 1. \quad (22)$$

In connection with P_W and with the reduction (21), (22) we consider the following version of Problem 1, where we assume that W is known.

Problem 2. Given $P_W f$ and W , find f .

General uniqueness and reconstruction results on Problem 2 were given, in particular, in [LB], [Be], [MQ], [F], [BQ], [Ku1], [N7], [GuN2], [I].

For some W exact and simultaneously explicit formulas for solving Problem 2 are also known, see [R], [N1], [BS], [Gi], [N6] and references therein.

Note that Problem 2 is nonoverdetermined for $d = 2$ and is overdetermined for $d \geq 3$. Indeed, $P_W f$ is a function on $T\mathbb{S}^{d-1}$, whereas f is a function on \mathbb{R}^d and

$$\begin{aligned} \dim T\mathbb{S}^{d-1} &= 2d - 2, \quad \dim \mathbb{R}^d = d, \\ 2d - 2 &= d \quad \text{for } d = 2, \quad 2d - 2 > d \quad \text{for } d \geq 3. \end{aligned}$$

Nevertheless, Problem 2 is not uniquely solvable, in general, even for $d \geq 3$.

An example of non-uniqueness for Problem 2 for $d = 2$ was constructed in [Bo]. In this example $W \in C^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$, $f \in C_0^\infty(\mathbb{R}^2)$.

An example of non-uniqueness for Problem 2 for $d \geq 3$ was constructed in [GN]. In this example $W \in C^\alpha(\mathbb{R}^d \times \mathbb{S}^{d-1})$ for some $\alpha > 0$, $f \in C_0^\infty(\mathbb{R}^d)$.

In these examples assumptions (19) are also fulfilled. The notation C_0^∞ stands for infinitely smooth compactly supported functions.

For more information on the theory and applications of the transforms P_W we refer to [LB], [Be], [MQ], [F], [Na], [BQ], [Bo], [Ku1], [N7], [GuN2], [I], [GN] and references therein.

5. Neutron polarization tomography (NPT)

In NPT one considers a medium with spatially varying magnetic field. The polarization data consist in changes of the polarization (spin) between incoming and outgoing neutrons. The basic problem of NPT consists in finding the magnetic field from the polarization data. See, e.g., [DMKHSB], [LDS] and references therein.

Problem 1 arises as a problem of NPT in the framework of the following reduction:

$$\begin{aligned} n &= 3, \quad A_{11} = A_{22} = A_{33} = 0, \\ A_{12} &= -A_{21} = -g B_3(x), \quad A_{13} = -A_{31} = g B_2(x), \\ A_{23} &= -A_{32} = -g B_1(x), \end{aligned} \tag{23}$$

where $B = (B_1, B_2, B_3)$ is the magnetic field, g is the gyromagnetic ratio of the neutron; in addition, S for equation (1) with A given by (23) describes the polarization data (but, in general, S can not be given explicitly in this case).

In this case S on $T\mathbb{S}^2$ uniquely determines B on \mathbb{R}^3 as a corollary of items (1), (2) of Theorem 6.1 of [N1]. In addition, the related 3D - reconstruction is based on local 2D - reconstructions based on solving Riemann conjugation problems (going back to [MZ]) and on the layer by layer reconstruction approach. The final 3D uniqueness and reconstruction results are global.

For the related 2D global uniqueness, see [E].

6. Electromagnetic polarization tomography (EPT)

In EPT one considers a medium with zero conductivity, unit magnetic permeability, and small anisotropic perturbation of some known (for example, uniform) dielectric permeability. The polarization data consist in changes of the polarization between incoming

and outcoming monochromatic electromagnetic waves. The basic problem of EPT consists in finding the anisotropic perturbation of the dielectric permeability from the polarization data. See [Sh1], [NS], [Sh4], [N5] and references therein.

Problem 1 arises as a problem of EPT (with uniform background dielectric permeability) in the framework of the following reduction (see [Sh1], [NS]):

$$n = 3, \quad A(x, \theta) = -\pi_\theta f(x) \pi_\theta, \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \quad (24)$$

where $\pi_\theta \in M(3, \mathbb{R})$, $\pi_{\theta,ij} = \delta_{ij} - \theta_i \theta_j$, f takes values in $M(3, \mathbb{C})$ and describes the anisotropic perturbation of the dielectric permeability tensor; by some physical arguments f must be skew-Hermitian, $f_{ij} = -\bar{f}_{ji}$; in addition, S for equation (1) with A given by (24) describes the polarization data (but, in general, S can not be given explicitly in this case).

In this case S on TS^2 does not determine f on \mathbb{R}^3 uniquely, in general, (in spite of the fact that $\dim TS^2 = 4 > \dim \mathbb{R}^3 = 3$), in particular, if

$$f_{11} = f_{22} = f_{33} \equiv 0, \quad (25)$$

$$\begin{aligned} f_{12}(x) &= \partial u(x) / \partial x_3, \quad f_{13}(x) = -\partial u(x) / \partial x_2, \quad f_{23}(x) = \partial u(x) / \partial x_1, \\ f_{21} &= -f_{12}, \quad f_{31} = -f_{13}, \quad f_{32} = -f_{23}, \end{aligned}$$

where u is a real smooth compactly supported function, then $S \equiv I$ on TS^2 ; see [NS].

On the other hand, a very natural additional physical assumption is that f is an imaginary-valued symmetric matrix: $f = -\bar{f}$, $f_{ij} = f_{ji}$. According to [N4], in this case

$$S \text{ on } \Lambda \text{ uniquely determines } f, \text{ at least, if } f \text{ is sufficiently small,} \quad (26)$$

where Λ is an appropriate $3d$ subset of TS^2 , for example,

$$\Lambda = \cup_{i=1}^6 \Gamma_{\omega^i}, \quad \Gamma_{\omega^i} = \{\gamma = (x, \theta) \in TS^2 : \theta \omega^i = 0\}, \quad (27)$$

$$\omega^1 = e_1, \quad \omega^2 = e_2, \quad \omega^3 = e_3,$$

$$\omega^4 = (e_1 + e_2) / \sqrt{2}, \quad \omega^5 = (e_1 + e_3) / \sqrt{2}, \quad \omega^6 = (e_2 + e_3) / \sqrt{2},$$

where e_1, e_2, e_3 is the basis in \mathbb{R}^3 . In addition, this determination is based on a convergent iterative reconstruction algorithm.

For more information on EPT and for more results on related non-abelian ray transforms we refer to [Sh1], [NS], [Sh4], [N5] and references therein.

7. Inverse connection problem

Let

$$A(x, \theta) = a_0(x) + \sum_{j=1}^d \theta_j a_j(x), \quad x \in \mathbb{R}^d, \quad \theta = (\theta_1, \dots, \theta_d) \in \mathbb{S}^{d-1}, \quad (28)$$

where a_j are sufficiently regular $M(n, \mathbb{C})$ -valued functions on \mathbb{R}^d with sufficient decay as $|x| \rightarrow \infty$, $j = 0, 1, \dots, d$. Then Problem 1 arises in differential geometry.

In particular, for $a_0 \equiv 0$ equation (1) with A given by (28) describes the parallel transport of the fibre in the trivial vector bundle with the base \mathbb{R}^d and the fibre \mathbb{C}^n and with the connection $a = (a_1, \dots, a_d)$ along the Euclidean geodesics in \mathbb{R}^d ; in addition, $S(\gamma)$ for fixed $\gamma \in T\mathbb{S}^{d-1}$ is the operator of this parallel transport along γ (from $-\infty$ to $+\infty$ on γ); see [Sh2], [N1].

Besides, for $a_0 \not\equiv 0$ equation (1) with A given by (28) describes the parallel transport of the fibre in the trivial vector bundle with the base $\mathbb{R}_{1,d}^{d+1}$ and the fibre \mathbb{C}^n and with the connection $a = (a_0, a_1, \dots, a_d)$ (independent of time) along the light rays in the Minkowski space $\mathbb{R}_{1,d}^{d+1}$; in addition, $S(\gamma)$ for fixed $\gamma = (x, \theta) \in T\mathbb{S}^{d-1}$ is the operator of this parallel transport along the light rays

$$l(\gamma, \tau) = \{(t, y) \in \mathbb{R}^{d+1} : t = 2^{-1/2}s + \tau, y = 2^{-1/2}s\theta + x, s \in \mathbb{R}\}, \quad \tau \in \mathbb{R},$$

with the orientation given by the vector $2^{-1/2}(1, \theta)$ (from $-\infty$ to $+\infty$ on $l(\gamma, \tau)$ for an arbitrary $\tau \in \mathbb{R}$); see [N1].

In these cases Problem 1 is an inverse connection problem. The determination in this problem is considered modulo gauge transforms

$$\begin{aligned} a &= (a_0, a_1, \dots, a_d) \rightarrow a' = (a'_0, a'_1, \dots, a'_d), \\ a'_0 &= g^{-1}a_0g, \quad a'_i = g^{-1}a_i g + g^{-1}\partial_i g, \quad \partial_i g(x) = \frac{\partial g(x)}{\partial x_i}, \quad i = 1, \dots, d, \end{aligned} \quad (29)$$

where g is a sufficiently regular $GL(n, \mathbb{C})$ -valued function on \mathbb{R}^d and $g \rightarrow I$ sufficiently fast as $|x| \rightarrow \infty$.

Global uniqueness and reconstruction results on this inverse connection problem in dimension $d \geq 3$ were given for the first time in [N1]. The related reconstruction is based on local $2D$ -reconstructions based on solving Riemann conjugation problems (going back to [MZ]) and on the layer by layer reconstruction approach.

In addition, counter examples to the global uniqueness for the aforementioned inverse connection problem for $a_0 \equiv 0$ in dimension $d = 2$ were also given for the first time in [N1]. These counter examples use the soliton solutions constructed in [Wa], [V] for equation (38) mentioned below.

In addition, for the global uniqueness in dimension $d = 2$ for the case of compactly supported $a = (a_0, a_1, \dots, a_d)$, see [E].

Note that [N1] was stimulated by [Sh2], where [Sh2] was preceded by [We].

For more information on the inverse connection problem we refer to [MZ], [Sh2], [N1], [E], [N4], [P], [GPSU] and references therein.

8. Vector field tomography

The inverse connection problem of Section 7 arises as a problem of the vector ultrasonic tomography in the framework of the following reduction:

$$n = 2, \quad a_0 = \begin{pmatrix} a(x) & 0 \\ 0 & 0 \end{pmatrix}, \quad a_j = \begin{pmatrix} 0 & u_j(x) \\ 0 & 0 \end{pmatrix}, \quad x \in \mathbb{R}^d, \quad j = 1, \dots, d, \quad (30)$$

$$S_{11} = \exp[-P_0 a], \quad S_{12} = \exp[-P_a u], \quad S_{21} = 0, \quad S_{22} = 1, \quad (31)$$

where $P_0 a$ is defined as in (7), (10),

$$P_a u(\gamma) = \int_{\mathbb{R}} \exp[-Da(x + s\theta, \theta)] \theta u(x + s\theta) ds, \quad \gamma = (x, \theta) \in T\mathbb{S}^{d-1}, \quad (32)$$

$\theta u = \sum_{j=1}^d \theta_j u_j$, Da is defined as in (12), a is the attenuation coefficient, $u = (u_1, \dots, u_d)$ is the flow velocity, $P_a u$ is the attenuated vectorial Radon transform of u along oriented straight lines.

The transform $P_a u$ for $a = 0$ is the standard vectorial Radon transform of u and is related to time-of-flight measurements or to Doppler measurements; $P_a u$ for $a \neq 0$ is related to the attenuated Doppler measurements; see [Sch] and references therein.

In connection with mathematics of vector field tomography we refer to [GGG], [Sh1], [N1], [Sh3], [KB], [Sch] and references therein.

9. Theory of the Yang-Mills fields

A. The inverse connection problem of Section 7 for $a_0 \equiv 0$ arises, in particular, in the framework of studies on inverse problems for the Schrödinger equation

$$\sum_{j=1}^d -\left(\frac{\partial}{\partial x_j} + a_j(x)\right)^2 \psi + v(x)\psi = E\psi \quad (33)$$

in the Yang-Mills field $a = (a_1, \dots, a_d)$ at high energies E (i.e., for $E \rightarrow +\infty$); see [N1] and references therein. The reason of this consists in the fact that for ψ of the form

$$\psi = e^{is\theta x} (\mu_0(x, \theta) + O(s^{-1})), \quad x \in \mathbb{R}^d, \quad \theta \in \mathbb{S}^{d-1}, \quad s = \sqrt{E} \rightarrow +\infty, \quad (34)$$

equation (33) in its leading part is reduced to equation (1) with μ_0 in place of ψ , where A is given by (28) with $a_0 \equiv 0$.

B. The inverse connection problem of Section 7 for $d = 2$ arises, in particular, in the framework of integrating the self-dual Yang-Mills equations; see [MZ], [Wa], [V], [N1] and references therein.

Actually, Problem 1 for

$$A(x, \theta) = a_0(x) + \theta_1 a_1(x) + \theta_2 a_2(x), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad \theta = (\theta_1, \theta_2) \in \mathbb{S}^1, \quad (35)$$

with $M(n, \mathbb{C})$ -valued a_0, a_1, a_2 (and some linear relation between a_1 and a_2) was considered for the first time in [MZ] in the framework of integration by the inverse scattering method of the evolution equation

$$(\chi^{-1} \chi_t)_t = (\chi^{-1} \chi_z)_{\bar{z}}, \quad (36)$$

where t, z, \bar{z} in (36) denote partial derivatives with respect to $t, z = x_1 + ix_2, \bar{z} = x_1 - ix_2$ and where χ is $SU(n)$ -valued function. Equation (36) is a (2+1)-dimensional reduction of the self-dual Yang-Mills equations in 2+2 dimensions.

To our knowledge, the terminology "non-abelian Radon transform" was introduced namely in [MZ] where it was used for S in (2) corresponding to the aforementioned A of (35).

The inverse scattering transform in [MZ] is based on Riemann conjugation problems. Related analysis was significantly developed, in particular, in [N1].

In addition, Problem 1 for

$$A(x, \theta) = \theta_2 a_2(x), \quad x = (x_1, x_2) \in \mathbb{R}^2, \quad \theta = (\theta_1, \theta_2) \in \mathbb{S}^1, \quad (37)$$

with $M(n, \mathbb{C})$ -valued a_2 arises in the framework of the inverse scattering method for the equation

$$(J^{-1} J_{x_1})_{x_1} - (J^{-1} J_{x_2})_t = 0, \quad (38)$$

where t, x_1, x_2 in (38) denote partial derivatives with respect to t, x_1, x_2 and where J is $SU(n)$ -valued function; see [Wa], [V], [N1], at least, for $n = 2$. Equation (38) is also a (2+1)-dimensional reduction of the self-dual Yang-Mills equations in 2+2 dimensions. This reduction is different from (36).

The aforementioned counter examples to the global uniqueness for the inverse connection problem of Section 7 for $a_0 \equiv 0$ in dimension $d = 2$ were constructed in [N1] using results of [Wa] and subsequent results of [V] concerning soliton solutions for equation (38).

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