

Extremes of independent stochastic processes: a point process approach

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Abstract For each $n \geq 1$, let $\{X_{in}, i \geq 1\}$ be independent copies of a nonnegative continuous stochastic process $X_n = (X_n(s))_{s \in S}$ indexed by a compact metric space S . We are interested in the process of partial maxima $\tilde{M}_n(t, s) = \max\{X_{in}(s), 1 \leq i \leq [nt]\}$, $t \geq 0, s \in S$, where the brackets $[\cdot]$ denote the integer part. Under a regular variation condition on the sequence of processes X_n , we prove that the partial maxima process \tilde{M}_n weakly converges to a superextremal process \tilde{M} as $n \rightarrow \infty$. We use a point process approach based on the convergence of empirical measures. Properties of the limit process are investigated: we characterize its finite-dimensional distributions, prove that it satisfies an homogeneous Markov property, and show in some cases that it is max-stable and self-similar. Convergence of further order statistics is also considered. We illustrate our results on the class of log-normal processes in connection with some recent results on the extremes of Gaussian processes established by Kabluchko.

Keywords Extreme value theory · Partial maxima process · Superextremal process · Functional regular variations · Weak convergence

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1 Introduction

A classical problem in extreme value theory (EVT) is to determine the asymptotic behavior of the maximum of independent and identically distributed (i.i.d.) random variables $(Z_i)_{i \geq 1}$. What are the assumptions that ensure the weak convergence of the rescaled maximum

$$\max_{1 \leq i \leq n} \frac{Z_i - b_n}{a_n}, \quad a_n > 0, b_n \in \mathbb{R},$$

and what are the possible limit distributions? These questions were at the basis of the development of EVT and found their answers in the Theorem by Fisher and Tippett (1928) characterizing all the max-stable distributions and in the description of their domain of attraction by Gnedenko (1943) and de Haan (1970). Another stimulating point of view developed by Lamperti (1964) is to introduce a time variable and consider the asymptotic behavior of the partial maxima

$$\max_{1 \leq i \leq [nt]} \frac{Z_i - b_n}{a_n}, \quad t \geq 0.$$

The corresponding limit process is known as an extremal process: it is a pure jump Markov process which is also max-stable, see Resnick (1975).

Since then, EVT has known many developments. Among several other directions, the extension of the theory to multivariate and spatial settings is particularly important, as well as the statistical issues raised by the applications on real data sets. For excellent reviews of such developments, the reader is invited to refer to the monographies by Resnick (2008), de Haan and Ferreira (2006) or Beirlant et al. (2004) and the references therein.

Our purpose here is to focus on the functional framework and investigate the asymptotic behavior of the partial maxima processes based on a doubly infinite array of independent random processes. Let S be a compact metric space and, for each $n \geq 1$, let $\{X_{in}, i \geq 1\}$ be independent copies of a sample continuous stochastic process $(X_n(s))_{s \in S}$. Without loss of generality, we will always suppose that X_n is non-negative (otherwise consider $X'_n(s) = e^{X_n(s)}$) and we denote by $\mathbb{C}^+ = \mathbb{C}^+(S)$ be the set of non-negative continuous functions on S . We are mainly interested in the process of pointwise maxima

$$M_n(s) = \max\{X_{in}(s), 1 \leq i \leq n\}, \quad s \in S, \tag{1}$$

and the process of partial maxima

$$\tilde{M}_n(t, s) = \max\{X_{in}(s), 1 \leq i \leq [nt]\}, \quad t \geq 0, s \in S. \tag{2}$$

We use the convention that the maximum over an empty set is equal to 0, so $\tilde{M}_n(0, s) \equiv 0$. Clearly, we also have $\tilde{M}_n(1, s) = M_n(s)$. In this framework, the parameter $s \in S$ is thought as a space parameter and $t \in [0, +\infty)$ as a time variable.

Our approach relies on the convergence of the following empirical measures

$$\beta_n = \sum_{i=1}^n \delta_{X_{in}} \quad \text{and} \quad \tilde{\beta}_n = \sum_{i \geq 1} \delta_{(X_{in}, i/n)} \tag{3}$$

on \mathbb{C}^+ and $\mathbb{C}^+ \times [0, +\infty)$ respectively, where δ_a denotes the Dirac measure at point a . The maxima processes M_n and \tilde{M}_n can be written as functionals of the empirical measure β_n and $\tilde{\beta}_n$ respectively. We will prove the continuity of the underlying functionals and then use the Continuous Mapping Theorem to deduce convergence of the maxima processes from convergence of empirical measures.

Connections between EVT and point processes are well known. In the case when the underlying state space is locally compact, the following result holds (see Proposition 3.21 in Resnick (2008)): if $n\mathbb{P}[X_n \in \cdot]$ vaguely converges to some measure μ , the empirical measures β_n and $\tilde{\beta}_n$ converge to Poisson random measures with intensity μ and $\mu \otimes \ell$ respectively, ℓ being the Lebesgue measure on $[0, +\infty)$. However, in our framework the state space \mathbb{C}^+ is not locally compact and we need a suitable generalization of the above result. To this aim, we follow the approach by Davis and Mikosch (2008) based on the notion of boundedly finite measures and \sharp -weak convergence detailed by Daley and Vere-Jones in Daley and Vere-Jones (2003).

The paper is organized as follows. In Section 2, we introduce the technical material needed on boundedly finite measures, \sharp -weak convergence and convergence of empirical measures. Our main result is the convergence of the partial maxima process \tilde{M}_n which is stated and proved in Section 3. Then, in Section 4, we investigate some properties of the limit process \tilde{M} , known as a superextremal process. A brief extension of our results to further order statistics is considered in Section 5. The last section is devoted to an application of our results to the class of log-normal processes, based on a recent work by Kabluchko (2011) on the extremes of Gaussian processes.

2 Preliminaries on boundedly finite measures and point processes

In this section, we present general results on boundedly finite measures and point processes that will be useful in the sequel. The reader should refer to Appendix 2.6 in Daley and Vere-Jones (2003) or to Section 2 of Davis and Mikosch (2008). This has also close connections with the theory of regular variation, see Hult and Lindskog (2005) and Hult and Lindskog (2006).

2.1 Boundedly finite measures

Let (E, d) be a complete separable metric space and \mathcal{E} the Borel σ -algebra of E . We denote by $M_b(E)$ the set of all finite measures on (E, \mathcal{E}) . A sequence of finite measures $\{\mu_n, n \geq 1\}$ is said to converge weakly to $\mu \in M_b(E)$ if and only if $\int f d\mu_n \rightarrow \int f d\mu$ for all bounded continuous function f on E . We write $\mu_n \xrightarrow{w} \mu$ to denote weak convergence. It is well known that this notion of convergence is metrized by the Prokhorov metric

$$p(\mu_1, \mu_2) = \inf\{\varepsilon > 0; \forall A \in \mathcal{E}, \mu_1(A) \leq \mu_2(A^\varepsilon) + \varepsilon \text{ and } \mu_2(A) \leq \mu_1(A^\varepsilon) + \varepsilon\}$$

where $A^\varepsilon = \{x \in E; \exists a \in A, d(a, x) < \varepsilon\}$ is the ε -neighborhood of A . Endowed with this metric, $M_b(E)$ is a complete separable metric space.

A measure μ on (E, \mathcal{E}) is called boundedly finite if it assigns finite measure to bounded sets, i.e. $\mu(B) < \infty$ for all bounded $B \in \mathcal{E}$. Let $M_b^\sharp(E)$ be the set of all boundedly finite measures μ on (E, \mathcal{E}) . A sequence of boundedly finite measures $(\mu_n)_{n \geq 1}$ is said to converge \sharp -weakly to $\mu \in M_b^\sharp(E)$ if and only if $\int f d\mu_n \rightarrow \int f d\mu$ for all bounded continuous function f on E with bounded support. There exists a metric p^\sharp on $M_b^\sharp(E)$ that makes $M_b^\sharp(E)$ a complete and separable metric space and such that convergence in the metric sense coincides with \sharp -weak convergence. Such a metric can be constructed as follows. Fix an origin $e_0 \in E$ and, for $r > 0$, let $\bar{B}_r = \{x \in E; d(x, e_0) \leq r\}$ be the closed ball of center e_0 and radius r . For any $\mu_1, \mu_2 \in M_b^\sharp(E)$, let $\mu_1^{(r)}$ and $\mu_2^{(r)}$ be the restriction of μ_1 and μ_2 to \bar{B}_r . Note that μ_1 and μ_2 are finite measures on \bar{B}_r and denote by p_r the Prokhorov metric on $M_b(\bar{B}_r)$. Define

$$p^\sharp(\mu_1, \mu_2) = \int_0^\infty e^{-r} \frac{p_r(\mu_1^{(r)}, \mu_2^{(r)})}{1 + p_r(\mu_1^{(r)}, \mu_2^{(r)})} dr.$$

There are several equivalent characterizations of \sharp -weak convergence, see Daley and Vere-Jones (2003) Proposition A2.6.II. Let $(\mu_n)_{n \geq 1}$ and μ be boundedly finite measures on E , the following statements are equivalent:

- i) $\int f d\mu_n \rightarrow \int f d\mu$ for all f bounded continuous real valued function on E with bounded support;
- ii) $p^\sharp(\mu_n, \mu) \rightarrow 0$;
- iii) there exist a sequence $r_k \nearrow +\infty$ such that, for all $k \geq 1$, $\mu_n^{(r_k)} \xrightarrow{w} \mu^{(r_k)}$;
- iv) $\mu_n(B) \rightarrow \mu(B)$ for all bounded $B \in \mathcal{E}$ such that $\mu(\partial B) = 0$.

We write $\mu_n \xrightarrow{w^\sharp} \mu$ to denote \sharp -weak convergence.

A boundedly finite point measure on E is a measure μ of the form

$$\mu = \sum_{i \in I} \delta_{x_i}$$

with $\{x_i, i \in I\}$ a finite or countable family of points in E such that any bounded set $B \subseteq E$ contains at most a finite number of the x_i 's. The set of boundedly finite point measures on E is denoted by $M_{(b,p)}^\sharp(E)$. It is a closed subset of $M_b^\sharp(E)$ and hence is complete and separable when endowed with the induced metric (see Daley and Vere-Jones (2008) Proposition 9.1.IV and Lemma 9.1.V).

The following characterization of \sharp -weak convergence of boundedly finite point measures will be useful.

Lemma 2.1 *Let $\{\mu_n = \sum_i \delta_{x_i^n}, n \geq 1\}$ and $\mu = \sum_i \delta_{x_i}$ be elements of $M_{(b,p)}^\sharp(E)$.*

Then $\mu_n \xrightarrow{w^\sharp} \mu$ if and only if there exist some sequence $r_k \nearrow +\infty$ such that for all $k \geq 1$, there exist $N \geq 0$ and $m \geq 0$ such that

$$\forall n \geq N, \quad \mu_n^{(r_k)} = \sum_{i=1}^m \delta_{x_i^{n,k}} \quad \text{and} \quad \mu^{(r_k)} = \sum_{i=1}^m \delta_{x_i^k}$$

for some $\{x_i^{n,k}, x_i^k; 1 \leq i \leq m, n \geq N\}$ in \bar{B}_{r_k} such that $x_i^{n,k} \rightarrow x_i^k$, where $\mu_n^{(r_k)}$ and $\mu^{(r_k)}$ are the restrictions of μ_n and μ to \bar{B}_{r_k} and the $x_i^{n,k}$'s and x_i^k 's are suitable relabellings of the points falling in \bar{B}_{r_k} .

This Lemma is in fact an extension of Proposition 3.13 in Resnick (2008) where vague convergence of locally finite point measures on a locally compact metric space is considered. Extension of the proof in the setting of boundedly finite point measures on a complete and separable metric space is straightforward.

2.2 Convergence of the empirical measures

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A boundedly finite point process on E is a measurable mapping $N : \Omega \rightarrow M_{(b,p)}^\sharp(E)$. A typical example of boundedly finite point process on E is a Poisson point process Π_ν with intensity $\nu \in M_b^\sharp(E)$. We will also consider the following empirical point processes β_n and $\tilde{\beta}_n$ defined by Eq. 3, where, for each $n \geq 1$, $\{X_{in}, i \geq 1\}$ are independent copies of an E -valued random variable X_n . The random variable β_n is a finite point process on E , while $\tilde{\beta}_n$ is a boundedly finite point process on $\tilde{E} = E \times [0, +\infty)$ endowed with the metric

$$\tilde{d}((x_1, u_1), (x_2, u_2)) = d(x_1, x_2) + |u_2 - u_1|, \quad x_1, x_2 \in E, \quad u_1, u_2 \in [0, +\infty).$$

The following Proposition will play a key role in the sequel.

Proposition 2.2 *The following statements are equivalent, where the limits are taken as $n \rightarrow \infty$:*

- i) $n\mathbb{P}[X_n \in \cdot] \xrightarrow{w^\sharp} \nu$;
- ii) $\beta_n \xrightarrow{d} \Pi_\nu$ with Π_ν a Poisson point process on E with intensity ν and \xrightarrow{d} standing for convergence in distribution in $M_{(b,p)}^\sharp(E)$.
- iii) $\tilde{\beta}_n \Rightarrow \tilde{\Pi}_\nu$ with $\tilde{\Pi}_\nu$ a Poisson point process on $E \times [0, +\infty)$ with intensity $\nu \otimes \ell$, ℓ being the Lebesgue measure on $[0, +\infty)$, and \xrightarrow{d} standing for convergence in distribution in $M_{(b,p)}^\sharp(E \times [0, +\infty))$.

The proof of this Proposition follows by an adaptation of Proposition 3.21 in Resnick (2008) which states a similar result in the case of a locally compact state space and in terms of vague convergence. As noticed by Davis and Mikosch (2008), the proof remains valid for a complete separable metric space E if we change vague convergence by \sharp -weak convergence. See also Theorem 4.3 in Davydov et al. (2008).

In the terminology of Hult and Lindskog (2005) and Hult and Lindskog (2006), the condition i) in Proposition 2.2 means that the sequence X_n is *regularly varying*. A particularly important case is when E is endowed with a structure of cone, i.e. a multiplication by positive scalars. If there exists a sequence $(a_n)_{n \geq 1}$ of positive reals such that

$$n\mathbb{P}[a_n^{-1} X \in \cdot] \xrightarrow{w^\sharp} \nu$$

for some nonzero $\nu \in M_b^\#(E)$, then the random variable X is said to be *regularly varying*. Under some technical assumptions on the structure of the convex cone E (e.g. continuity properties of the multiplication), the limit measure ν is proved to be homogeneous of order $-\alpha < 0$, i.e.

$$\nu(\lambda A) = \lambda^{-\alpha} \nu(A), \quad \lambda > 0, A \in \mathcal{E}.$$

Furthermore, the sequence a_n is regularly varying with index $1/\alpha$. In this framework, if $\{X_i, i \geq 1\}$ are i.i.d. copies of a regularly varying random variables X , then the triangular array $X_{in} = a_n^{-1} X_i$ satisfies condition i) since

$$n\mathbb{P}[X_{1n} \in \cdot] = n\mathbb{P}[a_n^{-1} X \in \cdot] \xrightarrow{w^\#} \nu.$$

3 Extremes of independent stochastic processes

We go back to our original problem where the X_{in} 's are non negative continuous processes on S . We endow the set $\mathbb{C}^+ = \mathbb{C}^+(S)$ of non negative continuous functions on S with the uniform norm $\|x\| = \sup_{s \in S} |x(s)|$, $x \in \mathbb{C}^+$. Recall that S is assumed to be a compact metric space and hence $(\mathbb{C}^+, \|\cdot\|)$ is a complete separable metric space. It turns out that, when working with maxima, the uniform metric is not adapted, mainly because sets such as $\{x \in \mathbb{C}^+; \|x\| \geq \varepsilon\}$ are not bounded for this metric. For this reason, following the seminal work of de Haan and Lin (2001), we introduce $\overline{\mathbb{C}}_0^+ = (0, +\infty] \times \mathbb{S}_{\mathbb{C}^+}$ where $\mathbb{S}_{\mathbb{C}^+} = \{x \in \mathbb{C}^+ : \|x\|_\infty = 1\}$ is the unit sphere. We define the metric

$$d((r_1, s_1), (r_2, s_2)) = |1/r_1 - 1/r_2| + \|s_1 - s_2\|, \quad (r_1, s_1), (r_2, s_2) \in \overline{\mathbb{C}}_0^+.$$

The metric space $(\overline{\mathbb{C}}_0^+, d)$ is complete and separable as the product of the complete and separable metric spaces $(0, +\infty]$ (with distance $|1/r_1 - 1/r_2|$) and $\mathbb{S}_{\mathbb{C}^+}$. The *polar decomposition* defined on $\mathbb{C}_0^+ = \mathbb{C}^+ \setminus \{0\}$ by

$$T : \begin{cases} \mathbb{C}_0^+ \rightarrow (0, \infty) \times \mathbb{S}_{\mathbb{C}^+} \\ x \mapsto (\|x\|, x/\|x\|) \end{cases}.$$

is an homeomorphism and we identify in the sequel \mathbb{C}_0^+ and $(0, +\infty) \times \mathbb{S}_{\mathbb{C}^+}$. In this metric, a subset B of \mathbb{C}_0^+ is bounded if and only if it is bounded away from zero, i.e. included in a set of the form $\{x \in \mathbb{C}^+; \|x\| \geq \varepsilon\}$ for some $\varepsilon > 0$.

3.1 Spatial maximum process

For the sake of clarity, we present first our results for the spatial maximum process M_n defined by Eq. 1. The following theorem is a natural extension of Theorem 2.4 from de Haan and Lin (2001) where $X_{in} = n^{-1} X_i$ is obtained from an i.i.d. sequence of continuous (or càd-làg) processes $(X_i)_{i \geq 1}$ and the limit maximum process M is simple max-stable. In our triangular array setting, we obtain in the limit a max-infinitely divisible process. We recall that a random element ξ of $\mathbb{C}(S)$ is called max-infinitely divisible if for every $n \geq 1$, there exists independent and identically

distributed elements $\xi_{1n}, \dots, \xi_{nn}$ in $\mathbb{C}(S)$ such that $\xi \stackrel{d}{=} \bigvee_{i=1}^n \xi_{in}$ where $\stackrel{d}{=}$ stands for equality in distribution (see Giné et al. (1990) or Kabluchko and Stoev (Preprint)).

Theorem 3.1 *Assume that for each $n \geq 1$, $\{X_{in}, i \geq 1\}$ are i.i.d. copies of a nonnegative continuous stochastic process X_n . Then the following statements are equivalent, where the limits are taken as $n \rightarrow \infty$:*

i) $(M_n(s))_{s \in S} \xrightarrow{d} (M(s))_{s \in S}$ where the limiting process M is assumed to satisfy the condition

$$\text{essinf}\|M\| := \inf\{y > 0; \mathbb{P}(\|M\| > y) < 1\} \equiv 0 \tag{4}$$

and \xrightarrow{d} stands for convergence in distribution in $\mathbb{C}^+(S)$;

ii) $n\mathbb{P}[X_n \in \cdot] \xrightarrow{w^\sharp} \nu$ in $M_b^\sharp(\overline{\mathbb{C}}_0^+)$ with $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$;

iii) $\beta_n \xrightarrow{d} \Pi_\nu$ with Π_ν a Poisson point process on \mathbb{C}_0^+ with intensity ν and \xrightarrow{d} standing for convergence in distribution in $M_{(b,p)}^\sharp(\overline{\mathbb{C}}_0^+)$.

Furthermore, M is a max-infinitely divisible process with exponent measure ν , i.e. it admits the representation

$$M(s) \stackrel{d}{=} \sup_{i \geq 1} Y_i(s), \quad s \in S, \tag{5}$$

where $\sum_{i \geq 1} \delta_{Y_i}$ is a Poisson point process on \mathbb{C}_0^+ with intensity ν , and the exponent measure ν is uniquely determined by the relations

$$\begin{aligned} & \mathbb{P}[M(s_1) \leq y_1, \dots, M(s_k) \leq y_k] \\ &= \exp\left(-\nu\{f \in \mathbb{C}_0^+; f(s_i) > y_i \text{ for some } i = 1, \dots, k\}\right) \end{aligned} \tag{6}$$

for all $k \geq 1, s_1, \dots, s_k \in S$ and $y_1, \dots, y_k > 0$.

Remark 3.2 The normalisation condition (4) might be stronger than necessary but is required in the proof. Since $\mathbb{P}[\|M_n\| \leq y] = \mathbb{P}[\|X_n\| \leq y]^n$, condition (4) is fulfilled as soon as

$$\sup_{n \geq 1} n\mathbb{P}[\|X_n\| > y] < \infty \quad \text{for all } y > 0.$$

Giné et al consider the weaker condition $\text{essinf}M(s) \equiv 0$ and prove that every continuous max-infinitely divisible process M satisfying this weaker condition admits a Poisson point process representation (5). This weak condition is not restrictive because if $b(s) = \text{essinf}M(s)$ is not identically equal to 0, one can consider $M(s) - b(s)$. It remains here an open question whether the convergence $M_n \xrightarrow{d} M$ with $\text{essinf}M(s) \equiv 0$ implies $\text{essinf}\|M\| = 0$ or not.

Remark 3.3 Working on the space $\mathbb{C}(S)$ of continuous functions is important in Theorem 3.1. The case of the space of càd-làg functions $\mathbb{D}[0, 1]$ endowed with the Skorokhod J_1 topology is studied by Gentric (2008). He shows that regular variation does not always imply convergence of the partial maxima process. This is related

to the fact that the operation $(f, g) \mapsto \max(f, g)$ is not continuous on $\mathbb{D}[0, 1]$. We believe that the good framework to work with non-continuous process is the space of semi-continuous functions as is the seminal work by Giné et al. (1990). For the sake of simplicity, we do not consider this general framework here.

For the proof of Theorem 3.1, note that the equivalence $ii) \Leftrightarrow iii)$ is a straightforward application of Proposition 2.2. For the implication $i) \Rightarrow ii)$, we follows the lines of the proof of Theorem 2.4 in de Haan and Lin (2001). Finally, for the implication $ii) \Rightarrow i)$ we will need the following Lemma. With a slight abuse of notation, we define

$$M_{(b,p)}^\sharp(\mathbb{C}_0^+) = \{\mu \in M_{(b,p)}^\sharp(\overline{\mathbb{C}}_0^+) : \mu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0\}.$$

Note $M_{(b,p)}^\sharp(\mathbb{C}_0^+)$ is an open subset of $M_{(b,p)}^\sharp(\overline{\mathbb{C}}_0^+)$.

Lemma 3.4 *The mapping $\theta : M_{(b,p)}^\sharp(\mathbb{C}_0^+) \rightarrow \mathbb{C}^+$ defined by $\theta(0) \equiv 0$ and*

$$\theta\left(\sum_{i \in I} \delta_{x_i}\right) = \left(s \mapsto \sup\{x_i(s); i \in I\}\right)$$

is well-defined and continuous.

Proof of Theorem 3.1 Proof of $ii) \Rightarrow i)$: Under the assumption $n\mathbb{P}[X_n \in \cdot] \xrightarrow{w^\sharp} \nu$, we know from Proposition 2.2 that the empirical measure β_n defined by Eq. 3 converges in distribution in $M_{(b,p)}(\overline{\mathbb{C}}_0^+)$ to a Poisson point process Π_ν with intensity ν . The assumption $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$ ensures that Π_ν lies almost surely in $M_{(b,p)}(\mathbb{C}_0^+)$. Noting that $M_n = \theta(\beta_n)$ with θ continuous (cf Lemma 3.4), the continuous mapping Theorem (see e.g. Theorem 5.1 in Billingsley (1968)) entails $\theta(\beta_n) \xrightarrow{d} \theta(\Pi_\nu)$. This proves the convergence $M_n \xrightarrow{r} M$ with $M = \theta(\Pi_\nu)$, i.e. $i)$ holds together with the representation (5). The normalisation condition (4) is easily checked since

$$\mathbb{P}[\|M\| > y] = 1 - \exp\left(-\nu\{f \in \mathbb{C}_0^+; \|f\| > y\}\right) < 1$$

for all $y > 0$.

Proof of $i) \Rightarrow ii)$: this is a direct adaptation of the proof of Theorem 2.4 in de Haan and Lin (2001) and we give only the main lines. We use the notation $\nu_n = n\mathbb{P}[X_n \in \cdot]$ and we want to prove ν_n converges in $M_b^\sharp(\overline{\mathbb{C}}_0^+)$. The starting point is the convergence $M_n \xrightarrow{d} M$ which entails for all $k \geq 1, s_1, \dots, s_k \in S$ and $y_1, \dots, y_k > 0$ continuity points

$$\mathbb{P}[M_n(s_1) \leq y_1, \dots, M_n(s_k) \leq y_k] \rightarrow \mathbb{P}[M(s_1) \leq y_1, \dots, M(s_k) \leq y_k]. \tag{7}$$

Setting

$$A_{s_1, y_1, \dots, s_k, y_k} = \{f \in \mathbb{C}_0^+; f(s_i) \leq y_i, i = 1, \dots, k\},$$

we have

$$\begin{aligned} \mathbb{P}[M_n(s_1) \leq y_1, \dots, M_n(s_k) \leq y_k] &= \mathbb{P}[X_n(s_1) \leq y_1, \dots, X_n(s_k) \leq y_k]^n \\ &= \left(1 - \frac{1}{n} v_n(A_{s_1, y_1, \dots, s_k, y_k}^c)\right)^n \end{aligned}$$

and we deduce from Eq. 7 that

$$\begin{aligned} \lim_{n \rightarrow \infty} v_n(A_{s_1, y_1, \dots, s_k, y_k}^c) &= \lim_{n \rightarrow +\infty} -\log \mathbb{P}[M_n(s_1) \leq y_1, \dots, M_n(s_k) \leq y_k] \\ &= -\log \mathbb{P}[M(s_1) \leq y_1, \dots, M(s_k) \leq y_k] \\ &:= v(A_{s_1, y_1, \dots, s_k, y_k}^c). \end{aligned}$$

This last quantity is finite thanks to the normalisation condition (4). This shows that if the limit v exists, it is necessarily given by Eq. 6. To prove the existence of the limit, one needs to show that the sequence v_n is tight in $M_b^\sharp(\overline{\mathbb{C}}_0^+)$. This implies that the relations (4) define a measure $\nu \in M_b^\sharp(\overline{\mathbb{C}}_0^+)$ and that $v_n \xrightarrow{w^\sharp} \nu$ as $n \rightarrow \infty$.

Tightness of the sequence v_n is the purpose of Lemma 3.2 in de Haan and Lin (2001). The proof there assumes that $M_n \xrightarrow{d} M$ in Skohorod space with the limit M simple max-stable. Here we consider convergence in $\mathbb{C}^+(S)$ (which simplifies the proof thanks to simpler tightness criterion) but we do not assume max-stability of the limit. A close inspection of the proof reveals that max-stability is not needed but only that

$$-\log \mathbb{P}(\|M\| \leq y) < \infty \quad \text{for all } y > 0$$

and that is the reason why we require the normalization condition (4). □

Proof of Lemma 3.4

- First, we show that the mapping θ is well defined. This is not obvious since the pointwise supremum of a countable family of functions is not necessarily finite nor continuous.

Let $\mu \in M_{(b,p)}^\sharp(\mathbb{C}_0^+)$. For $\varepsilon > 0$, define $\mu^{(\varepsilon)}$ as the restriction of μ to the set $\bar{B}^\varepsilon = [\varepsilon, \infty] \times \mathbb{S}_\mathbb{C}$ and $\theta_\varepsilon = \theta(\mu^{(\varepsilon)})$. Since \bar{B}^ε is bounded for the metric d , the point measure $\mu^{(\varepsilon)}$ has only a finite number of atoms and therefore $\theta(\mu^{(\varepsilon)})$ is the maximum of a finite number of non negative continuous functions and is hence a non negative continuous function.

Furthermore, for all $s \in S$, $|\theta(\mu^{(\varepsilon)})(s) - \theta(\mu)(s)| \leq \varepsilon$. So $\theta(\mu)$ is the uniform limit as $\varepsilon \rightarrow 0$ of the continuous functions $\theta_\varepsilon(\mu)$, and hence $\theta(\mu) \in \mathbb{C}^+$ and satisfies

$$\|\theta(\mu^{(\varepsilon)}) - \theta(\mu)\| \leq \varepsilon. \tag{8}$$

- Second, we show that the mapping θ is continuous.

Let $\{\mu_n = \sum_{i \in I_n} \delta_{x_i^n}, \quad n \geq 1\}$ be a sequence of measures in $M_{(b,p)}^\sharp(\mathbb{C}_0^+)$ converging to $\mu = \sum_{i \in I} \delta_{x_i}$. For all $\varepsilon > 0$, there exists $\varepsilon' < \varepsilon$, $N \geq 0$ and $m \geq 0$ such that

$$\forall n \geq N, \quad \mu_n^{(\varepsilon')} = \sum_{i=1}^m \delta_{x_i^{n\varepsilon'}} \quad \text{and} \quad \mu^{(\varepsilon')} = \sum_{i=1}^m \delta_{x_i^{\varepsilon'}}$$

with $x_i^{n\varepsilon'}, x_i^{\varepsilon'} \in \bar{B}^{\varepsilon'}$ and $x_i^{n\varepsilon'} \rightarrow x_i^{\varepsilon'}$ as $n \rightarrow \infty$. Clearly, this implies that $\theta(\mu_n^{(\varepsilon')})$ converges in \mathbb{C}^+ to $\theta(\mu^{(\varepsilon')})$ as $n \rightarrow \infty$. Hence there exists an integer n_0 such that $\|\theta(\mu_n^{(\varepsilon')}) - \theta(\mu^{(\varepsilon')})\| \leq \varepsilon$ for all $n \geq n_0$. Then, thanks to Eq. 8, we obtain for $n \geq n_0$

$$\|\theta(\mu_n) - \theta(\mu)\| \leq \|\theta(\mu_n) - \theta(\mu_n^{(\varepsilon')})\| + \|\theta(\mu_n^{(\varepsilon')}) - \theta(\mu^{(\varepsilon')})\| + \|\theta(\mu^{(\varepsilon')}) - \theta(\mu)\| \leq 3\varepsilon.$$

This proves that $\theta(\mu_n) \rightarrow \theta(\mu)$ as $n \rightarrow \infty$ and that the mapping θ is continuous. □

3.2 Spatio-temporal maximum process

We consider now convergence of the space-time process \tilde{M}_n defined by Eq. 2. For fixed $t \geq 0$, the space process $s \mapsto \tilde{M}_n(t, s)$ is sample continuous and non negative, i.e. a random elements of \mathbb{C}^+ . Furthermore, the time process $t \mapsto \tilde{M}_n(t, \cdot)$ can be seen as a \mathbb{C}^+ -valued càd-làg process on $[0, +\infty)$; it is indeed constant on intervals of the form $[k/n, k/n + 1/n)$, $k \in \mathbb{N}$. Hence, we will consider the process \tilde{M}_n as a random element of the Skohorod space $\mathbb{D}([0, +\infty), \mathbb{C}^+)$ endowed with the J_1 -topology (see for example Ethier and Kurtz (1986) for the definition and properties of Skohorod space).

Theorem 3.5 *Assume that $n\mathbb{P}[X_n \in \cdot] \xrightarrow{w^\sharp} \nu$ in $M_b^\sharp(\bar{\mathbb{C}}_0^+)$ and that $\nu(\bar{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$. Then, the process \tilde{M}_n weakly converges in $\mathbb{D}([0, +\infty), \mathbb{C}^+)$ as $n \rightarrow \infty$ to the superextremal process \tilde{M} defined by*

$$\tilde{M}(t, s) = \sup\{Y_i(s)\mathbf{1}_{[T_i, +\infty)}(t); \quad i \geq 1\}, \quad t > 0, \quad s \in S, \tag{9}$$

where $\tilde{\Pi}_\nu = \sum_{i \geq 1} \delta_{(Y_i, T_i)}$ is a Poisson point process on $\mathbb{C}_0^+ \times [0, +\infty)$ with intensity $\nu \otimes \ell$.

For the proof, we will need the following analog of Lemma 3.4 in the space-time framework. Let $M_{b,p}^\sharp(\mathbb{C}_0^+ \times [0, +\infty))$ be the subset of measures $\mu \in M_{(b,p)}^\sharp(\bar{\mathbb{C}}_0^+ \times [0, +\infty))$ such that $\mu((\bar{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) \times [0, +\infty)) = 0$ and define

$$\tilde{\mathcal{C}} = \left\{ \mu \in M_{(b,p)}^\sharp(\mathbb{C}_0^+ \times [0, +\infty)); \quad \mu(\bar{\mathbb{C}}_0^+ \times \{t\}) \leq 1 \quad \text{for all } t \geq 0 \right\}.$$

In other words, a measure $\mu = \sum_{i \in I} \delta_{(x_i, t_i)}$ belongs to $\tilde{\mathcal{C}}$ if and only if $r_i < +\infty$ for all $i \in I$ and the u_i 's are pairwise distinct.

Lemma 3.6 *The mapping $\tilde{\theta} : M_{(b,p)}^\sharp(\mathbb{C}_0^+ \times [0, +\infty)) \rightarrow \mathbb{D}([0, +\infty), \mathbb{C}^+)$ defined by $\tilde{\theta}(0) \equiv 0$ and*

$$\tilde{\theta}\left(\sum_{i \in I} \delta_{(x_i, t_i)}\right) = \left((t, s) \mapsto \sup\{x_i(s)1_{[t_i, +\infty)}(t); i \in I\}\right)$$

is well defined and continuous on \tilde{C} .

Proof of Lemma 3.6

The proof is similar to the proof of Lemma 3.4 and we give only the main lines.

- First we show that $\tilde{\theta}$ is well defined. Recall that $\overline{\mathbb{C}_0^+} \times [0, +\infty)$ is endowed with the metric

$$\tilde{d}((r_1, s_1, t_1), (r_2, s_2, t_2)) = d((r_1, s_1), (r_2, s_2)) + |t_2 - t_1|.$$

Let $\mu \in M_{(b,p)}^\sharp(\mathbb{C}_0^+ \times [0, +\infty))$. For $\varepsilon > 0$ and $M > 0$, let $\mu^{(\varepsilon, M)}$ be its restriction to $\bar{B}^{\varepsilon, M} = [\varepsilon, +\infty) \times \mathbb{S}_{\mathbb{C}^+} \times [0, M]$. Since $\bar{B}^{\varepsilon, M}$ is bounded, $\mu^{(\varepsilon, M)}$ has only a finite number of atoms and we easily check that $\tilde{\theta}(\mu^{(\varepsilon, M)})$ belongs to $\mathbb{D}([0, +\infty), \mathbb{C}^+)$. Furthermore, for $t < M$ and $s \in S$,

$$|\tilde{\theta}(\mu^{(\varepsilon, M)})(t, s) - \tilde{\theta}(\mu)(t, s)| \leq \varepsilon$$

so that $\tilde{\theta}(\mu^{(\varepsilon, M)})$ converges uniformly on $[0, M] \times T$ to $\tilde{\theta}(\mu)$ as $\varepsilon \rightarrow 0$. The constant M being arbitrary, this implies that $\tilde{\theta}(\mu) \in \mathbb{D}([0, +\infty), \mathbb{C}^+)$ and that the application $\tilde{\theta}$ is well defined. It also holds that

$$\sup_{(t,s) \in [0, M] \times S} |\tilde{\theta}(\mu^{(\varepsilon, M)})(t, s) - \tilde{\theta}(\mu)(t, s)| \leq \varepsilon. \tag{10}$$

- Next, we show that $\tilde{\theta}$ is continuous on \tilde{C} .

Let $\{\mu_n = \sum_{i \in I_n} \delta_{(x_i^n, t_i^n)}; n \geq 1\}$ be a sequence of $M_{(b,p)}^\sharp(\mathbb{C}_0^+ \times [0, +\infty))$ converging to $\mu = \sum_{i \in I} \delta_{(x_i, t_i)} \in \tilde{C}$. For all $\varepsilon > 0$ and $M > 0$, there exist $\varepsilon' < \varepsilon$, $M' > M$, $N \geq 1$ and some $m \geq 1$, such that

$$\forall n \geq N, \quad \mu_n^{(\varepsilon', M')} = \sum_{i=1}^m \delta_{(x_i^n, t_i^n)} \quad \text{and} \quad \mu^{(\varepsilon', M')} = \sum_{i=1}^m \delta_{(x_i, t_i)}$$

with $x_i^n \rightarrow x_i$ and $t_i^n \rightarrow t_i$ as $n \rightarrow \infty$. The condition $\mu \in \tilde{C}$ ensures that the t_i 's are pairwise distinct and we can suppose without loss of generality that $t_1 < \dots < t_m$. We then also have, for large enough n , $t_1^n < \dots < t_m^n$. Define $\delta_{M'}$ the metric associated with the J_1 -topology on $\mathbb{D}([0, M'], \mathbb{C}^+)$ by

$$\delta_{M'}(x, y) = \inf_{\lambda} \sup_{(t,s) \in [0, M'] \times S} |x(\lambda t, s) - y(t, s)|$$

where the infimum is taken over the set of non-decreasing homeomorphisms λ of $[0, M']$. Since for large n the t_i 's and the t_i^n 's are in the same relative order, there exists a non-decreasing homeomorphism $\lambda_{M'}^n$ of $[0, M']$ such that

$\lambda_{M'}^n(t_i^n) = t_i$ for all $1 \leq i \leq m$. We then have

$$\begin{aligned} & \delta_{M'}(\tilde{\theta}(\mu_n^{(\varepsilon', M')}), \tilde{\theta}(\mu^{(\varepsilon', M')})) \\ & \leq \delta_{M'}(\tilde{\theta}(\mu_n), \tilde{\theta}(\mu_n^{(\varepsilon', M')})) + \delta_{M'}(\tilde{\theta}(\mu_n^{(\varepsilon', M')}), \tilde{\theta}(\mu^{(\varepsilon', M')})) \\ & \quad + \delta_{M'}(\tilde{\theta}(\mu^{(\varepsilon', M')}), \tilde{\theta}(\mu)) \\ & \leq \sup_{(t,s) \in [0, M'] \times S} | \max_{1 \leq i \leq m} (x_i^n(\cdot) \mathbf{1}_{[\lambda_{M'}^n t_i^n, +\infty)}(t)) - \max_{1 \leq i \leq m} (x_i(\cdot) \mathbf{1}_{[t_i, +\infty)}(t)) | \\ & = \max_{1 \leq i \leq m} \|x_i^n - x_i\| \\ & \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Hence, for sufficiently large n , $\delta_{M'}(\tilde{\theta}(\mu_n^{(\varepsilon', M')}), \tilde{\theta}(\mu^{(\varepsilon', M')})) \leq \varepsilon$ and thanks to Eq. 10, this entails

$$\begin{aligned} & \delta_{M'}(\tilde{\theta}(\mu_n), \tilde{\theta}(\mu)) \\ & \leq \delta_{M'}(\tilde{\theta}(\mu_n), \tilde{\theta}(\mu_n^{(\varepsilon', M')})) + \delta_{M'}(\tilde{\theta}(\mu_n^{(\varepsilon', M')}), \tilde{\theta}(\mu^{(\varepsilon', M')})) \\ & \quad + \delta_{M'}(\tilde{\theta}(\mu^{(\varepsilon', M')}), \tilde{\theta}(\mu)) \\ & \leq 3\varepsilon. \end{aligned}$$

Since M' is arbitrary large and ε arbitrary small, this proves the convergence $\tilde{\theta}(\mu_n)$ to $\tilde{\theta}(\mu)$ in $\mathbb{D}([0, +\infty), \mathbb{C}^+)$. □

Proof of Theorem 3.5

The proof is very similar to the proof of Theorem 3.1. Note that $\tilde{M}_n = \tilde{\theta}(\tilde{\beta}_n)$ where $\tilde{\beta}_n$ is defined by Eq. 3. According to Proposition 2.2, the empirical measure $\tilde{\beta}_n$ weakly converges in $M_{(b,p)}(\overline{\mathbb{C}}_0^+ \times [0, +\infty))$ to a Poisson point process $\tilde{\Pi}_\nu$ with intensity $\nu \otimes \ell$. The assumption $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+)$ and the fact that the Lebesgue measure ℓ has no atoms ensure that $\tilde{\Pi}_\mu$ lies almost surely in \tilde{C} . From Lemma 3.6, the mapping $\tilde{\theta}$ is continuous on \tilde{C} , so that the Continuous Mapping Theorem implies $\tilde{\theta}(\tilde{\beta}_n) \Rightarrow \tilde{\theta}(\tilde{\Pi}_\nu)$ which is equivalent to $\tilde{M}_n \Rightarrow \tilde{M}$. □

4 Properties of the limit process

In this section, we give some properties of the limiting process \tilde{M} defined by Eq. 9 in Theorem 3.5. These results extend the classical theory on extremal processes (see Dwass 1964, 1966, 1974, Tiago de Oliveira 1968, Resnick and Rubinovitch 1973, Resnick 1975 ...) from a univariate or multivariate setting to a functional framework. Note also the work by Resnick and Roy (1994) in a fonctionnal setting with applications to the dynamic continuous choice problem.

First, we characterize the finite dimensional distributions of \tilde{M} . We use vectorial notations: for $l \geq 1$, $\mathbf{s} = (s_1, \dots, s_l) \in S^l$, $\mathbf{y} = (y_1, \dots, y_l) \in [0, +\infty)^l$ and $t > 0$, we write $\tilde{M}(t, \mathbf{s}) \leq \mathbf{y}$ if and only if $\tilde{M}(t, s_i) \leq y_i$ for all $i \in \{1, \dots, l\}$.

Proposition 4.1 For $k \geq 1$, $0 = t_0 < t_1 < \dots < t_k$, $\mathbf{s} \in S^l$ and $\mathbf{y}_1, \dots, \mathbf{y}_k \in [0, +\infty)^l$, it holds

$$\mathbb{P}\left[\tilde{M}(t_1, \mathbf{s}) \leq \mathbf{y}_1, \dots, \tilde{M}(t_k, \mathbf{s}) \leq \mathbf{y}_k\right] = \prod_{i=1}^k \exp[-(t_j - t_{j-1})\nu(A_j)]$$

with $A_j = \{f \in \mathbb{C}_0^+; \exists i \in 1, \dots, l, f(s_i) \geq \min_{k \geq j} y_{k,i}\}$.

Proof of Proposition 4.1

Let $j \in \{1, \dots, k\}$. Note that $\tilde{M}(t_j, \mathbf{s}) \leq \mathbf{y}_j$ if and only if $\tilde{\Pi}_\nu$ doesn't intersect the set

$$B_j = \{(f, t) \in \mathbb{C}_0^+ \times [0, +\infty); t \leq t_j \text{ and } \exists i \in \{1, \dots, l\} f(s_i) > y_{j,i}\}$$

So, we have

$$\begin{aligned} \mathbb{P}\left[\tilde{M}(t_1, \mathbf{s}) \leq \mathbf{y}_1, \dots, \tilde{M}(t_k, \mathbf{s}) \leq \mathbf{y}_k\right] &= \mathbb{P}\left[\tilde{\Pi}_\nu \cap (\cup_{j=1}^k B_j) = \emptyset\right] \\ &= \exp[-(\nu \otimes \ell)(\cup_{j=1}^k B_j)]. \end{aligned}$$

The B_j 's are not pairwise disjoint. To compute the measure $(\nu \otimes \ell)(\cup_{j=1}^k B_j)$, we observe that

$$\cup_{j=1}^k B_j = \cup_{j=1}^k ([t_{j-1}, t_j] \times A_j) \cup (\{t_k\} \times A_k)$$

where the sets in the right hand side are pairwise disjoint. From this, we deduce

$$(\nu \otimes \ell)(\cup_{j=1}^k B_j) = \sum_{j=1}^k (t_j - t_{j-1})\nu(A_j).$$

This proves the Proposition. □

Next we prove that the process $t \mapsto \tilde{M}(t, \cdot)$ is a $\mathbb{C}^+(S)$ -valued homogeneous Markov process and consider its sample path properties. Let \mathcal{F}_t be the σ -algebra generated by $\{\tilde{M}(t', s); t' \in [0, t], s \in S\}$. The symbol \vee stands for pointwise maximum.

- Proposition 4.2** • (Markov property) For each $t \geq 0$, the conditional distribution of $(\tilde{M}(t+h, \cdot))_{h \geq 0}$ given \mathcal{F}_t is equal to the distribution of $(\tilde{M}(t, \cdot) \vee \tilde{M}'(h, \cdot))_{h \geq 0}$ where \tilde{M}' is an independent copy of \tilde{M} .
- (regularity) The $\mathbb{C}^+(S)$ -valued process $t \mapsto \tilde{M}(t, \cdot)$ is a pure jump process with finitely many jumps in each interval $[t_1, t_2]$, $0 < t_1 < t_2$.

The first point of the proposition states that the process $t \mapsto \tilde{M}(t, \cdot)$ has “independent and stationary increments” with respect to the maximum: for $0 = t_0 < t_1 < \dots < t_k$, the distribution of $(\tilde{M}(t_i, \cdot))_{1 \leq i \leq k}$ is equal to the distribution of

$$\left(\vee_{j=1}^i \tilde{M}^j(t_j - t_{j-1}, \cdot)\right)_{1 \leq i \leq k},$$

where $\tilde{M}^1, \dots, \tilde{M}^k$ are i.i.d. copies of \tilde{M} . This property is similar to the property of independence and stationarity of increments (with respect to the addition) of Lévy

processes. For a fixed point $s \in S$, the process $\{M(t, s), t \geq 0\}$ is known as an extremal process (see Proposition 4.7 of Resnick (2008)).

Proof of Proposition 4.2

- Consider the decomposition $\tilde{\Pi}_\nu = \tilde{\Pi}_\nu^{[0,t]} \cup \tilde{\Pi}_\nu^{(t,\infty)}$ where

$$\tilde{\Pi}_\nu^{[0,t]} = \tilde{\Pi}_\nu \cap (\mathbb{C}_0^+ \times [0, t]) \quad \text{and} \quad \tilde{\Pi}_\nu^{(t,\infty)} = \tilde{\Pi}_\nu \cap (\mathbb{C}_0^+ \times (t, \infty)).$$

By the independence properties of the Poisson point process, $\tilde{\Pi}_\nu^{[0,t]}$ and $\tilde{\Pi}_\nu^{(t,\infty)}$ are independent. Furthermore,

$$\begin{aligned} \tilde{M}(t+h, \cdot) = & \sup\{x(\cdot)1_{[u,+\infty)}(t+h); (x, u) \in \tilde{\Pi}_\nu^{[0,t]}\} \\ & \vee \sup\{x(\cdot)1_{[u,+\infty)}(t+h); (x, u) \in \tilde{\Pi}_\nu^{(t,\infty)}\} \end{aligned}$$

and the two terms in the r.h.s. are independent. It is easily seen that the first term is equal to $\tilde{M}(t, \cdot)$, and that the invariance of the Lebesgue measure ℓ implies that the second term has the same distribution as $\tilde{M}(h, \cdot)$.

- The definition (9) implies that conditionally on $\tilde{M}(t_1, \cdot)$, the evolution of the process $\tilde{M}(t, \cdot)$ for $t \in [t_1, t_2]$ depends only on the points

$$\tilde{\Pi}_\nu \cap \{(x, t) \in \mathbb{C}_0^+ \times [t_1, t_2]; x(s) > \tilde{M}(t_1, s) \text{ for some } s \in S\}.$$

Each of this point may give rise to a jump of $\tilde{M}(t, \cdot)$ in $[t_1, t_2]$. The result follows since the number of such points follows a Poisson distribution and is hence finite.

□

In the particular case when the measure ν is homogeneous of order $-\alpha < 0$, the process M enjoys further interesting properties.

Proposition 4.3 *Suppose ν is homogeneous with index $-\alpha < 0$. Then:*

- \tilde{M} is max-stable with index $\alpha > 0$, i.e. if $\tilde{M}^1, \dots, \tilde{M}^n$ are independent copies of \tilde{M} , the maximum $\vee_{i=1}^n \tilde{M}^i$ has the same law as $n^{1/\alpha} \tilde{M}$;
- \tilde{M} is self-similar with index $1/\alpha$, i.e. for all $c > 0$, the rescaled process $(\tilde{M}(ct, \cdot))_{t \geq 0}$ has the same distribution as $(c^{1/\alpha} \tilde{M}(ct, \cdot))_{t \geq 0}$

Proof of Proposition 4.3

We check that the the two processes have the same finite dimensional distributions. To this aim, we use Proposition 4.1 and the notations of the proposition.

- Let $\tilde{M}^1, \dots, \tilde{M}^n$ be independent copies of \tilde{M} . By Proposition 4.1 and the homogeneity of ν ,

$$\begin{aligned} \mathbb{P}\left[\bigvee_{i=1}^n \tilde{M}^i(t_j, \mathbf{s}) \leq \mathbf{y}_j, \forall j \in \{1, \dots, k\}\right] &= \mathbb{P}\left[\tilde{M}(t_j, \mathbf{s}) \leq \mathbf{y}_j, \forall j \in \{1, \dots, k\}\right]^n \\ &= \prod_{i=1}^k \exp[-(t_j - t_{j-1})\nu(A_j)]^n \\ &= \prod_{i=1}^k \exp[-(t_j - t_{j-1})\nu(n^{-1/\alpha} A_j)] \\ &= \mathbb{P}\left[n^{1/\alpha} \tilde{M}(t_j, \mathbf{s}) \leq \mathbf{y}_j, \forall j \in \{1, \dots, k\}\right]. \end{aligned}$$

This proves the max-stability.

- Similarly, the self-similarity is proven as follows:

$$\begin{aligned} \mathbb{P}\left[\tilde{M}(ct_j, \mathbf{s}) \leq \mathbf{y}_j, \forall j \in \{1, \dots, k\}\right] &= \prod_{i=1}^k \exp[-c(t_j - t_{j-1})\nu(A_j)] \\ &= \prod_{i=1}^k \exp[-(t_j - t_{j-1})\nu(c^{-1/\alpha} A_j)] \\ &= \mathbb{P}\left[c^{1/\alpha} \tilde{M}(t_j, \mathbf{s}) \leq \mathbf{y}_j, \forall j \in \{1, \dots, k\}\right]. \end{aligned}$$

□

5 Convergence of order statistics

The point process approach for extremes is powerful: the convergence of the empirical measure β_n entails not only the convergence of the maxima M_n but also of all the order statistics, and a similar result hold for the space-time version of these processes.

For $1 \leq r \leq n$ and $s \in S$, define $M_n^r(s)$ as the r -th largest value among $X_{1n}(s), \dots, X_{nn}(s)$. Note $M_n^1(s) = M_n(s)$ is simply the maximum. For $r > n$, we use the convention $M_n^r(s) = 0$. The process $M_n^r = (M_n^r(s))_{s \in S}$ is referred to as the r -th order statistic of the sample $\{X_{1n}, \dots, X_{1n}\}$. Similarly, for $r \geq 1, s \in S$ and $t \geq 0$ define $\tilde{M}_n^r(t, s)$ as the r -th largest value among $X_{1n}(s), \dots, X_{[nt]n}(s)$ with the convention $\tilde{M}_n^r(t, s) = 0$ if $r > [nt]$. An alternative definition in terms of the empirical measure β_n is given by

$$\tilde{M}_n^r(t, s) = \sup\{y \geq 0; \tilde{\beta}_n(B_{t,s}^y) \geq r\}$$

with $B_{t,s}^y = \{(f, u) \in \mathbb{C}_0^+ \times [0, +\infty); f(s) \geq y, u \leq t\}$ and the convention that the supremum over an empty set is equal to zero.

With these notations, we can strengthen Theorem 3.5 as follows.

Theorem 5.1 Assume that $n\mathbb{P}[X_n \in \cdot] \xrightarrow{w^\dagger} \nu$ in $M_b^\dagger(\overline{\mathbb{C}}_0^+)$ and that $\nu(\overline{\mathbb{C}}_0^+ \setminus \mathbb{C}_0^+) = 0$. Then, for each $r \geq 1$, the vector of order statistics $(\tilde{M}_n^1, \dots, \tilde{M}_n^r)$ weakly converges in $\mathbb{D}([0, +\infty), (\mathbb{C}^+)^r)$ as $n \rightarrow \infty$ to $(\tilde{M}^1, \dots, \tilde{M}^r)$ defined by

$$\tilde{M}^j(t, s) = \sup\{y \geq 0; \tilde{\Pi}_\nu \cap B_{t,s}^y \geq r\}, \quad 1 \leq j \leq r, t \geq 0, s \in S,$$

with $\tilde{\Pi}_\nu$ a Poisson point process on $\mathbb{C}_0^+ \times [0, +\infty)$ with intensity $\nu \otimes \ell$.

The limit process \tilde{M}^j corresponds to the r -th space-time order statistic associated to the point process $\tilde{\Pi}_\nu$.

Proof of Theorem 5.1

Using once again Proposition 2.2 and the Continuous Mapping Theorem, it is enough to prove the following generalization of Lemma 3.6: the mapping

$$\tilde{\theta}^r : M_{(b,p)}^\dagger(\mathbb{C}_0^+ \times [0, +\infty)) \rightarrow \mathbb{D}([0, +\infty), (\mathbb{C}^+)^r)$$

defined by $\tilde{\theta}(0) \equiv 0$ and

$$\tilde{\theta}(\beta)(u, t) = (\sup\{y \geq 0; \tilde{\beta}(B_{u,t}^y) \geq j\})_{1 \leq j \leq r}$$

is well defined and continuous on $\tilde{\mathcal{C}}$. The proof is very similar to the proof of Lemma 3.6 and the details are omitted for the sake of brevity. □

6 An application to maxima of Gaussian processes

In geostatistics, Gaussian processes are often used as spatial models. In the finite-dimensional setting, the asymptotic behavior of the maxima of Gaussian random vectors was first investigated by Hüsler and Reiss (1989). Recently, Kabluchko et al. (2009) and Kabluchko (2011) consider the functional setting and prove convergence of the maxima of i.i.d. centered Gaussian processes under suitable conditions on their covariance structure. The family of Brown-Resnick processes has quickly become a popular model for spatial extremes due to its simple characterization by a negative definite functions. Our purpose here is to revisit their results and apply the present framework: we put the emphasis on regular variations and point processes. In the sequel, we follow the approach by Kabluchko, see Theorems 2 and 6 in Kabluchko (2011).

We suppose that (S, d) is a compact metric space satisfying the entropy condition

$$\int_0^1 (\log N(\varepsilon))^{1/2} d\varepsilon < \infty,$$

where $N(\varepsilon)$ is the smallest number of balls of radius ε needed to cover S .

Let $Z_n = \{Z_n(s); s \in S\}$, $n \geq 1$, be a sequence of continuous zero-mean unit-variance Gaussian processes with covariance function

$$r_n(s_1, s_2) = \mathbb{E}[Z_n(s_1)Z_n(s_2)], \quad s_1, s_2 \in S.$$

Define the scaling sequence

$$b_n = (2 \log n)^{1/2} - (2 \log n)^{-1/2}((1/2) \log \log n + \log(2\sqrt{\pi})), \quad n \geq 1 \quad (11)$$

and the rescaled process

$$Y_n(s) = b_n(Z_n(s) - b_n), \quad s \in S.$$

We consider the log-normal process

$$X_n(s) = \exp Y_n(s), \quad s \in S.$$

The following theorem provides a non trivial example of Gaussian random fields satisfying condition ii) in Theorem 3.1. Thanks to the equivalence i) \Leftrightarrow ii), it essentially follows from the convergence of maxima due to Kabluchko (2011). Since convergence of point process is conceptually at a higher level than convergence of maxima, we wish to provide here a direct proof of the \sharp -convergence $\nu_n \xrightarrow{w^\sharp} \nu$ without considering convergence of maxima $M_n \xrightarrow{d} M$. To this aim, we use a criterion for \sharp -convergence due to Hult and Lindskog (2006).

Theorem 6.1 Fix $s_0 \in S$ and suppose that:

i) Uniformly in $s_1, s_2 \in S$:

$$\Gamma(s_1, s_2) = \lim_{n \rightarrow \infty} 4 \log n(1 - r_n(s_1, s_2)) \in [0, \infty). \quad (12)$$

ii) For all $s_1, s_2 \in S$, there exists $C > 0$ such that

$$\sup_{n \geq 1} \log n(1 - r_n(s_1, s_2)) \leq Cd(s_1, s_2). \quad (13)$$

Then,

$$n\mathbb{P}[X_n \in \cdot] \xrightarrow{w^\sharp} \nu(\cdot) \quad \text{in } M_b^\sharp(\mathbb{C}_0^+),$$

where

$$\nu(A) = \int_0^\infty \mathbb{P}[we^{W(\cdot) - \frac{1}{2}\Gamma(s_0, \cdot)} \in A]w^{-2}dw, \quad A \subseteq \mathbb{C}_0^+ \text{ Borel set,}$$

and $\{W(s), s \in S\}$ a centered Gaussian process such that $W(s_0) = 0$ and with incremental variance Γ .

Following (Kabluchko et al. 2009; Kabluchko 2011), we define Y_n^w the process Y_n conditioned by $Y_n(s_0) = w$ and we note $\mu_n^w(s) = \mathbb{E}[Y_n^w(s)]$. We will need the following two Lemmas.

Lemma 6.2 Under conditions i) and ii) of Theorem 6.1, the family $\{Y_n^w; n \geq 1\}$ is tight in $\mathbb{C}(S)$ for all fixed $\omega \in \mathbb{R}$, as well as the family $\{Y_n^w - \mu_n^w; w \in \mathbb{R}, n \geq 1\}$.

Lemma 6.2 follows from the proof of Theorem 6 in Kabluchko (2011) (see also the proof of Theorem 17 in Kabluchko et al. (2009)). Details are omitted here.

Lemma 6.3 *Under condition i) of Theorem 6.1, the following convergence in the sense of finite dimensional distributions holds: as $n \rightarrow \infty$,*

$$\{Y_n^w(t), s \in S\} \xrightarrow{f.d.d.} \{w + W(s) - \frac{1}{2}\Gamma(s, s_0), s \in S\}$$

with W defined in Theorem 6.1.

Proof of Lemma 6.3

Standard computations for Gaussian processes entails that Y_n^w is a Gaussian process with mean and covariance

$$\mu_n^w(s) = wr_n(s, s_0) + b_n^2(r_n(s, s_0) - 1), \tag{14}$$

$$\text{Cov}[Y_n^w(s_1), Y_n^w(s_2)] = b_n^2(r_n(s_1, s_2) - r_n(s_1, s_0)r_n(s_2, s_0)). \tag{15}$$

Under assumption i), it holds

$$\lim_{n \rightarrow \infty} \mu_n^w(s) = w - \frac{1}{2}\Gamma(s, s_0)$$

and

$$\lim_{n \rightarrow \infty} \text{Cov}[Y_n^w(s_1), Y_n^w(s_2)] = \frac{1}{2}(\Gamma(s_1, s_0) + \Gamma(s_2, s_0) - \Gamma(s_1, s_2)).$$

This implies that $\lim_{n \rightarrow \infty} \text{Var}(Y_n^w(s_2) - Y_n^w(s_1)) = \Gamma(s_1, s_2)$ and proves the Lemma. \square

Proof of Theorem 6.1

Our proof relies on the following criterion for \sharp -convergence in $\overline{\mathbb{C}}_0^+$ by Hult and Lindskog (2006). According to (Hult and Lindskog 2006, Theorem 4.4), it is enough to prove that:

- i) for every $r > 0$, $\sup_{n \geq 1} n\mathbb{P}[\sup_{s \in S} X_n(s) > r] < \infty$;
- ii) for every $\varepsilon > 0$, $\lim_{\delta \rightarrow 0} \sup_{n \geq 1} n\mathbb{P}[\omega_{X_n}(\delta) \geq \varepsilon] = 0$, where $\omega_f(\delta) = \sup\{|f(s_1) - f(s_2)|; s_1, s_2 \in T, d(s_1, s_2) \leq \delta\}$ denotes the modulus of continuity of $f \in \mathbb{C}(S)$;
- iii) $n\mathbb{P}[X_n \in \cdot] \rightarrow \nu$ in the sense of finite-dimensional convergence.

These three points are proven below. At several places, we will use that the scaling sequence b_n defined by Eq. 11 satisfies $\sqrt{2\pi}b_n e^{b_n^2/2} \sim n$ as $n \rightarrow \infty$. As a consequence, the sequence $\frac{n}{\sqrt{2\pi}b_n e^{b_n^2/2}}$ is bounded by some constant $M > 0$.

- Proof of i): Let $r > 0$ and $\tilde{r} = \ln r$. We have

$$\begin{aligned} n\mathbb{P}[\sup_{s \in S} X_n(s) > r] &= n\mathbb{P}[\sup_{s \in S} Y_n(s) > \tilde{r}] \\ &= \frac{n}{\sqrt{2\pi} b_n e^{b_n^2/2}} \int_{\mathbb{R}} e^{-w-w^2/2b_n^2} \mathbb{P}[\sup_{s \in S} Y_n^w(s) > \tilde{r}] dw \\ &\leq M \int_{\mathbb{R}} e^{-w} \mathbb{P}[\sup_{s \in S} Y_n^w(s) > \tilde{r}] dw, \end{aligned}$$

so that

$$\sup_{n \geq 1} n\mathbb{P}[\sup_{s \in S} X_n(s) > r] \leq M \int_{\mathbb{R}} e^{-w} \sup_{n \geq 1} \mathbb{P}[\sup_{s \in S} Y_n^w(s) > \tilde{r}] dw. \tag{16}$$

First, we have

$$\int_0^{+\infty} e^{-w} \sup_{n \geq 1} \mathbb{P}[\sup_{s \in S} Y_n^w(s) > \tilde{r}] dw \leq \int_0^{+\infty} e^{-w} dw < \infty. \tag{17}$$

By tightness of the family $\{Y_n^w - \mu_n^w; w \in \mathbb{R}, n \geq 1\}$ (see Lemma 6.2), there exists $c_1 > 0$ such that for all $n \geq 1$ and $w \in \mathbb{R}$,

$$\mathbb{P}[\sup_{s \in S} (Y_n^w(s) - \mu_n^w(s)) > c_1] < \frac{1}{2}.$$

Furthermore, Eqs. 14–15 and assumption ii) of Theorem 6.1 together imply that there are some $c_2, c_3 > 0$ and $n_0 \geq 1$ such that for all $n \geq n_0$ and all $w < 0$,

$$\sup_{s \in S} \mu_n^w(s) \leq \frac{1}{2}w + c_2, \text{ and } \sup_{s \in S} \text{Var}[Y_n^w(s)] \leq c_3^2.$$

Applying Borell’s inequality (see Theorem D.1 in Piterbarg (1996)), we obtain

$$\mathbb{P}[\sup_{s \in S} Y_n^w(s) > \tilde{r}] < 2\psi\left(-\frac{\tilde{r} - w/2 - c_1 - c_2}{c_3}\right) \leq 2e^{-\left(\frac{\tilde{r} - w/2 - c_1 - c_2}{c_3}\right)^2}, \quad w < 0,$$

where ψ is the tail of the standard Gaussian distribution. Consequently,

$$\int_{-\infty}^0 e^{-w} \sup_{n \geq n_0} \mathbb{P}[\sup_{s \in S} Y_n^w(s) > \tilde{r}] dw < 2 \int_{-\infty}^0 e^{-w} e^{-\left(\frac{\tilde{r} - w/2 - c_1 - c_2}{c_3}\right)^2} dw < \infty. \tag{18}$$

Eqs. 17 and 18 together imply

$$\int_{\mathbb{R}} e^{-w} \sup_{n \geq 1} \mathbb{P}[\sup_{s \in S} Y_n^w(s) > \tilde{r}] dw < \infty. \tag{19}$$

In view of Eq. 16, this proves i).

- Proof of ii): For $\delta > 0$, we have

$$\begin{aligned} \sup_{n \geq 1} n \mathbb{P}[\omega_{X_n}(\delta) \geq \varepsilon] &= \sup_{n \geq 1} \frac{n}{\sqrt{2\pi} b_n e^{b_n^2/2}} \int_{\mathbb{R}} e^{-w - \frac{w^2}{2b_n^2}} \mathbb{P}[\omega_{e^{Y_n^w}}(\delta) \geq \varepsilon] dw \\ &\leq M \int_{\mathbb{R}} e^{-w} \sup_{n \geq 1} \mathbb{P}[\omega_{e^{Y_n^w}}(\delta) \geq \varepsilon] dw. \end{aligned}$$

We will apply Lebesgue dominated convergence Theorem to prove that this has limit zero as $\delta \rightarrow 0$. Lemma 6.2 implies that for every fixed w , the sequence $\{e^{Y_n^w}; n \geq 1\}$ is tight so that

$$\lim_{\delta \rightarrow 0} e^{-w} \sup_{n \geq 1} \mathbb{P}[\omega_{e^{Y_n^w}}(\delta) \geq \varepsilon] = 0.$$

It remains to prove a suitable domination condition: since $\delta \mapsto \mathbb{P}[\omega_{e^{Y_n^w}}(\delta) \geq \varepsilon]$ is a nondecreasing function, it is enough to prove that for some $\delta_0 > 0$

$$\int_{\mathbb{R}} e^{-w} \sup_{n \geq 1} \mathbb{P}[\omega_{e^{Y_n^w}}(\delta_0) \geq \varepsilon] dw < \infty.$$

Since $\omega_{e^{Y_n^w}}(\delta_0) \leq \exp[\sup_{t \in T} Y_n^w(t)]$, we have

$$e^{-w} \sup_{n \geq 1} \mathbb{P}[\omega_{e^{Y_n^w}}(\delta_0) \geq \varepsilon] \leq e^{-w} \sup_{n \geq 1} \mathbb{P}[\sup_{t \in T} Y_n^w(t) \geq \ln \varepsilon].$$

Equation 19 with $\tilde{r} = \ln \varepsilon$ provides the required domination condition.

- Proof of iii): We have to prove that for all set $A \subseteq \overline{C}_0^+$ of the form

$$A = \{f \in \mathbb{C}_0^+; \exists j \in \llbracket 1, k \rrbracket, f(s_j) > x_k\}, \quad k \geq 1, s_1, \dots, s_k \in S, x_1, \dots, x_k \in (0, +\infty),$$

it holds

$$\lim_{n \rightarrow \infty} n \mathbb{P}[X_n \in A] = \nu(A).$$

Letting $y_j = \ln x_j$, we have

$$\begin{aligned} n \mathbb{P}[X_n \in A] &= n \mathbb{P}[\exists j \in \llbracket 1, k \rrbracket, X_n(s_j) > x_j] \\ &= n \mathbb{P}[\exists j \in \llbracket 1, k \rrbracket, Y_n(s_j) > y_j] \\ &= \frac{n}{\sqrt{2\pi} b_n e^{b_n^2/2}} \int_{\mathbb{R}} e^{-w - \frac{w^2}{2b_n^2}} \mathbb{P}[\exists j \in \llbracket 1, k \rrbracket, Y_n^w(s_j) > y_j] dw \\ &\sim \int_{\mathbb{R}} e^{-w - \frac{w^2}{2b_n^2}} \mathbb{P}[\exists j \in \llbracket 1, k \rrbracket, Y_n^w(s_j) > y_j] dw. \end{aligned}$$

We will apply once again Lebesgue’s dominated convergence Theorem. Lemma 6.3 entails

$$\mathbb{P}[\exists j \in \llbracket 1, k \rrbracket, Y_n^w(s_j) > y_j] \rightarrow \mathbb{P}[\exists j \in \llbracket 1, k \rrbracket, w + W(s_j) - \frac{1}{2} \Gamma(s_j, s_0) > y_j],$$

so that we expect

$$\begin{aligned} n\mathbb{P}[X_n \in A] &\rightarrow \int_{\mathbb{R}} e^{-w} \mathbb{P}[\exists j \in \llbracket 1, k \rrbracket, w + W(s_j) - \frac{1}{2}\Gamma(s_j, s_0) > y_j] dw \\ &= \int_0^\infty \mathbb{P}[\exists j \in \llbracket 1, k \rrbracket, we^{W(s_j) - \frac{1}{2}\Gamma(s_j, s_0)} > x_j] w^{-2} dw \\ &= \nu(A). \end{aligned}$$

We are left to prove a suitable domination condition: for all $n \geq 1$, we have

$$e^{-w - \frac{w^2}{2b_n^2}} \mathbb{P}[\exists j \in \llbracket 1, k \rrbracket, Y_n^w(s_j) > y_j] \leq e^{-w} \sup_{n \geq 1} \mathbb{P}[\sup_{s \in S} Y_n^w(s) > \tilde{r}]$$

with $\tilde{r} = \min_{1 \leq j \leq k} y_j$. Equation 19 yields the required domination condition. \square

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