The Hardy-Schrödinger operator on the Poincaré Ball: compactness and multiplicity

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THE HARDY–SCHRÖDINGER OPERATOR ON THE POINCARÉ BALL: COMPACTNESS AND MULTIPLICITY FOR INTERIOR SINGULARITY.

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ABSTRACT. Let $\Omega_{\mathbb{B}^n}$ be a a compact smooth domain in the Poincaré ball model of the Hyperbolic space $\mathbb{B}^n$, $n \geq 5$. Let $0 < s < 2$ and write $2^\star(s) := \frac{2(n-s)}{n-2}$ for the corresponding critical Sobolev exponent. We show that if $\gamma < \frac{(n-2)^2}{4} - 4$ and $\lambda > \frac{n-2}{n-4}\left(\frac{n(n-4)}{4} - \gamma\right)$, then the following Dirichlet boundary value problem:

$$\begin{cases} -\Delta_{\mathbb{B}^n} u - \gamma V_2 u - \lambda u &= V_{2^\star(s)} |u|^{2^\star(s)-2} u \quad \text{in } \Omega_{\mathbb{B}^n} \\ u &= 0 \quad \text{on } \partial\Omega_{\mathbb{B}^n}, \end{cases}$$

has infinitely many solutions. Here $-\Delta_{\mathbb{B}^n}$ is the Laplace-Beltrami operator associated with the metric $g_{\mathbb{B}^n} = \frac{4}{(1-|x|^2)^2} g_{\text{Eucl}}$, V_2 is the corresponding Hardy-type potential that behaves like $\frac{1}{r^2}$ at the origin, while $V_{2^\star(s)}$ is the Hardy-Sobolev weight, which behaves like $\frac{1}{r^s}$ at the origin. The solutions belong to $C^2(\overline{\Omega}_{\mathbb{B}^n} \setminus \{0\})$ while around 0 they behave like

$$u(x) \sim \frac{K}{|x|^{\frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - \gamma}}} \text{ for some } K \in \mathbb{R}.$$

CONTENTS

1. INTRODUCTION

Consider the Poincaré ball model of the Hyperbolic space $\mathbb{B}^n$, $n \geq 3$, which is the Euclidean unit ball $B_1(0) := \{x \in \mathbb{R}^n : |x| < 1\}$ endowed with the metric

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$g_{\mathbb{B}^n} = \left(\frac{2}{1-|x|^2}\right)^2 g_{\text{Eucl}}$. Let

$$f(r) := \frac{(1-r^2)^{n-2}}{r^{n-1}} \text{ and } G(r) := \int_r^1 f(t)dt, \tag{1}$$

where $r = \sqrt{\sum_{i=1}^{n} x_i^2}$ denotes the Euclidean distance from a point x to the origin. The function $\frac{1}{n\omega_{n-1}} G(r)$ is then a fundamental solution of the Hyperbolic Laplacian $\Delta_{\mathbb{B}^n} u = div_{\mathbb{B}^n}(\nabla_{\mathbb{B}^n} u)$. Moreover, if we consider the hyperbolic scaling of a given function $u : \mathbb{B}^n \to \mathbb{R}$, defined for $\lambda > 0$ by

$$u_\lambda(r) = \lambda^{-\frac{1}{2}} u\left(G^{-1}(\lambda G(r))\right),$$

then for any radially symmetric $u \in H^1_r(\mathbb{B}^n)$ and $p \geq 1$, one has the following invariance property:

$$\int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u_\lambda|^p \, dv_{g_{\mathbb{B}^n}} = \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^p dv_{g_{\mathbb{B}^n}} \quad \text{and} \quad \int_{\mathbb{B}^n} V_p |u_\lambda|^p dv_{g_{\mathbb{B}^n}} = \int_{\mathbb{B}^n} V_p |u|^p dv_{g_{\mathbb{B}^n}}, \tag{2}$$

where

$$V_p(r) := \frac{f(r)^2(1-r^2)^2}{4(n-2)^2 G(r)^{\frac{p+2}{2}}}. \tag{3}$$

The weights V_p have the following asymptotic behaviors: for $n \geq 3$ and $p > 1$,

$$V_p(r) = \frac{c_0(n,p)}{r^{n(1-p/2^*)}}(1+o(1)) \qquad \text{as } r \to 0$$

$$V_p(r) = \frac{c_1(n,p)}{(1-r)^{(n-1)(p-2)/2}}(1+o(1)) \qquad \text{as } r \to 1. \tag{4}$$

In particular for $n \geq 3$, the weight $V_2(r) = \frac{1}{4(n-2)^2}\left(\frac{f(r)(1-r^2)}{G(r)}\right)^2 \sim_{r\to 0} \frac{1}{4r^2}$, while at $r = 1$ it has a finite positive value. In other words, V_2 is qualitatively similar to the Euclidean Hardy potential, which led Sandeep–Tintarev to establish the following Hyperbolic Hardy inequality on $\mathbb{B}^n$ (Theorem 3.4 of [32]):

$$\frac{(n-2)^2}{4} \int_{\mathbb{B}^n} V_2 |u|^2 \, dv_{g_{\mathbb{B}^n}} \leq \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} \quad \text{for any } u \in H^1(\mathbb{B}^n). \tag{5}$$

They also show the following Hyperbolic Sobolev inequality: ffor some constant $C > 0$.

$$C\left(\int_{\mathbb{B}^n} V_{2^\star} |u|^{2^\star} \, dv_{g_{\mathbb{B}^n}}\right)^{2/2^\star} \leq \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} \quad \text{for any } u \in H^1(\mathbb{B}^n). \tag{6}$$

By interpolating between these two inequalities, then one easily obtain for $0 \leq s \leq 2$, the following Hyperbolic Hardy-Sobolev inequality [11]:
If $\gamma < \frac{(n-2)^2}{4}$, then there exists a constant $C > 0$ such that for any $u \in H^1(\mathbb{B}^n)$,

$$C\left(\int_{\mathbb{B}^n} V_{2^*(s)} |u|^{2^*(s)} \, dv_{g_{\mathbb{B}^n}}\right)^{2/2^*(s)} \leq \int_{\mathbb{B}^n} |\nabla_{\mathbb{B}^n} u|^2 \, dv_{g_{\mathbb{B}^n}} - \gamma \int_{\mathbb{B}^n} V_2 |u|^2 \, dv_{g_{\mathbb{B}^n}}, \tag{7}$$

where $2^\star(s) := \frac{2(n-s)}{(n-2)}$. Note that $V_{2^\star(s)}$ behaves like $\frac{1}{r^s}$ at the origin, making (7) the exact analogue of the Euclidean Hardy-Sobolev inequality:

$$C\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx\right)^{2/2^*(s)} \leq \int_\Omega |\nabla u|^2\,dx - \gamma\int_\Omega \frac{u^2}{|x|^2}\,dx \quad \text{for any } u\in C_c^\infty(\Omega).$$

In this paper, we are interested in the question of existence and multiplicity of solutions to the following Dirichlet boundary value problem:

$$\left\{\begin{array}{rll} -\Delta_{\mathbb{B}^n}u - \gamma V_2 u - \lambda u &= V_{2^\star(s)}|u|^{2^\star(s)-2}u & \text{in } \Omega_{\mathbb{B}^n}\\ u &= 0 & \text{on } \partial\Omega_{\mathbb{B}^n},\end{array}\right. \tag{8}$$

where $\Omega_{\mathbb{B}^n}$ is a compact smooth subdomain of $\mathbb{B}^n$, $n\geq 3$, such that $0\in\Omega_{\mathbb{B}^n}$, but $\overline{\Omega}_{\mathbb{B}^n}$ does not touch the boundary of $\mathbb{B}^n$. It is clear that (8) is the Euler-Lagrange equation for the following energy functional on $H_0^1(\Omega_{\mathbb{B}^n})$:

$$\mathcal{J}_{\gamma,s,\lambda}(u) := \frac{\int_{\Omega_{\mathbb{B}^n}}\left(|\nabla_{\mathbb{B}^n}u|^2 - \gamma V_2 u^2 - \lambda u^2\right)dv_{g_{\mathbb{B}^n}}}{\left(\int_{\Omega_{\mathbb{B}^n}}|u|^{2^*(s)}V_{2^\star(s)}dv_{g_{\mathbb{B}^n}}\right)^{2/2^\star(s)}},$$

where $H_0^1(\Omega_{\mathbb{B}^n})$ is the completion of $C_c^\infty(\Omega_{\mathbb{B}^n})$ with respect to the norm given by $\|u\| = \sqrt{\int_{\Omega_{\mathbb{B}^n}}|\nabla_{\mathbb{B}^n}u|^2 dv_{g_{\mathbb{B}^n}}}$.

The existence of a positive ground state solution for (8) has already been addressed in [11] and is stated below for comparison purposes. This paper is dedicated to the proof of the second part, which is concerned with the multiplicity of higher energy solutions.

Theorem 1.1. *Let $\Omega_{\mathbb{B}^n}\Subset\mathbb{B}^n$, $n\geq 5$, be a smooth compact domain containing 0 and let $0<s<2$. Suppose $\lambda > \frac{n-2}{n-4}\left(\frac{n(n-4)}{4}-\gamma\right)$, then*

1. *If $\gamma \leq \frac{(n-2)^2}{4}-1$, then (8) has a positive solution that is a ground state for $\mathcal{J}_{\gamma,s}$.*
2. *If $\gamma < \frac{(n-2)^2}{4}-4$, then (8) has an infinite number of solutions corresponding to higher energy critical levels for $\mathcal{J}_{\gamma,s}$.*

These solutions belong to $C^2(\overline{\Omega}_{\mathbb{B}^n}\setminus\{0\})$ while around 0 they behave like

$$u(x)\sim K\ G(|x|)^{\frac{1}{2}-\sqrt{\frac{1}{4}-\frac{\gamma}{(n-2)^2}}} \sim \frac{K}{|x|^{\frac{n-2}{2}-\sqrt{\frac{(n-2)^2}{4}-\gamma}}} \quad \text{for some } K\in\mathbb{R}. \tag{9}$$

We also note that for $\gamma > \frac{(n-2)^2}{4}-1$, the existence of positive ground state solutions in Part 1) was also established in [11] under a global condition requiring the positivity of the *Hardy-singular boundary mass* of the compact subdomain $\Omega_{\mathbb{B}^n}$. We do not know whether there is a corresponding condition that yields the multiplicity result of Part 2) in low dimensions. This said, we can show that compactness holds for non-negative solutions of the subcritical problem, when the non-linearities approach $2^*(s)$.

In order to prove Theorem 1.1, we shall use a conformal transformation

$$g_{\mathbb{B}^n} = \varphi^{\frac{4}{n-2}}\,\text{Eucld} \quad \text{where} \quad \varphi = \left(\frac{2}{1-r^2}\right)^{\frac{n-2}{2}},$$

to reduce equation (8) to a Dirichlet boundary value problem on Euclidean space. We recall the following result from [11]

Lemma 1. *$u \in H_0^1(\Omega_{\mathbb{B}^n})$ satisfies (8) if and only if $v := \varphi u \in H_0^1(\Omega)$ satisfies*

$$\left\{ \begin{array}{rll} -\Delta v - \left(\frac{\gamma}{|x|^2} + h_{\gamma,\lambda}(x)\right) v &= b(x)\frac{v^{2^\star(s)-1}}{|x|^s} & \text{in } \Omega \\ v &= 0 & \text{on } \partial\Omega, \end{array} \right. \tag{10}$$

where $b(x)$ is a positive function in $C^0(\overline{\Omega})$ with $b(0) = \dfrac{(n-2)^{\frac{2-s}{n-2}}}{2^{2-s}}$. Moreover, when $n \geq 5$, $b \in C^1(\overline{\Omega})$ and $\nabla b(0) = 0$. In addition, there exists $c_3, c_4 \in \mathbb{R}$ such that

$$h_{\gamma,\lambda}(x) = h_{\gamma,\lambda}(r) = \left\{ \begin{array}{ll} \frac{4\gamma}{r} + c_3 + O(r) & \text{when } \ n = 3, \\ 8\gamma \log\frac{1}{r} + c_4 + O(r) & \text{when } \ n = 4, \\ \frac{4(n-2)}{n-4}\gamma + 4\lambda - n(n-2) & \text{when } \ n \geq 5, \end{array} \right. \tag{11}$$

Moreover, the hyperbolic operator $-\Delta_{\mathbb{B}^n} - \gamma V_2 - \lambda$ is coercive if and only if the corresponding Euclidean operator $-\Delta - \left(\frac{\gamma}{|x|^2} + h_{\gamma,\lambda}(x)\right)$ is coercive.

Our multiplicity result will therefore follow from the following more general Euclidean statement. In this paper $D^{1,2}(\Omega)$ – or $H_0^1(\Omega)$ if the domain is bounded – is the completion of $C_c^\infty(\Omega)$ with respect to the norm given by $||u||^2 = \int\limits_\Omega |\nabla u|^2\, dx$.

Theorem 1.2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $0 \leq \theta < 2$. Let h and b be two real-valued functions with the following properties:*

- *h is in $C^1(\overline{\Omega} \setminus \{0\})$ with $\lim\limits_{x\to 0} |x|^\theta h(x) = c$ and $\lim\limits_{x\to 0} |x|^\theta (x, \nabla h(x)) = -c\theta$ for some $c \in \mathbb{R}$, and the operator $-\Delta - \frac{\gamma}{|x|^2} - h(x)$ is coercive in Ω.*
- *b is a non-negative function in $C^1(\overline{\Omega})$ such that $b(0) > 0$ and $\nabla b(0) = 0$, while if $\theta = 0$, we shall simply assume that b is a positive constant.*

If $\gamma < \frac{(n-2)^2}{4} - (2-\theta)^2$ and $\lim\limits_{|x|\to 0} |x|^\theta h(x) > 0$, then the boundary value problem

$$\left\{ \begin{array}{rcll} -\Delta u - \gamma\frac{u}{|x|^2} - h(x)u &=& b(x)\frac{|u|^{2^\star(s)-2}u}{|x|^s} & \text{in } \Omega \setminus \{0\}, \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{12}$$

has an infinite number of possibly sign-changing solutions in $H_0^1(\Omega)$. Moreover, these solutions belong to $C^2(\overline{\Omega} \setminus \{0\})$ while around 0 they behave like

$$u(x) \sim \frac{K}{|x|^{\frac{n-2}{2} - \sqrt{\frac{(n-2)^2}{4} - \gamma}}} \quad \text{for some } K \in \mathbb{R}.$$

The multiplicity result will follow from standard min-max arguments once we prove the required compactness, which relies on blow-up analysis techniques. The proof consists of analyzing the asymptotic behaviour of a family of solutions to the related

subcritical equations –potentially developing a singularity at zero– as we approach the critical exponent.

Theorem 1.3. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $0 \leq \theta \leq 2 - 2/2^\star(s)$. Suppose that the functions $h(x)$ and $b(x)$ satisfies the hypothesis of Theorem (1.2), and let $(p_\varepsilon)_{\varepsilon>0}$ be in the interval $[0, 2^\star(s) - 2)$ in such a way that $\lim_{\varepsilon\to 0} p_\varepsilon = 0$. Consider a sequence of functions $(u_\varepsilon)_{\varepsilon>0}$ that is uniformly bounded in $H_0^1(\Omega)$ and such that for each $\epsilon > 0$, u_ϵ is a solution to the equation:*

$$\left\{ \begin{array}{rcll} -\Delta u_\varepsilon - \gamma \frac{u_\varepsilon}{|x|^2} - h(x)u_\varepsilon &=& b(x)\frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon}u_\varepsilon}{|x|^s} & \text{in } \Omega \setminus \{0\}, \\ u_\varepsilon &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{13}$$

If $\gamma < \frac{(n-2)^2}{4} - (2-\theta)^2$ and $c = \lim_{|x|\to 0} |x|^\theta h(x) > 0$, then the sequence $(u_\varepsilon)_{\varepsilon>0}$ is pre-compact in the space $H_0^1(\Omega)$.

As to the regularity of the solutions, this will follow from the following result established by Ghoussoub-Robert in [21, 22]. Assuming that $\gamma < \frac{(n-2)^2}{4}$, note that the function $x \mapsto |x|^{-\beta}$ is a solution of

$$(-\Delta - \frac{\gamma}{|x|^2})u = 0 \qquad \text{on } \mathbb{R}^n \setminus \{0\}, \tag{14}$$

if and only if $\beta \in \{\beta_-(\gamma), \beta_+(\gamma)\}$, where

$$\beta_\pm(\gamma) := \frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} - \gamma}. \tag{15}$$

Actually, one can show that any non-negative solution $u \in C^2(\mathbb{R}^n \setminus \{0\})$ of (14) is of the form

$$u(x) = C_-|x|^{-\beta_-(\gamma)} + C_+|x|^{-\beta_+(\gamma)} \text{ for all } x \in \mathbb{R}^n \setminus \{0\}, \tag{16}$$

where $C_-, C_+ \geq 0$.

We collect the following important results from the papers [21, 22] which we shall use repeatedly in our work.

Theorem 1.4 (Optimal regularity and Hopf Lemma)**.** *Let $\gamma < \frac{(n-2)^2}{4}$ and let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Caratheodory function such that*

$$|f(x,v)| \leq C|v|\left(1 + \frac{|v|^{2^\star(s)-2}}{|x|^s}\right) \text{ for all } x \in \Omega \text{ and } v \in \mathbb{R}.$$

(1) Let $u \in D^{1,2}(\Omega)$ be a weak solution of

$$\Delta u - \frac{\gamma + O(|x|^\tau)}{|x|^2}u = f(x,u), \tag{17}$$

for some $\tau > 0$. Then, there exists $K \in \mathbb{R}$ such that

$$\lim_{x\to 0} \frac{u(x)}{|x|^{-\beta_-(\gamma)}} = K. \tag{18}$$

Moreover, if $u \geq 0$ and $u \not\equiv 0$, we have that $K > 0$.

(2) As a consequence, one gets that if $u \in D^{1,2}(\mathbb{R}^n)$ is a weak solution for

$$-\Delta u - \frac{\gamma}{|x|^2}u = \frac{|u|^{2^\star(s)-2}u}{|x|^s} \qquad \text{in } \mathbb{R}^n \setminus \{0\},$$

then there exists $K_1, K_2 \in \mathbb{R}$ such that

$$u(x) \sim_{|x|\to 0} \frac{K_1}{|x|^{\beta_-(\gamma)}} \quad \text{and} \quad u(x) \sim_{|x|\to+\infty} \frac{K_1}{|x|^{\beta_+(\gamma)}},$$

and therefore there exists a constant $C > 0$ such that for all x in $\mathbb{R}^n \setminus \{0\}$,

$$|u(x)| \le \frac{C}{|x|^{\beta_-(\gamma)} + |x|^{\beta_+(\gamma)}}. \tag{19}$$

The next theorem describes the properties of the Green's function of the Hardy-Schrödinger operator in a bounded smooth domain. To prove this theorem one can argue as in the case $\theta = 0$ like in [22].

Theorem 1.5 (Green's function)**.** *Let Ω be a smooth bounded domain of $\mathbb{R}^n$ such that $0 \in \Omega$ is an interior point. Let $\gamma < \frac{(n-2)^2}{4}$ and let $h \in C^1(\overline{\Omega} \setminus \{0\})$ satisfy the hypothesis of Theorem* (1.2) *such that the operator $-\Delta - \frac{\gamma}{|x|^2} - h(x)$ is coercive in Ω.*

(1) There exists then $H \in C^\infty(\overline{\Omega} \setminus \{0\})$ such that

$$\begin{cases} -\Delta H - \frac{\gamma}{|x|^2}H - h(x)H = 0 & \text{in } \Omega \setminus \{0\} \\ H > 0 & \text{in } \Omega \setminus \{0\} \\ H = 0 & \text{on } \partial\Omega. \end{cases} \tag{20}$$

These solutions are unique up to a positive multiplicative constant, and there exists $\tilde{C} > 0$ such that $H(x) \simeq_{x\to 0} \frac{\tilde{C}}{|x|^{\beta_+(\gamma)}}$.

(2) Then there exists

$$G : (\Omega \setminus \{0\})^2 \setminus \{(x,x)/\, x \in \Omega \setminus \{0\}\} \to \mathbb{R}$$

such that

(i) *For any $p \in \Omega \setminus \{0\}$, $G_p := G(p,\cdot) \in H^2_1(\Omega \setminus B_\delta(p))$ for all $\delta > 0$, $G_p \in C^{2,\theta}(\overline{\Omega} \setminus \{0,p\})$*

(ii) *For all $f \in L^p(\Omega)$, $p > n/2$, and all $\varphi \in H^2_{1,0}(\Omega)$ such that*

$$-\Delta\varphi - \left(\frac{\gamma}{|x|^2} + h(x)\right)\varphi = f \text{ in } \Omega\,;\ \varphi_{|\partial\Omega} = 0,$$

we have

$$\varphi(p) = \int_\Omega G(p,x) f(x)\, dx \tag{21}$$

In addition, $G > 0$ is unique and

(iii) *For all $p \in \Omega \setminus \{0\}$, there exists $C_0(p) > 0$ such that*

$$G_p(x) \sim_{x\to 0} \frac{C_0(p)}{|x|^{\beta_-(\gamma)}} \text{ and } G_p(x) \sim_{x\to p} \frac{1}{(n-2)\omega_{n-1}|x-p|^{n-2}} \tag{22}$$

(iv) *There exists $C > 0$ such that*

$$0 < G_p(x) \leq C \left(\frac{\max\{|p|,|x|\}}{\min\{|p|,|x|\}} \right)^{\beta_-(\gamma)} |x-p|^{2-n} \tag{23}$$

(v) *For all $\omega \Subset \Omega$, there exists $C(\omega) > 0$ such that*

$$C(\omega) \left(\frac{\max\{|p|,|x|\}}{\min\{|p|,|x|\}} \right)^{\beta_-(\gamma)} |x-p|^{2-n} \leq G_p(x) \text{ for all } p, x \in \omega \setminus \{0\}. \tag{24}$$

Via the conformal transformation the above regularity result will yield that the corresponding solutions for equation (8) in the Hyperbolic Sobolev Space $H_0^1(\Omega_{\mathbb{B}^n})$ will satisfy

$$\lim_{|x|\to 0} \frac{u(x)}{G(|x|)^{\alpha_-}} = K \in \mathbb{R}, \tag{25}$$

where

$$\alpha_-(\gamma) = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{\gamma}{(n-2)^2}}, \tag{26}$$

which amounts to the regularity claimed in Theorem 1.1.

A general remark, in this paper every convergence is up to a subsequence.

2. Setting the blow-up

Throughout this paper, Ω will denote a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$. We will always assume that $\gamma < \frac{(n-2)^2}{4}$, $s \in (0,2)$, and $\beta_\pm(\gamma) := \frac{n-2}{2} \pm \sqrt{\frac{(n-2)^2}{4} - \gamma}$. We now describe the compactness result that will be needed for the proof:
For $\varepsilon > 0$, we let $p_\varepsilon \in [0, 2^\star(s) - 2)$ be such that

$$\lim_{\varepsilon \to 0} p_\varepsilon = 0. \tag{27}$$

We also consider

(28)
$h \in C^1(\overline{\Omega} \setminus \{0\})$ with $\lim_{x\to 0} |x|^\theta h(x) = c$ and $\lim_{x\to 0} |x|^\theta (x, \nabla h(x)) = -c\theta$ for some $0 \leq \theta < 2$ and $c \in \mathbb{R}$, such that the operator $-\Delta - \frac{\gamma}{|x|^2} - h(x)$ is coercive in Ω.

and

(29) a non-negative function b in $C^1(\overline{\Omega})$ with $b(0) > 0$ and $\nabla b(0) = 0$.

Consider a sequence of functions $(u_\varepsilon)_{\varepsilon>0}$ in $H_0^1(\Omega)$ such that for all $\varepsilon > 0$ the function u_ε is a solution to the Dirichlet boundary value problem:

$$\left\{ \begin{array}{rcll} -\Delta u_\varepsilon - \gamma \frac{u_\varepsilon}{|x|^2} - h(x) u_\varepsilon &=& b(x) \frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon} u_\varepsilon}{|x|^s} & \text{in } D^{1,2}(\Omega), \\ u_\varepsilon &=& 0 & \text{on } \partial\Omega. \end{array} \right. \tag{E_ε}$$

where (p_ε), $h(x)$ and $b(x)$ is such that (27), (28) and (29) holds.

By the regularity theorem (1.4), $u_\varepsilon \in C^2(\overline{\Omega}\setminus\{0\})$ and there exists $K_\varepsilon \in \mathbb{R}$ such that $\lim_{x\to 0} |x|^{\beta_-(\gamma)} u_\varepsilon(x) = K_\epsilon$. In addition, we assume that the sequence $(u_\varepsilon)_{\varepsilon>0}$ is bounded in $H_0^1(\Omega)$ and we let $\Lambda > 0$ be such that

$$\int_\Omega \frac{|u_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s} dx \le \Lambda \qquad \text{for all } \varepsilon > 0. \tag{30}$$

It then follows from the weak compactness of the unit ball of $H_0^1(\Omega)$ that there exists $u_0 \in H_0^1(\Omega)$ such that

$$u_\varepsilon \rightharpoonup u_0 \qquad \text{weakly in } H_0^1(\Omega) \text{ as } \varepsilon \to 0. \tag{31}$$

Then u_0 s a solution to the Dirichlet boundary value problem

$$\begin{cases} -\Delta u_0 - \gamma \frac{u_0}{|x|^2} - h(x)u_0 &= b(x)\frac{|u_0|^{2^\star(s)-2}u_0}{|x|^s} & \text{in } \Omega\setminus\{0\}, \\ u_0 &= 0 & \text{on } \partial\Omega. \end{cases}$$

Again from the regularity theorem (1.4), $u_0 \in C^{2,\theta}(\overline{\Omega}\setminus\{0\})$ and $\lim_{x\to 0} |x|^{\beta_-(\gamma)} u_0(x) = K_0 \in \mathbb{R}$. Fix now $\tau \in \mathbb{R}$ such that

$$\beta_-(\gamma) < \tau < \frac{n-2}{2}. \tag{32}$$

The following Proposition shows that the sequence $(u_\varepsilon)_\varepsilon$ is pre-compact in $H_0^1(\Omega)$ if $|x|^\tau u_\varepsilon$ is uniformly bounded in $L^\infty(\Omega)$.

Proposition 1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \ge 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$, $\gamma < \frac{(n-2)^2}{4}$. We let (u_ε), (p_ε), $h(x)$ and $b(x)$ be such that (E_ε), (27), (28), (29) and (30) holds. Suppose that there exists $C > 0$ such that $|x|^\tau |u_\varepsilon(x)| \le C$ for all $x \in \Omega$ and for all $\varepsilon > 0$. Then up to a subsequence, $\lim_{\varepsilon\to 0} u_\varepsilon = u_0$ in $H_0^1(\Omega)$ where u_0 is as in (31).*

Proof: We have assumed that $|x|^\tau |u_\varepsilon(x)| \le C$ for all $x \in \Omega$ and for all $\varepsilon > 0$. So the sequence (u_ε) is uniformly bounded in $L^\infty(\Omega')$ for any $\Omega' \subset\subset \overline{\Omega}\setminus\{0\}$. Then by standard elliptic estimates and from (31) it follows that $u_\varepsilon \to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$. Now since $|x|^\tau |u_\varepsilon(x)| \le C$ for all $x \in \Omega$ and for all $\varepsilon > 0$ and since $\tau < \frac{n-2}{2}$, we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \int_{B_\delta(0)} b(x)\frac{|u_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s} dx = 0 \ \text{ and } \ \lim_{\delta\to 0}\lim_{\varepsilon\to 0} \int_{B_\delta(0)} \frac{|u_\varepsilon|^2}{|x|^2} dx = 0. \tag{33}$$

Therefore

$$\lim_{\varepsilon\to 0}\int_\Omega b(x)\frac{|u_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s} dx = \int_\Omega b(x)\frac{|u_0|^{2^\star(s)}}{|x|^s} dx \ \text{ and } \ \lim_{\varepsilon\to 0}\int_\Omega \frac{|u_\varepsilon|^2}{|x|^2} dx = \int_\Omega \frac{|u_0|^2}{|x|^2} dx.$$

From (E_ε) and (31) we then obtain

$$\lim_{\varepsilon\to 0}\int_\Omega \left(|\nabla u_\varepsilon|^2 - \gamma\frac{u_\varepsilon^2}{|x|^2} - h(x)u_\varepsilon^2\right) dx = \int_\Omega \left(|\nabla u_0|^2 - \gamma\frac{u_0^2}{|x|^2} - h(x)u_0^2\right) dx$$

$$\text{and so then } \lim_{\varepsilon\to 0}\int_\Omega |\nabla u_\varepsilon|^2 = \lim_{\varepsilon\to 0}\int_\Omega |\nabla u_0|^2.$$

And hence $\lim_{\varepsilon\to 0} u_\varepsilon = u_0$ in $H_0^1(\Omega)$. □

From now on, we shall assume that

$$\lim_{\varepsilon\to 0} \||x|^\tau u_\varepsilon\|_{L^\infty(\Omega)} = +\infty, \tag{34}$$

and work towards a contradiction. We shall say that blow-up occurs whenever (34) holds.

3. Some Scaling Lemmas

In this section we state and prove two scaling lemmas which we shall use many times in our analysis.

Lemma 2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $\gamma < \frac{(n-2)^2}{4}$. Let (u_ε), (p_ε), $h(x)$ and $b(x)$ be such that (E_ε), (27), (28), (29) and (30) holds. Let $(y_\varepsilon)_\varepsilon \in \Omega \setminus \{0\}$ and let*

$$\nu_\varepsilon^{-\frac{n-2}{2}} := |u_\varepsilon(y_\varepsilon)|, \quad \ell_\varepsilon := \nu_\varepsilon^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \quad \text{and} \quad \kappa_\varepsilon := |y_\varepsilon|^{s/2}\, \ell_\varepsilon^{\frac{2-s}{2}} \qquad \text{for } \varepsilon > 0$$

Suppose $\lim_{\varepsilon\to 0} y_\varepsilon = 0$ and $\lim_{\varepsilon\to 0} \nu_\varepsilon = 0$. Assume that for any $R > 0$ there exists $C(R) > 0$ such that for all $\varepsilon > 0$

$$|u_\varepsilon(x)| \leq C(R)\, \frac{|y_\varepsilon|^\tau}{|x|^\tau} |u_\varepsilon(y_\varepsilon)| \qquad \text{for all } x \in B_{R\kappa_\varepsilon}(y_\varepsilon) \setminus \{0\}. \tag{35}$$

Then

$$|y_\varepsilon| = O(\ell_\varepsilon) \qquad \text{as } \varepsilon \to 0. \tag{36}$$

Proof of Lemma 2: We proceed by contradiction and assume that

$$\lim_{\varepsilon\to 0} \frac{|y_\varepsilon|}{\ell_\varepsilon} = +\infty. \tag{37}$$

Then it follows from the definition of κ_ε that

$$\lim_{\varepsilon\to 0} \kappa_\varepsilon = 0,\ \lim_{\varepsilon\to 0} \frac{\kappa_\varepsilon}{\ell_\varepsilon} = +\infty \text{ and } \lim_{\varepsilon\to 0} \frac{\kappa_\varepsilon}{|y_\varepsilon|} = 0. \tag{38}$$

Fix a $\rho > 0$. We define for all $\varepsilon > 0$

$$v_\epsilon(x) := \nu_\epsilon^{\frac{n-2}{2}} u_\varepsilon(y_\varepsilon + \kappa_\varepsilon x) \qquad \text{for } x \in B_{2\rho}(0)$$

Note that this is well defined since $\lim_{\epsilon\to 0} |y_\varepsilon| = 0 \in \Omega$ and $\lim_{\varepsilon\to 0} \frac{\kappa_\varepsilon}{|y_\varepsilon|} = 0$. It follows from (35) that there exists $C(\rho) > 0$ such that all $\varepsilon > 0$

$$|v_\epsilon(x)| \leq C(\rho) \frac{1}{\left|\frac{y_\varepsilon}{|y_\epsilon|} + \frac{\kappa_\varepsilon}{|y_\epsilon|}x\right|^\tau} \qquad \forall x \in B_{2\rho}(0) \tag{39}$$

using (38) we then get as $\varepsilon \to 0$

$$|v_\epsilon(x)| \leq C(\rho)\,(1 + o(1)) \qquad \forall x \in B_{2\rho}(0).$$

From equation (E_ε) we obtain that v_ε satisfies the equation

$$-\Delta v_\epsilon - \frac{\kappa_\varepsilon^2}{|y_\varepsilon|^2} \frac{\gamma}{\left|\frac{y_\varepsilon}{|y_\varepsilon|} + \frac{\kappa_\varepsilon}{|y_\varepsilon|}x\right|^2} v_\epsilon - \kappa_\varepsilon^2\, h(y_\varepsilon + \kappa_\varepsilon x)\, v_\epsilon = b(y_\varepsilon + \kappa_\varepsilon x) \frac{|v_\epsilon|^{2^\star(s)-2-p_\varepsilon} v_\epsilon}{\left|\frac{y_\varepsilon}{|y_\varepsilon|} + \frac{\kappa_\varepsilon}{|y_\varepsilon|}x\right|^s}$$

weakly in $B_{2\rho}(0)$ for all $\varepsilon > 0$. With the help of (38), (28) and standard elliptic theory it then follows that there exists $v \in C^1(B_{2\rho}(0))$ such that

$$\lim_{\varepsilon\to 0} v_\epsilon = v \qquad \text{in } C^1(B_\rho(0)).$$

In particular,

$$v(0) = \lim_{\varepsilon\to 0} v_\epsilon(0) = 1 \tag{40}$$

and therefore $v \not\equiv 0$.

On the other hand, change of variables and the definition of κ_ε yields

$$\begin{aligned}\int\limits_{B_{\rho\kappa_\varepsilon}(y_\varepsilon)} \frac{|u_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s}\,dx &= \frac{|u_\varepsilon(y_\varepsilon)|^{2^\star(s)-p_\varepsilon}\kappa_\varepsilon^n}{|y_\varepsilon|^s}\int\limits_{B_\rho(0)} \frac{|v_\epsilon|^{2^\star(s)-p_\varepsilon}}{\left|\frac{y_\varepsilon}{|y_\varepsilon|}+\frac{\kappa_\varepsilon}{|y_\varepsilon|}x\right|^s}\,dx \\ &= \ell_\varepsilon^{-\left(1+\frac{2(2-s)}{2^\star(s)-2-p_\varepsilon}\right)}\left(\frac{|y_\varepsilon|}{\ell_\varepsilon}\right)^{s\left(\frac{n-2}{2}\right)}\int\limits_{B_\rho(0)} \frac{|v_\epsilon|^{2^\star(s)-p_\varepsilon}}{\left|\frac{y_\varepsilon}{|y_\varepsilon|}+\frac{\kappa_\varepsilon}{|y_\varepsilon|}x\right|^s}\,dx \\ &\geq \left(\frac{|y_\varepsilon|}{\ell_\varepsilon}\right)^{s\left(\frac{n-2}{2}\right)}\int\limits_{B_\rho(0)} \frac{|v_\epsilon|^{2^\star(s)-p_\varepsilon}}{\left|\frac{y_\varepsilon}{|y_\varepsilon|}+\frac{\kappa_\varepsilon}{|y_\varepsilon|}x\right|^s}\,dx.\end{aligned}$$

Using the equation (E_ε), (30), (37), (38) and passing to the limit $\varepsilon \to 0$ we get that $\int_{B_\rho(0)} |v|^{2^\star(s)}\,dx = 0$, and so then $v \equiv 0$ in $B_\rho(0)$, a contradiction with (40). Thus (37) cannot hold. This proves that $y_\varepsilon = O(\ell_\varepsilon)$ when $\varepsilon \to 0$, which proves the lemma. □

Lemma 3. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $\gamma < \frac{(n-2)^2}{4}$. Let (u_ε), (p_ε), $h(x)$ and $b(x)$ be such that (E_ε), (27), (28), (29) and (30) holds. Let $(y_\varepsilon)_\varepsilon \in \Omega \setminus \{0\}$ and let*

$$\nu_\varepsilon^{-\frac{n-2}{2}} := |u_\varepsilon(y_\varepsilon)| \quad \text{and} \quad \ell_\varepsilon := \nu_\varepsilon^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \qquad \text{for } \varepsilon > 0$$

Suppose $\nu_\varepsilon \to 0$ and $|y_\varepsilon| = O(\ell_\varepsilon)$ as $\varepsilon \to 0$.

For $\varepsilon > 0$ we rescale and define

$$w_\varepsilon(x) := \nu_\varepsilon^{\frac{n-2}{2}} u_\varepsilon(\ell_\varepsilon x) \qquad \text{for } x \in \ell_\varepsilon^{-1}\Omega \setminus \{0\}.$$

Assume that for any $R > \delta > 0$ there exists $C(R,\delta) > 0$ such that for all $\varepsilon > 0$

$$|w_\varepsilon(x)| \leq C(R,\delta) \qquad \text{for all } x \in B_R(0) \setminus \overline{B_\delta(0)}. \tag{41}$$

Then there exists $w \in D^{1,2}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ such that

$$\begin{aligned} w_\varepsilon &\rightharpoonup w \qquad \text{weakly in } D^{1,2}(\mathbb{R}^n) \quad \text{as } \varepsilon \to 0 \\ w_\varepsilon &\to w \qquad \text{in } C^1_{loc}(\mathbb{R}^n \setminus \{0\}) \quad \text{as } \varepsilon \to 0\end{aligned}$$

And w satisfies weakly the equation

$$-\Delta w - \frac{\gamma}{|x|^2}w = b(0)\frac{|w|^{2^\star(s)-2}w}{|x|^s} \text{ in } \mathbb{R}^n \setminus \{0\}.$$

Moreover if $w \not\equiv 0$, then

$$\int\limits_{\mathbb{R}^n} \frac{|w|^{2^\star(s)}}{|x|^s} \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)} \right)^{\frac{2^\star(s)}{2^\star(s)-2}}$$

and there exists $t \in (0,1]$ such that $\lim\limits_{\varepsilon \to 0} \nu_\varepsilon^{p_\varepsilon} = t$.

Proof of Lemma 3: The proof proceeds in four steps.
Step 3.1: Let $\eta \in C_c^\infty(\mathbb{R}^n)$. One has that $\eta w_\varepsilon \in H_0^1(\mathbb{R}^n)$ for $\varepsilon > 0$ sufficiently small. We claim that there exists $w_\eta \in D^{1,2}(\mathbb{R}^n)$ such that upto a subsequence

$$\begin{cases} \eta w_\varepsilon \rightharpoonup w_\eta & \text{weakly in } D^{1,2}(\mathbb{R}^n) \text{ as } \varepsilon \to 0, \\ \eta w_\varepsilon \to w_\eta(x) & a.e \text{ in } \mathbb{R}^n \text{ as } \varepsilon \to 0. \end{cases}$$

We prove the claim. Let $x \in \mathbb{R}^n$, then

$$\nabla\left(\eta w_\varepsilon\right)(x) = w_\varepsilon(x)\nabla\eta(x) + \nu_\varepsilon^{\frac{n-2}{2}} \ell_\varepsilon \ \eta(x)\nabla u_\varepsilon(\ell_\varepsilon x).$$

Now for any $\theta > 0$, there exists $C(\theta) > 0$ such that for any $a, b > 0$

$$(a+b)^2 \leq C(\theta)a^2 + (1+\theta)b^2$$

With this inequality we then obtain

$$\int\limits_{\mathbb{R}^n} |\nabla\left(\eta w_\varepsilon\right)|^2 \ dx \leq C(\theta) \int\limits_{\mathbb{R}^n} |\nabla\eta|^2 w_\varepsilon^2 \ dx + (1+\theta)\nu_\epsilon^{n-2}\ell_\varepsilon^2 \int\limits_{\mathbb{R}^n} \eta^2 \left|\nabla u_\varepsilon(\ell_\varepsilon x)\right|^2 \ dx$$

With Hölder inequality and a change of variables this becomes

$$\begin{aligned} \int\limits_{\mathbb{R}^n} |\nabla\left(\eta w_\varepsilon\right)|^2 dx \leq C(\theta) \left\|\nabla\eta\right\|_{L^n}^2 \left(\frac{\nu_\varepsilon}{\ell_\varepsilon}\right)^{n-2} \left(\int\limits_\Omega |u_\varepsilon|^{2^\star} \ dx \right)^{\frac{n-2}{n}} \\ + (1+\theta)\left(\frac{\nu_\varepsilon}{\ell_\varepsilon}\right)^{n-2} \int\limits_\Omega \left(\eta\left(\frac{x}{\ell_\varepsilon}\right)\right)^2 |\nabla u_\varepsilon|^2 \ dx. \end{aligned} \tag{42}$$

Since $\|u_\epsilon\|_{H_0^1(\Omega)} = O(1)$, so for $\varepsilon > 0$ small enough

$$\|\eta w_\varepsilon\|_{D^{1,2}(\mathbb{R}^n)} \leq C_\eta$$

Where C_η is a constant depending on the function η. The claim then follows from the reflexivity of $D^{1,2}(\mathbb{R}^n)$.

Step 3.2: Let $\eta_1 \in C_c^\infty(\mathbb{R}^n)$, $0 \leq \eta_1 \leq 1$ be a smooth cut-off function, such that

$$\eta_1 = \begin{cases} 1 & \text{for} \quad x \in B_0(1) \\ 0 & \text{for} \quad x \in \mathbb{R}^n \backslash B_0(2) \end{cases} \tag{43}$$

For any $R > 0$ we let $\eta_R = \eta_1(x/R)$. Then with a diagonal argument we can assume that upto a subsequence for any $R > 0$ there exists $w_R \in D^{1,2}(\mathbb{R}^n)$ such that

$$\begin{cases} \eta_R w_\varepsilon \rightharpoonup w_R & \text{weakly in } D^{1,2}(\mathbb{R}^n) \text{ as } \varepsilon \to 0 \\ \eta_R w_\varepsilon(x) \to w_R(x) & a.e\ x \text{ in } \mathbb{R}^n \text{ as } \varepsilon \to 0 \end{cases}$$

Since $\|\nabla\eta_R\|_n^2 = \|\nabla\eta_1\|_n^2$ for all $R > 0$, letting $\varepsilon \to 0$ in (42) we obtain that

$$\int_{\mathbb{R}^n} |\nabla w_R|^2 \, dx \leq C \qquad \text{for all } R > 0$$

where C is a constant independent of R. So there exists $w \in D^{1,2}(\mathbb{R}^n)$ such that

$$\begin{cases} w_R \rightharpoonup w & \text{weakly in } D^{1,2}(\mathbb{R}^n) \text{ as } R \to +\infty \\ w_R(x) \to w(x) & a.e\ x \text{ in } \mathbb{R}^n \text{ as } R \to +\infty \end{cases}$$

Step 3.3: We claim that $w \in C^1(\mathbb{R}^n \setminus \{0\})$ and it satisfies weakly the equation

$$-\Delta w - \frac{\gamma}{|x|^2} w = b(0) \frac{|w|^{2^\star(s)-2} w}{|x|^s} \text{ in } \mathbb{R}^n \setminus \{0\}.$$

We prove the claim. From (E_ε) it follows that for any $\varepsilon > 0$ and $R > 0$, $\eta_R w_\varepsilon$ satisfies weakly the equation

(44)

$$-\Delta(\eta_R w_\varepsilon) - \frac{\gamma}{|x|^2}(\eta_R w_\varepsilon) - \ell_\varepsilon^2\, h(\ell_\varepsilon x)(\eta_R w_\varepsilon) = b(\ell_\varepsilon x) \frac{|(\eta_R w_\varepsilon)|^{2^\star(s)-2-p_\varepsilon}(\eta_R w_\varepsilon)}{|x|^s}.$$

From (41) and (28), using the standard elliptic estimates it follows that $w_R \in C^1(B_R(0) \setminus \{0\})$ and that up to a subsequence

$$\lim_{\epsilon \to 0} \eta_R w_\varepsilon = w_R \qquad \text{in } C^1_{loc}(B_R(0) \setminus \{0\}).$$

Letting $\varepsilon \to 0$ in eqn (44) gives that w_R satisfies weakly the equation

$$-\Delta w_R - \frac{\gamma}{|x|^2} w_R = b(0) \frac{|w_R|^{2^\star(s)-2-p_\varepsilon} w_R}{|x|^s}.$$

Again we have that $|w_R(x)| \le C(R,\delta)$ for all $x \in \overline{B_{R/2}(0)} \setminus \overline{B_{2\delta}(0)}$ and then again from standard elliptic estimates it follows that $w \in C^1(\mathbb{R}^n \setminus \{0\})$ and $\lim_{R \to +\infty} \tilde{w}_R = \tilde{w}$ in $C^1_{loc}(\mathbb{R}^n \setminus \{0\})$, up to a subsequence. Letting $R \to +\infty$ we obtain that w satisfies weakly the equation

$$-\Delta w - \frac{\gamma}{|x|^2} w = b(0) \frac{|w|^{2^\star(s)-2} w}{|x|^s}.$$

This proves our claim.

Step 3.4: Coming back to equation (42) we have for $R > 0$

$$\int_{\mathbb{R}^n} |\nabla(\eta_R w_\varepsilon)|^2\ dx \le C(\theta) \left(\int_{B_0(2R) \setminus B_0(R)} (\eta_{2R} w_\varepsilon)^{2^*}\ dx \right)^{\frac{n-2}{n}} + (1+\theta) \left(\frac{\nu_\varepsilon}{\ell_\varepsilon} \right)^{n-2} \int_\Omega |\nabla u_\varepsilon|^2\ dx. \tag{45}$$

Since the sequence $(u_\varepsilon)_\varepsilon$ is bounded in $H^1_0(\Omega)$, letting $\varepsilon \to 0$ and then $R \to +\infty$ we obtain for some constant C

$$\int_{\mathbb{R}^n} |\nabla w|^2\ dx \le C \left(\lim_{\epsilon \to 0} \left(\frac{\nu_\varepsilon}{\ell_\varepsilon} \right) \right)^{n-2}. \tag{46}$$

Now if $w \not\equiv 0$ weakly satisfies the equation

$$-\Delta w - \frac{\gamma}{|x|^2} w = b(0) \frac{|w|^{2^\star(s)-2} w}{|x|^s}$$

using the definition of $\mu_{\gamma,s,0}(\mathbb{R}^n)$ it then follows that

$$\int\limits_{\mathbb{R}^n} \frac{|w|^{2^\star(s)}}{|x|^s} \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}}.$$

Hence $\lim\limits_{\epsilon\to 0}\left(\dfrac{\nu_\varepsilon}{\ell_\varepsilon}\right) > 0$ which implies that

$$t := \lim_{\epsilon\to 0} \nu_\varepsilon^{p_\varepsilon} > 0. \tag{47}$$

Since $\lim\limits_{\varepsilon\to 0}\nu_\varepsilon = 0$, therefore we have that $0 < t \leq 1$. This completes the lemma. □

4. Construction and Exhaustion of the blow-up scales

In this section we prove the following proposition:

Proposition 2. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $\gamma < \frac{(n-2)^2}{4}$. Let (u_ε), (p_ε), $h(x)$ and $b(x)$ be such that (E_ε), (27), (28), (29) and (30) holds. Assume that blow-up occurs, that is*

$$\lim_{\varepsilon\to 0} \||x|^\tau u_\varepsilon\|_{L^\infty(\Omega)} = +\infty \quad \text{where} \quad \beta_-(\gamma) < \tau < \frac{n-2}{2}.$$

Then, there exists $N \in \mathbb{N}^\star$ families of scales $(\mu_{i,\varepsilon})_{\varepsilon>0}$ such that we have:

(A1) $\lim\limits_{\varepsilon\to 0} u_\varepsilon = u_0$ *in* $C^2_{loc}(\overline{\Omega}\setminus\{0\})$ *where* u_0 *is as in*(31).

(A2) $0 < \mu_{1,\varepsilon} < ... < \mu_{N,\varepsilon}$, *for all* $\varepsilon > 0$.

(A3)

$$\lim_{\varepsilon\to 0}\mu_{N,\varepsilon} = 0 \text{ and } \lim_{\varepsilon\to 0}\frac{\mu_{\varepsilon,i+1}}{\mu_{i,\varepsilon}} = +\infty \text{ for all } 1 \leq i \leq N-1.$$

(A4) *For any $1 \leq i \leq N$ and for $\varepsilon > 0$ we rescale and define*

$$\tilde{u}_{i,\varepsilon}(x) := \mu_{i,\varepsilon}^{\frac{n-2}{2}} u_\varepsilon(k_{i,\varepsilon}x) \qquad \text{for } x \in k_{i,\varepsilon}^{-1}\Omega\setminus\{0\}$$

where $k_{i,\varepsilon} = \mu_{i,\varepsilon}^{1-\frac{p_\varepsilon}{2^\star(s)-2}}$.
Then there exists $\tilde{u}_i \in D^{1,2}(\mathbb{R}^n)\cap C^1(\mathbb{R}^n\setminus\{0\})$, $\tilde{u}_i \not\equiv 0$ such that $\tilde{u}_i$ weakly solves the equation

$$-\Delta\tilde{u}_i - \frac{\gamma}{|x|^2}\tilde{u}_i = b(0)\frac{|\tilde{u}_i|^{2^\star(s)-2}\tilde{u}_i}{|x|^s}$$

and

$$\tilde{u}_{i,\varepsilon} \longrightarrow \tilde{u}_i \qquad \text{in } C^1_{loc}(\mathbb{R}^n\setminus\{0\}) \qquad \text{as } \varepsilon\to 0,$$
$$\tilde{u}_{i,\varepsilon} \rightharpoonup \tilde{u}_i \qquad \text{weakly in } D^{1,2}(\mathbb{R}^n) \quad \text{as } \varepsilon\to 0.$$

(A5) *There exists $C > 0$ such that*

$$|x|^{\frac{n-2}{2}}|u_\varepsilon(x)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \leq C$$

for all $\varepsilon > 0$ and all $x \in \Omega \setminus \{0\}$.

(A6) $\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{\Omega\setminus B_{Rk_{N,\varepsilon}}(0)}|x|^{\frac{n-2}{2}}|u_\varepsilon(x)-u_0(x)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}}=0.$

(A7) $\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\sup_{B_{\delta k_{1,\varepsilon}}(0)\setminus\{0\}}|x|^{\frac{n-2}{2}}\left|u_\varepsilon(x)-\mu_{1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{x}{k_{1,\varepsilon}}\right)\right|^{1-\frac{p_\varepsilon}{2^\star(s)-2}}=0.$

(A8) *For any $\delta > 0$ and any $1 \leq i \leq N-1$, we have*

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{\delta k_{i+1,\varepsilon}\geq|x|\geq Rk_{i,\varepsilon}}|x|^{\frac{n-2}{2}}\left|u_\varepsilon(x)-\mu_{i+1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\varepsilon}}\right)\right|^{1-\frac{p_\varepsilon}{2^\star(s)-2}}=0.$$

(A9) *For any $i \in \{1,...,N\}$ there exists $t_i \in (0,1]$ such that $\lim_{\varepsilon\to 0}\mu_{i,\varepsilon}^{p_\varepsilon} = t_i$.*

The proof of this proposition proceeds in five steps.
Since $s > 0$, the subcriticality $2^\star(s) < 2^\star := 2^\star(0)$ of equations (E_ε) in $\Omega \setminus \{0\}$ along with (31) yields that $u_\varepsilon \to u_0$ in $C^2_{loc}(\overline{\Omega} \setminus \{0\})$. So the only blow-up point is the origin.

Step 4.1: The construction of the $\mu_{i,\varepsilon}$'s proceeds by induction. This step is the initiation.

By the regularity Theorem (1.4) and the definition of τ it follows that for any $\varepsilon > 0$ there exists $x_{1,\varepsilon} \in \Omega \setminus \{0\}$ such that

$$\sup_{x\in\Omega\setminus\{0\}}|x|^\tau|u_\varepsilon(x)| = |x_{1,\varepsilon}|^\tau|u_\varepsilon(x_{1,\varepsilon})|. \tag{48}$$

We define $\mu_{1,\varepsilon}$ and $k_{1,\varepsilon} > 0$ as follows

$$\mu_{1,\varepsilon}^{-\frac{n-2}{2}} := |u_\varepsilon(x_{1,\varepsilon})| \quad \text{and} \quad k_{1,\varepsilon} := \mu_{1,\varepsilon}^{1-\frac{p_\varepsilon}{2^\star-2}}. \tag{49}$$

Since blow-up occurs, that is (34) holds, we have

$$\lim_{\varepsilon\to 0}\mu_{1,\varepsilon} = 0$$

It follows that u_ε satisfies the hypothesis (35) of lemma (2) with $y_\varepsilon = x_{1,\varepsilon}$, $\nu_\varepsilon = \mu_{1,\varepsilon}$. Therefore

$$|x_{1,\varepsilon}| = O\left(k_{1,\varepsilon}\right) \quad \text{as } \varepsilon \to 0.$$

Infact, we claim that there exists $c_1 > 0$ such that

$$\lim_{\varepsilon\to 0}\frac{|x_{1,\varepsilon}|}{k_{1,\varepsilon}} = c_1. \tag{50}$$

We argue by contradiction and we assume that $|x_{1,\varepsilon}| = o(k_{1,\varepsilon})$ as $\varepsilon \to 0$. We define for $\varepsilon > 0$

$$\tilde{v}_\varepsilon(x) := \mu_{1,\varepsilon}^{\frac{n-2}{2}}u_\varepsilon(|x_{1,\varepsilon}|x) \qquad \text{for } x \in |x_{1,\varepsilon}|^{-1}\Omega \setminus \{0\}$$

Using (E_ε) we obtain that $\tilde{v}_\varepsilon$ weakly satisfies the equation in $|x_{1,\varepsilon}|^{-1}\Omega \setminus \{0\}$

$$-\Delta\tilde{v}_\varepsilon - \frac{\gamma}{|x|^2}\tilde{v}_\varepsilon - |x_{1,\varepsilon}|^2h(|x_{1,\varepsilon}|x)\ \tilde{v}_\varepsilon = b(|x_{1,\varepsilon}|x)\left(\frac{|x_{1,\varepsilon}|}{k_{1,\varepsilon}}\right)^{2-s}\frac{|\tilde{v}_\varepsilon|^{2^\star(s)-2}\tilde{v}_\varepsilon}{|x|^s}.$$

The definition (48) yields $|x|^{\tau}|\tilde{v}_{\varepsilon}(x)| \leq 1$ for all $x \in |x_{1,\varepsilon}|^{-1}\Omega \setminus \{0\}$. Standard elliptic theory then yield the existence of $\tilde{v} \in C^2(\mathbb{R}^n \setminus \{0\})$ such that $\tilde{v}_{\varepsilon} \to \tilde{v}$ in $C^2_{loc}(\mathbb{R}^n \setminus \{0\})$ where

$$-\Delta \tilde{v} - \frac{\gamma}{|x|^2}\tilde{v} = 0 \qquad \text{in } \mathbb{R}^n \setminus \{0\}.$$

In addition, we have that $\left|\tilde{v}_{\varepsilon}\left(|x_{1,\varepsilon}|^{-1}x_{1,\varepsilon}\right)\right| = 1$ and so $\tilde{v} \not\equiv 0$. Also since $|x|^{\tau}|\tilde{v}(x)| \leq 1$ in $\mathbb{R}^n \setminus \{0\}$, we have the bound

$$|\tilde{v}(x)| < 2|x|^{-\beta_+(\gamma)} + 2|x|^{-\beta_-(\gamma)} \qquad \text{in } \mathbb{R}^n \setminus \{0\}.$$

The classification of positive solutions of $-\Delta v - \frac{\gamma}{|x|^2}v = 0$ in $\mathbb{R}^n \setminus \{0\}$ (see (16)) yields the existence of $A, B \in \mathbb{R}$ such that $\tilde{v}(x) = A|x|^{-\beta_+(\gamma)} + B|x|^{-\beta_-(\gamma)}$ in $\mathbb{R}^n \setminus \{0\}$. Then the pointwise control $|x|^{\tau}|\tilde{v}(x)| \leq 1$ in $\mathbb{R}^n \setminus \{0\}$ yields $A = B = 0$, contradicting $\tilde{v} \not\equiv 0$. This proves the claim (50).

We rescale and define

$$\tilde{u}_{1,\varepsilon}(x) := \mu_{1,\varepsilon}^{\frac{n-2}{2}} u_{\varepsilon}(k_{1,\varepsilon}x) \qquad \text{for } x \in k_{1,\epsilon}^{-1}\Omega \setminus \{0\}$$

It follows from (48) and (50) that $\tilde{u}_{1,\varepsilon}$ satisfies the hypothesis (41) of lemma (3) with $y_{\varepsilon} = x_{1,\varepsilon}$, $\nu_{\varepsilon} = \mu_{1,\varepsilon}$. Then using lemma (3) we get that there exists $\tilde{u}_1 \in D^{1.2}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ weakly satisfying the equation:

$$-\Delta \tilde{u}_1 - \frac{\gamma}{|x|^2}\tilde{u}_1 = b(0)\frac{|\tilde{u}_1|^{2^{\star}(s)-2}\tilde{u}_1}{|x|^s} \text{ in } \mathbb{R}^n \setminus \{0\}.$$

and

$$\tilde{u}_{1,\varepsilon} \rightharpoonup \tilde{u}_1 \qquad \text{weakly in } D^{1,2}(\mathbb{R}^n) \quad \text{as } \varepsilon \to 0$$
$$\tilde{u}_{1,\varepsilon} \to \tilde{u}_1 \qquad \text{in } C^1_{loc}(\mathbb{R}^n \setminus \{0\}) \qquad \text{as } \varepsilon \to 0$$

It follows from the definition that $\left|\tilde{u}_{1,\varepsilon}\left(\frac{x_{1,\varepsilon}}{k_{1,\varepsilon}}\right)\right| = 1$. From (50) we therefore have that $\tilde{u}_1 \not\equiv 0$. And hence again from lemma (3) we get that

$$\int\limits_{\mathbb{R}^n} \frac{|\tilde{u}_1|^{2^{\star}(s)}}{|x|^s} \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^{\star}(s)}{2^{\star}(s)-2}}$$

and there exists $t_1 \in (0,1]$ such that $\lim\limits_{\varepsilon\to 0} \mu_{1,\varepsilon}^{p_{\varepsilon}} = t_1$.

Next, since $|x|^{\beta_-(\gamma)}\tilde{u}_1 \in C^0(\mathbb{R}^n)$, we have

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \sup_{B_{\delta k_{1,\varepsilon}}(0)\setminus\{0\}} |x|^{\frac{n-2}{2}} \left|\mu_{1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{x}{k_{1,\varepsilon}}\right)\right|^{1-\frac{p_{\varepsilon}}{2^{\star}(s)-2}} = 0$$

and then using the definitions (48), (49) it follows that

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0} \sup_{B_{\delta k_{1,\varepsilon}}(0)\setminus\{0\}} |x|^{\frac{n-2}{2}} \left|u_{\varepsilon}(x) - \mu_{1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{x}{k_{1,\varepsilon}}\right)\right|^{1-\frac{p_{\varepsilon}}{2^{\star}(s)-2}} = 0.$$

□

Step 4.2: We claim that there exists $C > 0$ such that

$$|x|^{\frac{n-2}{2}}|u_{\varepsilon}(x)|^{1-\frac{p_{\varepsilon}}{2^{\star}(s)-2}} \leq C \tag{51}$$

for all $\varepsilon > 0$ and all $x \in \Omega \setminus \{0\}$.

Proof of Step 4.3: We argue by contradiction and let $(y_\varepsilon)_{\varepsilon>0} \in \Omega\setminus\{0\}$ be such that

$$\sup_{x\in\Omega\setminus\{0\}} |x|^{\frac{n-2}{2}}|u_\varepsilon(x)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} = |y_\varepsilon|^{\frac{n-2}{2}}|u_\varepsilon(y_\varepsilon)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \to +\infty \quad \text{as } \varepsilon\to 0. \tag{52}$$

By the regularity Theorem (1.4) it follows that the sequence $(y_\varepsilon)_{\varepsilon>0}$ is well-defined and moreover $\lim_{\varepsilon\to 0} y_\varepsilon = 0$, since $u_\varepsilon \to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$. For $\varepsilon>0$ we let

$$\nu_\varepsilon := |u_\varepsilon(y_\varepsilon)|^{-\frac{2}{n-2}},\ \ell_\varepsilon := \nu_\varepsilon^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \text{ and } \kappa_\varepsilon := |y_\epsilon|^{s/2}\,\ell_\varepsilon^{\frac{2-s}{2}}.$$

Then it follows from (52) that

$$\lim_{\varepsilon\to 0}\nu_\varepsilon = 0,\ \lim_{\varepsilon\to 0}\frac{|y_\varepsilon|}{\ell_\varepsilon} = +\infty \text{ and } \lim_{\varepsilon\to 0}\frac{\kappa_\varepsilon}{|y_\varepsilon|} = 0. \tag{53}$$

Let $R>0$ and let $x\in B_R(0)$ be such that $y_\varepsilon+\kappa_\varepsilon x\in\Omega\setminus\{0\}$. It follows from the definition (52) of y_ε that for all $\varepsilon>0$

$$|y_\varepsilon+\kappa_\varepsilon x|^{\frac{n-2}{2}}|u_\varepsilon(y_\varepsilon+\kappa_\varepsilon x)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \le |y_\varepsilon|^{\frac{n-2}{2}}|u_\varepsilon(y_\varepsilon)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}}$$

and then, for all $\varepsilon>0$

$$\left(\frac{|u_\varepsilon(y_\varepsilon+\kappa_\varepsilon x)|}{|u_\varepsilon(y_\varepsilon)|}\right)^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \le \left(\frac{1}{1-\frac{\kappa_\varepsilon}{|y_\varepsilon|}R}\right)^{\frac{n-2}{2}}$$

for all $x\in B_R(0)$ such that $y_\varepsilon+\kappa_\varepsilon x\in\Omega\setminus\{0\}$. Using (53), we get that there exists $C(R)>0$ such that the hypothesis (35) of lemma (2) is satisfied and therefore one has $|y_\varepsilon| = O(\ell_\varepsilon)$ when $\varepsilon\to 0$, contradiction to (53). This proves (51). □

Let $\mathcal{I}\in\mathbb{N}^\star$. We consider the following assertions:

(B1) $0<\mu_{1,\varepsilon}<...<\mu_{\mathcal{I},\varepsilon}$.

(B2) $\lim_{\varepsilon\to 0}\mu_{\varepsilon,\mathcal{I}} = 0$ and $\lim_{\varepsilon\to 0}\frac{\mu_{\varepsilon,i+1}}{\mu_{i,\varepsilon}} = +\infty$ for all $1\le i\le \mathcal{I}-1$

(B3) For all $1\le i\le\mathcal{I}$ there exists $\tilde{u}_i\in D^{1,2}(\mathbb{R}^n)\cap C^2(\mathbb{R}^n\setminus\{0\})$ such that $\tilde{u}_i$ weakly solves the equation

$$-\Delta\tilde{u}_i - \frac{\gamma}{|x|^2}\tilde{u}_i = b(0)\frac{|\tilde{u}_i|^{2^\star(s)-2}\tilde{u}_i}{|x|^s} \qquad \text{in } \mathbb{R}^n\setminus\{0\}$$

with

$$\int_{\mathbb{R}^n}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s} \ge \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}},$$

and

$$\tilde{u}_{i,\varepsilon}\longrightarrow\tilde{u}_i \qquad \text{in } C^1_{loc}(\mathbb{R}^n\setminus\{0\}) \qquad \text{as } \varepsilon\to 0,$$
$$\tilde{u}_{i,\varepsilon}\rightharpoonup\tilde{u}_i \qquad \text{weakly in } D^{1,2}(\mathbb{R}^n) \quad \text{as } \varepsilon\to 0.$$

where for $\varepsilon>0$

$$\tilde{u}_{i,\varepsilon}(x) := \mu_{i,\varepsilon}^{\frac{n-2}{2}}u_\varepsilon(k_{i,\varepsilon}x) \qquad \text{for } x\in k_{i,\epsilon}^{-1}\Omega\setminus\{0\}$$

with $k_{i,\varepsilon} = \mu_{i,\varepsilon}^{1-\frac{p_\varepsilon}{2^\star(s)-2}}$.

(B4) For all $1\le i\le\mathcal{I}$, there exists $t_i\in(0,1]$ such that $\lim_{\varepsilon\to 0}\mu_{i,\varepsilon}^{p_\varepsilon} = t_i$.

We say that $\mathcal{H}_{\mathcal{I}}$ holds if there exists $\mathcal{I}$ sequences $(\mu_{i,\varepsilon})_{\varepsilon>0}$, $i = 1, ..., \mathcal{I}$ such that points (B1), (B2) (B3) and (B4) holds. Note that it follows from Step 4.1 that $\mathcal{H}_1$ holds. Next we show the following holds:

Step 4.3 Let $I \geq 1$. We assume that $\mathcal{H}_{\mathcal{I}}$ holds. Then either $\mathcal{H}_{\mathcal{I}+1}$ holds or

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{\Omega\setminus B_0(Rk_{\mathcal{I},\varepsilon})} |x|^{\frac{n-2}{2}}|u_\varepsilon(x)-u_0(x)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} = 0.$$

Proof of Step 4.3: Suppose

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{\Omega\setminus B_0(Rk_{\mathcal{I},\varepsilon})} |x|^{\frac{n-2}{2}}|u_\varepsilon(x)-u_0(x)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \neq 0.$$

Then there exists a sequence of points $(y_\varepsilon)_{\varepsilon>0} \in \Omega \setminus \{0\}$ such that

$$\lim_{\varepsilon\to 0}\frac{|y_\varepsilon|}{k_{\mathcal{I},\varepsilon}} = +\infty \text{ and } \lim_{\varepsilon\to 0}|y_\varepsilon|^{\frac{n-2}{2}}|u_\varepsilon(y_\varepsilon)-u_0(y_\varepsilon)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} = a > 0. \tag{54}$$

Since $u_\varepsilon \to u_0$ in $C^2_{loc}(\overline{\Omega}\setminus\{0\})$ it follows that $\lim_{\varepsilon\to 0} y_\varepsilon = 0$. Then by the regularity Theorem (1.4) and since $\beta_-(\gamma) < \frac{n-2}{2}$, we get

$$\lim_{\varepsilon\to 0}|y_\varepsilon|^{\frac{n-2}{2}}|u_\varepsilon(y_\varepsilon)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} = a > 0 \tag{55}$$

for some positive constant a. In particular, $\lim_{\varepsilon\to 0}|u_\varepsilon(y_\varepsilon)| = +\infty$. Let

$$\mu_{\mathcal{I}+1,\varepsilon} := |u_\varepsilon(y_\varepsilon)|^{-\frac{2}{n-2}} \text{ and } k_{\mathcal{I}+1,\varepsilon} := \mu_{\mathcal{I}+1,\varepsilon}^{1-\frac{p_\varepsilon}{2^\star(s)-2}}.$$

As a consequence we have

$$\lim_{\varepsilon\to 0}\mu_{\mathcal{I}+1,\varepsilon} = 0 \qquad \text{and} \qquad \lim_{\varepsilon\to 0}\frac{|y_\varepsilon|}{k_{\mathcal{I}+1,\varepsilon}} = a > 0. \tag{56}$$

We rescale and define

$$\tilde{u}_{\mathcal{I}+1,\varepsilon}(x) := \mu_{\mathcal{I}+1,\varepsilon}^{\frac{n-2}{2}} u_\varepsilon(k_{\mathcal{I}+1,\varepsilon}\, x) \qquad \text{for } x \in k^{-1}_{\mathcal{I}+1,\epsilon}\Omega\setminus\{0\}$$

It follows from (51) that for all $\varepsilon > 0$

$$|x|^{\frac{n-2}{2}}|\tilde{u}_{\mathcal{I}+1,\varepsilon}(x)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \leq C \qquad \text{for } x \in k^{-1}_{\mathcal{I}+1,\epsilon}\Omega\setminus\{0\}.$$

so hypothesis (41) of lemma (3) is satisfied. Then using lemma (3) we get that there exists $\tilde{u}_{\mathcal{I}+1} \in D^{1.2}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n\setminus\{0\})$ weakly satisfying the equation:

$$-\Delta\tilde{u}_{\mathcal{I}+1} - \frac{\gamma}{|x|^2}\tilde{u}_{\mathcal{I}+1} = b(0)\frac{|\tilde{u}_{\mathcal{I}+1}|^{2^\star(s)-2}\tilde{u}_{\mathcal{I}+1}}{|x|^s} \text{ in } \mathbb{R}^n\setminus\{0\}.$$

and

$$\begin{aligned} &\tilde{u}_{\mathcal{I}+1,\varepsilon} \rightharpoonup \tilde{u}_{\mathcal{I}+1} && \text{weakly in } D^{1,2}(\mathbb{R}^n) && \text{as } \varepsilon\to 0\\ &\tilde{u}_{\mathcal{I}+1,\varepsilon} \to \tilde{u}_{\mathcal{I}+1} && \text{in } C^1_{loc}(\mathbb{R}^n\setminus\{0\}) && \text{as } \varepsilon\to 0 \end{aligned}$$

We denote $\tilde{y}_\varepsilon := \dfrac{y_\varepsilon}{k_{\mathcal{I}+1,\varepsilon}}$. From (56) it follows that that $\lim\limits_{\varepsilon\to 0}|\tilde{y}_\varepsilon| := |\tilde{y}_0| = a \neq 0$. Therefore $|\tilde{u}_{\mathcal{I}+1}(\tilde{y}_0)| = \lim_{\varepsilon\to 0}|\tilde{u}_{\mathcal{I}+1,\varepsilon}(\tilde{y}_\varepsilon)| = 1$, and hence $\tilde{u}_{\mathcal{I}+1} \not\equiv 0$. And hence again from lemma (3) we get

$$\int\limits_{\mathbb{R}^n} \frac{|\tilde{u}_{\mathcal{I}+1}|^{2^\star(s)}}{|x|^s} \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}}$$

and there exists $t_{\mathcal{I}+1} \in (0,1]$ such that $\lim\limits_{\varepsilon\to 0}\mu^{p_\varepsilon}_{\mathcal{I}+1,\varepsilon} = t_{\mathcal{I}+1}$. Moreover, it follows from (54) and (56) that

$$\lim_{\varepsilon\to 0}\frac{\mu_{\mathcal{I}+1,\varepsilon}}{\mu_{\mathcal{I},\varepsilon}} = +\infty \text{ and } \lim_{\varepsilon\to 0}\mu_{\mathcal{I}+1,\varepsilon} = 0.$$

Hence the families $(\mu_{i,\varepsilon})_{\varepsilon>0}$, $1 \leq i \leq \mathcal{I}+1$ satisfy $\mathcal{H}_{\mathcal{I}+1}$. □

The next step is equivalent to step 4.3 at intermediate scales.

Step 4.4 Let $I \geq 1$. We assume that $\mathcal{H}_{\mathcal{I}}$ holds. Then for any $1 \leq i \leq \mathcal{I}-1$ and for any $\delta > 0$, either $\mathcal{H}_{\mathcal{I}+1}$ holds or

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{B_{\delta k_{i+1,\varepsilon}}(0)\setminus\overline{B}_{Rk_{i,\varepsilon}}(0)} |x|^{\frac{n-2}{2}}\left|u_\varepsilon(x) - \mu_{i+1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\varepsilon}}\right)\right|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} = 0.$$

Proof of Step 4.4: We assume that there exists an $i \leq \mathcal{I}-1$ and $\delta > 0$ such that

$$\lim_{R\to+\infty}\lim_{\varepsilon\to 0}\sup_{B_{\delta k_{i+1,\varepsilon}}(0)\setminus\overline{B}_{Rk_{i,\varepsilon}}(0)} |x|^{\frac{n-2}{2}}\left|u_\varepsilon(x) - \mu_{i+1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\varepsilon}}\right)\right|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} > 0.$$

It then follows that there exists a sequence $(y_\varepsilon)_{\varepsilon>0} \in \Omega$ such that

$$\lim_{\varepsilon\to 0}\frac{|y_\varepsilon|}{k_{i,\varepsilon}} = +\infty, \qquad |y_\varepsilon| \leq \delta k_{i+1,\varepsilon} \text{ for all } \varepsilon > 0 \tag{57}$$

$$|y_\varepsilon|^{\frac{n-2}{2}}\left|u_\varepsilon(y_\varepsilon) - \mu_{i+1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{y_\varepsilon}{k_{i+1,\varepsilon}}\right)\right|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} = a > 0. \tag{58}$$

for some positive constant a. Note that $a < +\infty$ since

$$|x|^{\frac{n-2}{2}}\left|u_\varepsilon(x) - \mu_{i+1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\varepsilon}}\right)\right|^{1-\frac{p_\varepsilon}{2^\star(s)-2}}$$

is uniformly bounded for all $x \in B_{\delta k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0)$.

We let $\tilde{y}^*_\varepsilon \in \mathbb{R}^n$ be such that $y_\varepsilon = k_{i+1,\varepsilon}\ \tilde{y}^*_\varepsilon$. It follows from (57) that $|\tilde{y}^*_\varepsilon| \leq \delta$ for all $\varepsilon > 0$. . We rewrite (58) as

$$\lim_{\varepsilon\to 0}|\tilde{y}^*_\varepsilon|^{\frac{n-2}{2}}\ |\tilde{u}_{i+1,\varepsilon}(\tilde{y}^*_\varepsilon) - \tilde{u}_{i+1}(\tilde{y}^*_\varepsilon)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} = a > 0.$$

Then from point (B3) of $\mathcal{H}_{\mathcal{I}}$ it follows that $\tilde{y}^*_\varepsilon \to 0$ as $\varepsilon \to 0$. And since $|x|^{\beta_-(\gamma)}\tilde{u}_{i+1} \in C^0(\mathbb{R}^n)$, we get as $\varepsilon \to 0$

$$|y_\varepsilon|^{\frac{n-2}{2}}\left|\mu_{i+1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{y_\varepsilon}{k_{i+1,\varepsilon}}\right)\right|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} = O\left(\frac{|y_\varepsilon|}{k_{i+1,\varepsilon}}\right)^{\frac{n-2}{2}-\beta_-(\gamma)} = o(1)$$

Then (58) becomes

$$\lim_{\varepsilon\to 0} |y_\varepsilon|^{\frac{n-2}{2}} |u_\varepsilon(y_\varepsilon)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} = a > 0. \tag{59}$$

In particular, $\lim_{\varepsilon\to 0} |u_\varepsilon(y_\varepsilon)| = +\infty$. We let

$$\nu_\varepsilon := |u_\varepsilon(y_\varepsilon)|^{-\frac{2}{n-2}} \text{ and } \ell_\varepsilon := \nu_\varepsilon^{1-\frac{p_\varepsilon}{2^\star(s)-2}}.$$

Then we have

$$\lim_{\varepsilon\to 0} \nu_\varepsilon = 0 \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \frac{|y_\varepsilon|}{\ell_\varepsilon} = a > 0. \tag{60}$$

We rescale and define

$$\tilde{u}_\varepsilon(x) := \nu_\varepsilon^{\frac{n-2}{2}} u_\varepsilon(\ell_\varepsilon\, x) \qquad \text{for } x \in \ell_\varepsilon^{-1}\Omega \setminus \{0\}$$

It follows from (51) that for all $\varepsilon > 0$

$$|x|^{\frac{n-2}{2}} |\tilde{u}_\varepsilon(x)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \le C \qquad \text{for } x \in \ell_\varepsilon^{-1}\Omega \setminus \{0\}.$$

so hypothesis (41) of lemma (3) is satisfied. Then using lemma (3) we get that there exists $\tilde{u} \in D^{1.2}(\mathbb{R}^n) \cap C^1(\mathbb{R}^n \setminus \{0\})$ weakly satisfying the equation:

$$-\Delta\tilde{u} - \frac{\gamma}{|x|^2}\tilde{u} = b(0)\frac{|\tilde{u}|^{2^\star(s)-2}\tilde{u}}{|x|^s} \text{ in } \mathbb{R}^n \setminus \{0\}.$$

and

$$\begin{aligned} &\tilde{u}_\varepsilon \rightharpoonup \tilde{u} \qquad \text{weakly in } D^{1,2}(\mathbb{R}^n) \quad \text{as } \varepsilon \to 0 \\ &\tilde{u}_\varepsilon \to \tilde{u} \qquad \text{in } C^1_{loc}(\mathbb{R}^n \setminus \{0\}) \qquad \text{as } \varepsilon \to 0 \end{aligned}$$

We denote $\tilde{y}_\varepsilon := \dfrac{y_\varepsilon}{\ell_\varepsilon}$. From (59) it follows that that $\lim_{\varepsilon\to 0} |\tilde{y}_\varepsilon| := |\tilde{y}_0| = a \neq 0$. Therefore $|\tilde{u}(\tilde{y}_0)| = \lim_{\varepsilon\to 0} |\tilde{u}_\varepsilon(\tilde{y}_\varepsilon)| = 1$, and hence $\tilde{u} \not\equiv 0$. And hence again from lemma (3) we get

$$\int_{\mathbb{R}^n} \frac{|\tilde{u}|^{2^\star(s)}}{|x|^s} \ge \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}}$$

and there exists $t \in (0,1]$ such that $\lim_{\varepsilon\to 0} \nu_\varepsilon^{p_\varepsilon} = t$. Moreover it follows from(59), (57) and since $\lim_{\varepsilon\to 0} \frac{|y_\varepsilon|}{k_{i+1,\varepsilon}} = 0$, that

$$\lim_{\varepsilon\to 0} \frac{\nu_\varepsilon}{\mu_{i,\varepsilon}} = +\infty \text{ and } \lim_{\varepsilon\to 0} \frac{\mu_{i+1,\varepsilon}}{\nu_\varepsilon} = +\infty.$$

Hence the families $(\mu_{1,\varepsilon})$,..., $(\mu_{i,\varepsilon})$, (ν_ε), $(\mu_{i+1,\varepsilon})$,..., $(\mu_{\mathcal{I},\varepsilon})$ satisfy $\mathcal{H}_{\mathcal{I}+1}$. □

The last step tells us that family $\{\mathcal{H}_\mathcal{I}\}$ is finite.

Step 4.5: Let $N_0 = \max\{\mathcal{I} : \mathcal{H}_\mathcal{I} \text{ holds }\}$. Then $N_0 < +\infty$ and the conclusion of Proposition 2 holds with $N = N_0$.

Proof of Step 4.5: Indeed, assume that $\mathcal{H}_\mathcal{I}$ holds. Since $\mu_{i,\varepsilon} = o(\mu_{i+1,\varepsilon})$ for all $1 \le i \le N-1$, we get with a change of variable and the definition of $\tilde{u}_{i,\varepsilon}$ that for

any $R > \delta > 0$

$$\int_\Omega \frac{|u_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s}dx \geq \sum_{i=1}^{\mathcal{I}} \int_{B_{Rk_{i,\varepsilon}}(0)\setminus \overline{B}_{\delta k_{i,\varepsilon}}(0)} \frac{|u_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s}dx$$
$$\geq \sum_{i=1}^{\mathcal{I}} \int_{B_R(0)\setminus \overline{B}_\delta(0)} \frac{|\tilde{u}_{i,\varepsilon}|^{2^\star(s)-p_\varepsilon}}{|x|^s}dx.$$

(61) Then from (30) we have $\Lambda \geq \sum_{i=1}^{\mathcal{I}} \int_{B_R(0)\setminus \overline{B}_\delta(0)} \frac{|\tilde{u}_{i,\varepsilon}|^{2^\star(s)-p_\varepsilon}}{|x|^s}dx.$

Passing to the limit $\varepsilon \to 0$ and then $\delta \to 0$, $R \to +\infty$ we obtain using point (B3) of $\mathcal{H}_{\mathcal{I}}$, that

$$\Lambda \geq \left(\frac{\mu_{\gamma,s,0}(\mathbb{R}^n)}{b(0)}\right)^{\frac{2^\star(s)}{2^\star(s)-2}} \mathcal{I}.$$

It then follows that $N_0 < +\infty$. □

We let families $(\mu_{1,\varepsilon})_{\varepsilon>0}$,..., $(\mu_{N_0,\varepsilon})_{\varepsilon>0}$ such that $\mathcal{H}_{N_0}$ holds. We argue by contradiction and assume that the conclusion of Proposition 2 does not hold with $N = N_0$. Assertions (A1), (A2), (A3),(A4), (A5), (A7) and (A9) holds. Assume that (A6) or (A8) does not hold. It then follows from Steps (4.3), (4.4) and (4.5) that $\mathcal{H}_{N+1}$ holds. A contradiction with the choice of $N = N_0$ and the proposition is proved. □

5. Strong Pointwise Estimates

The objective of this section is the proof of the following strong pointwise control.

Proposition 3. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $\gamma < \frac{(n-2)^2}{4}$. Let (u_ε), (p_ε), $h(x)$ and $b(x)$ be such that (E_ε), (27), (28), (29) and (30) holds. Assume that blow-up occurs, that is*

$$\lim_{\varepsilon\to 0} \||x|^\tau u_\varepsilon\|_{L^\infty(\Omega)} = +\infty \quad \text{where} \quad \beta_-(\gamma) < \tau < \frac{n-2}{2}.$$

Consider the $\mu_{1,\varepsilon}, ..., \mu_{N,\varepsilon}$ from Proposition 2. Then there exists $C > 0$ such that for all $\varepsilon > 0$

$$|u_\varepsilon(x)| \leq C\left(\sum_{i=1}^N \frac{\mu_{i,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}{\mu_{i,\varepsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}|x|^{\beta_-(\gamma)} + |x|^{\beta_+(\gamma)}} + \frac{\||x|^{\beta_-(\gamma)}u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)}}\right) \tag{62}$$

for all $x \in \Omega \setminus \{0\}$.

The proof of this estimate proceeds in seven steps.

Step 5.1: We claim that for any $\alpha > 0$ small and any $R > 0$, there exists $C(\alpha, R) > 0$ such that for all $\varepsilon > 0$ sufficiently small

$$|u_\varepsilon(x)| \leq C(\alpha,R)\left(\frac{\mu_{N,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha}}{|x|^{\beta_+(\gamma)-\alpha}} + \frac{\||x|^{\beta_-(\gamma)}u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)+\alpha}}\right) \text{ for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0). \tag{63}$$

Proof of Step 5.1: We fix γ' such that $\gamma < \gamma' < \frac{(n-2)^2}{4}$. Since the operator $-\Delta - \frac{\gamma}{|x|^2} - h(x)$ is coercive, taking γ' close to γ it follows that the operator $-\Delta - \frac{\gamma'}{|x|^2} - h(x)$ is also coercive in Ω. From Theorem (20) it get that there exists $H \in C^\infty(\overline{\Omega} \setminus \{0\})$ such that

$$\begin{cases} -\Delta H - \frac{\gamma'}{|x|^2} H - h(x)H = 0 & \text{in } \Omega \setminus \{0\} \\ H > 0 & \text{in } \Omega \setminus \{0\} \\ H = 0 & \text{on } \partial\Omega. \end{cases} \tag{64}$$

And we have the following bound on H: there exists $\delta_1, C_1 > 0$ such that

$$\frac{1}{C_1} \frac{1}{|x|^{\beta_+(\gamma')}} \leq H(x) \leq C_1 \frac{1}{|x|^{\beta_+(\gamma')}} \qquad \text{for all } x \in B_{2\delta_1}(0). \tag{65}$$

We let $\lambda_1^{\gamma'} > 0$ be the first eigenvalue of the coercive operator $-\Delta - \frac{\gamma'}{|x|^2} - h$ on Ω and we let $\varphi \in H_0^1(\Omega)$ be the unique eigenfunction such that

$$\begin{cases} -\Delta\varphi - \frac{\gamma'}{|x|^2}\varphi - h(x)\varphi &= \lambda_1^{\gamma'}\varphi \quad \text{in } \Omega \\ \varphi &> 0 \quad \text{in } \Omega \setminus \{0\} \\ \varphi &= 0 \quad \text{on } \partial\Omega. \end{cases} \tag{66}$$

It follows from the regularity result, Theorem (1.4) that there exists $C_2, \delta_2 > 0$ such that

$$\frac{1}{C_2} \frac{1}{|x|^{\beta_-(\gamma')}} \leq \varphi(x) \leq C_2 \frac{1}{|x|^{\beta_-(\gamma')}} \qquad \text{for all } x \in \Omega \cap B_{2\delta_2}(0). \tag{67}$$

We define the operator

$$\mathcal{L}_\varepsilon := -\Delta - \left(\frac{\gamma}{|x|^2} + h\right) - b(x)\frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon}}{|x|^s}.$$

Step 5.1.1: We claim that given any $\gamma < \gamma' < \frac{(n-2)^2}{4}$ there exist $\delta_0 > 0$ and $R_0 > 0$ such that for any $0 < \delta < \delta_0$ and $R > R_0$, we have for $\varepsilon > 0$ sufficiently small

$$\mathcal{L}_\varepsilon H(x) > 0, \text{ and } \mathcal{L}_\varepsilon\varphi(x) > 0 \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0), \text{ if } u_0 \not\equiv 0.$$

$$\mathcal{L}_\varepsilon H(x) > 0 \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0), \text{ if } u_0 \equiv 0. \tag{68}$$

As one checks for all $\varepsilon > 0$ and $x \neq 0$

$$\frac{\mathcal{L}_\varepsilon H(x)}{H(x)} = \frac{\gamma' - \gamma}{|x|^2} - b(x)\frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon}}{|x|^s}$$

and

$$\frac{\mathcal{L}_\varepsilon\varphi(x)}{\varphi(x)} = \frac{\gamma' - \gamma}{|x|^2} - b(x)\frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon}}{|x|^s} + \lambda_1^{\gamma'}.$$

Choose $0 < \delta_0 < \min\{1, \delta_1, \delta_2\}$ such that

$$||b||_{L^\infty(\Omega)}\delta_0^{(2^\star(s)-2)\left(\frac{n-2}{2}-\beta_-(\gamma)\right)}\||x|^{\beta_-(\gamma)}u_0||_{L^\infty(\Omega)}^{2^\star(s)-2} \leq \frac{\gamma' - \gamma}{2^{2^\star(s)+3}} \tag{69}$$

It follows from point (A6) of Proposition 2 that, there exists $R_0 > 0$ such that for any $R > R_0$, we have for all $\varepsilon > 0$ sufficiently small

$$|b(x)|^{\frac{1}{2^\star(s)-2}}|x|^{\frac{n-2}{2}}|u_\varepsilon(x) - u_0(x)|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \leq \left(\frac{\gamma' - \gamma}{2^{2^\star(s)+2}}\right)^{\frac{1}{2^\star(s)-2}} \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)$$

With this choice of δ_0 and R_0 we get that for any $0 < \delta < \delta_0$ and $R > R_0$, we have for $\varepsilon > 0$ small enough

$$\begin{aligned} |b(x)|\ |x|^{2-s}|u_\varepsilon(x)|^{2^\star(s)-2-p_\varepsilon} &\leq 2^{2^\star(s)-1-p_\varepsilon}|x|^{2-s}|b(x)||u_\varepsilon(x)-u_0(x)|^{2^\star(s)-2-p_\varepsilon} \\ &\quad + 2^{2^\star(s)-1-p_\varepsilon}|x|^{2-s}|b(x)||u_0(x)|^{2^\star(s)-2-p_\varepsilon} \\ &\leq \frac{\gamma'-\gamma}{4} \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0), \text{ if } u_0 \not\equiv 0 \end{aligned}$$

$$\text{and } |b(x)|\ |x|^{2-s}|u_\varepsilon(x)|^{2^\star(s)-2-p_\varepsilon} \leq \frac{\gamma'-\gamma}{4} \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0), \text{ if } u_0 \equiv 0.$$

Hence we obtain that for $\varepsilon > 0$ small enough

$$\begin{aligned} \frac{\mathcal{L}_\varepsilon H(x)}{H(x)} &= \frac{\gamma'-\gamma}{|x|^2} - b(x)\frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon}}{|x|^s} \\ &\geq \frac{\gamma'-\gamma}{|x|^2} - \frac{\gamma'-\gamma}{4|x|^2} \\ &> 0 \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0), \text{ if } u_0 \not\equiv 0 \end{aligned}$$

$$\text{and } \frac{\mathcal{L}_\varepsilon H(x)}{H(x)} > 0 \qquad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0), \text{ if } u_0 \equiv 0. \tag{70}$$

And

$$\begin{aligned} \frac{\mathcal{L}_\varepsilon \varphi(x)}{\varphi(x)} &\geq \frac{\gamma'-\gamma}{|x|^2} - b(x)\frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon}}{|x|^s}. \\ &\geq \frac{\gamma'-\gamma}{|x|^2} - \frac{\gamma'-\gamma}{4|x|^2} \\ &> 0 \qquad \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)). \end{aligned} \tag{71}$$

□

Step 5.1.2: It follows from point (A4) of Proposition 2 that there exists $C_1'(R) > 0$ such that for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq C_1'(R)\frac{\mu_{N,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}{|x|^{\beta_+(\gamma')}} \qquad \text{for all } x \in \partial B_{Rk_{N,\varepsilon}}(0) \tag{72}$$

By estimate (65) on H, we then have for some constant $C_1(R) > 0$

$$|u_\varepsilon(x)| \leq C_1(R)\mu_{N,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) \qquad \text{for all } x \in \partial B_{Rk_{N,\varepsilon}}(0). \tag{73}$$

It follows from point (A1) of Proposition 2 and the regularity result (1.4), that there exists $C_2'(\delta) > 0$ such that for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq C_2'(\delta)\frac{||x|^{\beta_-(\gamma)}u_0||_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)}} \qquad \text{for all } x \in \partial B_\delta(0), \text{ if } u_0 \not\equiv 0. \tag{74}$$

And then by the estimate (67) on φ we then have for some constant $C_2(\delta) > 0$

$$|u_\varepsilon(x)| \leq C_2(\delta)\||x|^{\beta_-(\gamma)}u_0\|_{L^\infty(\Omega)}\ \varphi(x) \qquad \text{for all } x \in \partial B_\delta(0) \text{ if } u_0 \not\equiv 0.. \tag{75}$$

We now let for $\varepsilon > 0$,

$$\Psi_\varepsilon(x) := C_1(R)\mu_{N,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x)+C_2(\delta)\||x|^{\beta_-(\gamma)}u_0||_{L^\infty(\Omega)}\varphi(x) \qquad \text{for } x \in \Omega\backslash\{0\}.$$

Then (75) and (73) implies that for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \le \Psi_\varepsilon(x) \qquad \text{for all } x \in \partial(B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)), \text{ if } u_0 \not\equiv 0. \tag{76}$$

and if $u_0 \equiv 0$ then

$$|u_\varepsilon(x)| \le \Psi_\varepsilon(x) \qquad \text{for all } x \in \partial(\Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)). \tag{77}$$

Therefore when $u_0 \not\equiv 0$ it follows from (68)) and (76) that for all $\varepsilon > 0$ sufficiently small

$$\begin{cases} \mathcal{L}_\varepsilon\Psi_\varepsilon \ge 0 = \mathcal{L}_\varepsilon u_\varepsilon & \text{in } B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0) \\ \Psi_\varepsilon \ge u_\varepsilon & \text{on } \partial(B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)) \\ \mathcal{L}_\varepsilon\Psi_\varepsilon \ge 0 = -\mathcal{L}_\varepsilon u_\varepsilon & \text{in } B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0) \\ \Psi_\varepsilon \ge -u_\varepsilon & \text{on } \partial(B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)). \end{cases}$$

and from (68) and (77), in case $u_0 \equiv 0$, we have for $\varepsilon > 0$ sufficiently small

$$\begin{cases} \mathcal{L}_\varepsilon\Psi_\varepsilon \ge 0 = \mathcal{L}_\varepsilon u_\varepsilon & \text{in } \Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0) \\ \Psi_\varepsilon \ge u_\varepsilon & \text{on } \partial(\Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)) \\ \mathcal{L}_\varepsilon\Psi_\varepsilon \ge 0 = -\mathcal{L}_\varepsilon u_\varepsilon & \text{in } \Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0) \\ \Psi_\varepsilon \ge -u_\varepsilon & \text{on } \partial(\Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)). \end{cases}$$

Since $\Psi_\varepsilon > 0$ and $\mathcal{L}_\varepsilon\Psi_\varepsilon > 0$, it follows from the comparison principle of Berestycki-Nirenberg-Varadhan [5] that the operator $\mathcal{L}_\varepsilon$ satisfies the comparison principle on $B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)$. Therefore

$$\begin{aligned} |u_\varepsilon(x)| \le \Psi_\varepsilon(x) \qquad & \text{for all } x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0), \\ \text{and } \ |u_\varepsilon(x)| \le \Psi_\varepsilon(x) \qquad & \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0) \ \text{ if } u_0 \equiv 0. \end{aligned}$$

Therefore when $u_0 \not\equiv 0$, we have for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \le C_1(R)\mu_{N,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) + C_2(\delta)\||x|^{\beta_-(\gamma)}u_0||_{L^\infty(\Omega)}\varphi(x)$$

for all $x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)$, for R large and δ small.

Then when $u_0 \not\equiv 0$, using the estimates (65) and (67) we have or all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \ \le C_1(R)\frac{\mu_{N,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}{|x|^{\beta_+(\gamma')}} + C_2(\delta)\frac{\||x|^{\beta_-(\gamma)}u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma')}}$$

for all $x \in B_\delta(0) \setminus \overline{B}_{Rk_{N,\varepsilon}}(0)$, for R large and δ small.

And if $u_0 \equiv 0$, then all $\varepsilon > 0$ small and $R > 0$ large

$$|u_\varepsilon(x)| \le C_1(R)\frac{\mu_{N,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}{|x|^{\beta_+(\gamma')}} \quad \text{for all } x \in \Omega \setminus \overline{B}_{Rk_{N,\varepsilon}}(0).$$

Taking γ' close to γ, along with points (A1) and (A4) of Proposition (2) it then follows that estimate (63) holds on $\Omega \setminus \overline{B}_{Rk_{\varepsilon,N}}(0)$ for all $R > 0$. □

Step 5.2: Let $1 \le i \le N-1$. We claim that for any $\alpha > 0$ small and any $R, \rho > 0$, there exists $C(\alpha, R, \rho) > 0$ such that all $\varepsilon > 0$.

$$|u_\varepsilon(x)| \leq C(\alpha,R,\rho)\left(\frac{\mu_{i,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha}}{|x|^{\beta_+(\gamma)-\alpha}}+\frac{1}{\mu_{i+1,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha}|x|^{\beta_-(\gamma)+\alpha}}\right) \tag{78}$$

for all $x \in B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0)$.

Proof of Step 5.2: We let $i \in \{1,...,N-1\}$. We emulate the proof of Step 5.1. Fix γ' such that $\gamma < \gamma' < \frac{(n-2)^2}{4}$. Consider the functions H and φ defined in Step 5.1 satisfying (64) and (64) respectively. We define the operator

$$\mathcal{L}_\varepsilon := -\Delta - \left(\frac{\gamma}{|x|^2}+h\right) - b(x)\frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon}}{|x|^s}.$$

Step 5.2.1: We claim that given any $\gamma < \gamma' < \frac{(n-2)^2}{4}$ there exist $\rho_0 > 0$ and $R_0 > 0$ such that for any $0 < \rho < \rho_0$ and $R > R_0$, we have for $\varepsilon > 0$ sufficiently small

$$\mathcal{L}_\varepsilon H(x) > 0, \text{ and } \mathcal{L}_\varepsilon\varphi(x) > 0 \qquad \text{for all } x \in B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0) \tag{79}$$

As one checks for all $\varepsilon > 0$ and $x \neq 0$

$$\begin{aligned}\frac{\mathcal{L}_\varepsilon H(x)}{H(x)} &= \frac{\gamma'-\gamma}{|x|^2} - b(x)\frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon}}{|x|^s},\\ \frac{\mathcal{L}_\varepsilon\varphi(x)}{\varphi(x)} &\geq \frac{\gamma'-\gamma}{|x|^2} - b(x)\frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon}}{|x|^s}.\end{aligned}$$

We choose $0 < \rho_0 < 1$ such that

$$||b||_{L^\infty}\rho_0^{(2^\star(s)-2)\left(\frac{n-2}{2}-\beta_-(\gamma)\right)}\||x|^{\beta_-(\gamma)}\tilde{u}_{i+1}\|^{2^\star(s)-2}_{L^\infty(B_2(0)\cap\mathbb{R}^n)} \leq \frac{\gamma'-\gamma}{2^{2^\star(s)+3}} \tag{80}$$

It follows from point (A8) of Proposition (2) that there exists $R_0 > 0$ such that for any $R > R_0$ and any $0 < \rho < \rho_0$, we have for all $\varepsilon > 0$ sufficiently small

$$|b(x)|^{\frac{1}{2^\star(s)-2}}|x|^{\frac{n-2}{2}}\left|u_\varepsilon(x) - \mu_{i+1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\varepsilon}}\right)\right|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \leq \left(\frac{\gamma'-\gamma}{2^{2^\star(s)+2}}\right)^{\frac{1}{2^\star(s)-2}}$$

for all $x \in B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0)$.

With this choice of ρ_0 and R_0 we get that for any $0 < \rho < \rho_0$ and $R > R_0$, we have for $\varepsilon > 0$ small enough

$$\begin{aligned}|b(x)||x|^{2-s}|u_\varepsilon(x)|^{2^\star(s)-2-p_\varepsilon} &\leq 2^{2^\star(s)-1-p_\varepsilon}|x|^{2-s}|b(x)|\left|u_\varepsilon(x) - \mu_{i+1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\varepsilon}}\right)\right|^{2^\star(s)-2-p_\varepsilon}\\ &\quad + 2^{2^\star(s)-1-p_\varepsilon}\left(\frac{|x|}{k_{i+1,\varepsilon}}\right)^{2-s}|b(x)|\left|\tilde{u}_{i+1}\left(\frac{x}{k_{i+1,\varepsilon}}\right)\right|^{2^\star(s)-2-p_\varepsilon}\\ &\leq \frac{\gamma'-\gamma}{4} \qquad \text{for all } x \in B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0).\end{aligned} \tag{81}$$

Hence as in Step 5.1 we have that for $\varepsilon > 0$ small enough

$$\frac{\mathcal{L}_\varepsilon H(x)}{H(x)} > 0 \text{ and } \frac{\mathcal{L}_\varepsilon\varphi(x)}{\varphi(x)} > 0 \text{ for all } x \in B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0).$$

□

Step 5.2.2: let $i \in \{1, ..., N-1\}$. It follows from point (A4) of Proposition 2 that there exists $C_1'(R) > 0$ such that for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq C_1'(R) \frac{\mu_{i,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}{|x|^{\beta_+(\gamma')}} \qquad \text{for all } x \in \partial B_{Rk_{i,\varepsilon}}(0) \tag{82}$$

And then by the estimate (65) on H we have for some constant $C_1(R) > 0$

$$|u_\varepsilon(x)| \leq C_1(R) \mu_{i,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) \qquad \text{for all } x \in \partial B_{Rk_{i,\varepsilon}}(0). \tag{83}$$

From point (A4) of Proposition 2 it follows that there exists $C_2'(\rho) > 0$ such that for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq C_2'(\rho) \frac{1}{\mu_{i+1,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} |x|^{\beta_-(\gamma')}} \qquad \text{for all } x \in \partial B_{\rho k_{i+1,\varepsilon}}(0). \tag{84}$$

Then by the estimate (67) on φ we have for some constant $C_2(\delta) > 0$

$$|u_\varepsilon(x)| \leq C_2(\rho) \frac{\varphi(x)}{\mu_{i+1,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \qquad \text{for all } x \in \partial B_{\rho k_{i+1,\varepsilon}}(0). \tag{85}$$

We let for all $\varepsilon > 0$

$$\tilde{\Psi}_\varepsilon(x) := C_1(R) \mu_{i,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) + C_2(\rho) \frac{\varphi(x)}{\mu_{i+1,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \qquad \text{for } x \in \Omega \setminus \{0\}$$

Then (83) and (85) implies that for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq \tilde{\Psi}_\varepsilon(x) \qquad \text{for all } x \in \partial \left(B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0) \right). \tag{86}$$

Therefore it follows from (79) and (86) that $\varepsilon > 0$ sufficiently small

$$\begin{cases} \mathcal{L}_\varepsilon \tilde{\Psi}_\varepsilon \geq 0 = \mathcal{L}_\varepsilon u_\varepsilon & \text{in } B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0) \\ \tilde{\Psi}_\varepsilon \geq u_\varepsilon & \text{on } \partial \left(B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0) \right) \\ \mathcal{L}_\varepsilon \tilde{\Psi}_\varepsilon \geq 0 = -\mathcal{L}_\varepsilon u_\varepsilon & \text{in } B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0) \\ \tilde{\Psi}_\varepsilon \geq -u_\varepsilon & \text{on } \partial \left(B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0) \right). \end{cases}$$

Since $\tilde{\Psi}_\varepsilon > 0$ and $\mathcal{L}_\varepsilon \tilde{\Psi}_\varepsilon > 0$, it follows from the comparison principle of Berestycki-Nirenberg-Varadhan [5] that the operator $\mathcal{L}_\varepsilon$ satisfies the comparison principle on $B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0)$. Therefore

$$|u_\varepsilon(x)| \leq \tilde{\Psi}_\varepsilon(x) \qquad \text{for all } x \in B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0)).$$

So for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq C_1(R) \mu_{i,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} H(x) + C_2(\rho) \frac{\varphi(x)}{\mu_{i+1,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}$$

for all $x \in B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0)$, for R large and ρ small. Then using the estimates (65) and (67) we have or all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq C_1(R) \frac{\mu_{i,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}}{|x|^{\beta_+(\gamma')}} + \frac{C_2(\rho)}{\mu_{i+1,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} |x|^{\beta_-(\gamma')}} \quad \text{for all } x \in B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0).$$

for R large and ρ small.

Taking γ' close to γ, along with point (A4) of Proposition (2) it then follows that estimate (78) holds on $B_{\rho k_{i+1,\varepsilon}}(0) \setminus \overline{B}_{Rk_{i,\varepsilon}}(0)$ for all $R, \rho > 0$. □

Step 5.3: We claim that for any $\alpha > 0$ small and any $\rho > 0$, there exists $C(\alpha, \rho) > 0$ such that all $\varepsilon > 0$.

$$|u_\varepsilon(x)| \leq C(\alpha,\rho)\frac{1}{\mu_{1,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha}|x|^{\beta_-(\gamma)+\alpha}} \qquad \text{for all } x \in B_{\rho k_{1,\varepsilon}}(0) \setminus \{0\}. \tag{87}$$

Proof of Step 5.3: Fix γ' such that $\gamma < \gamma' < \frac{(n-2)^2}{4}$. Consider the function φ defined in Step 5.1 satisfying (64). We define

$$\mathcal{L}_\varepsilon := -\Delta - \left(\frac{\gamma}{|x|^2} + h\right) - b(x)\frac{|u_\varepsilon|^{2^\star(s)-2-p_\varepsilon}}{|x|^s}.$$

Step 5.3.1: We claim that given any $\gamma < \gamma' < \frac{(n-2)^2}{4}$ there exist $\rho_0 > 0$ such that for any $0 < \rho < \rho_0$ we have for $\varepsilon > 0$ sufficiently small

$$\mathcal{L}_\varepsilon\varphi(x) > 0 \qquad \text{for all } x \in B_{\rho k_{1,\varepsilon}}(0) \setminus \{0\} \tag{88}$$

We choose $0 < \rho_0 < 1$ such that

$$\rho_0^{2-\theta} \sup_\Omega ||x|^\theta h| \leq \frac{\gamma'-\gamma}{4} \quad \text{for all } \varepsilon > 0 \text{ small and}$$

$$||b||_{L^\infty}\rho_0^{(2^\star(s)-2)\left(\frac{n-2}{2}-\beta_-(\gamma)\right)}\||x|^{\beta_-(\gamma)}\tilde{u}_1\|^{2^\star(s)-2}_{L^\infty(B_2(0)\cap\mathbb{R}^n)} \leq \frac{\gamma'-\gamma}{2^{2^\star(s)+3}} \tag{89}$$

It follows from point (A7) of Proposition (2) that for any $0 < \rho < \rho_0$, we have for all $\varepsilon > 0$ sufficiently small

$$|b(x)|^{\frac{1}{2^\star(s)-2}}|x|^{\frac{n-2}{2}}\left|u_\varepsilon(x) - \mu_{1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{x}{k_{1,\varepsilon}}\right)\right|^{1-\frac{p_\varepsilon}{2^\star(s)-2}} \leq \left(\frac{\gamma'-\gamma}{2^{2^\star(s)+2}}\right)^{\frac{1}{2^\star(s)-2}}$$

for all $x \in B_{\rho k_{1,\varepsilon}}(0) \setminus \{0\}$.

With this choice of ρ_0 we get that for any $0 < \rho < \rho_0$ we have for $\varepsilon > 0$ small enough

$$\begin{aligned}|b(x)||x|^{2-s}|u_\varepsilon(x)|^{2^\star(s)-2-p_\varepsilon} &\leq 2^{2^\star(s)-1-p_\varepsilon}|x|^{2-s}|b(x)|\left|u_\varepsilon(x) - \mu_{1,\varepsilon}^{-\frac{n-2}{2}}\tilde{u}_1\left(\frac{x}{k_{1,\varepsilon}}\right)\right|^{2^\star(s)-2-p_\varepsilon} \\ &\quad + 2^{2^\star(s)-1-p_\varepsilon}\left(\frac{|x|}{k_{1,\varepsilon}}\right)^{2-s}|b(x)|\left|\tilde{u}_1\left(\frac{x}{k_{1,\varepsilon}}\right)\right|^{2^\star(s)-2-p_\varepsilon} \\ &\leq \frac{\gamma'-\gamma}{4} \qquad \text{for all } x \in B_{\rho k_{1,\varepsilon}}(0) \setminus \{0\}. \end{aligned} \tag{90}$$

Hence similarly as in Step 5.1, we obtain that for $\varepsilon > 0$ small enough

$$\frac{\mathcal{L}_\varepsilon\varphi(x)}{\varphi(x)} > 0 > 0 \qquad \text{for all } x \in x \in B_{\rho k_{1,\varepsilon}}(0) \setminus \{0\}. \tag{91}$$

□

Step 5.3.2: It follows from point (A4) of Proposition 2 that there exists $C_2'(\rho) > 0$ such that for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq C_2'(\rho) \frac{1}{\mu_{1,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} |x|^{\beta_-(\gamma')}} \qquad \text{for all } x \in \partial B_{\rho k_{1,\varepsilon}}(0) \tag{92}$$

and then by the estimate (67) on φ we have for some constant $C_2(\delta) > 0$

$$|u_\varepsilon(x)| \leq C_2(\rho) \frac{\varphi(x)}{\mu_{1,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \qquad \text{for all } x \in \partial B_{\rho k_{1,\varepsilon}}(0). \tag{93}$$

We let for all $\varepsilon > 0$

$$\Psi_\varepsilon^0(x) := C_2(\rho) \frac{\varphi(x)}{\mu_{1,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \qquad \text{for } x \in \Omega \setminus \{0\}.$$

Then (93) implies that for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq \Psi_\varepsilon^0(x) \qquad \text{for all } x \in \partial B_{\rho k_{1,\varepsilon}}(0). \tag{94}$$

Therefore it follows from (88) and (94) that $\varepsilon > 0$ sufficiently small

$$\begin{cases} \mathcal{L}_\varepsilon \Psi_\varepsilon^0 \geq 0 = \mathcal{L}_\varepsilon u_\varepsilon & \text{in } B_{\rho k_{1,\varepsilon}}(0) \setminus \{0\} \\ \Psi_\varepsilon^0 \geq u_\varepsilon & \text{on } \partial B_{\rho k_{1,\varepsilon}}(0) \setminus \{0\} \\ \mathcal{L}_\varepsilon \Psi_\varepsilon^0 \geq 0 = -\mathcal{L}_\varepsilon u_\varepsilon & \text{in } B_{\rho k_{1,\varepsilon}}(0) \\ \Psi_\varepsilon^0 \geq -u_\varepsilon & \text{on } \partial B_{\rho k_{1,\varepsilon}}(0). \end{cases}$$

Since the operator $\mathcal{L}_\varepsilon$ satisfies the comparison principle on $B_{\rho k_{1,\varepsilon}}(0)$. Therefore

$$|u_\varepsilon(x)| \leq \Psi_\varepsilon^0(x) \qquad \text{for all } x \in B_{\rho k_{1,\varepsilon}}(0).$$

And so for all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq C_2(\rho) \frac{\varphi(x)}{\mu_{1,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}}} \quad \text{for all } x \in B_{\rho k_{1,\varepsilon}}(0) \setminus \{0\}.$$

for ρ small. Using the estimate (67) we have or all $\varepsilon > 0$ small

$$|u_\varepsilon(x)| \leq \frac{C_2(\rho)}{\mu_{1,\varepsilon}^{\frac{\beta_+(\gamma')-\beta_-(\gamma')}{2}} |x|^{\beta_-(\gamma')}} \quad \text{for all } x \in B_{\rho k_{1,\varepsilon}}(0) \setminus \{0\}.$$

for ρ small. It then follows from point (A4) of Proposition (2) that estimate (87) holds on $x \in B_{\rho k_{1,\varepsilon}}(0)$ for all $\rho > 0$. □

Step 5.4: Combining the previous three steps, it follows from (63), (78), (87) and Proposition 2 that for any $\alpha > 0$ small, there exists $C(\alpha) > 0$ such that for all $\varepsilon > 0$ we have

$$|u_\varepsilon(x)| \leq C(\alpha) \left(\sum_{i=1}^N \frac{\mu_{i,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha}}{\mu_{i,\varepsilon}^{(\beta_+(\gamma)-\beta_-(\gamma))-2\alpha} |x|^{\beta_-(\gamma)+\alpha} + |x|^{\beta_+(\gamma)-\alpha}} + \frac{\||x|^{\beta_-(\gamma)} u_0\|_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)+\alpha}} \right) \tag{95}$$

for all $x \in \Omega \setminus \{0\}$.

Next we improve the above estimate and show that one can take $\alpha = 0$ in (95).

We let G_0 be the Green's function of the coercive operator $-\Delta - \frac{\gamma}{|x|^2} - h$ on Ω with Dirichlet boundary condition. Green's representation formula, the pointwise

bounds on the Green's function (23) [see Ghoussoub-Robert [22]] yields for any $z \in \Omega$

$$u_\varepsilon(z) = \int_\Omega G_0(z,x)b(x)\frac{|u_\varepsilon(x)|^{2^\star(s)-2-p_\varepsilon}\ u_\varepsilon(x)}{|x|^s}\ dx$$

$$|u_\varepsilon(z)| \leq \int_\Omega G_0(z,x)|b(x)|\frac{|u_\varepsilon(x)|^{2^\star(s)-1-p_\varepsilon}}{|x|^s}\ dx$$

$$\leq C\int_\Omega \left(\frac{\max\{|z|,|x|\}}{\min\{|z|,|x|\}}\right)^{\beta_-(\gamma)} \frac{1}{|x-z|^{n-2}}\frac{|u_\varepsilon(x)|^{2^\star(s)-1-p_\varepsilon}}{|x|^s}\ dx. \tag{96}$$

Using (95) we then obtain with $0 < \alpha < \frac{2^\star(s)-2}{2^\star(s)-1}\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)$ that

$$|u_\varepsilon(z)| \leq$$

$$C\sum_{i=1}^N \int_\Omega \left(\frac{\max\{|z|,|x|\}}{\min\{|z|,|x|\}}\right)^{\beta_-(\gamma)} \frac{1}{|x-z|^{n-2}|x|^s}\left(\frac{\mu_{i,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}-\alpha}}{\mu_{i,\varepsilon}^{(\beta_+(\gamma)-\beta_-(\gamma))-2\alpha}|x|^{\beta_-(\gamma)+\alpha}+|x|^{\beta_+(\gamma)-\alpha}}\right)^{2^\star(s)-1-p_\varepsilon} dx$$

$$+ C\||x|^{\beta_-(\gamma)}u_0\|_{L^\infty(\Omega)}^{2^\star(s)-1-p_\varepsilon}\int_\Omega \left(\frac{\max\{|z|,|x|\}}{\min\{|z|,|x|\}}\right)^{\beta_-(\gamma)} \frac{1}{|x-z|^{n-2}|x|^s}\frac{1}{|x|^{(\beta_-(\gamma)+\alpha)(2^\star(s)-1-p_\varepsilon)}}\ dx. \tag{97}$$

The first term in the above integral was computed for each bubble in Ghoussoub-Robert in [22] when $p_\varepsilon = 0$. The proof goes exactly the same with $p_\varepsilon > 0$. The last last term in straightforward to estimate.

We then get that there exists a constant $C > 0$ such that for any sequence of points (z_ε) in $\Omega \setminus \{0\}$ we have

$$|u_\varepsilon(z_\varepsilon)| \leq\ C\left(\sum_{i=1}^N \frac{\mu_{i,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}{\mu_{i,\varepsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}|z_\varepsilon|^{\beta_-(\gamma)}+|z_\varepsilon|^{\beta_+(\gamma)}} + \frac{\||x|^{\beta_-(\gamma)}u_0\|_{L^\infty(\Omega)}}{|z_\varepsilon|^{\beta_-(\gamma)}}\right). \tag{98}$$

This completes the proof of Proposition (3). □

6. Proof of Compactness

In this section we prove our compactness result, Theorem (1.3).

Proposition 4. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $\gamma < \frac{(n-2)^2}{4}$. Let (u_ε), (p_ε), $h(x)$ and $b(x)$ be such that (E_ε), (27), (28), (29) and (30) holds. Assume that blow-up occurs, that is*

$$\lim_{\varepsilon\to 0}\||x|^\tau u_\varepsilon\|_{L^\infty(\Omega)} = +\infty \quad \textit{where} \quad \beta_-(\gamma) < \tau < \frac{n-2}{2}.$$

Then we have following rate of blow-up when $\gamma < \dfrac{(n-2)^2}{4} - (2-\theta)^2$.

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon}{\mu_{N,\varepsilon}^{2-\theta}} = -\left(\frac{2-\theta}{2}\right) \frac{c}{t_N^{\frac{n-\theta}{2^\star(s)-2}}} \frac{4(n-s)}{(n-2)^2} \frac{\int\limits_{\mathbb{R}^n} \frac{\tilde{u}_N^2}{|x|^\theta} dx}{\sum\limits_{i=1}^N \frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int\limits_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}}$$

Proof of Proposition (4)*:* The proof proceeds in several steps.

Note that $\gamma < \frac{(n-2)^2}{4} - (2-\theta)$ is equivalent to $\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2} > 2-\theta$.

Step 6.1: Let $r_\varepsilon := \sqrt{\mu_{N,\varepsilon}}$. We rescale and define for all $\varepsilon > 0$

$$\tilde{v}_\varepsilon(x) := r_\varepsilon^{\beta_-(\gamma)} u_\varepsilon(r_\varepsilon\, x) \qquad \text{for } x \in r_\varepsilon^{-1}\Omega \setminus \{0\}$$

Then there exists $\tilde{v} \in C^1(\mathbb{R}^n \setminus \{0\})$ such that

$$\lim_{\varepsilon\to 0} \tilde{v}_\varepsilon(x) = \tilde{v} \ \text{ in } C^1_{loc}(\mathbb{R}^n \setminus \{0\})$$
$$\text{where} \qquad -\Delta\tilde{v} - \frac{\gamma}{|x|^2}\tilde{v} = 0 \ \text{ in } \mathbb{R}^n \setminus \{0\}$$

Proof of step 6.1: From (E_ε) it follows that $\tilde{v}_\varepsilon$ weakly satisfies the equation

(99)
$$-\Delta\tilde{v}_\varepsilon - \frac{\gamma}{|x|^2}\tilde{v}_\varepsilon - r_\varepsilon^2\ h(r_\varepsilon x)\ \tilde{v}_\varepsilon = r_\varepsilon^{(2^\star(s)-2)\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}+p_\varepsilon\beta_-(\gamma)} b(r_\varepsilon x) \frac{|\tilde{v}_\varepsilon|^{2^\star(s)-2-p_\varepsilon}\tilde{v}_\varepsilon}{|x|^s}$$

for all $\varepsilon > 0$. Using the pointwise estimate (62) we obtain the bound, that there exists a constant $C > 0$ such that for all $\varepsilon > 0$

$$|\tilde{v}_\varepsilon(x)| \le C \left(r_\varepsilon^{\beta_-(\gamma)} \sum_{i=1}^N \frac{\mu_{i,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}{\mu_{i,\varepsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}|r_\varepsilon x|^{\beta_-(\gamma)} + |r_\varepsilon x|^{\beta_+(\gamma)}} + r_\varepsilon^{\beta_-(\gamma)} \frac{\||x|^{\beta_-(\gamma)}u_0||_{L^\infty(\Omega)}}{|r_\varepsilon x|^{\beta_-(\gamma)}} \right)$$
$$\le C \left(\sum_{i=1}^N \frac{\left(\frac{\mu_{i,\varepsilon}}{\mu_{N,\varepsilon}}\right)^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}}{\left(\frac{\mu_{i,\varepsilon}}{\sqrt{\mu_{N,\varepsilon}}}\right)^{\beta_+(\gamma)-\beta_-(\gamma)}|x|^{\beta_-(\gamma)} + |x|^{\beta_+(\gamma)}} + \frac{\||x|^{\beta_-(\gamma)}u_0||_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)}} \right)$$

(100)
$$\le C \left(\frac{1}{|x|^{\beta_+(\gamma)}} + \frac{\||x|^{\beta_-(\gamma)}u_0||_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)}} \right) \qquad \text{for all } x \in r_\varepsilon^{-1}\Omega \setminus \{0\}.$$

Standard elliptic theory then yields the existence of $\tilde{v} \in C^2(\mathbb{R}^n \setminus \{0\})$ such that $\tilde{v}_\varepsilon \to \tilde{v}$ in $C^2_{loc}(\mathbb{R}^n \setminus \{0\})$ and then passing to limits in equation (99) we then obtain that $\tilde{v}$ satisfies the equation

$$-\Delta\tilde{v} - \frac{\gamma}{|x|^2}\tilde{v} = 0 \ \text{ in } \mathbb{R}^n \setminus \{0\}.$$

and

$$|\tilde{v}(x)| \le C \left(\frac{1}{|x|^{\beta_+(\gamma)}} + \frac{\||x|^{\beta_-(\gamma)}u_0||_{L^\infty(\Omega)}}{|x|^{\beta_-(\gamma)}} \right) \qquad \text{for all } x \text{ in } \mathbb{R}^n \setminus \{0\}.$$

□

Step 6.2: From the Pohozaev identity (119) with $y_0 = 0$ and $U_\varepsilon = B_{r_\varepsilon}(0) \setminus B_{k_{1,\varepsilon}^2}(0)$ we get

$$
\begin{aligned}
&- \int\limits_{B_{r_\varepsilon}(0)\setminus B_{k_{1,\varepsilon}^2}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx \\
&- \frac{p_\varepsilon}{2^\star(s)} \left(\frac{n-s}{2^\star(s) - p_\varepsilon} \right) \int\limits_{B_{r_\varepsilon}(0)\setminus B_{k_{1,\varepsilon}^2}(0)} b(x) \frac{|u_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s} \, dx \\
&\qquad - \frac{1}{2^\star(s) - p_\varepsilon} \int\limits_{B_{r_\varepsilon}(0)\setminus B_{k_{1,\varepsilon}^2}(0)} \frac{|u|^{2^\star(s)-p_\varepsilon}}{|x|^s} (x, \nabla b) \, dx \\
&\qquad = \int\limits_{\partial B_{r_\varepsilon}(0)} F_\varepsilon(x) \, d\sigma - \int\limits_{\partial B_{k_{1,\varepsilon}^2}(0)} F_\varepsilon(x) \, d\sigma \qquad (101)
\end{aligned}
$$

where

$$
\begin{aligned}
F_\varepsilon(x) := (x, \nu) &\left(\frac{|\nabla u_\varepsilon|^2}{2} - \frac{\gamma}{2} \frac{u_\varepsilon^2}{|x|^2} - \frac{h(x)}{2} u_\varepsilon^2 - \frac{b(x)}{2^\star(s) - p_\varepsilon} \frac{|u_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s} \right) \\
&- \left(x^i \partial_i u_\varepsilon + \frac{n-2}{2} u_\varepsilon \right) \partial_\nu u_\varepsilon. \qquad (102)
\end{aligned}
$$

We will estimate each of the terms in the above integral identity and calculate the limit as $\varepsilon \to 0$.

Step 6.3: When $\beta_+(\gamma) - \beta_-(\gamma) > 2 - \theta$, we have as $\varepsilon \to 0$

(103)

$$
\int\limits_{B_{r_\varepsilon}(0)\setminus B_{k_{1,\varepsilon}^2}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx = \mu_{N,\varepsilon}^{2-\theta} \left[\left(\frac{2-\theta}{2} \right) \frac{c}{t_N^{\frac{n-\theta}{2^\star(s)-2}}} \int\limits_{\mathbb{R}^n} \frac{\tilde{u}_N^2}{|x|^\theta} \, dx + o(1) \right].
$$

where $0 \le \theta < 2$ and $c \in \mathbb{R}$ are such that: $\lim\limits_{x\to 0} |x|^\theta h(x) = c$ and $\lim\limits_{x\to 0} |x|^\theta (x, \nabla h(x)) = -c\theta$.

Proof of Step 6.3: When $\beta_+(\gamma) - \beta_-(\gamma) > 2 - \theta$ it follows from point (2) of Theorem (1.4) that the for all $1 \le i \le N$, the function $\frac{\tilde{u}_i}{|x|^{\theta/2}} \in L^2(\mathbb{R}^n)$. For any $R, \rho > 0$ we decompose the integral as

$$
\begin{aligned}
&\int\limits_{B_{r_\varepsilon}(0)\setminus B_{k_{1,\varepsilon}^2}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx = \\
&\int\limits_{B_{r_\varepsilon}(0)\setminus \overline{B}_{Rk_{N,\varepsilon}}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx + \sum_{i=1}^{N} \int\limits_{B_{Rk_{i,\varepsilon}}(0)\setminus \overline{B}_{\rho k_{i,\varepsilon}}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx \\
&+ \sum_{i=1}^{N-1} \int\limits_{B_{\rho k_{i+1,\varepsilon}}(0)\setminus \overline{B}_{Rk_{i,\varepsilon}}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx + \int\limits_{B_{\rho k_{1,\varepsilon}}(0)\setminus B_{k_{1,\varepsilon}^2}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx.
\end{aligned}
$$

From the estimate (62), we get as $\varepsilon \to 0$

$$\begin{aligned}
&\mu_{N,\varepsilon}^{-(2-\theta)} \int\limits_{B_{r_\varepsilon}(0)\setminus\overline{B}_{Rk_{N,\varepsilon}}(0)} \left| h(x) + \frac{(\nabla h, x)}{2} \right| u_\varepsilon^2 \, dx \\
&\leq C \, \mu_{N,\varepsilon}^{-(2-\theta)} \int\limits_{B_{r_\varepsilon}(0)\setminus\overline{B}_{Rk_{N,\varepsilon}}(0)} \frac{1}{|x|^\theta} \left[\frac{\mu_{N,\varepsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}}{|x|^{2\beta_+(\gamma)}} + \frac{1}{|x|^{2\beta_-(\gamma)}} \right] dx \\
&\leq C \int\limits_{B_{\frac{r_\varepsilon}{k_{N,\varepsilon}}}(0)\setminus\overline{B}_R(0)} \frac{1}{|x|^\theta} \frac{1}{|x|^{n+(\beta_+(\gamma)-\beta_-(\gamma)-2)}} \, dx \\
&+ C \int\limits_{B_1(0)\setminus\overline{B}_{\frac{Rk_{N,\varepsilon}}{r_\varepsilon}}(0)} \frac{1}{|x|^\theta} \frac{\mu_{N,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)-(2-\theta)}{2}}}{|x|^{n-(\beta_+(\gamma)-\beta_-(\gamma)+2)}} \, dx \\
&\leq C \left(R^{-(\beta_+(\gamma)-\beta_-(\gamma)-(2-\theta))} + \mu_{N,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)-(2-\theta)}{2}} \right).
\end{aligned}$$

Therefore when $\beta_+(\gamma) - \beta_-(\gamma) > 2 - \theta$

$$\lim_{R\to+\infty} \lim_{\varepsilon\to 0} \mu_{N,\varepsilon}^{-(2-\theta)} \int\limits_{B_{r_\varepsilon}(0)\setminus\overline{B}_{Rk_{N,\varepsilon}}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx = 0. \tag{104}$$

Since in this case $\frac{\tilde{u}_i}{|x|^{\theta/2}} \in L^2(\mathbb{R}^n_-)$ for any $1 \leq i \leq N$, it follows from Proposition 2 that

$$\begin{aligned}
&\lim_{R\to+\infty} \lim_{\rho\to 0} \lim_{\varepsilon\to 0} \left(\mu_{i,\varepsilon}^{-(2-\theta)} \int\limits_{B_{Rk_{i,\varepsilon}}(0)\setminus\overline{B}_{\rho k_{i,\varepsilon}}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx \right) \\
&\qquad = \left(\frac{2-\theta}{2} \right) \frac{c}{t_i^{\frac{n-\theta}{2^\star(s)-2}}} \int\limits_{\mathbb{R}^n} \frac{\tilde{u}_i^2}{|x|^\theta} \, dx.
\end{aligned} \tag{105}$$

where $0 \leq \theta < 2$ and $c \in \mathbb{R}$ are such that: $\lim\limits_{x\to 0} |x|^\theta h(x) = c$ and $\lim\limits_{x\to 0} |x|^\theta (x, \nabla h(x)) = -c\theta$.

Let $1 \leq i \leq N-1$. Using the pointwise estimates of theorem (3), for any $R, \rho > 0$ and all $\varepsilon > 0$ we have as $\varepsilon \to 0$

$$\begin{aligned}
&\mu_{i+1,\varepsilon}^{-(2-\theta)} \int\limits_{B_{\rho k_{i+1,\varepsilon}}(0)\setminus\overline{B}_{Rk_{i,\varepsilon}}(0)} \left| h(x) + \frac{(\nabla h, x)}{2} \right| u_\varepsilon^2 \, dx \\
&\leq C \, \mu_{i+1,\varepsilon}^{-(2-\theta)} \int\limits_{B_{\rho k_{i+1,\varepsilon}}(0)\setminus\overline{B}_{Rk_{i,\varepsilon}}(0)} \frac{1}{|x|^\theta} \left[\frac{\mu_{i,\varepsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}}{|x|^{2\beta_+(\gamma)}} + \frac{\mu_{i+1,\varepsilon}^{-(\beta_+(\gamma)-\beta_-(\gamma))}}{|x|^{2\beta_-(\gamma)}} \right] dx \\
&\leq C \, \mu_{i+1,\varepsilon}^{-(2-\theta)} \int\limits_{B_{\frac{\rho k_{i+1,\varepsilon}}{k_{i,\varepsilon}}}(0)\setminus\overline{B}_R(0)} \frac{1}{|x|^\theta} \frac{\mu_{i,\varepsilon}^{2-\theta}}{|x|^{n+(\beta_+(\gamma)-\beta_-(\gamma)-2)}} \, dx \\
&\quad + \, C \, \mu_{i+1,\varepsilon}^{-(2-\theta)} \int\limits_{B_\rho(0)\setminus\overline{B}_{\frac{Rk_{i,\varepsilon}}{k_{i+1,\varepsilon}}}(0)} \frac{1}{|x|^\theta} \frac{\mu_{i+1,\varepsilon}^{2-\theta}}{|x|^{n-(\beta_+(\gamma)-\beta_-(\gamma)+2)}} \, dx \\
&\quad \leq C \, \mu_{i+1,\varepsilon}^{-(2-\theta)} \left(\mu_{i,\varepsilon}^{2-\theta} R^{-(\beta_+(\gamma)-\beta_-(\gamma)-(2-\theta))} + \mu_{i+1,\varepsilon}^{2-\theta} \rho^{\beta_+(\gamma)-\beta_-(\gamma)+(2-\theta)} \right).
\end{aligned}$$

And so as $\beta_+(\gamma) - \beta_-(\gamma) > 2$

$$\lim_{R\to+\infty} \lim_{\rho\to 0} \lim_{\varepsilon\to 0} \mu_{i+1,\varepsilon}^{-(2-\theta)} \int\limits_{B_{\rho k_{i+1,\varepsilon}}(0)\setminus\overline{B}_{Rk_{i,\varepsilon}}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx = 0. \tag{106}$$

And from the pointwise estimates of theorem (3), we have as $\varepsilon \to 0$

$$\begin{aligned}
&\mu_{1,\varepsilon}^{-(2-\theta)} \int\limits_{B_{\rho k_{1,\varepsilon}}(0)\setminus B_{k_{1,\varepsilon}^2}(0)} \left| h(x) + \frac{(\nabla h, x)}{2} \right| u_\varepsilon^2 \, dx \\
&\leq C \, \mu_{1,\varepsilon}^{-(2-\theta)} \int\limits_{B_{\rho k_{1,\varepsilon}}(0)\setminus\overline{B}_{k_{1,\varepsilon}^2}(0)} \frac{1}{|x|^\theta} \frac{\mu_{1,\varepsilon}^{-(\beta_+(\gamma)-\beta_-(\gamma))}}{|x|^{2\beta_-(\gamma)}} \, dx \\
&\leq C \int\limits_{B_\rho(0)\setminus\overline{B}_{k_{1,\varepsilon}}(0)} \frac{1}{|x|^\theta} \frac{1}{|x|^{n-(\beta_+(\gamma)-\beta_-(\gamma)+2)}} \, dx \\
&\quad \leq C \, \rho^{\beta_+(\gamma)-\beta_-(\gamma)+(2-\theta)}.
\end{aligned}$$

Therefore

$$\lim_{\rho\to 0} \lim_{\varepsilon\to 0} \mu_{1,\varepsilon}^{-(2-\theta)} \int\limits_{B_{\rho k_{1,\varepsilon}}(0)\setminus B_{k_{1,\varepsilon}^2}(0)} \left(h(x) + \frac{(\nabla h, x)}{2} \right) u_\varepsilon^2 \, dx = 0. \tag{107}$$

From (104), (105), (106), (107) and Proposition (2) we then obtain (103).

□

Similarly with the pointwise control of Proposition (3) and by Proposition (2), we get as $\varepsilon \to 0$

$$\int\limits_{B_{r_\varepsilon}(0)\setminus B_{k_{1,\varepsilon}^2}(0)} b(x)\frac{|u_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s}\,dx = \sum_{i=1}^{N}\frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^n}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx + o(1). \tag{108}$$

And using the condition (29) on b we obtain as $\varepsilon \to 0$

$$\mu_{N,\varepsilon}^{-(2-\theta)}\int\limits_{B_{r_\varepsilon}(0)\setminus B_{k_{1,\varepsilon}^2}(0)} (x,\nabla b)\frac{|u_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s}\,dx = o(1). \tag{109}$$

Next, with a change of variable and the definition of $\tilde{v}_\varepsilon$, we get

$$\int\limits_{\partial B_{r_\varepsilon}(0)} F_\varepsilon(x)\,d\sigma =$$

$$r_\varepsilon^{\beta_+(\gamma)-\beta_-(\gamma)}\int\limits_{\partial B_1(0)}(x,\nu)\left(\frac{|\nabla\tilde{v}_\varepsilon|^2}{2}-\frac{\gamma}{2}\frac{\tilde{v}_\varepsilon^2}{|x|^2}\right)-\left(x^i\partial_i\tilde{v}_\varepsilon+\frac{n-2}{2}\tilde{v}_\varepsilon\right)\partial_\nu\tilde{v}_\varepsilon\,d\sigma$$

$$-\,r_\varepsilon^{\beta_+(\gamma)-\beta_-(\gamma)}\int\limits_{\partial B_1(0)} r_\varepsilon^2\frac{h(r_\varepsilon x)}{2}u_\varepsilon^2-\frac{r_\varepsilon^{(2^\star(s)-2)\left(\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}\right)+\beta_-(\gamma)p_\varepsilon}}{2^\star(s)-p_\varepsilon}\frac{|\tilde{v}_\varepsilon|^{2^\star(s)-p_\varepsilon}}{|x|^s}\,d\sigma$$

From the convergence result of *Step 6.1* we then have as $\varepsilon \to 0$

$$\int\limits_{\partial B_{r_\varepsilon}(0)} F_\varepsilon(x)\,d\sigma = O\left(\mu_{N,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}\right). \tag{110}$$

Similarly we have

$$\int\limits_{\partial B_{k_{1,\varepsilon}}(0)} F_\varepsilon(x)\,d\sigma = O\left(\mu_{N,\varepsilon}^{\beta_+(\gamma)-\beta_-(\gamma)}\right). \tag{111}$$

□

Plugging (103), (108), (109) (110) and (111) into the Pohozaev identity (101) yields as $\varepsilon \to 0$

$$-\mu_{N,\varepsilon}^{2-\theta}\left[\left(\frac{2-\theta}{2}\right)\frac{c}{t_N^{\frac{n-\theta}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^n}\frac{\tilde{u}_N^2}{|x|^\theta}\,dx+o(1)\right]-$$

$$\frac{p_\varepsilon}{2^\star(s)}\left(\frac{n-s}{2^\star(s)-p_\varepsilon}\right)\left(\sum_{i=1}^{N}\frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}}\int\limits_{\mathbb{R}^n}\frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}\,dx+o(1)\right)$$

$$= O\left(\mu_{N,\varepsilon}^{\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2}}\right).$$

In case $\frac{\beta_+(\gamma)-\beta_-(\gamma)}{2} > 2-\theta$ and since $\tilde{u}_N \not\equiv 0$, we obtain that

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon}{\mu_{N,\varepsilon}^{2-\theta}} = -\left(\frac{2-\theta}{2}\right) \frac{c}{t_N^{\frac{n-\theta}{2^\star(s)-2}}} \frac{4(n-s)}{(n-2)^2} \frac{\int_{\mathbb{R}^n} \frac{\tilde{u}_N^2}{|x|^\theta} dx}{\sum_{i=1}^N \frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}} \tag{112}$$

This completes the proof of Proposition (4). □

As a by-product of Proposition (4) one obtains the compactness result for sign-changing solutions.

Theorem 6.1. *Let Ω be a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$, such that $0 \in \Omega$ and assume that $0 < s < 2$ and $\gamma < \frac{(n-2)^2}{4}$. Suppose that $h(x)$ and $b(x)$ satisfies (28) and (29) respectively. Let $(p_\varepsilon)_{\varepsilon>0}$ be such that $p_\varepsilon \in [0, 2^\star(s)-2)$ for all $\varepsilon > 0$ and $\lim_{\varepsilon\to 0} p_\varepsilon = 0$. Consider a sequence of functions $(u_\varepsilon)_{\varepsilon>0}$ that is uniformly bounded in $H_0^1(\Omega)$ and satisfies the equation:*

$$\begin{cases} -\Delta u - \gamma \frac{u}{|x|^2} - h(x)u &= b(x) \frac{|u|^{2^\star(s)-2-p_\varepsilon} u}{|x|^s} & \text{in } \Omega \setminus \{0\}, \\ u &= 0 & \text{on } \partial\Omega. \end{cases}$$

Then the sequence $(u_\varepsilon)_{\varepsilon>0}$ is pre-compact in the $H_0^1(\Omega)$ if $\lim_{|x|\to 0} |x|^\theta h(x) > 0$ for $\gamma < \frac{(n-2)^2}{4} - (2-\theta)^2$.

Proof of Theorem (6.1)**:** (u_ε), (p_ε), $h(x)$ and $b(x)$ be such that (E_ε), (27), (28), (29) and (30) holds. Assume that blow-up occurs, that is

$$\lim_{\varepsilon\to 0} \||x|^\tau u_\varepsilon\|_{L^\infty(\Omega)} = +\infty \quad \text{where} \quad \beta_-(\gamma) < \tau < \frac{n-2}{2}.$$

When $\gamma < \frac{(n-2)^2}{4} - (2-\theta)^2$, by Proposition (4) we have

$$\lim_{\varepsilon\to 0} \frac{p_\varepsilon}{\mu_{N,\varepsilon}^{2-\theta}} = -\left(\frac{2-\theta}{2}\right) \frac{c}{t_N^{\frac{n-\theta}{2^\star(s)-2}}} \frac{4(n-s)}{(n-2)^2} \frac{\int_{\mathbb{R}^n} \frac{\tilde{u}_N^2}{|x|^\theta} dx}{\sum_{i=1}^N \frac{b(0)}{t_i^{\frac{n-2}{2^\star(s)-2}}} \int_{\mathbb{R}^n} \frac{|\tilde{u}_i|^{2^\star(s)}}{|x|^s}}$$

Since $p_\varepsilon \geq 0$ this a contradiction if $\lim_{|x|\to 0} |x|^\theta h(x) = c > 0$. So blow-up (34) cannot occur in this case and by Proposition (1) we have up to a subsequence $\lim_{\varepsilon\to 0} u_\varepsilon = u_0$ in $H_0^1(\Omega)$ where u_0 is as in (31). This completes the proof of Theorem (6.1). □

7. Proof of Existence and Multiplicity

We are now ready to prove Theorem (1.2). For each $2 < p \leq 2^\star(s)$, we consider the C^2-functional

$$I_{p,\gamma,h}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 \, dx - \frac{\gamma}{2} \int_\Omega \frac{u^2}{|x|^2} \, dx - \frac{1}{2} \int_\Omega hu \, dx - \frac{1}{p} \int_\Omega \frac{b(x)|u|^p}{|x|^s} \, dx \tag{113}$$

on $H_0^1(\Omega)$, whose critical points are the weak solutions of

$$\left\{ \begin{array}{rl} -\Delta u - \frac{\gamma}{|x|^2}u - h(x)u &= \frac{b(x)|u|^{p-2}u}{|x|^s} \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega. \end{array} \right. \tag{114}$$

Recall that $b(x)$ is a positive function in $C^1(\overline{\Omega})$ with $b(0) > 0$ and $\nabla b(0) = 0$ and while $h(x)$ satisfies (28).

For a fixed $u \in H_0^1(\Omega)$, we have since

$$I_{p,\gamma,h}(\lambda u) = \frac{\lambda^2}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\gamma\lambda^2}{2}\int_\Omega \frac{u^2}{|x|^2}\,dx - \frac{\lambda^2}{2}\int_\Omega h(x)u^2 - \frac{\lambda^p}{p}\int_\Omega \frac{b(x)|u|^p}{|x|^s}\,dx$$

that $\text{limit}_{\lambda\to\infty} I_{p,\gamma,h}(\lambda u) = -\infty$, which means that for each finite dimensional subspace $E_k \subset E := H_0^1(\Omega)$, there exists $R_k > 0$ such that

$$\sup\{I_{p,\gamma,h}(u); u \in E_k, \|u\| > R_k\} < 0, \tag{115}$$

when $p \to 2^\star(s)$. Let $(E_k)_{k=1}^\infty$ be an increasing sequence of subspaces of $H_0^1(\Omega)$ such that $\dim E_k = k$ and $\overline{\cup_{k=1}^\infty E_k} = E := H_0^1(\Omega)$ and define the min-max values:

$$c_{p,k} = \inf_{g\in\mathbf{H}_k}\sup_{x\in E_k} I_{p,\gamma,h}(g(x)),$$

where

$$\mathbf{H}_k = \{g \in C(E,E);\ g \text{ is odd and } g(v) = v \text{ for } \|v\| > R_k \text{ for some } R_k > 0\}.$$

Proposition 5. *With the above notation and assuming $n \geq 3$, we have:*

1. *For each $k \in \mathbb{N}$, $c_{p,k} > 0$ and $\lim_{p\to 2^\star(s)} c_{p,k} = c_{2^\star(s),k} := c_k$.*
2. *If $2 < p < 2^\star(s)$, there exists for each k, functions $u_{p,k} \in H_0^1(\Omega)$ such that $I'_{p,\gamma}(u_{p,k}) = 0$, and $I_{p,\gamma,h}(u_{p,k}) = c_{p,k}$.*
3. *For each $2 < p < 2^\star(s)$, we have $c_{p,k} \geq D_{n,p}k^{\frac{p+1}{p-1}\frac{2}{n}}$ where $D_{n,p} > 0$ is such that $\lim_{p\to 2^\star(s)} D_{n,p} = 0$.*
4. *$\lim_{k\to\infty} c_k = \lim_{k\to\infty} c_{2^\star(s),k} = +\infty$.*

Proof: First, note that since the operator $-\Delta - \frac{\gamma}{|x|^2} - h(x)$ is coercive, we have for some constant $\Lambda_0 > 0$

$$\int_\Omega \left(|\nabla u|^2 - \gamma\frac{u^2}{|x|^2} - h(x)u^2\right)dx \geq \Lambda_0\int_\Omega |\nabla u|^2\,dx \quad \text{for all } u \in H_0^1(\Omega). \tag{116}$$

Then in view of the Hardy-Sobolev inequalities, we have

$$I_{p,\gamma,h}(u) \geq \frac{\Lambda_0}{2}\|\nabla u\|_2^2 - C\|\nabla u\|_2^p = \|\nabla u\|_2^2\left(\frac{\Lambda_0}{2} - C\|\nabla u\|_2^{p-2}\right) \geq \alpha > 0,$$

provided $\|u\|_{H_0^1(\Omega)} = \rho$ for some $\rho > 0$ small enough. Then the $S_\rho = \{u \in E; \|u\|_{H_0^1(\Omega)} = \rho\}$ must intersect every image $g(E_k)$ by an odd continuous function g. It follows that

$$c_{p,k} \geq \inf\{I_{p,\gamma,h}(u); u \in S_\rho\} \geq \alpha > 0.$$

In view of (115), it follows that for each $g \in \mathbf{H}_k$, we have that

$$\sup_{x\in E_k} I_{p_i,\gamma,h}(g(x)) = \sup_{x\in D_k} I_{p,\gamma,h}(g(x))$$

where D_k denotes the ball in E_k of radius R_k. Consider now a sequence $p_i \to 2^\star(s)$ and note first that for each $u \in E$, we have that $I_{p_i,\gamma,h}(u) \to I_{2^\star(s),\gamma.h}(u)$. Since $g(D_k)$ is compact and the family of functionals $(I_{p,\gamma,h})_p$ is equicontinuous, it follows that $\sup_{x\in E_k} I_{p,\gamma}(g(x)) \to \sup_{x\in E_k} I_{2^\star(s),\gamma,h}(g(x))$, from which follows that $\limsup_{i\in\mathbb{N}} c_{p_i,k} \leq \sup_{x\in E_k} I_{2^\star(s),\gamma,h}(g(x))$. Since this holds for any $g \in \mathbf{H}_k$, it follows that

$$\limsup_{i\in\mathbb{N}} c_{p_i,k} \leq c_{2^\star(s),k} = c_k.$$

On the other hand, the function $f(r) = \frac{1}{p}r^p - \frac{1}{2^\star(s)}r^{2^\star(s)}$ attains its maximum on $[0,+\infty)$ at $r = 1$ and therefore $f(r) \leq \frac{1}{p} - \frac{1}{2^\star(s)}$ for all $r > 0$. It follows

$$\begin{aligned} I_{2^\star(s),\gamma,h}(u) &= I_{p,\gamma,h}(u) + \int_\Omega \frac{b(x)}{|x|^s}\left(\frac{1}{p}|u(x)|^p - \frac{1}{2^\star(s)}|u(x)|^{2^\star(s)}\right) dx \\ &\leq I_{p,\gamma,h}(u) + \int_\Omega \frac{b(x)}{|x|^s}\left(\frac{1}{p} - \frac{1}{2^\star(s)}\right) dx \end{aligned}$$

from which follows that $c_k \leq \liminf_{i\in\mathbb{N}} c_{p_i,k}$, and claim (1) is proved.

If now $p < 2^\star(s)$, we are in the subcritical case, that is we have compactness in the Sobolev embedding $H_0^1(\Omega) \to L^p(\Omega; |x|^{-s}dx)$ and therefore $I_{p,\gamma,h}$ has the Palais-Smale condition. It is then standard to find critical points $u_{p,k}$ for $I_{p,\gamma,h}$ at each level $c_{p,k}$ (see for example the book [17]).

Consider now the functional

$$I_{p,0,0}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{p}\int_\Omega \frac{|u|^p}{|x|^s}\, dx \tag{117}$$

and its critical values

$$c^0_{p,k} = \inf_{g\in\mathbf{H}_k}\sup_{x\in E_k} I_{p,0,0}(g(x)).$$

It has been shown in [20] (for the case $0 \in \partial\Omega$, but the same arguments work here) that (1), (2) and (3) of Proposition 5 hold, with $c^0_{p,k}$ and c^0_k replacing $c_{p,k}$ and c_k respectively. In particular, $\lim_{k\to\infty} c^0_k = \lim_{k\to\infty} c^0_{2^\star(s),k} = +\infty$.
On the other hand, note that

$$I_{p,\gamma,h}(u) \geq \Lambda_0^{\frac{p}{p-2}} I_{p,0,0}(v) \quad \text{for every } u \in H_0^1(\Omega),$$

where Λ_0 is the coercivity constant from (116)and $v = \Lambda_0^{-\frac{1}{p-2}}u$. It then follows that $\lim_{k\to\infty} c_k = \lim_{k\to\infty} c_{2^\star(s),k} = +\infty$. □

To complete the proof of Theorem (1.2), notice that since for each k, we have $\lim_{p_i\to 2^\star(s)} I_{p_i,\gamma,h}(u_{p_i,k}) = \lim_{p_i\to 2^\star(s)} c_{p_i,k} = c_k$, it follows that the sequence $(u_{p_i,k})_i$ is uniformly bounded in $H_0^1(\Omega)$. Moreover, since $I'_{p_i,\gamma,h}(u_{p_i,k}) = 0$, it follows from the compactness Theorem 6.1 that by letting $p_i \to 2^\star(s)$, we get a solution u_k of (12) in such a way that $I_{2^\star(s),\gamma,h}(u_k) = \lim_{p\to 2^\star(s)} I_{p,\gamma,h}(u_{p,k}) = \lim_{p\to 2^\star(s)} c_{p,k} = c_k$. Since the latter sequence goes to infinity, it follows that (12) has an infinite number of critical levels. □

8. Appendix: Pohozaev identity

Let $U \subset \mathbb{R}^n$ be a smooth bounded domain and let $u \in C^2(\overline{U})$. For any $y_0 \in \mathbb{R}^n$, the classical Pohozaev identity yields

(118)
$$-\int_U \left((x-y_0)^i\partial_i u + \frac{n-2}{2}u\right)\Delta u\, dx = \int_{\partial U}\left[(x-y_0,\nu)\frac{|\nabla u|^2}{2} - \left((x-y_0)^i\partial_i u + \frac{n-2}{2}u\right)\partial_\nu u\right]d\sigma,$$

where ν is the outer normal to the boundary ∂U.

One has for $1 \le j \le n$ and $b \in C^1(\overline{U})$

$$\partial_j\left(b(x)\frac{|u|^{2^\star(s)-p}}{|x|^s}\right) = -s\, b(x)\frac{x^j}{|x|^{s+2}}|u|^{2^\star(s)-p} + (2^\star(s)-p)b(x)\frac{|u|^{2^\star(s)-2-p}}{|x|^s}u\partial_j u + \frac{|u|^{2^\star(s)-p}}{|x|^s}\partial_j b(x).$$

So

$$(x-y_0,\nabla u)\, b(x)\frac{|u|^{2^\star(s)-2-p}}{|x|^s}u = \frac{1}{2^\star(s)-p}(x-y_0)^j\partial_j\left(b(x)\frac{|u|^{2^\star(s)-p}}{|x|^s}\right) + \frac{s}{2^\star(s)-p}b(x)\frac{|u|^{2^\star(s)-p}}{|x|^s} - \frac{s}{2^\star(s)-p}\frac{(x,y_0)}{|x|^{s+2}}b(x)|u|^{2^\star(s)-p} - \frac{1}{2^\star(s)-p}\frac{|u|^{2^\star(s)-p}}{|x|^s}(x-y_0,\nabla b).$$

Then integration by parts gives us

$$\int_U (x-y_0,\nabla u)\, b(x)\frac{|u|^{2^\star(s)-2-p}}{|x|^s}u\, dx = \frac{1}{2^\star(s)-p}\int_U (x-y_0)^j\partial_j\left(b(x)\frac{|u|^{2^\star(s)-p}}{|x|^s}\right)dx + \frac{s}{2^\star(s)-p}\int_U b(x)\frac{|u|^{2^\star(s)-p}}{|x|^s}dx - \frac{s}{2^\star(s)-p}\int_U \frac{(x,y_0)}{|x|^{s+2}}b(x)|u|^{2^\star(s)-p}dx - \frac{1}{2^\star(s)-p}\int_U \frac{|u|^{2^\star(s)-p}}{|x|^s}(x-y_0,\nabla b)\, dx.$$

$$= -\frac{n-s}{2^\star(s)-p}\int_U b(x)\frac{|u|^{2^\star(s)-p}}{|x|^s}\, dx - \frac{s}{2^\star(s)-p}\int_U b(x)\frac{(x,y_0)}{|x|^{s+2}}|u|^{2^\star(s)-p}dx - \frac{1}{2^\star(s)-p}\int_U \frac{|u|^{2^\star(s)-p}}{|x|^s}(x-y_0,\nabla b)\, dx + \frac{1}{2^\star(s)-p}\int_{\partial U}(x-y_0,\nu)b(x)\frac{|u|^{2^\star(s)-p}}{|x|^s}\, d\sigma$$

And similarly we obtain

$$(x-y_0,\nabla u)\frac{u}{|x|^2} = \frac{1}{2}(x-y_0)^j\partial_j\left(\frac{u^2}{|x|^2}\right) + \frac{u^2}{|x|^2} - \frac{(x,y_0)}{|x|^4}u^2$$

$$\int_U (x-y_0,\nabla u)\,\frac{u}{|x|^2}\,dx = -\frac{n-2}{2}\int_U \frac{u^2}{|x|^2}\,dx - \int_U \frac{(x,y_0)}{|x|^4}u^2dx + \frac{1}{2}\int_{\partial U}(x-y_0,\nu)\frac{u^2}{|x|^2}\,d\sigma$$

and

$$\int_U (x-y_0,\nabla u)\,h(x)u\,dx = -\frac{n}{2}\int_U h(x)u^2\,dx - \frac{1}{2}\int_U (\nabla h, x-y_0)\,u^2\,dx + \frac{1}{2}\int_{\partial U}(x-y_0,\nu)h(x)u^2\,d\sigma.$$

Combining, we obtain

(119)

$$\begin{aligned}
&\int_U \left((x-y_0)^i\partial_i u + \frac{n-2}{2}u\right)\left(-\Delta u - \gamma\frac{u}{|x|^2} - hu - b(x)\frac{|u|^{2^\star(s)-2-p}}{|x|^s}u\right)\,dx \\
&\quad - \int_U h(x)u^2\,dx - \frac{1}{2}\int_U (\nabla h, x-y_0)\,u^2\,dx - \frac{p}{2^\star(s)}\left(\frac{n-s}{2^\star(s)-p}\right)\int_U b(x)\frac{|u|^{2^\star(s)-p}}{|x|^s}\,dx \\
&\quad - \frac{1}{2^\star(s)-p}\int_U \frac{|u|^{2^\star(s)-p}}{|x|^s}\,(x-y_0,\nabla b)\,dx - \gamma\int_U \frac{(x,y_0)}{|x|^4}u^2dx \\
&\quad - \frac{s}{2^\star(s)-p}\int_U \frac{(x,y_0)}{|x|^{s+2}}b(x)|u|^{2^\star(s)-p}\,dx \\
&= \int_{\partial U}\left[(x-y_0,\nu)\left(\frac{|\nabla u|^2}{2} - \frac{\gamma}{2}\frac{u^2}{|x|^2} - \frac{h(x)}{2}u^2 - \frac{b(x)}{2^\star(s)-p}\frac{|u|^{2^\star(s)-p}}{|x|^s}\right)\right]d\sigma \\
&\quad - \int_{\partial U}\left[\left((x-y_0)^i\partial_i u + \frac{n-2}{2}u\right)\partial_\nu u\right]d\sigma.
\end{aligned}$$

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