Core congestion is inherent in hyperbolic networks

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Victor Chepoi* Feodor F. Dragan† Yann Vaxès*

Abstract. We investigate the impact the negative curvature has on the traffic congestion in large-scale networks. We prove that every Gromov hyperbolic network G admits a core, thus answering in the positive a conjecture by Jonckheere, Lou, Bonahon, and Baryshnikov, Internet Mathematics, 7 (2011) which is based on the experimental observation by Narayan and Saniee, Physical Review E, 84 (2011) that real-world networks with small hyperbolicity have a core congestion. Namely, we prove that for every subset X of vertices of a graph with δ-thin geodesic triangles (in particular, of a δ-hyperbolic graph) G there exists a vertex m of G such that the ball $B(m, 4\delta)$ of radius 4δ centered at m intercepts at least one half of the total flow between all pairs of vertices of X, where the flow between two vertices $x, y \in X$ is carried by geodesic (or quasi-geodesic) (x, y)-paths. Moreover, we prove a primal-dual result showing that, for any commodity graph R on X and any $r \geq 8\delta$, the size $\sigma_r(R)$ of the least r-multi-core (i.e., the number of balls of radius r) intercepting all pairs of R is upper bounded by the maximum number of pairwise $(2r-5\delta)$-apart pairs of R and that an r-multi-core of size $\sigma_{r-5\delta}(R)$ can be computed in polynomial time for every finite set X.

Our result about total r-multi-cores is based on a Helly-type theorem for quasiconvex sets in δ-hyperbolic graphs (this is our second main result). Namely, we show that for any finite collection $\mathcal{Q}$ of pairwise intersecting ϵ-quasiconvex sets of a δ-hyperbolic graph G there exists a single ball $B(c, 2\epsilon + 5\delta)$ intersecting all sets of $\mathcal{Q}$. More generally, we prove that if $\mathcal{Q}$ is a collection of $2r$-close (i.e., any two sets of $\mathcal{Q}$ are at distance $\leq 2r$) ϵ-quasiconvex sets of a δ-hyperbolic graph G, then there exists a ball $B(c, r^*)$ of radius $r^* := \max\{2\epsilon + 5\delta, r + \epsilon + 3\delta\}$ intersecting all sets of $\mathcal{Q}$. These kind of Helly-type results are also useful in geometric group theory.

Using the Helly theorem for quasiconvex sets and a primal-dual approach, we show algorithmically that the minimum number of balls of radius $2\epsilon + 5\delta$ intersecting all sets of a family $\mathcal{Q}$ of ϵ-quasiconvex sets does not exceed the packing number of $\mathcal{Q}$ (maximum number of pairwise disjoint sets of $\mathcal{Q}$). We extend the covering and packing result to set-families ${}^{\kappa}\mathcal{Q}$ in which each set is a union of at most κ ϵ-quasiconvex sets of a δ-hyperbolic graph G. Namely, we show that if $r \geq \epsilon + 2\delta$ and $\pi_r({}^{\kappa}\mathcal{Q})$ is the maximum number of mutually $2r$-apart members of ${}^{\kappa}\mathcal{Q}$, then the minimum number of balls of radius $r + 2\epsilon + 6\delta$ intersecting all members of ${}^{\kappa}\mathcal{Q}$ is at most $2\kappa^2\pi_r({}^{\kappa}\mathcal{Q})$ and such a hitting set and a packing can be constructed in polynomial time for every finite ${}^{\kappa}\mathcal{Q}$ (this is our third main result). For set-families consisting of unions of κ balls in δ-hyperbolic graphs a similar result was obtained by Chepoi and Estellon (2007). In case of $\delta = 0$ (trees) and $\epsilon = r = 0$, (subtrees of a tree) we recover the result of Alon (2002) about the transversal and packing numbers of a set-family in which each set is a union of at most κ subtrees of a tree.

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1 Introduction

Understanding key structural properties of large-scale data networks is crucial for analyzing and optimizing their performance, as well as improving their reliability and security. In prior empirical and theoretical studies researchers have mainly focused on features such as small world phenomenon, power law degree distribution, navigability, and high clustering coefficients (see [7, 8, 10, 16, 19, 27, 28, 30, 37]). Those nice features were observed in many real-world complex networks and their underlying graphs arising in Internet applications, in biological and social sciences, and in chemistry and physics. Although those features are interesting and important, as noted in [32], the impact of intrinsic geometric and topological features of large-scale data networks on performance, reliability and security is of much greater importance.

Recently, there has been a surge of empirical works measuring and analyzing geometric characteristics of real-world networks, namely the *hyperbolicity* (sometimes called also the *negative curvature*) of the network (see, e.g., [1, 2, 13, 26, 31, 32, 36]). It has been shown that a number of data networks, including Internet application networks, web networks, collaboration networks, social networks, and others, have small hyperbolicity. It has been suggested (see [25, 32]) that the property, observed in real-world networks, in which traffic between vertices (nodes) tends to go through a relatively small core of the network, as if the shortest path between them is curved inwards, may be due to global curvature of the network.

In this paper, we prove that any finite subset X of vertices in a locally finite δ-hyperbolic graph G admits a core, namely there exists a vertex m of G such that the ball $B(m, 4\delta)$ centered at m of radius 4δ intersects all geodesics (shortest paths) between at least one half of all pairs of vertices of X. This solves in the positive and in the stronger form the first part of Conjecture 1 of [25], asserting: *"Consider a large but finite negatively curved graph G, subject to the uniformly distributed demand. Then there are very*

few nodes v that have very high traffic rate...". This phenomenon was observed experimentally in [32] in some real-world networks with small hyperbolicity. On the other hand, we show that the vertex m is not a center of mass as conjectured in [25] (*"...furthermore, the vertices of highest traffic rate are in a small neighborhood of the vertices of minimum inertia"*) but is a vertex of G close to a median point of X in the injective hull of G. This confirms the experimental observation of [32] that *"... the core is close to the geometric center, defined as the node whose average (geodesic) distance to all other nodes in the graph is the smallest."* Notice also that the authors of [25] established their conjecture for a particular case of graphs that are quasi-isometric to the balls of the n-dimensional hyperbolic space $\mathbb{H}^n$.

We also consider the case of non-uniform traffic between vertices of X. In this case, a unit demand of flow exists only between certain pairs of vertices of X defined by a commodity graph R; as in the previous case, the traffic between any pair of vertices defining an edge of R is evenly distributed over all geodesics connecting them. We prove a primal-dual result showing that for any $r \geq 8\delta$ the size of an r-multi-core (i.e., the number of balls of radius r) intercepting all pairs of R is upper bounded by the maximum number of pairwise $(2r-5\delta)$-apart pairs of R. Finally, if R consists of all mutually distant vertex pairs of a finite δ-hyperbolic graph G, then a single ball $B(m, 2\delta)$ of radius 2δ intercepts all pairs of R.

The proofs of all our results about cores implicitly or explicitly use various Helly type properties for balls, geodesics, and intervals in δ-hyperbolic graphs. For example, the proof of our main result about existence of cores is based on the fact that, for any metric space (X, d), there exists the smallest hyperconvex space $E(X)$ (i.e., geodesic metric space in which balls satisfy the Helly property) into which (X, d) isometrically embeds; $E(X)$ is called the injective hull of X [17, 23]. We use a result of Lang [29] asserting that if (X, d) is δ-hyperbolic, then $E(X)$ is also δ-hyperbolic and if, in addition, X is geodesic or a graph, then any point of $E(X)$ is within distance δ from some point of X. This last result is also a consequence of the Helly property for balls establishes in [15].

The second main result of our paper is a general Helly-type theorem for quasiconvex sets in δ-hyperbolic graphs, extending similar results for balls, geodesics, and intervals. Namely, we show that for any finite collection $\mathcal{Q}$ of pairwise intersecting ϵ-quasiconvex sets of a δ-hyperbolic graph G there exists a single ball B^* of radius $2\epsilon + 5\delta$ intersecting all sets of $\mathcal{Q}$. More generally, we prove that if $\mathcal{Q}$ is a collection of $2r$-close (i.e., any two sets of $\mathcal{Q}$ are at distance $\leq 2r$) ϵ-quasiconvex sets of a δ-hyperbolic graph G, then there exists a ball B^* of radius $r^* := \max\{2\epsilon + 5\delta, r + \epsilon + 3\delta\}$ intersecting all sets of $\mathcal{Q}$. Niblo and Reeves [33, Lemma 7] and implicitly Sageev [35] established this kind of Helly-type property for ϵ-quasiconvex sets in δ-hyperbolic graphs (see also [22, Proposition 7.7] for a generalization to relatively hyperbolic groups), but in their result the radius of the ball B^* hitting the sets of $\mathcal{Q}$ depends also on the number of sets in $\mathcal{Q}$. This statement plays a fundamental role in the cubulation process in proving the cocompactness of the cube complex associated with a finite set of quasiconvex codimension-1 subgroups [22, 33, 35]. The Helly property for balls proved in [15] is also important in the dismantlability and cop-and-robber game characterizations of hyperbolic graphs established in [12].

Using the Helly theorem for quasiconvex sets and a primal-dual approach, we show algorithmically that the minimum number of balls of radius $2\epsilon + 5\delta$ intersecting all sets of a family $\mathcal{Q}$ of ϵ-quasiconvex sets does not exceed the packing number of $\mathcal{Q}$ (maximum number of pairwise disjoint sets of $\mathcal{Q}$). The Helly property for geodesics and intervals is used to establish the existence of total beam cores and the covering and packing result is used in the computation of total multi-cores. Then we extend the covering and packing result from set-families $\mathcal{Q}$ consisting of quasiconvex sets to set-families ${}^{\kappa}\mathcal{Q}$ in which each set is a union of at most κ ϵ-quasiconvex sets of a δ-hyperbolic graph G. Namely, we show that if $r \geq \epsilon + 2\delta$ and $\pi_r({}^{\kappa}\mathcal{Q})$ is the maximum number of mutually $2r$-apart members of ${}^{\kappa}\mathcal{Q}$, then the minimum number of balls of radius $r+2\epsilon+6\delta$ intersecting all members of ${}^{\kappa}\mathcal{Q}$ is at most $2\kappa^2\pi_r({}^{\kappa}\mathcal{Q})$ and such a hitting set and a packing can be constructed in polynomial time for every finite ${}^{\kappa}\mathcal{Q}$ (this is our third main result). For set-families consisting of unions of κ balls in δ-hyperbolic graphs a similar result was obtained in [15] (and we closely follow the local-ratio proof-techniques of [15] and [9]). In case of $\delta = 0$ (trees) and $\epsilon = 0$ (subtrees of a tree) we recover the result of Alon [4] about the transversal and packing numbers of a set-family in which each set is a union of at most κ subtrees of a tree (for intervals of a line a similar inequality was proved in [3, 9]). Thus our result can be viewed as a far-reaching generalization of the result of [4] in which trees are replaced by hyperbolic graphs and subtrees by quasiconvex subgraphs.

2 Preliminaries

2.1 Graphs All graphs $G = (V, E)$ occurring in this paper are undirected, connected, without loops

or multiple edges, but not necessarily finite. For a subset $A \subseteq V$, the subgraph of $G = (V, E)$ *induced by* A is the graph $G(A) = (A, E')$ such that $uv \in E'$ if and only if $u, v \in A$ and $uv \in E$. The *distance* $d(u, v) := d_G(u, v)$ between two vertices u and v of G is the length (number of edges) of a (u, v)-*geodesic*, i.e., a shortest (u, v)-path. For a vertex v of G and an integer $r \geq 0$, we will denote by $B(v, r)$ the *ball* in G of radius r centered at v, i.e., $B(v, r) = \{x \in V : d(v, x) \leq r\}$. The *interval* $I(u, v)$ between u and v consists of all vertices on (u, v)-geodesics, that is, of all vertices (metrically) *between* u and v:

$$I(u, v) = \{x \in V : d(u, x) + d(x, v) = d(u, v)\}.$$

Let $d(X, Y) = \min\{d(x, y) : x \in X, y \in Y\}$ denote the distance between two subsets X, Y of vertices of G. We will say that two sets X and Y are r-*close* if $d(X, Y) \leq r$ and that X and Y are r-*apart* if $d(X, Y) > r$. In particular, two intersecting sets are 0-close.

We will call any finite subset X of vertices of a graph G a *profile*. Given a profile X, any vertex v of G minimizing the distance sum $\Psi_X(v) := \sum_{x \in X} d(v, x)$ is called a *median vertex* of X. Analogously, any vertex v of G minimizing the sum $\Phi_X(v) := \sum_{x \in X} d^2(v, x)$ is called a *center of mass* or a *centroid* of X.

Given a finite set X of vertices of a graph G, the *diameter* $\mathrm{diam}(X)$ of X is the maximum distance between any two vertices of X. A *diametral pair* of X is any pair of vertices $x, y \in X$ such that $d(x, y) = \mathrm{diam}(X)$. For a vertex x of a graph G of finite diameter, the set $P(x)$ of *furthest neighbors* of x (or of *peripheral with respect* to x vertices) consists of all vertices of G located at the maximum distance from x. The *eccentricity* $\mathrm{ecc}(x)$ of a vertex x is the distance from x to any vertex of $P(x)$. The *center* $C(G)$ of G is the set of all vertices of G having minimum eccentricity; the vertices of $C(G)$ are called central vertices. The radius $\mathrm{rad}(G)$ of G is the eccentricity of its central vertices. A geodesic $[x, y]$ between two vertices x, y such that $y \in P(x)$ is called a *beam* and $\{x, y\}$ is called a *beam pair* of G. Two vertices x, y of a graph G are called *mutually distant* if $x \in P(y)$ and $y \in P(x)$.

Given a graph $G = (V, E)$, a finite subset X of vertices of G, and a set of pairs $F \subseteq X \times X$, analogously to the multicommodity flow terminology (see also the next subsection), the pair $R = (X, F)$ will be called a *commodity graph*.

2.2 Cores We say that a ball $B(v, r)$ *intercepts* a geodesic $[x, y]$ of G if $B(v, r) \cap [x, y] \neq \varnothing$. More generally, we will say that a ball $B(v, r)$ *intercepts* a pair of vertices x, y if $B(v, r)$ intercepts all geodesics $[x, y]$ between x and y.

Given $0 < \alpha \leq 1$ and $r \geq 0$, we will say that a graph G has an (α, r)-*core* if for any profile X in G there exists a vertex $m := m(X)$ such that the ball $B(m, r)$ intercepts strictly more than the fraction of α of all pairs of X, i.e., there exist more than $\frac{\alpha|X|(|X|-1)}{2}$ pairs $\{x, y\}$ of X intercepted by $B(m, r)$.

Given an integer $r \geq 0$, we will say that a graph G admits a *total beam* r-*core* if there exists a ball $B(v, r)$ of radius r intercepting all beam pairs of G. More generally, given a graph $G = (V, E)$, an integer $r \geq 0$, and a commodity graph $R = (X, F)$ with a profile $X \subseteq V$, we will say that $R = (X, F)$ has a *total* r-*multi-core of size* k if all pairs of F can be intercepted with k balls of radius r. This last definition of multi-core corresponds to the model in which the traffic is not uniform but is performed only among the pairs of vertices of X defined by the commodity graph R. We will denote by $\sigma_r(R)$ the least integer k such that the commodity graph $R = (X, F)$ has a total r-multi-core of size k.

2.3 Traffic metrics and cores Following [25], let us consider a network in which the traffic is driven by a demand measure $\Lambda_d : V \times V \leftarrow \mathbb{R}^+$, where the demand $\Lambda_d(s, t)$ is the traffic rate (e.g. the number of packets per second) to be transmitted from the source s to the destination target t. Assume that the routing protocol sends packets from source s to target t along the geodesic $[s, t]$ with probability $Pr([s, t])$. It is customary as a load balancing strategy to randomize the Dijkstra algorithm so as to distribute the traffic more evenly. Under this scheme, the geodesic $[s, t]$ inherits a traffic rate measure $\mu([s, t]) := \Lambda_d(s, t)Pr([s, t])$. A subset of S of vertices crossed by a path $[s, t]$ inherits from that path a traffic $\mu([s, t])$. Aggregating this traffic over all source-target pairs and all geodesics traversing S, yields the traffic rate sustained by the subset S:

$$\mu(S) := \sum_{(s,t) \in V \times V} \sum_{[s,t] \cap S \neq \varnothing} \mu([s, t]).$$

In this paper, we will consider both uniform and non uniform traffic. In case of uniform traffic, we show that in any δ-hyperbolic network G, there exists a ball of radius $O(\delta)$ that has an extremely high traffic load in the sense that the majority of the traffic passes through this ball. In case of non uniform traffic, we consider a family of geodesics on which the traffic is sent and show that the minimum number of balls of radius $O(\delta)$ needed to collectively intercepts all these geodesics is bounded by the maximum number

of pairwise $O(\delta)$-apart geodesics in this family.

2.4 Hitting and packing problems The hitting and packing problems are classical problems in computer science and combinatorics. Let $\mathcal{S}$ be a finite collection of subsets of a domain V. A subset T of V is called a *hitting set* of $\mathcal{S}$ if $T \cap S \neq \varnothing$ for any $S \in \mathcal{S}$. The *minimum hitting set problem* asks to find a hitting set of $\mathcal{S}$ of smallest cardinality $\tau(\mathcal{S})$. The *set packing problem* (dual to the hitting set problem) asks to find a maximum number $\pi(\mathcal{S})$ of pairwise disjoint subsets of $\mathcal{S}$. We will call $\tau(\mathcal{S})$ and $\pi(\mathcal{S})$ the *transversal* (or *hitting*) and the *packing numbers* of $\mathcal{S}$. Obviously, the inequality $\tau(\mathcal{S}) \geq \pi(\mathcal{S})$ holds for any set-family $\mathcal{S}$.

In this paper, the domain V is the set of vertices of a connected graph $G = (V, E)$ or the set of points of a metric space (V, d). It this case, we can formulate the following relaxed hitting set problem. For $r \geq 0$, the *r-neighborhood* of S is the set $N_r(S) := \bigcup_{v \in S} B_r(v)$. For a collection of sets $\mathcal{S}$, let $\mathcal{S}_r := \{N_r(S) : S \in \mathcal{S}\}$; we will sometime refer to $\mathcal{S}_r$ as to the *r-inflation* of the collection $\mathcal{S}$. For $r \geq 0$, a subset T of V is called an *r-hitting set* of $\mathcal{S}$ if for any $S \in \mathcal{S}$ there exists $t \in T$ such that $B(t, r) \cap S \neq \varnothing$. The *minimum r-hitting set problem* asks to find an r-hitting set of $\mathcal{S}$ of smallest cardinality $\tau_r(\mathcal{S})$. Notice that $\tau_r(\mathcal{S}) = \tau(\mathcal{S}_r)$. Analogously, a subfamily $\mathcal{P}$ of $\mathcal{S}$ is called an *r-packing* if $N_r(S) \cap N_r(S') = \varnothing$ for any $S, S' \in \mathcal{P}$, i.e., if any two sets of $\mathcal{P}$ are $2r$-apart. We will be interested in set-families $\mathcal{S}$ such that for any $r \geq 0$ and for some constant α not depending on the family $\mathcal{S}$, $\tau(\mathcal{S}_{r+\alpha})$ is upper bounded by $\pi(\mathcal{S}_r)$. In our case, if $\mathcal{S}$ is a collection of ϵ-quasiconvex sets of a δ-hyperbolic graph G, then α will be a constant depending only on ϵ and δ.

3 Gromov hyperbolicity

3.1 Definition, characterizations, and properties Let (X, d) be a metric space and $w \in X$. The *Gromov product* of $y, z \in X$ with respect to w is defined to be

$$(y|z)_w = \frac{1}{2}(d(y, w) + d(z, w) - d(y, z)).$$

Let $\delta \geq 0$. A metric space (X, d) is said to be δ-*hyperbolic* [21] if

$$(x|y)_w \geq \min\{(x|z)_w, (y|z)_w\} - \delta$$

for all $w, x, y, z \in X$. Equivalently, (X, d) is δ-hyperbolic if for any four points u, v, x, y of X, the two larger of the three distance sums $d(u, v)+d(x, y)$, $d(u, x) + d(v, y)$, $d(u, y) + d(v, x)$ differ by at most $2\delta \geq 0$. In case of geodesic metric spaces and graphs, there exist several equivalent definitions of δ-hyperbolicity involving different but comparable values of δ [5, 11, 20, 21].

Let (X, d) be a metric space. A *geodesic segment* joining two points x and y from X is a (continuous) map ρ from the segment $[a, b]$ of $\mathbb{R}^1$ of length $|a-b| = d(x, y)$ to X such that $\rho(a) = x, \rho(b) = y$, and $d(\rho(s), \rho(t)) = |s-t|$ for all $s, t \in [a, b]$. A metric space (X, d) is *geodesic* if every pair of points in X can be joined by a geodesic segment. Every (combinatorial) graph $G = (V, E)$ equipped with its standard distance $d := d_G$ can be transformed into a geodesic (network-like) space (X_G, d) by replacing every edge $e = (u, v)$ by a segment $\gamma_{uv} = [u, v]$ of length 1; the segments may intersect only at common ends. Then (V, d_G) is isometrically embedded in a natural way in (X_G, d). X_G is often called a *metric graph*. The restrictions of geodesics of X_G to the set of vertices V of G are the shortest paths of G. For simplicity of notation and brevity (and if not said otherwise), in all subsequent results, by a geodesic $[x, y]$ in a graph G we will mean an arbitrary shortest path between two vertices x, y of G.

Let (X, d) be a geodesic metric space. A *geodesic triangle* $\Delta(x, y, z)$ with $x, y, z \in X$ is the union $[x, y] \cup [x, z] \cup [y, z]$ of three geodesic segments connecting these vertices. A geodesic triangle $\Delta(x, y, z)$ is called δ-*slim* if for any point u on the side $[x, y]$ the distance from u to $[x, z] \cup [z, y]$ is at most δ. Let m_x be the point of the geodesic segment $[y, z]$ located at distance $\alpha_y := (x|z)_y = (d(y, x)+d(y, z)-d(x, z))/2$ from y. Then m_x is located at distance $\alpha_z := (y|x)_z = (d(z, y) + d(z, x) - d(y, x))/2$ from z because $\alpha_y + \alpha_z = d(y, z)$. Analogously, define the points $m_y \in [x, z]$ and $m_z \in [x, y]$ both located at distance $\alpha_x := (y|z)_x = (d(x, y) + d(x, z) - d(y, z))/2$ from x; see Fig. 1 for an illustration. There exists a unique isometry φ which maps $\Delta(x, y, z)$ to a star $\Upsilon(x', y', z')$ consisting of three solid segments $[x', m'], [y', m']$, and $[z', m']$ of lengths α_x, α_y, and α_z, respectively. This isometry maps the vertices x, y, z of $\Delta(x, y, z)$ to the respective leaves x', y', z' of $\Upsilon(x', y', z')$ and the points m_x, m_y, and m_z to the center m of this tripod. Any other point of $\Upsilon(x', y', z')$ is the image of exactly two points of $\Delta(x, y, z)$. A geodesic triangle $\Delta(x, y, z)$ is called δ-*thin* if for all points $u, v \in \Delta(x, y, z)$, $\varphi(u) = \varphi(v)$ implies $d(u, v) \leq \delta$. The notions of geodesic triangles, δ-slim and δ-thin triangles can be also defined in case of graphs. The single difference is that for graphs, the center of the tripod is not necessarily the image of any vertex on the geodesic of $\Delta(x, y, z)$. Nevertheless, if a point of the tripod is the image of a vertex of one side of $\Delta(x, y, z)$, then it is also the image of another

vertex located on another side of $\Delta(x,y,z)$. A *graph with δ-thin triangles* is a graph G where each geodesic triangle is δ-thin.

The following results show that hyperbolicity of a geodesic space is equivalent to having thin or slim geodesic triangles (the same result holds for graphs).

PROPOSITION 3.1. *[5, 11, 20, 21] Geodesic triangles of geodesic δ-hyperbolic spaces are 4δ-thin and 3δ-slim.*

We will use the following converse given in [11, p. 411, Proposition 1.22] (since we often use the fact that δ-thin triangles imply the δ-hyperbolicity, we will present a proof for the completeness):

LEMMA 3.1. *A geodesic space (X,d) or a graph with δ-thin triangles is δ-hyperbolic.*

Proof. We will prove that for any four points w,x,y,z, we have $(x|y)_w \geq \min\{(x|z)_w,(y|z)_w\} - \delta$. Consider two geodesic triangles $\Delta(x,y,w)$ and $\Delta(y,z,w)$ sharing the common geodesic $[y,w]$. Suppose without loss of generality that $\alpha := (y|z)_w \leq (x|y)_w$. Let x' and y' be two points on the geodesics $[w,x]$ and $[w,y]$, respectively, located at distance $(x|y)_w$ from w. Analogously, let z' and y'' be two points on the geodesics $[w,z]$ and $[w,y]$, respectively, located at distance α from w. Since $\alpha \leq (x|y)_w$, the point y'' is located on the geodesic $[w,y]$ between w and y'. Let x'' be a point of $[w,x]$ located at distance α from w. Again, x'' is located on $[w,x]$ between w and x'. From the definition of the points x'',y'',z' we conclude that $d(x'',y'') \leq \delta$ and $d(y'',z') \leq \delta$. Hence, by the triangle inequality, we obtain

$$\begin{aligned} d(x,z) &\leq d(x,x'') + d(x'',y'') + d(y'',z') + d(z',z) \\ &= (d(x,w) - \alpha) + 2\delta + (d(z,w) - \alpha). \end{aligned}$$

By definition, $(x|z)_w = \frac{1}{2}(d(x,w)+d(z,w)-d(x,z))$. Replacing in the right-hand side the previous inequality for $d(x,z)$, we obtain that

$$(x|z)_w \geq \frac{1}{2}\left(d(x,w)+d(z,w)-d(x,w)-d(z,w)+2\alpha - 2\delta\right)$$

, whence

$$(x|z)_w \geq \frac{2\alpha - 2\delta}{2} = \min\{(x|z)_w,(y|z)_w\} - \delta.$$

□

An interval $I(u,v)$ of a graph (or a geodesic metric space) is called *ν-thin*, if $d(x,y) \leq \nu$ for any two points $x,y \in I(u,v)$ such that $d(u,x) = d(u,y)$ and $d(v,x) = d(v,y)$. From the definition of δ-hyperbolicity easily follows that intervals of δ-hyperbolic geodesic metric spaces or graphs are 2δ-thin. In case of graphs (or geodesic spaces) with δ-thin triangles, a better bound holds:

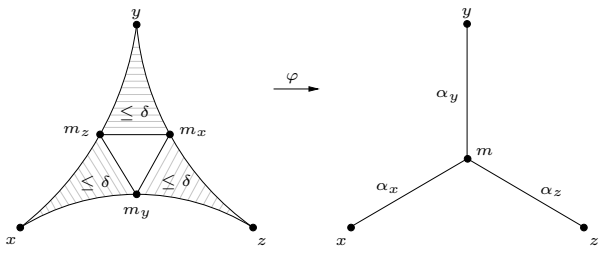

Figure 1: A geodesic triangle $\Delta(x,y,z)$, the points m_x,m_y,m_z, and the tripod $\Upsilon(x',y',z')$

LEMMA 3.2. *Intervals of a graph G (or geodesic space) with δ-thin geodesic triangles are δ-thin.*

Proof. Let u,v be two arbitrary vertices of G and let $x,y \in I(u,v)$ such that $d(u,x) = d(u,y)$. Let $[u,v]$ be any (u,v)-geodesic passing via x and let $[u,x],[x,v]$ be two arbitrary (u,x)- and (x,v)-geodesics. Consider the geodesic triangle $\Delta(x,u,v) := [u,x] \cup [x,v] \cup [v,u]$ and define the points $v' \in [u,x], u' \in [x,v]$, and $x' \in [u,v]$ such that $d(u,v') = d(u,x') = (x|v)_u, d(x,v') = d(x,u') = (u|v)_x$, and $d(v,u') = d(v,x') = (u|x)_v$. Since $d(u,x)+d(x,v) = d(u,v) = d(u,x')+d(x',v)$, necessarily $v' = x = u'$. Consequently, $d(u,x) = d(u,x')$ and $d(v,x) = d(v,x')$, i.e., $x' = y$. Since $\Delta(x,u,v)$ is δ-thin, $d(v',x') \leq \delta$, yielding $d(x,y) \leq \delta$. □

By this lemma, any result about cores intercepting families of geodesics can be transformed into a result about cores intercepting all pairs of vertices corresponding to ends of those geodesics.

3.2 Quasiconvexity A subset C of a geodesic metric space or graph is called *convex* if for all $x,y \in C$ each geodesic joining x and y is contained in C. The following "quasification" of this notion due to Gromov [21] plays an important role in the study of hyperbolic and cubical groups [11,22,33,35]. For $\epsilon \geq 0$, a subset C of a geodesic metric space (X,d) or graph $G = (V,E)$ is called *ϵ-quasiconvex* if for all $x,y \in C$ each geodesic joining x and y is contained in the ϵ-neighborhood $N_\epsilon(C)$ of C. C is said to be *quasiconvex* if there exists a constant $\epsilon \geq 0$ such that C is ϵ-quasiconvex. It turns out that in δ-hyperbolic spaces the collection of quasiconvex sets is abundant and it contains, in particular, geodesics, intervals, and balls:

LEMMA 3.3. *Let G be a graph (or geodesic space) with δ-thin geodesic triangles. Then the geodesics, the intervals, and the balls of G are δ-quasiconvex, and the neighborhoods of ϵ-quasiconvex sets are $(\epsilon + 2\delta)$-quasiconvex.*

Proof. That geodesics are δ-quasiconvex immediately follows from the fact that the intervals are δ-thin (Lemma 3.2). To prove that any interval $I(u, v)$ is δ-quasiconvex, pick any two points $x, y \in I(u, v)$ and any geodesic $[x, y]$ between x and y. Let $[u, x]$ and $[u, y]$ be two arbitrary geodesics between u and x and u and y, respectively. Since the resulting geodesic triangle $\Delta(u, x, y)$ is δ-thin, any point of $[x, y]$ is at distance at most δ from a point of $[u, x]$ or $[u, y]$. Since $[u, x] \cup [u, y] \subset I(u, v)$, $[x, y]$ is contained in the δ-neighborhood of $I(u, v)$ and we are done. The proof that balls are δ-quasiconvex is analogous.

Finally suppose that C is an ϵ-quasiconvex set of (X, d) and for $r \geq 0$ let $N_r(C)$ be the r-neighborhood of C. Let $x, y \in N_r(C)$ and $x', y' \in C$ such that $d(x, x'), d(y, y') \leq r$. Pick any geodesics $[x, y], [x, y'], [x, x'], [x', y']$, and $[y', y]$. Notice that $[x, x'] \cup [y, y'] \subset N_r(C)$. Let $\Delta(x, y, y')$ be the geodesic triangle with sides $[x, y], [x, y'], [y', y]$ and $\Delta(x, x', y')$ be the geodesic triangle with sides $[x, x'], [x', y'], [x, y']$. Let z be any point of $[x, y]$. Since $\Delta(x, y, y')$ is δ-thin, z is at distance at most δ from some point $z' \in [x, y'] \cup [y, y']$. If $z' \in [y, y'] \subset N_r(C)$, then $d(z, z') \leq \delta$ and we are done. So, suppose that $z' \in [x, y']$. Since $\Delta(x, x', y')$ is δ-thin, z' is at distance at most δ from a point $z'' \in [x, x'] \cup [x', y']$. Again, if $z'' \in [x, x'] \subset N_r(C)$, then $d(z, z'') \leq d(z, z') + d(z', z') \leq 2\delta$ and we are done. Finally, if $z'' \in [x', y']$, since C is ϵ-quasiconvex, there exists a point $p \in C$ such that $d(z'', p) \leq \epsilon$. Consequently, $d(z, p) \leq d(z, z'') + d(z'', p) \leq \epsilon + 2\delta$. □

3.3 Injective hulls of Gromov hyperbolic spaces A metric space (X, d) is said to be *injective* if, whenever X is isometric to a subspace Z of a metric space (Y, d'), then the subspace Z is a retract of Y, i.e., there exists a map $f : Y \to Z$ such that $f(z) = z$ for any $z \in Z$ and $d'(f(x), f(y)) \leq d'(x, y)$ for any $x, y \in X$. As shown in [6], injectivity of a metric space (X, d) is equivalent to its hyperconvexity. A metric space (X, d) is said to be *hyperconvex* if it is a geodesic metric space and its closed balls satisfy the *Helly property*, i.e., if $\mathcal{B}$ is any family of closed balls of X such that each pair of balls in $\mathcal{B}$ meet, then there exists a point x common to all the balls in $\mathcal{B}$. By a construction of Isbell [23], for any metric space (X, d) there exists an essentially unique *injective hull* $(e, E(X))$, that is $E(X)$ is an injective metric space, $e : X \to E(X)$ is an isometric embedding, and every isometric embedding of X into some injective metric space Z implies an isometric embedding of $E(X)$ into Z (thus $E(X)$ is the smallest injective space containing an image of an isometric embedding of X). This construction was rediscovered later by Dress [17]. It was noticed without any proof in [18] that the injective hull of a δ-hyperbolic space is δ-hyperbolic. This result was rediscovered recently by Lang [29], who also proved that if (X, d) is a geodesic space and a graph, then any point of $E(X)$ is located at distance at most 2δ (respectively, $2\delta + \frac{1}{2}$) from a point of X. Since we use this Lang's result, we briefly recall the basic definitions about injective hulls (in which we closely follow [29]).

Let (X, d) be a metric space. Denote by $\mathbb{R}^X$ the vector space of all real valued functions on X, and define

$$\Delta(X) := \{f \in \mathbb{R}^X : f(x) + f(y) \geq d(x, y) \text{ for all } x, y \in X\}.$$

Notice that if $f \in \Delta(X)$, then $\mathcal{B}(f) = \{B(x, f(x)) : x \in X\}$ is a family of pairwise intersecting balls. For a point $z \in X$ define the distance function $d_z \in \mathbb{R}^X$ by setting $d_z(x) = d(x, z)$ for any $x \in X$. By the triangle inequality, each d_z belongs to $\Delta(X)$. A function $f \in \Delta(X)$ is called *extremal* if it is a minimal element of the partially ordered set $(\Delta(X), \leq)$, where $g \leq f$ means $g(x) \leq f(x)$ for all $x \in X$. Let

$$E(X) = \{f \in \Delta(X) : \text{if } g \in \Delta(X) \text{ and } g \leq f, \text{ then } g = f\}$$

denote the set of all extremal functions on X. Then the *injective hull* of X is the set $E(X)$ equipped with the ℓ_∞-metric $||f - g||_\infty = \sup_{x \in X} |f(x) - g(x)|$. It can be easily seen that the map $e : X \to E(X)$ defined by $e(z) = d_z$ for any $z \in X$ is a canonical isometric embedding of X into $E(X)$. Moreover, it was shown in [17, 23, 29] that $E(X)$ is an injective (and thus hyperconvex) space and it is minimal in this sense.

A similar construction can be done if X is a graph G; there exists a smallest Helly graph (i.e., a graph satisfying the Helly property for balls) comprising G as an isometric subgraph [24,34]. In this case, instead of taking the set $E(X)$ of all extremal functions, one can take the subset $E_0(X)$ of $E(X)$ consisting only of integer-valued extremal functions; endow it with the ℓ_∞-metric, and consider the graph $H(G)$ having $E_0(X)$ as the vertex-set and all pairs of vertices having ℓ_∞-distance 1 as edges. This graph $H(G)$ is called the *Hellyfication* of G.

Returning to δ-hyperbolic spaces and graphs, in what follows, we will use the following result of Lang [29]:

PROPOSITION 3.2. *[29, Proposition 1.3] If (X, d) is a δ-hyperbolic metric space, then its injective hull is δ-hyperbolic. If, in addition, X is a geodesic space or a graph, then any point x^* of $E(X)$ (respectively, $E_0(X)$) is within distance 2δ from some point (respectively, some vertex) x of X.*

For geodesic spaces or graphs with δ-thin triangles, the second assertion of Proposition 3.2 with (δ instead of 2δ) also follows from the following *Helly property for balls*:

PROPOSITION 3.3. *[15, Corollary 2] Let X be a geodesic space or graph with δ-thin triangles and let $B(x_i, r_i)$ be a collection of pairwise intersecting balls of G. Then the balls $B(x_i, r_i + \delta)$ have a nonempty intersection.*

PROPOSITION 3.4. *If X is a geodesic space or a graph with δ-thin triangles, then any point x^* of $E(X)$ is within distance δ from some point x of X.*

Proof. Pick any point $f \in E(X)$. Since $f(x) + f(y) \geq d(x, y)$, $\{B(x, f(x)) : x \in X\}$ is a collection of pairwise intersecting balls of X. By Proposition 3.3, there exists a point z belonging to all balls $B(x, f(x) + \delta), x \in X$. Consider the extremal map d_z, i.e., the point of $E(X)$ corresponding to z. Recall, that $d_z(x) = d(x, z)$ for any $x \in X$. By definition of z, for any $x \in X$ we have $d_z(x) = d(x, z) \leq f(x) + \delta$. On the other hand, if there exists $x \in X$ such that $f(x) > d_z(x) + \delta$, then we assert that f is not an extremal map. Indeed, for any $y \neq x$ of X, we will obtain that $f(x) + f(y) > d_z(x) + \delta + f(y) \geq d_z(x) + d_z(y) \geq d(x, y)$, showing that f is not extremal. Consequently, $f(x) \leq d_z(x) + \delta$ for any $x \in X$, whence $||f - d_z||_\infty = \sup_{x \in X} |f(x) - d_z(x)| \leq \delta$. □

4 Existence of cores

The goal of this section is to prove the following result:

THEOREM 4.1. (EXISTENCE OF CORES) *Let G be a δ-hyperbolic graph (respectively, a graph with δ-thin triangles). Then any finite subset X of vertices of G has a $(\frac{1}{2}, 4\delta)$-core (respectively, a $(\frac{1}{2}, 3\delta + \frac{1}{2})$-core).*

Proof. Let $n := |X|$. First suppose that G is a graph with δ-thin triangles. By Lemma 3.1, G is δ-hyperbolic. Let $E(G)$ be the injective hull of G and let $H(G)$ be the Hellification of G (induced by all points of $E(G)$ with integer coordinates). By the first part of Proposition 3.2, $E(G)$ is δ-hyperbolic, thus $H(G)$ is also δ-hyperbolic. Let m^* be a median vertex of the profile X in the Helly graph $H(G)$. By Proposition 3.4, m^* is at distance at most δ from a vertex m of G.

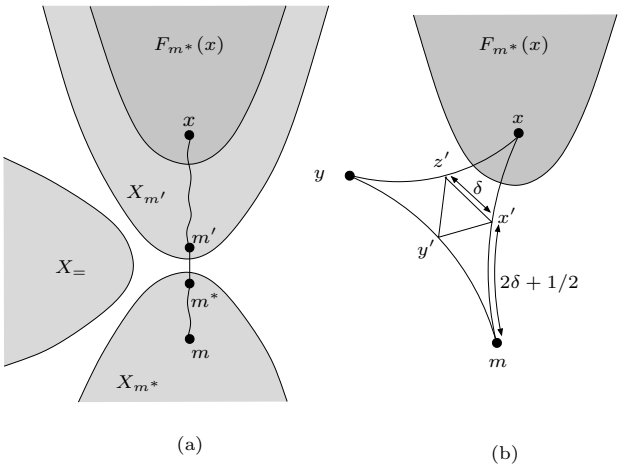

Figure 2: To the proof of Claims 1 and 3.

For a vertex $x \in X$, let

$$F_{m^*}(x) = \{y \in X : (x|y)_{m^*} \geq \delta + 1\}$$

and call $F_{m^*}(x)$ the *fiber* of x with respect to m^*.

Claim 1. For any vertex $x \in X$, the fiber $F_{m^*}(x)$ contains at most $n/2$ vertices.

Proof of Claim 1. Suppose, by way of contradiction, that $|F_{m^*}(x)| > n/2$. For each vertex $v \in F_{m^*}(x)$, set $r_v := d(m^*, v) - 1$. Consider the following collection of balls:

$$\mathcal{B} = \{B(m^*, 1)\} \cup \{B(v, r_v) : v \in F_{m^*}(x)\}.$$

We assert that the balls from $\mathcal{B}$ pairwise intersect. From the definition of r_v, this is obviously true for $B(m^*, 1)$ and $B(v, r_v)$ for any $v \in F_{m^*}(x)$. Now, pick two arbitrary vertices $u, v \in F_{m^*}(x)$. We assert that $d(u, v) \leq r_u + r_v$. Since $H(G)$ is δ-hyperbolic and $u, v \in F_{m^*}(x)$,

$$(u|v)_{m^*} \geq \min\{(x|u)_{m^*}, (x|v)_{m^*}\} - \delta \geq \delta + 1 - \delta = 1.$$

Hence, $d(u, m^*) + d(v, m^*) - d(u, v) = 2(u|v)_{m^*} \geq 2$. Consequently, $d(u, v) \leq d(u, m^*) + d(v, m^*) - 2 = r_u + r_v$, showing that the balls $B(u, r_u)$ and $B(v, r_v)$ intersect.

Applying the Helly property to the collection $\mathcal{B}$, we can find a vertex m' of $H(G)$ belonging to all balls of $\mathcal{B}$. Since $r_v = d(m^*, v) - 1$ for any

$v \in F_{m^*}(x)$, m' is different from m^*. Consequently, m' is a neighbor of m^* belonging to all intervals $I(m^*, v), v \in F_{m^*}(x)$; see Fig. 2(a). Consider now the values of the median function Ψ_X on the vertices m^* and m' of $H(G)$. Consider a partition of the set X into three sets $X_{m^*}, X_{m'}$, and $X_=$, where $X_{m^*} := \{v \in X : d(m^*, v) < d(m', v)\}$, $X_{m'} := \{v \in X : d(m', v) < d(m^*, v)\}$ and $X_= := \{v \in X : d(m^*, v) = d(m', v)\}$. Since $m' \in I(m^*, v)$ for all $v \in F_{m^*}(x)$, necessarily $F_{m^*}(x) \subseteq X_{m'}$. Since $|F_{m^*}(x)| > n/2$, this implies that $|X_{m'}| - |X_{m^*}| > 0$. Since $d(m^*, v) = d(m', v)$ for any $v \in X_=$ and since m^* and m' are adjacent, one can easily deduce that $\Psi_X(m^*) - \Psi_X(m') = |X_{m'}| - |X_{m^*}| > 0$, contrary to the assumption that m^* is a median of X in $H(G)$. This concludes the proof of Claim 1.

Claim 2. For any vertex $x \in X$ and any vertex $y \notin F_{m^*}(x)$, we have $(x|y)_m < 2\delta + 1$, i.e., $(x|y)_m \leq 2\delta + \frac{1}{2}$.

Proof of Claim 2. Recall that m is a vertex of G at distance at most δ from m^*. By definition of $F_{m^*}(x)$ and since $y \notin F_{m^*}(x)$, we obtain $(x|y)_{m^*} < \delta + 1$. Suppose, by way of contradiction, that $(x|y)_m \geq 2\delta+1$. This implies that $d(x, m)+d(y, m)-d(x, y) \geq 4\delta + 2$, i.e., $(d(x, m) - \delta) + (d(y, m) - \delta) - d(x, y) \geq 2\delta+2$. Since $d(m, m^*) \leq \delta$, by the triangle inequality, we obtain that $d(x, m^*) \geq d(x, m)-\delta$ and $d(y, m^*) \geq d(y, m)-\delta$, yielding $(x|y)_{m^*} = (d(x, m^*)+d(y, m^*)-d(x, y))/2 \geq \delta + 1$, contrary to the assumption that $y \notin F_{m^*}(x)$. Hence $(x|y)_m < 2\delta + 1$. Since the Gromov product $(x|y)_m$ in graphs is an integer or a half-integer, we obtain $(x|y)_m \leq 2\delta + \frac{1}{2}$. This concludes the proof of Claim 2.

Claim 3. For any vertex $x \in X$ and any vertex $y \notin F_{m^*}(x)$, any geodesic $[x, y]$ of G intersects the ball $B(m, 3\delta + \frac{1}{2})$ of G of radius $3\delta + \frac{1}{2}$ and center m.

Proof of Claim 3. Consider a geodesic triangle $\Delta(x, y, m)$ and let $[m, x], [m, y]$, and $[x, y]$ be the sides of this triangle; see Fig. 2(b). Let x', y' be the points of $[m, x]$ and $[m, y]$, respectively, located at distance $(x|y)_m$ from m. Since the triangles of G are δ-thin, $d(x', y') \leq \delta$, moreover $d(x', z') \leq \delta$ and $d(y', z') \leq \delta$, where z' is the point of $[x, y]$ at distance $(y|m)_x$ from x and at distance $(x|m)_y$ from y. Since, by Claim 2, $(x|y)_m \leq 2\delta + \frac{1}{2}$, we conclude that $d(m, z') \leq d(m, x') + d(x', z') \leq 2\delta + \frac{1}{2} + \delta = 3\delta + \frac{1}{2}$. This establishes Claim 3.

Now, we can conclude the proof of the theorem for graphs with δ-thin triangles. Indeed, by Claim 1, for any vertex x of X, the fiber $F_{m^*}(x)$ contains at most $n/2$ vertices. By Claim 3, the ball $B(m, 3\delta + \frac{1}{2})$ intersects any geodesic $[x, y]$ between a vertex $x \in X$ and any vertex $y \notin F_{m^*}(x)$, i.e., $B(m, 3\delta + \frac{1}{2})$ intercepts any geodesic between any $x \in X$ and at least $n/2$ vertices of X. This implies that $B(m, 3\delta + \frac{1}{2})$ intercepts at least $n^2/4$ of the pairs of vertices of X.

Now, suppose that G is a δ-hyperbolic graph. Then the proof is exactly the same except the proof of Claim 3, in which we used δ-thin triangles. We replace Claim 3 by the following assertion:

Claim 4. For any vertex $x \in X$ and any vertex $y \notin F_{m^*}(x)$, any geodesic $[x, y]$ of G intersects the ball $B(m, 4\delta)$ of G of radius 4δ and center m.

Proof of Claim 4. Consider a geodesic triangle $\Delta(x, y, m)$ and let $[m, x], [m, y]$, and $[x, y]$ be the sides of this triangle. Let z be the point of $[x, y]$ at distance $(y|m)_x$ from x and at distance $(x|m)_y$ from y. Consider the three distance sums $d(x, y) + d(m, z), d(m, x)+d(z, y)$, and $d(m, y)+d(x, z)$. Notice that $d(x, y) + d(m, z) = (y|m)_x + (x|m)_y + d(m, z)$, $d(m, x) + d(z, y) = (x|y)_m + (y|m)_x + (x|m)_y$, and $d(m, y) + d(x, z) = (x|y)_m + (x|m)_y + (y|m)_x$. Therefore the distance sums $d(m, x)+d(z, y)$ and $d(m, y)+d(x, z)$ coincide. If these two sums are the largest distance sums, then from inequality $(y|m)_x + (x|m)_y + d(m, z) \leq (x|y)_m + (y|m)_x + (x|m)_y$ and Claim 2 we obtain that $d(m, z) \leq (x|y)_m \leq 2\delta + \frac{1}{2}$. On the other hand, if $d(x, y) + d(m, z)$ is the largest distance sum, then, since G is δ-hyperbolic, $d(x, y) + d(m, z) \leq d(m, x) + d(z, y) + 2\delta$. Consequently, $d(m, z) \leq (x|y)_m + 2\delta \leq 4\delta + \frac{1}{2}$. As $d(m, [x, y])$ is an integer, $d(m, [x, y]) \leq 4\delta$ holds. This establishes Claim 4.

This completes the proof of the theorem. □

In case of δ-hyperbolic Helly graphs, the radius of the intercepting core-ball can be decreased, because in this case $m^* = m$:

COROLLARY 4.1. *If G is a δ-hyperbolic Helly graph (respectively, a Helly graph with δ-thin triangles), then any finite subset X of vertices of G has a $(\frac{1}{2}, 3\delta + \frac{1}{2})$-core (respectively, a $(\frac{1}{2}, 2\delta)$-core).*

REMARK 1. *The analogue of Theorem 4.1 holds for all geodesic δ-hyperbolic spaces.*

REMARK 2. *Contrary to what was asserted in [25], for any constant α, the center of mass of a δ-hyperbolic network G can be arbitrarily far from the center of any $(\alpha, O(\delta))$-core of G.*

Proof. Consider the family of trees (0-hyperbolic graphs) T_n, $n \in \mathbb{N}$, consisting of a path P on $3\sqrt{n}$ vertices and a star S with $n - 3\sqrt{n}$ leaves centered at

the end-vertex x of the path P. It can be shown by simple computations that the distance between the center of mass of T_n and x grows with n. However, any $(\alpha, O(1))$-core must contain the vertex x. Therefore, the distance between the center of the core and the center of the star S is $O(1)$. This implies that the distance between the center of the core and the center of mass of T_n can be made arbitrarily large by taking n large enough. □

REMARK 3. *For a finite n-vertex m-edge (δ-hyperbolic) graph $G = (V, E)$, a $(\frac{1}{2}, \rho)$-core with minimum ρ can be found in at most $\rho n O(nm) = O(\rho n^2 m)$ time by iterating over each vertex $v \in V$ and computing the smallest radius ρ such that $d_G(x, y) < d_{G'}(x, y)$ holds for at least $n^2/4$ pairs $x, y \in V$, where G' is a graph obtained from G by removing the vertices of the ball $B(v, \rho)$. Here, $O(nm)$ stands for the time needed to compute the distance matrices of G and G'.*

REMARK 4. *The analogue of Theorem 4.1 (with a larger radius of the intercepting ball) holds if the traffic is performed not only along geodesics but also along quasi-geodesics. A (λ, ϵ)-quasi-geodesic in a metric space (X, d) is a (λ, ϵ)-quasi-isometric embedding $c : I \to X$ (where I is an interval of the real line), i.e., $\frac{1}{\lambda}|t - t'| - \epsilon \le d(c(t), c(t')) \le \lambda|t - t'| + \epsilon$ holds for all $t, t' \in I$. By the well-known Morse Lemma [11, Theorem 1.7, Part III], there exists $R := R(\delta, \lambda, \epsilon)$ such that for any quasi-geodesic c and any geodesic segment $[p, q]$ joining the endpoints of c, any point of c is at distance at most R from a point of $[p, q]$. Therefore, the ball of radius $R + 4\delta$ centered at m will intercept at least one half of the total traffic sent along quasi-geodesics.*

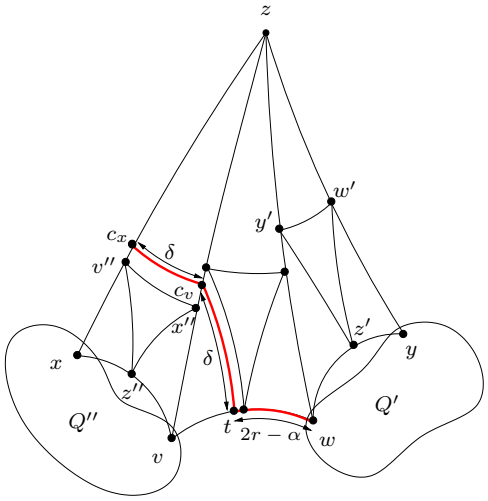

Figure 3: Case 1 in the proof of Lemma 5.1.

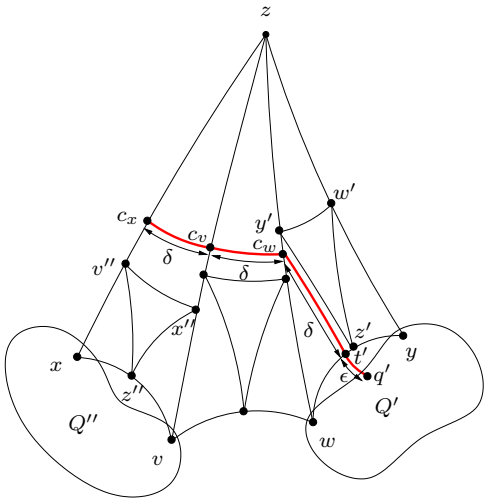

Figure 4: Case 2 in the proof of Lemma 5.1.

5 Helly theorem for quasiconvex sets

The Helly property for balls established in Proposition 3.3 of [15] was an important tool in the proof of the existence of cores from the previous section. In this section, we extend this result to quasiconvex sets of δ-hyperbolic graphs. As a consequence of this result, for any finite collection $\mathcal{Q}$ of ϵ-quasiconvex sets we obtain a relationship between the packing and transversal numbers of $\mathcal{Q}$ by showing that $\tau_{2\epsilon+5\delta}(\mathcal{Q}) \le \pi(\mathcal{Q})$.

We start with a fundamental lemma, which can be viewed as an extension of [15, Lemma 1] from balls to all quasiconvex sets. For nonnegative integers r, ϵ, δ, let $r^* := r^*(r, \epsilon, \delta) := \max\{2\epsilon + 5\delta, r + \epsilon + 3\delta\}$.

LEMMA 5.1. *Let G be a graph with δ-thin triangles, r be a nonnegative integer, z be a vertex of G, and Q', Q'' be two $2r$-close ϵ-quasiconvex sets of G such that $d(z, Q'') \ge d(z, Q')$. If x is a vertex of Q'' closest to z and c is the vertex at distance r from x on a (x, z)-geodesic $[x, z]$, then $d(c, Q') \le r^*$. If Q' and Q'' are two $2r$-close geodesics of G, then $d(c, Q') \le \max\{r + 3\delta, 5\delta\}$.*

Proof. Let $w \in Q'$ and $v \in Q''$ such that $d(v, w) \le 2r$. Let y be a vertex of Q' closest to z. Consider two geodesic triangles $\Delta(z, w, y) := [z, w] \cup [w, y] \cup [y, z]$ and $\Delta(x, v, z) := [x, v] \cup [v, z] \cup [z, x]$, where $[y, z], [z, w], [w, y], [x, v], [z, x]$, and $[v, z]$ are arbitrary geodesics connecting the corresponding vertices (if Q' and Q'' are geodesics, then we suppose that $[v, x] \subseteq Q''$ and $[w, y] \subseteq Q'$).

Define the points $w' \in [y, z]$ and $z' \in [w, y]$ both located at distance $\alpha_y := (w|z)_y$ from y and the point $y' \in [w, z]$ located at distance $(w|y)_z$ from z (and hence at distance $(z|y)_w = d(w, z) - (w|y)_z$ from

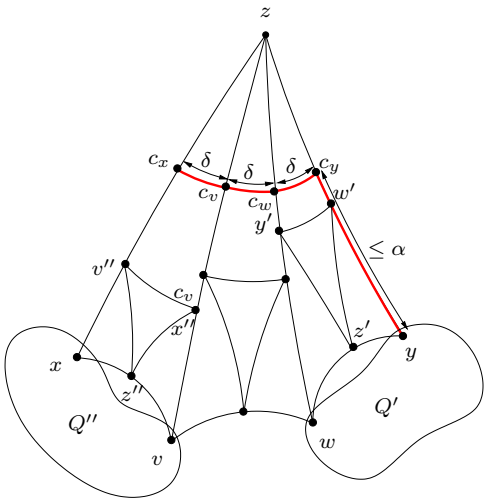

Figure 5: Case 3 in the proof of Lemma 5.1.

w). Define the points $v'' \in [x,z]$ and $z'' \in [v,x]$ both located at distance $\alpha_x := (v|z)_x$ from x and the point $x'' \in [v,z]$ located at distance $(v|x)_z$ from z (and hence at distance $(z|x)_v = d(v,z) - (v|x)_z$ from v). From the definition of y', z', w', any point $t \in [y',w] \cup [w',y]$ is at distance $\leq \delta$ from a point $t' \in [y,z'] \cup [z',w] = [y,w]$. Since $y, w \in Q'$ and Q' is ϵ-quasiconvex, $d(t',Q') \leq \epsilon$, i.e., there exists a point $q' \in Q'$ such that $d(t',q') \leq \epsilon$. Consequently, $d(t,Q') \leq d(t,t') + d(t',Q') \leq \delta + \epsilon$. Analogously, for any point $t \in [v'',x] \cup [x'',v]$ we have $d(t,Q'') \leq \delta + \epsilon$. In particular, $d(w',Q') \leq \delta + \epsilon$ and $d(v'',Q'') \leq \delta + \epsilon$. Since y is a vertex of Q' closest to z and $w' \in [z,y]$, y is also a vertex of Q' closest to w', thus $d(w',y) = d(w',Q')$. Consequently, $\alpha_y = d(w',Q') \leq \delta + \epsilon$. Analogously, since x is a vertex of Q'' closest to z and $v'' \in [x,z]$, we deduce that $\alpha_x = d(v'',x) = d(v'',Q'') \leq \delta + \epsilon$. If Q' and Q'' are geodesics, then one can easily see that $\alpha_x \leq \delta$ and $\alpha_y \leq \delta$.

Let $\alpha := \max(r, \alpha_x)$, $\beta := d(x,z) - \alpha$ and define the points $c_x \in [z,x]$, $c_v \in [z,v]$, $c_w \in [z,w]$, and $c_y \in [z,y]$ located at distance β from z. Notice that $\beta \leq (x|v)_z$. Recall also that c is the point of $[x,z]$ at distance r from x. Therefore, if $r \geq \alpha_x$, then c coincides with $c_x \in [v'',z]$ and if $r < \alpha_x$, then $c_x = v''$ and c is located between x and v''. Since $d(x,v'') \leq \delta + \epsilon$, in all cases we deduce that $d(c,c_x) \leq \delta + \epsilon$. Therefore, to bound $d(c,Q')$ it suffices to get a bound on $d(c_x,Q')$. We distinguish between three cases:

Case 1. $\beta > (v|w)_z$ (Fig. 3).

Since x is a vertex of Q'' closest to z and $d(z,c_x) = d(z,c_v) = \beta$, we have $d(v,c_v) \geq d(x,c_x) = \alpha$. Let t be the point on $[v,w]$ at distance $d(v,c_v) \geq \alpha$ from v. Since $d(v,w) \leq 2r$, $d(v,t) \geq \alpha$, and $w \in Q'$, we deduce that $d(t,Q') \leq d(t,w) \leq 2r - \alpha$. Again, since the geodesic triangles in G are δ-thin, we obtain that $d(c_x,Q') \leq d(c_x,c_v) + d(c_v,t) + d(t,Q') \leq 2\delta + 2r - \alpha$. If $\alpha = r$, then c_x coincide with c and we get $d(c,Q') \leq 2\delta + r$. Otherwise, $r < \alpha_x$ and consequently $d(c,Q') \leq d(c,c_x) + d(c_x,Q') \leq \epsilon + \delta + 2\delta + 2r - \alpha_x \leq \epsilon + 3\delta + r$.

Case 2. $\beta \leq (v|w)_z$ and $\beta \geq (w|y)_z$ (Fig. 4).

In this case, $c_v \in [x'',z]$ and $c_w \in [y',w]$. By what has been shown above, in this case we have $d(c_w,Q') \leq \delta + \epsilon$. Since the geodesic triangles of G are δ-thin, $d(c_x,c_v) \leq \delta$ and $d(c_v,c_w) \leq \delta$. Consequently, $d(c_x,Q') \leq d(c_x,c_v) + d(c_v,c_w) + d(c_w,Q') \leq \epsilon + 3\delta$, i.e., in this case $d(c,Q') \leq 2\epsilon + 4\delta$.

Case 3. $\beta \leq (v|w)_z$ and $\beta < (w|y)_z$ (Fig. 5).

In this case, $c_v \in [x'',z]$, while $c_w \in [y',z]$ and $c_y \in [w',z]$. Since $d(z,x) = d(z,Q'') \geq d(z,Q') = d(z,y)$, $d(z,x) = d(z,c_x) + d(c_x,x) = \beta + \alpha$, and $d(z,y) = d(z,c_y) + d(c_y,y) = \beta + d(c_y,y)$, we conclude that $d(c_y,y) \leq \alpha$. Since the geodesic triangles in G are δ-thin and $y \in Q'$, we derive that $d(c_x,Q') \leq d(c_x,c_v) + d(c_v,c_w) + d(c_w,c_y) + d(c_y,y) \leq 3\delta + \alpha$. If $\alpha = r$, then c_x coincide with c and we get $d(c,Q') \leq r + 3\delta$. Otherwise, $\alpha = \alpha_x \leq \epsilon + \delta$ and consequently, $d(c,Q') \leq d(c,c_x) + d(c_x,Q') \leq \epsilon + \delta + 3\delta + \epsilon + \delta \leq 2\epsilon + 5\delta$.

This proves that in all cases we have $d(c,Q') \leq r^*$, where $r^* := \max\{2\epsilon + 5\delta, r + \epsilon + 3\delta\}$. If Q' and Q'' are two $2r$-close geodesics, then $\alpha_x \leq \delta$ and $\alpha_y \leq \delta$ and one can see that $d(c,Q') \leq \max\{r + 3\delta, 5\delta\}$. □

THEOREM 5.1. *(Helly property for quasiconvex sets) Let G be a graph with δ-thin triangles and $\mathcal{Q}$ be a finite collection of ϵ-quasiconvex subsets of G. If the sets of $\mathcal{Q}$ are pairwise $2r$-close, then there exists a ball $B(c,r^*)$ of radius r^* intersecting all sets of $\mathcal{Q}$. In particular, if $\mathcal{Q}$ is a collection of pairwise intersecting ϵ-quasiconvex subsets of G, then there exists a ball of radius $2\epsilon + 5\delta$ intersecting all sets of $\mathcal{Q}$ ($r^* = \max\{r + 3\delta, 5\delta\}$ if the sets of $\mathcal{Q}$ are geodesics).*

Proof. Let z be an arbitrary vertex of G. Suppose that the sets of $\mathcal{Q}$ are ordered $Q_1, Q_2, \ldots, Q_n$ in such a way that $d(z,Q_1) \geq d(z,Q_2) \geq \ldots \geq d(z,Q_n)$. Let x be a vertex of Q_1 closest to z, i.e., $d(z,x) = d(z,Q_1)$. Let $[z,x]$ be any geodesic between z and x and let c be the point of $[z,x]$ located at distance r from x. Since $d(z,Q_1) \geq d(z,Q_i)$ for any $i = 2, \ldots, n$, applying for each such i Lemma 5.1 with $Q'' := Q_1$ and $Q' := Q_i$, we obtain that $d(c,Q_i) \leq r^* := \max\{2\epsilon + 5\delta, r + \epsilon + 3\delta\}$ and $d(x,Q_i) \leq r^* := \max\{r + 3\delta, 5\delta\}$ if all sets of $\mathcal{Q}$ are

geodesics. Consequently, $B(c, r^*)$ intersects all sets of $\mathcal{Q}$. □

REMARK 5. *In case when $\mathcal{Q}$ is a set of geodesics, with a different proof one can slightly improve the bounds in Theorem 5.1: 2δ instead of 5δ if the geodesics of $\mathcal{Q}$ pairwise intersect and $\max\{r+3\delta, 4\delta\}$ instead of $\max\{r+3\delta, 5\delta\}$ if the geodesics of $\mathcal{Q}$ are $2r$-close.*

PROPOSITION 5.1. *Let $\mathcal{Q}$ be a finite collection of ϵ-quasiconvex sets of a graph G with δ-thin triangles. Then the packing number $\pi(\mathcal{Q})$ and the transversal number $\tau(\mathcal{Q}_{2\epsilon+5\delta})$ satisfy the inequality $\tau(\mathcal{Q}_{2\epsilon+5\delta}) \leq \pi(\mathcal{Q})$. Moreover, a hitting set T of $\mathcal{Q}_{2\epsilon+5\delta}$ and a packing $\mathcal{P}$ of $\mathcal{Q}$ such that $|T| = |\mathcal{P}|$ can be constructed in polynomial time. More generally, for any integer $r \geq 0$, $\tau(\mathcal{Q}_{r^*}) \leq \pi(\mathcal{Q}_r)$ ($r^* := \max\{r+3\delta, 5\delta\}$ for geodesics and $r^* := \max\{r+4\delta, 7\delta\}$ for intervals); a hitting set T_{r^*} of $\mathcal{Q}_{r^*}$ and a packing $\mathcal{P}_r$ of $\mathcal{Q}_r$ such that $|T_{r^*}| = |\mathcal{P}_r|$ can be constructed in polynomial time.*

Proof. We start with the first assertion. The proof of this result is algorithmic: we construct the packing $\mathcal{P}$ and the hitting set T step by step ensuring that the following properties hold: (i) each time when a new point is inserted in T, then a new set of $\mathcal{Q}$ is also inserted in $\mathcal{P}$, and (ii) at the end, the sets of $\mathcal{P}$ are pairwise disjoint and T is a hitting set of $\mathcal{Q}_{2\epsilon+5\delta}$.

The algorithm starts with $\mathcal{Q}^* := \mathcal{Q}$, $T := \varnothing$, and $\mathcal{P} := \varnothing$. Let z be an arbitrary fixed vertex of G. While the set $\mathcal{Q}^*$ is nonempty, the algorithm computes the distances from z to the sets of $\mathcal{Q}^*$. Suppose that $\mathcal{Q}^* = \{Q_1, \ldots, Q_n\}$, where $d(z, Q_1) \geq d(z, Q_2) \geq \ldots \geq d(z, Q_n)$. Set $Q'' := Q_1$. Denote by $\mathcal{Q}'$ the subfamily of $\mathcal{Q}^*$ consisting of Q'' and all sets of $\mathcal{Q}^*$ intersecting the set Q''. Let x be a vertex of Q'' closest to z. Applying Lemma 5.1 with $r = 0$ we deduce that $d(x, Q') \leq 2\epsilon + 5\delta$ for any set $Q' \in \mathcal{Q}'$ (since $r = 0$, x plays the role of c), i.e, $x \in N_{2\epsilon+5\delta}(Q')$. Then we include the vertex x in the transversal T and the set Q' in the packing $\mathcal{P}$, and we update $\mathcal{Q}^*$ by setting $\mathcal{Q}^* \leftarrow \mathcal{Q}^* \setminus \mathcal{Q}'$. By construction, x belongs to all sets of $\mathcal{Q}'_{2\epsilon+5\delta}$ and Q'' is disjoint from all sets previously included in $\mathcal{P}$. This implies that at the end, each set of $\mathcal{Q}_{2\epsilon+5\delta}$ contains a point of T and that the sets of $\mathcal{P}$ are pairwise disjoint. Therefore, T is a transversal of $\mathcal{Q}_{2\epsilon+5\delta}$, $\mathcal{P}$ is a packing of $\mathcal{Q}$, and $|T| = |\mathcal{P}|$.

The second (general) assertion can be established in a similar way subject to the following changes. Initially, we set $\mathcal{Q}^* := \mathcal{Q}$, $T_{r^*} := \varnothing$, and $\mathcal{P}_r := \varnothing$. At each step, Q'' is a furthest from z set of $\mathcal{Q}^*$, x is a vertex of Q'' closest to z and c is a point on $[z, x]$ at distance r from x. The set-family $\mathcal{Q}'$ consists of Q'' and all sets of $\mathcal{Q}^*$ which are $2r$-close to Q''. By Lemma 5.1, we deduce that $d(x, Q') \leq r^*$ for any set $Q' \in \mathcal{Q}'$. Then we include the vertex c in T_{r^*} and the r-neighborhood $N_r(Q'')$ of Q'' in $\mathcal{P}_r$. Finally, $\mathcal{Q}^*$ is updated by setting $\mathcal{Q}^* := \mathcal{Q}^* \setminus \mathcal{Q}'$. From the construction it follows that T_{r^*} is a hitting set of $\mathcal{Q}_{r^*}$, $\mathcal{P}_r$ is a packing of $\mathcal{Q}_r$, and $|T_{r^*}| = |\mathcal{P}_r|$. □

REMARK 6. *All results of this section also hold for quasiconvex sets of geodesic δ-hyperbolic spaces.*

6 Total cores

In this section, we establish the results about total multi-cores and total beam cores in δ-hyperbolic graphs. We start with the problem of computation of total r-multi-cores of minimal size. Recall that for a commodity graph $R = (X, F)$ and an integer $r \geq 0$, $\sigma_r(R)$ denotes the smallest size of an r-multi-core for R. Let $\mathcal{I}(R) := \{I(x, y) : xy \in F\}$ denote the set-family in which the sets are the intervals defined by the edges of R. Denote by $\mathcal{I}_r(R)$ the r-inflation of $\mathcal{I}(R)$. The following result shows that for any $r \geq 8\delta$ it is possible to construct in polynomial time a total r-multi-core not of optimal size $\sigma_r(R)$ but of size $\sigma_{r-5\delta}(R)$.

PROPOSITION 6.1. (TOTAL MULTI-CORES) *Let $G = (V, E)$ be a graph with δ-thin triangles. For any commodity graph $R = (X, F)$ with finite $X \subseteq V$ and any integer $r \geq 8\delta$, the following inequalities hold:*

$$\pi(\mathcal{I}_r(R)) \leq \tau(\mathcal{I}_r(R)) \leq \sigma_r(R) \leq \tau(\mathcal{I}_{r-\delta}(R)) \leq \pi(\mathcal{I}_{r-5\delta}(R)) \leq \sigma_{r-5\delta}(R).$$

A total r-multi-core of R of size $\sigma_{r-5\delta}(R)$ can be constructed in polynomial time.

Proof. Let C be any total r-multi-core of the commodity graph R and let xy be any edge of R. For any geodesic $[x, y]$ between x and y there exists a vertex $c \in C$ such that the ball $B(c, r)$ intercepts $[x, y]$, i.e., $[x, y] \cap B(c, r) \neq \varnothing$. Consequently, $d(c, I(x, y)) \leq r$ and therefore C is a hitting set of $\mathcal{I}_r(R)$. This shows that $\tau(\mathcal{I}_r(R)) \leq \sigma_r(R)$. The inequality $\pi(\mathcal{I}_r(R)) \leq \tau(\mathcal{I}_r(R))$ is trivial.

To prove the inequality $\sigma_r(R) \leq \tau(\mathcal{I}_{r-\delta}(R))$, let T be a hitting set of $\mathcal{I}_{r-\delta}(R)$. This implies that for any interval $I(x, y)$ with $xy \in F$, the $(r-\delta)$-neighborhood $N_{r-\delta}(I(x, y))$ of $I(x, y)$ intersects T. Let $t \in T \cap N_{r-\delta}(I(x, y))$. Let z be a closest to t vertex of $I(x, y)$ and suppose that $d(z, x) = k'$ and $d(z, y) = k''$, where $k' + k'' = d(x, y)$. From the choice of t we conclude that $d(t, z) \leq r - \delta$. Since G is a graph with δ-thin triangles, by Lemma 3.2

the intervals of G are δ-thin. This implies that if z' is any other vertex of $I(x,y)$ with $d(z',x)=k'$ and $d(z',y)=k''$, then $d(z,z')\leq\delta$. Consequently, if L is any geodesic between x and y and z' is a vertex of L at distance k' from x, then $d(z,z')\leq\delta$, yielding $d(t,z')\leq d(t,z)+d(z,z')\leq r$. This implies that the ball $B(t,r)$ intercepts all geodesics L between x and y. Consequently, T is an r-multi-core for R, establishing the inequality $\sigma_r(R)\leq\tau(\mathcal{I}_{r-\delta}(R))$.

The inequality $\tau(\mathcal{I}_{r-\delta}(R))\leq\pi(\mathcal{I}_{r-5\delta}(R))$ is obtained by applying Proposition 5.1 for intervals with $r\geq 8\delta$ (since this proposition is used with $(r-5\delta)$ instead of r, we require $r\geq 8\delta$ to ensure $(r-5\delta)+4\delta\geq 7\delta$). Finally, the inequality $\pi(\mathcal{I}_{r-5\delta}(R))\leq\sigma_{r-5\delta}(R)$ is obtained by applying the inequality $\pi(\mathcal{I}_r(R))\leq\sigma_r(R)$ with $r-5\delta$ instead of r. By Proposition 5.1, a hitting set $T_{r-\delta}$ of $\mathcal{I}_{r-\delta}(R)$ and a packing $\mathcal{P}_{r-5\delta}$ of $\mathcal{I}_{r-5\delta}(R)$ such that $|T_{r-\delta}|=|\mathcal{P}_{r-5\delta}|\leq\sigma_{r-5\delta}(R)$ can be constructed in polynomial time. Since $T_{r-\delta}$ is also a total r-multi-core for R, we are done. □

Now, we will prove the existence of total beam cores in graphs G with δ-thin triangles. We proceed in two stages. First, we show that any two beams of G are 2δ-close (a result which may be of independent interest) and we use the Helly theorem for geodesics to show that G admits a total beam 5δ-core. Then we present a direct proof that G admits a total beam 2δ-core which can be found in linear time if G is finite.

LEMMA 6.1. *If G is a graph with δ-thin triangles, then any two beams of G are 2δ-close.*

Proof. Consider two arbitrary beams $[x,y]$ and $[u,v]$ with $y\in P(x), v\in P(u)$ and two geodesic triangles $\Delta(u,v,y):=[u,v]\cup[v,y]\cup[y,u]$ and $\Delta(x,v,y):=[x,y]\cup[v,y]\cup[v,x]$, where $[v,y],[y,u]$ and $[v,x]$ are arbitrary geodesics connecting v with y, y with u and v with x.

Let $\alpha:=(y|u)_v=(d(y,v)+d(v,u)-d(y,u))/2$, $\beta:=(v|u)_y=(d(v,y)+d(y,u)-d(v,u))/2=d(v,y)-\alpha$. Since $d(u,v)\geq d(u,y)$, we have $\alpha-\beta=(d(y,v)+d(v,u)-d(y,u)-d(v,y)-d(y,u)+d(v,u))/2=d(v,u)-d(y,u)\geq 0$, i.e., $\alpha\geq d(v,y)/2\geq\beta$. As geodesic triangles in G are δ-thin, all points of $[y,v]$ at distance at most α from v are at distance at most δ from the beam $[u,v]$.

Similarly, if $\alpha':=(y|x)_v=(d(y,v)+d(v,x)-d(y,x))/2$ and $\beta':=(v|x)_y=(d(v,y)+d(y,x)-d(v,x))/2=d(v,y)-\alpha'$, then $\alpha'\leq d(v,y)/2\leq\beta'$, since $d(x,v)\leq d(x,y)$. Hence, all points of $[y,v]$ at distance at most β' from y are at distance at most δ from the beam $[x,y]$.

As $\alpha\geq d(v,y)/2$ and $\beta'\geq d(v,y)/2$, there must exist a point in $[y,v]$ which is at distance at most δ from both beams $[u,v]$ and $[x,y]$. Hence, the beams $[u,v]$ and $[x,y]$ are 2δ-close, concluding the proof. □

By Lemma 6.1, any two beams of G are 2δ-close. Since any beam is a geodesic, by Theorem 5.1 for geodesics, we deduce that all beams of G are intercepted by a single ball of radius 5δ. The following result improves this:

PROPOSITION 6.2. *(Existence of total beam cores) Let G be a finite graph with δ-thin triangles, u,v be a pair of mutually distant vertices of G and m be a middle vertex of any (u,v)-geodesic $[u,v]$. Then the ball $B(m,2\delta)$ is a total beam core of G. Moreover, $B(m,2\delta)$ can be computed in linear time if δ is a constant.*

Proof. Let $[x,y]$ be an arbitrary beam of G with $y\in P(x)$ and let $\Delta(u,v,y):=[u,v]\cup[v,y]\cup[y,u]$ be a geodesic triangle, where $[v,y],[y,u]$ are arbitrary geodesics connecting y with v and u. Let m_y be a point on $[u,v]$ which is at distance $(y|u)_v=(d(y,v)+d(v,u)-d(y,u))/2$ from v and hence at distance $(y|v)_u=(d(y,u)+d(v,u)-d(y,v))/2$ from u. Since vertices u and v are mutually distant, we can assume, without loss of generality, that m is located on $[u,v]$ between v and m_y, i.e., $d(v,m)\leq d(v,m_y)=(y|u)_v$.

We claim that $d(v,m)\geq(x|y)_v$. Indeed, if $d(v,m)<(x|y)_v$, then $d(v,u)/2=d(v,m)<(x|y)_v=(d(x,v)+d(y,v)-d(x,y))/2\leq d(y,v)/2$ since $d(x,v)\leq d(x,y)$ (recall that y is a farthest from x vertex of G, i.e., $y\in P(x)$). But then $d(v,u)<d(y,v)$ contradicts with $u\in P(v)$.

Since $\Delta(u,v,y)$ is δ-thin, there is a point m' on $[v,y]$ with $(x|y)_v\leq d(v,m')\leq(y|u)_v$ and $d(m,m')\leq\delta$. Consider now a geodesic triangle $\Delta(x,v,y):=[x,y]\cup[v,y]\cup[v,x]$, where $[v,x]$ is an arbitrary geodesic connecting v with x. As $\Delta(x,v,y)$ is δ-thin and $(x|y)_v\leq d(v,m')$, we get $d(m',[x,y])\leq\delta$, i.e., $d(m,[x,y])\leq 2\delta$.

Finally, if m is not a vertex (i.e., it is the middle point of an edge connecting two middle vertices of $[u,v]$), then for any middle vertex $\bar{m}$ of $[u,v]$, ball $B(\bar{m},2\delta+1/2)=B(\bar{m},2\delta)$ (as the radius of any ball in graphs can be taken as an integer) of G intercepts all beams of G. This establishes the existence of total beam cores.

To compute a total beam core in linear time, recall that in δ-hyperbolic graphs (and hence in graphs with δ-thin triangles), if y is a most distant vertex from an arbitrary vertex z and x is a most distant vertex from y, then $d(x,y)\geq\mathrm{diam}(G)-2\delta$ [14, Proposition 3]. Hence, using at most $O(\delta)$ *breadth-first-searches*, one can generate a sequence of vertices $y:=v_1,x:=v_2,v_3,\ldots v_k$ with $k\leq 2\delta+2$

such that each v_i is most distant from v_{i-1} and v_k and v_{k-1} are mutually distant vertices (the initial value $d(x,y) \geq \mathrm{diam}(G) - 2\delta$ can be improved at most 2δ times). □

The following proposition shows that, in a graph G with δ-thin triangles, a middle vertex of any geodesic between two mutually distant vertices is not far from the center $C(G)$ of G. This shows that, while (α, r)-cores are close to median vertices, the total beam 2δ-cores are close to center vertices of G. We will need the following lemma of an independent interest.

LEMMA 6.2. *For every graph with δ-thin triangles,* $\mathrm{diam}(G) \geq 2\,\mathrm{rad}(G) - 2\delta - 1$ *holds.*

Proof. Assume $\mathrm{diam}(G) \leq 2\,\mathrm{rad}(G) - 2\delta - 2$ and consider a family of balls $\{B(v, \mathrm{rad}(G) - \delta - 1) : v \in V\}$. Each two balls of that family intersect. By Proposition 3.3, there must exist a vertex x which is at distance at most $\mathrm{rad}(G) - \delta - 1 + \delta = \mathrm{rad}(G) - 1$ from every vertex v of G. Since $\mathrm{ecc}(x) \leq \mathrm{rad}(G) - 1$, a contradiction arises. □

PROPOSITION 6.3. *Let G be a graph with δ-thin triangles, u, v be a pair of mutually distant vertices of G and m be a middle vertex of any (u,v)-geodesic. Then $C(G) \subseteq B(m, 4\delta + 1)$.*

Proof. As mentioned in the proof of Proposition 6.2, $d(u,v) \geq \mathrm{diam}(G) - 2\delta$ holds. Therefore, by Lemma 6.2, $d(u,v) \geq 2\,\mathrm{rad}(G) - 4\delta - 1$, and hence, without loss of generality, we may assume that $d(u,m) \geq \mathrm{rad}(G) - 2\delta$ and $d(v,m) \geq \mathrm{rad}(G) - 2\delta - 1$. Consider an arbitrary vertex $c \in C(G)$ and three distance sums: $S_1 = d(c,m) + d(u,v)$, $S_2 = d(u,m) + d(c,v)$, $S_3 = d(v,m) + d(c,u)$. If $S_2 > S_1$ (analogously, if $S_3 > S_1$), then $d(c,m) < d(u,m) + d(c,v) - d(u,v) = d(c,v) - d(v,m) \leq \mathrm{rad}(G) - \mathrm{rad}(G) + 2\delta + 1 = 2\delta + 1$, i.e., $d(c,m) \leq 2\delta$.

Thus, we may assume that S_1 is the largest sum. Without loss of generality, we may assume also that $S_2 \geq S_3$ (the case when $S_3 \geq S_2$ is similar). As, by Lemma 3.1, G is a δ-hyperbolic graph, we have $2\delta \geq S_1 - S_2 = d(c,m) + d(u,v) - d(u,m) - d(c,v) = d(c,m) + d(v,m) - d(c,v) \geq d(c,m) + \mathrm{rad}(G) - 2\delta - 1 - \mathrm{rad}(G) = d(c,m) - 2\delta - 1$, i.e., $d(c,m) \leq 4\delta + 1$. □

7 Hitting sets and packings for (κ, ϵ)-quasiconvex sets

Let $G = (V, E)$ be a graph with δ-thin triangles and $\kappa > 0, \epsilon \geq 0$, and $r \geq \epsilon + 2\delta$ be three nonnegative integers. Set as before $r^* := r + \epsilon + 3\delta$ and let $r' := r^* + \epsilon + 3\delta$. A (κ, ϵ)-*quasiconvex set* is a collection ${}^{\kappa}Q_i = \{Q_i^1, Q_i^2, \ldots, Q_i^{\kappa}\}$ of κ (not necessarily disjoint) ϵ-quasiconvex sets of G. Let ${}^{\kappa}\mathcal{Q} = \{{}^{\kappa}Q_1, {}^{\kappa}Q_2, \ldots {}^{\kappa}Q_m\}$ be a set-family whose members are (κ, ϵ)-quasiconvex sets of G. In this section, we establish a relationship between the maximum number $\pi_r({}^{\kappa}\mathcal{Q})$ of pairwise $2r$-apart (κ, ϵ)-quasiconvex sets in ${}^{\kappa}\mathcal{Q}$ and the minimum number $\tau_{r'}({}^{\kappa}\mathcal{Q})$ of balls of radius r' hitting all subsets of ${}^{\kappa}\mathcal{Q}$. More precisely, we prove the following result:

THEOREM 7.1. *Let ${}^{\kappa}\mathcal{Q} = \{{}^{\kappa}Q_1, \ldots, {}^{\kappa}Q_m\}$ be a family of (κ, ϵ)-quasiconvex sets of a finite graph $G = (V, E)$ with δ-thin geodesic triangles. Then $\tau_{r'}({}^{\kappa}\mathcal{Q}) \leq 2\kappa^2 \pi_r({}^{\kappa}\mathcal{Q})$. Moreover an r-packing $\mathcal{P}$ and an r'-hitting set T of ${}^{\kappa}\mathcal{Q}$ such that $|T| \leq 2\kappa^2|\mathcal{P}|$ can be constructed in polynomial time.*

The proof closely follows the proof of Theorem 2 of [15]. Denote by $\mathcal{Q}$ the collection of all ϵ-quasiconvex sets participating in the (κ, ϵ)-quasiconvex sets of ${}^{\kappa}\mathcal{Q}$ (obviously, $|\mathcal{Q}| = \kappa \cdot m$). For a vertex $v \in V$, let $\Gamma[v] := \{i : d(v, {}^{\kappa}Q_i) \leq r\}$ be the set of indices of all (κ, ϵ)-quasiconvex sets ${}^{\kappa}Q_i$ at distance at most r from v. For any $i = 1, \ldots, m$, let $\Gamma[i]$ be the set of indices of all (κ, ϵ)-quasiconvex sets which cannot be included in a packing containing ${}^{\kappa}Q_i$, i.e., $\Gamma[i] = \bigcup\{\Gamma[v] : v \in V, d(v, {}^{\kappa}Q_i) \leq r\}$. Clearly, if $j \in \Gamma[i]$, then $i \in \Gamma[j]$. Notice also that $i \in \Gamma[i]$.

Let $\pi'_r({}^{\kappa}\mathcal{Q})$ and $\tau'_r({}^{\kappa}\mathcal{Q})$ be respectively the optima of the following fractional packing and fractional hitting set problems (they can be solved in polynomial time as a pair of dual linear programs):

$$\Pi_r({}^{\kappa}\mathcal{Q}): \begin{cases} \max \sum_{i=1}^{m} x_i & \\ \text{s.t.} \sum_{i \in \Gamma[v]} x_i \leq 1 & \forall\, v \in V \\ x_i \geq 0 & \forall\, i = 1, \ldots, m \end{cases}$$

$$\Upsilon_r({}^{\kappa}\mathcal{Q}): \begin{cases} \min \sum_{v \in V} y_v & \\ \text{s.t.} \sum_{v \in N_r({}^{\kappa}Q_i)} y_v \geq 1 & \forall\, i = 1, \ldots, m \\ y_v \geq 0 & \forall\, v \in V. \end{cases}$$

LEMMA 7.1. *If $\mathbf{x} = \{x_i : i = 1, \ldots, m\}$ is an admissible solution of $\Pi_{r^*}({}^{\kappa}\mathcal{Q})$, then there exists a (κ, ϵ)-quasiconvex set ${}^{\kappa}Q_i$ such that $\sum_{j \in \Gamma[i]} x_j \leq 2\kappa$.*

Proof. The proof of this result is inspired by the averaging argument used in the proof of Lemma 4.1 of [9]. Define a graph $\mathbf{\Gamma}$ with $1, \ldots, m$ as the set of vertices and in which ij is an edge if and only if $j \in \Gamma[i]$ (and consequently $i \in \Gamma[j]$). For each edge ij of $\mathbf{\Gamma}$, set $z(i,j) = x_i \cdot x_j$. Since $i \in \Gamma[i]$, define $z(i,i) = x_i^2$. In the sum $\sum_{i=1}^{m} \sum_{j \in \Gamma[i]} z(i,j)$ every $z(i,j)$ is counted twice. On the other hand, an upper bound on this sum can be obtained in the

following way. Let t be any vertex of G. Pick any ϵ-quasiconvex set Q in the family $\mathcal{Q}$. By Lemma 5.1, there exists a vertex c_Q at distance at most r^* from every ϵ-quasiconvex set Q' in $\mathcal{Q}$ that is $2r$-close to Q and such that $d(t,Q') \leq d(t,Q)$. Let $\Gamma^*[c_Q]$ be the set of indices of all (κ,ϵ)-quasiconvex sets which are at distance at most r^* from c_Q. Now, for each (κ,ϵ)-quasiconvex set ${}^{\kappa}Q_i$ consider its collection of ϵ-quasiconvex sets, and for each ϵ-quasiconvex set Q in the collection ${}^{\kappa}Q_i$, add up $z(i,j)$ for all $j \in \Gamma^*[c_Q]$, and then multiply the total sum by 2. This way we computed the sum $2\sum_{i=1}^{m}\sum_{Q\in{}^{\kappa}Q_i}\sum_{j\in\Gamma^*[c_Q]} z(i,j)$. We assert that this suffices. Indeed, pick any $z(i,j)$ for an edge ij of the graph $\mathbf{\Gamma}$. Thus the (κ,ϵ)-quasiconvex sets ${}^{\kappa}Q_i$ and ${}^{\kappa}Q_j$ contain two $2r$-close ϵ-quasiconvex sets Q and Q'. Suppose without loss of generality that $d(t,Q') \leq d(t,Q)$. Then necessarily $j \in \Gamma^*[c_Q]$ because $d(c_Q,Q') \leq r^*$. Hence the term $z(i,j)$ will appear at least once in the triple sum, establishing the required inequality

$$\sum_{i=1}^{m}\sum_{j\in\Gamma[i]} z(i,j) \leq 2\sum_{i=1}^{m}\sum_{Q\in{}^{\kappa}Q_i}\sum_{j\in\Gamma^*[c_Q]} z(i,j).$$

Taking into account that $z(i,j) = x_i \cdot x_j = z(j,i)$, this inequality can be rewritten in the following way:

$$\sum_{i=1}^{m} x_i \sum_{j\in\Gamma[i]} x_j \leq 2\sum_{i=1}^{m} x_i \sum_{Q\in{}^{\kappa}Q_i}\sum_{j\in\Gamma^*[c_Q]} x_j.$$

Now, since c_Q is at distance at most r^* from any ϵ-quasiconvex sets in $\Gamma^*[c_Q]$ and $\mathbf{x}$ is an admissible solution of $\Pi_{r^*}({}^{\kappa}\mathcal{Q})$, we conclude that $\sum_{j\in\Gamma^*[c_Q]} x_j \leq 1$. Thus $\sum_{Q\in{}^{\kappa}Q_i}\sum_{j\in\Gamma^*[c_Q]} x_j \leq \kappa$ and we deduce that $\sum_{i=1}^{m} x_i \sum_{j\in\Gamma[i]} x_j \leq 2\kappa\sum_{i=1}^{m} x_i$. Hence, there exists ${}^{\kappa}Q_i$ such that $x_i\sum_{j\in\Gamma[i]} x_j \leq 2\kappa x_i$, yielding $\sum_{j\in\Gamma[i]} x_j \leq 2\kappa$. □

LEMMA 7.2. *It is possible to construct in polynomial time an integer admissible solution $\mathbf{x}^*$ of the linear program $\Pi_r({}^{\kappa}\mathcal{Q})$ of size at least $\pi'_{r^*}({}^{\kappa}\mathcal{Q})/(2\kappa)$.*

Proof. Let $\mathbf{x} = \{x_1,\ldots,x_m\}$ be an optimal (fractional) solution of the linear program $\Pi_{r^*}({}^{\kappa}\mathcal{Q})$ (it can be found in polynomial time). We will iteratively use Lemma 7.1 to $\mathbf{x}$ to derive an integer solution $\mathbf{x}^* = \{x_1^*,\ldots,x_m^*\}$ for the linear program $\Pi_r({}^{\kappa}\mathcal{Q})$. The algorithm starts by setting ${}^{\kappa}\mathcal{Q}' \leftarrow {}^{\kappa}\mathcal{Q}$. By Lemma 7.1 there exists a (κ,ϵ)-quasiconvex set ${}^{\kappa}Q_i \in {}^{\kappa}\mathcal{Q}'$ such that $\sum_{j\in\Gamma[i]} x_j \leq 2\kappa$. We set $x_i^* := 1$ and $x_j^* := 0$ for all $j \in \Gamma[i]\setminus\{i\}$, then we remove all (κ,ϵ)-quasiconvex sets ${}^{\kappa}Q_j$ with $j \in \Gamma[i]$ from ${}^{\kappa}\mathcal{Q}'$. The algorithm continues with the current set ${}^{\kappa}\mathcal{Q}'$ until it becomes empty. In all iterations of the algorithm the restriction of $\mathbf{x}$ to the current collection ${}^{\kappa}\mathcal{Q}'$ remains an admissible solution of the linear program $\Pi_{r^*}({}^{\kappa}\mathcal{Q}')$ defined by ${}^{\kappa}\mathcal{Q}'$. This justifies the use of Lemma 7.1 in all iterations of the algorithm.

To show that $\mathbf{x}^*$ is an admissible solution of $\Pi_r({}^{\kappa}\mathcal{Q})$, suppose by way of contradiction that there exist two $2r$-close (κ,ϵ)-quasiconvex sets ${}^{\kappa}Q_i$ and ${}^{\kappa}Q_j$ with $x_i^* = 1 = x_j^*$. Suppose that the algorithm selects ${}^{\kappa}Q_i$ before ${}^{\kappa}Q_j$. Consider the iteration when x_i^* becomes 1. Since $j \in \Gamma[i]$, at this iteration x_j^* becomes 0 and ${}^{\kappa}Q_j$ is removed from ${}^{\kappa}\mathcal{Q}'$. Thus x_j^* cannot become 1 at a later stage. This shows that the (κ,ϵ)-quasiconvex sets ${}^{\kappa}Q_i$ with $x_i^* = 1$ indeed constitute an r-packing for ${}^{\kappa}\mathcal{Q}$.

It remains to compare the costs of the solutions $\mathbf{x}$ and $\mathbf{x}^*$. For this, notice that according to the algorithm, for each (κ,ϵ)-quasiconvex set ${}^{\kappa}Q_i$ with $x_i^* = 1$ we can define a subset $\Gamma'[i]$ of $\Gamma[i]$ such that $i \in \Gamma'[i]$, $x_j^* = 0$ for all $j \in \Gamma'[i]\setminus\{i\}$, and $\sum_{j\in\Gamma'[i]\cup\{i\}} x_j \leq 2\kappa$. Hence, the (κ,ϵ)-quasiconvex sets of ${}^{\kappa}\mathcal{Q}$ can be partitioned into groups, such that each group contains a (κ,ϵ)-quasiconvex set selected in the integer solution and the total cost of the fractional solutions of the sets from each group is at most 2κ. This shows that $\sum_{i=1}^{m} x_i^* \geq (\sum_{i=1}^{m} x_i)/(2\kappa)$. □

LEMMA 7.3. *It is possible to construct in polynomial time an integer solution $\mathbf{y}^*$ of the linear program $\Upsilon_{r^*}({}^{\kappa}\mathcal{Q})$ of size at most $\kappa\pi'_r({}^{\kappa}\mathcal{Q})$.*

Proof. Let $\mathbf{y} = \{y_v : v \in V\}$ be an optimal (fractional) solution of the linear program $\Upsilon_r({}^{\kappa}\mathcal{Q})$. Since $\sum_{v\in\Gamma_r({}^{\kappa}Q_i)} y_v \geq 1$ for all $i = 1,\ldots,m$, each (κ,ϵ)-quasiconvex set ${}^{\kappa}Q_i$ contains an ϵ-quasiconvex set, which we will denote by Q_i, such that $\kappa\sum_{v\in N_r(Q_i)} y_v \geq 1$. Set $\mathcal{R} := \{Q_1,\ldots,Q_m\}$. Notice that $\mathbf{y}' = \{y'_v : v \in V\}$ defined by setting $y'_v = \kappa\cdot y_v$ if $v \in \bigcup_{i=1}^{m} Q_i$ and $y'_v = 0$ otherwise, is a fractional r-hitting set for the family $\{Q_1,\ldots,Q_m\}$. Thus the cost of $\mathbf{y}'$ is at least $\tau'_r(\mathcal{R}) = \pi'_r(\mathcal{R})$. Notice also that the cost of $\mathbf{y}'$ is at most κ times the cost of $\mathbf{y}$. By Proposition 5.1, we can construct in polynomial time a set T of size at most $\pi_r(\mathcal{R})$ which is an r^*-hitting set of $\mathcal{R}$. Let $\mathbf{y}^* = \{y_v : v \in V\}$ be defined by setting $y_v^* := 1$ if $v \in T$ and $y_v^* := 0$ otherwise. Since $\pi_r(\mathcal{R}) \leq \pi'_r(\mathcal{R})$, putting all things together, we obtain:

$$\sum_{v\in V} y_v^* = |T| \leq \pi_r(\mathcal{R}) \leq \pi'_r(\mathcal{R}) = \tau'_r(\mathcal{R}) \leq \sum_{v\in V} y'_v \leq \kappa\sum_{v\in V} y_v = \kappa\tau'_r({}^{\kappa}\mathcal{Q}).$$

□

Now, we are ready to complete the proof of Theorem 7.1. According to Lemma 7.2 we can construct in polynomial time an integer solution $\mathbf{x}^*$ for $\Pi_r({}^\kappa\mathcal{Q})$ of size at least $\pi'_{r^*}({}^\kappa\mathcal{Q})/(2\kappa)$. Let $\mathcal{P} = \{{}^\kappa Q_i : x_i^* = 1\}$. On the other hand, applying Lemma 7.3 with the radius r^* instead of r, we can construct in polynomial time an integer solution $\mathbf{y}^*$ of the linear program $\Upsilon_{r'}({}^\kappa\mathcal{Q})$ of size at most $\kappa\tau'_{r^*}({}^\kappa\mathcal{Q})$. Let $T = \{v \in V : y_v^* = 1\}$. Since, by duality, $\tau'_{r^*}({}^\kappa\mathcal{Q}) = \pi'_{r^*}({}^\kappa\mathcal{Q})$, we deduce that $|T| \leq 2\kappa^2|\mathcal{P}|$, as required.

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