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**SOME LOGARITHMICALLY COMPLETELY
MONOTONIC FUNCTIONS AND INEQUALITIES FOR
MULTINOMIAL COEFFICIENTS AND MULTIVARIATE
BETA FUNCTIONS**

*Feng Qi, Da-Wei Niu, Dongkyu Lim, Bai-Ni Guo**

Dedicated to people facing and battling COVID-19

In the paper, the authors extend a function arising from the Bernoulli trials in probability and involving the gamma function to its largest ranges, find logarithmically complete monotonicity of these extended functions, and, in light of logarithmically complete monotonicity of these extended functions, derive some inequalities for multinomial coefficients and multivariate beta functions. These results recover, extend, and generalize some known conclusions.

1. BACKGROUND AND MOTIVATION

Let us denote by $P_{n,k}(p)$ the probability of achieving exactly k successes in n Bernoulli trials with success probability p . Then

$$(1.1) \quad P_{n,k}(p) = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)} p^k (1-p)^{n-k} = \binom{n}{k} p^k (1-p)^{n-k},$$

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where $\Gamma(z)$ denotes the classical Euler gamma function which can be defined [1, 19, 24] by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0$$

or by

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{\prod_{k=0}^n (z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}.$$

In the technical report [16], Leblanc and Johnson considered a problem: which is more likely to happen: k successes in n trials or $2k$ successes in $2n$ trials? They proved that

$$(1.2) \quad P_{2n,2k}(p) \leq P_{n,k}(p), \quad 0 \leq k \leq n, \quad p \in (0, 1).$$

This means that k successes in n trials is more likely to happen than $2k$ successes in $2n$ trials. One year later, the same authors generalized the inequality (1.2) in [17, Corollary 2.4] by

$$(1.3) \quad P_{(j+1)n,(j+1)k}(p) \leq P_{jn,jk}(p), \quad j \in \mathbb{N}.$$

About ten years later, Alzer extended the inequality (1.3) in the note [2] by considering the function

$$(1.4) \quad G(x) = G_{k,n;p}(x) = \frac{\Gamma(nx+1)}{\Gamma(kx+1)\Gamma((n-k)x+1)} p^{kx} (1-p)^{(n-k)x}$$

and essentially proving that the function $G(x)$ is logarithmically completely monotonic on $(0, \infty)$, where k and n are integers with $0 \leq k \leq n$, $p \in (0, 1)$, and an infinitely differentiable and positive function $F(x)$ is said [4, 5, 27, 29, 37] to be logarithmically completely monotonic on an interval I if and only if $(-1)^m [\ln F(x)]^{(m)} \geq 0$ for all $m \geq 2$ and $x \in I$. We observe that we can write

$$G(x) = G_{k,n;p}(x) = \binom{nx}{kx} p^{kx} (1-p)^{(n-k)x} = \frac{n}{k(n-k)} \frac{p^{kx} (1-p)^{(n-k)x}}{x \text{B}(kx, (n-k)x)},$$

where

$$(1.5) \quad \binom{x}{y} = \frac{\Gamma(x+1)}{\Gamma(y+1)\Gamma(x-y+1)} = \frac{1}{(x+1) \text{B}(x-y+1, y+1)}, \quad x, y \in \mathbb{C}$$

and

$$\text{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad \Re(x), \Re(y) > 0$$

denotes the classical Euler beta function and $\text{B}(x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Recall from [18, Chapter XIII], [37, Chapter 1], and [38, Chapter IV] that an infinitely differentiable and nonnegative function $f(x)$ is said to be completely monotonic on an interval I if and only if

$$0 \leq (-1)^{m-1} f^{(m-1)}(x) < \infty, \quad m \geq 2, \quad x \in I.$$

The Bernstein–Widder theorem [38, p. 161, Theorem 12b] characterizes that a necessary and sufficient condition for $f(x)$ to be completely monotonic on $(0, \infty)$ is that

$$f(x) = \int_0^\infty e^{-xt} d\mu(t), \quad x \in (0, \infty),$$

where $\mu(t)$ is non-decreasing and the above integral converges for $x \in (0, \infty)$. In other words or simply speaking, a function is completely monotonic on $(0, \infty)$ if and only if it is a Laplace transform. Recall from [5, 11, 27, 37] that a logarithmically completely monotonic function must be completely monotonic on the same defined interval, but not conversely. This is why we restate here the main result in [2] in terms of the logarithmically complete monotonicity. For more information on new developments of this topic, please refer to [10, 12, 25, 30, 34, 35, 37] and closely related references therein.

In this paper, we first consider the function

$$\begin{aligned} (1.6) \quad Q(x) = Q_{\alpha, \beta; p}(x) &= \frac{\Gamma((\alpha + \beta)x + 1)}{\Gamma(\alpha x + 1)\Gamma(\beta x + 1)} p^{\alpha x} (1 - p)^{\beta x} \\ &= \binom{(\alpha + \beta)x}{\alpha x} p^{\alpha x} (1 - p)^{\beta x} = \frac{\alpha + \beta}{\alpha \beta} \frac{p^{kx} (1 - p)^{(n-k)x}}{x B(\alpha x, \beta x)} \end{aligned}$$

for $x \in (0, \infty)$, where $\alpha, \beta > 0$ and $p \in (0, 1)$. It is easy to see that the function $Q(x)$ is an extension of $G(x)$ and $P_{n, k}(p)$ and satisfies

$$Q_{\alpha, \beta; p}(x) = Q_{\beta, \alpha; 1-p}(x), \quad Q_{k, n-k; p}(x) = G_{k, n; p}(x), \quad Q_{k, n-k; p}(1) = P_{n, k}(p).$$

In Section , we will verify that the function $Q(x)$ is logarithmically completely monotonic on $(0, \infty)$.

More generally, we can consider the function

$$\begin{aligned} (1.7) \quad Q(x) = Q_{\mathbf{a}, \mathbf{p}; m}(x) &= \frac{\Gamma(1 + x \sum_{i=1}^m a_i)}{\prod_{i=1}^m \Gamma(1 + x a_i)} \prod_{i=1}^m p_i^{x a_i} \\ &= \binom{x \sum_{i=1}^m a_i}{x a_1, x a_2, \dots, x a_m} \prod_{i=1}^m p_i^{x a_i} = \frac{\sum_{i=1}^m a_i}{\prod_{i=1}^m a_i} \frac{\prod_{i=1}^m p_i^{x a_i}}{x^{m-1} B(x a_1, x a_2, \dots, x a_m)} \end{aligned}$$

for $x \in (0, \infty)$ and $m \geq 2$, where $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$, $\mathbf{p} = (p_1, p_2, \dots, p_m)$ with $p_i \in (0, 1)$ for $1 \leq i \leq m$ and $\sum_{i=1}^m p_i = 1$, the notation

$$\binom{\sum_{i=1}^m a_i}{a_1, a_2, \dots, a_m} = \prod_{i=1}^m \binom{\sum_{\ell=1}^i a_\ell}{a_i}$$

in terms of the notation in (1.5) is called the multinomial coefficient, and

$$B(a_1, a_2, \dots, a_m) = \frac{\Gamma(a_1)\Gamma(a_2) \cdots \Gamma(a_m)}{\Gamma(a_1 + a_2 + \cdots + a_m)}$$

is called the multivariate beta function. It is obvious that the function $\mathcal{Q}_{\alpha,p;m}(x)$ is a generalization and an extension of the functions $Q_{\alpha,\beta;p}(x)$, $G_{k,n;p}(x)$, and $P_{n,k}(p)$ defined in (1.1), (1.4), and (1.6) respectively. Concretely speaking,

$$\begin{aligned}\mathcal{Q}_{(\alpha,\beta),(p,1-p);2}(x) &= Q_{\alpha,\beta;p}(x), \\ \mathcal{Q}_{(k,n-k),(p,1-p);2}(x) &= G_{k,n;p}(x), \\ \mathcal{Q}_{(k,n-k),(p,1-p);2}(1) &= P_{n,k}(p).\end{aligned}$$

In Section , we will show that the function $\mathcal{Q}(x)$ is logarithmically completely monotonic on $(0, \infty)$.

In Section , in light of logarithmically complete monotonicity of $Q(x)$ and $\mathcal{Q}(x)$, we will offer some inequalities for multinomial coefficients. In Section , we will reformulate combinatorial inequalities obtained in Section in terms of multivariate beta functions, that is, we will present some inequalities for multivariate beta functions. In Section 4.3, the last section of this paper, we will recover some known results in [2, 17] from those inequalities obtained in Section for multinomial coefficients.

2. COMPLETELY MONOTONIC FUNCTIONS

We now start off to prove our first main result in this paper: the function $Q(x)$ is logarithmically completely monotonic on $(0, \infty)$.

Theorem 2.1. *For $\alpha, \beta > 0$ and $p \in (0, 1)$, the function $Q(x) = Q_{\alpha,\beta;p}(x)$ defined in (1.6) is logarithmically completely monotonic on $(0, \infty)$.*

Proof. Straightforward computation yields

$$\begin{aligned}\ln Q(x) &= \ln \Gamma((\alpha + \beta)x + 1) - \ln \Gamma(\alpha x + 1) \\ &\quad - \ln \Gamma(\beta x + 1) + \alpha x \ln p + \beta x \ln(1 - p), \\ [\ln Q(x)]' &= (\alpha + \beta)\psi((\alpha + \beta)x + 1) - \alpha\psi(\alpha x + 1) \\ &\quad - \beta\psi(\beta x + 1) + \alpha \ln p + \beta \ln(1 - p), \\ [\ln Q(x)]'' &= (\alpha + \beta)^2\psi'((\alpha + \beta)x + 1) - \alpha^2\psi'(\alpha x + 1) - \beta^2\psi'(\beta x + 1).\end{aligned}$$

From

$$\psi'(z) = \int_0^\infty \frac{t}{1 - e^{-t}} e^{-zt} dt, \quad \Re(z) > 0$$

in [1, p. 260, 6.4.1], it follows that

$$\begin{aligned}\psi'(\tau z + 1) &= \int_0^\infty \frac{t}{1 - e^{-t}} e^{-(\tau z + 1)t} dt \\ &= \int_0^\infty \frac{t}{e^t - 1} e^{-\tau z t} dt = \frac{1}{\tau} \int_0^\infty h\left(\frac{v}{\tau}\right) e^{-vz} dv,\end{aligned}$$

where $\tau > 0$ and $h(t) = \frac{t}{e^t - 1}$. Accordingly, we have

$$(2.8) \quad [\ln Q(x)]'' = \int_0^\infty \left[(\alpha + \beta)h\left(\frac{v}{\alpha + \beta}\right) - \alpha h\left(\frac{v}{\alpha}\right) - \beta h\left(\frac{v}{\beta}\right) \right] e^{-xv} dv.$$

Since the function $h(t)$ is decreasing on $(-\infty, \infty)$, we obtain

$$(\alpha + \beta)h\left(\frac{v}{\alpha + \beta}\right) = \alpha h\left(\frac{v}{\alpha + \beta}\right) + \beta h\left(\frac{v}{\alpha + \beta}\right) \geq \alpha h\left(\frac{v}{\alpha}\right) + \beta h\left(\frac{v}{\beta}\right).$$

Substituting this inequality into the equation (2.8) reveals that the second derivative $[\ln Q(x)]''$ is completely monotonic on $(0, \infty)$.

By the complete monotonicity of $[\ln Q(x)]''$, we see that the first derivative $[\ln Q(x)]'$ is strictly increasing on $(0, \infty)$, hence,

$$\begin{aligned} [\ln Q(x)]' &\leq \lim_{x \rightarrow \infty} [(\alpha + \beta)\psi((\alpha + \beta)x + 1) - \alpha\psi(\alpha x + 1) - \beta\psi(\beta x + 1)] \\ &\quad + \alpha \ln p + \beta \ln(1 - p) \\ &= \lim_{x \rightarrow \infty} \left[(\alpha + \beta)\psi((\alpha + \beta)x) - \alpha\psi(\alpha x) - \beta\psi(\beta x) - \frac{1}{x} \right] \\ &\quad + \alpha \ln p + \beta \ln(1 - p) \\ &= \lim_{x \rightarrow \infty} \left((\alpha + \beta)[\psi((\alpha + \beta)x) - \ln((\alpha + \beta)x)] - \alpha[\psi(\alpha x) - \ln(\alpha x)] \right. \\ &\quad \left. - \beta[\psi(\beta x) - \ln(\beta x)] - \frac{1}{x} + (\alpha + \beta) \ln((\alpha + \beta)x) \right. \\ &\quad \left. - \alpha \ln(\alpha x) - \beta \ln(\beta x) \right) + \alpha \ln p + \beta \ln(1 - p) \\ &= (\alpha + \beta) \ln(\alpha + \beta) - \alpha \ln \frac{\alpha}{p} - \beta \ln \frac{\beta}{1 - p} \\ &= \left[p \frac{\alpha}{p} + (1 - p) \frac{\beta}{1 - p} \right] \ln \left[p \frac{\alpha}{p} + (1 - p) \frac{\beta}{1 - p} \right] \\ &\quad - p \left(\frac{\alpha}{p} \ln \frac{\alpha}{p} \right) - (1 - p) \left(\frac{\beta}{1 - p} \ln \frac{\beta}{1 - p} \right) \\ &< 0, \end{aligned}$$

where we used the facts that the function $x \ln x$ is convex on $(0, \infty)$ and that

$$(2.9) \quad \lim_{x \rightarrow \infty} [\ln x - \psi(x)] = 0,$$

see [13, Theorem 1] and [14]. In conclusion, the function $Q(x)$ is logarithmically completely monotonic on $(0, \infty)$. The proof of Theorem 2.1 is complete. \square

We now prove our second main result in this paper: the function $\mathcal{Q}(x)$ is logarithmically completely monotonic on $(0, \infty)$.

Theorem 2.2. Let $m \geq 2$, $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$, and $\mathbf{p} = (p_1, p_2, \dots, p_m)$ with $p_i \in (0, 1)$ for $1 \leq i \leq m$ and $\sum_{i=1}^m p_i = 1$. Then the function $\mathcal{Q}(x) = \mathcal{Q}_{\mathbf{a}, \mathbf{p}; m}(x)$ defined in (1.7) is logarithmically completely monotonic on $(0, \infty)$.

Proof. Direct calculation gives

$$\begin{aligned} \ln \mathcal{Q}(x) &= \ln \Gamma\left(1 + x \sum_{i=1}^m a_i\right) - \sum_{i=1}^m \ln \Gamma(1 + a_i x) + x \sum_{i=1}^m a_i \ln p_i, \\ [\ln \mathcal{Q}(x)]' &= \left(\sum_{i=1}^m a_i\right) \psi\left(1 + x \sum_{i=1}^m a_i\right) - \sum_{i=1}^m a_i \psi(1 + a_i x) + \sum_{i=1}^m a_i \ln p_i, \end{aligned}$$

and

$$[\ln \mathcal{Q}(x)]'' = \left(\sum_{i=1}^m a_i\right)^2 \psi'\left(1 + x \sum_{i=1}^m a_i\right) - \sum_{i=1}^m a_i^2 \psi'(1 + a_i x).$$

As did in the proof of Theorem 2.1, we can obtain

$$(2.10) \quad [\ln \mathcal{Q}(x)]'' = \int_0^\infty \left[\left(\sum_{i=1}^m a_i\right) h\left(\frac{v}{\sum_{i=1}^m a_i}\right) - \sum_{i=1}^m a_i h\left(\frac{v}{a_i}\right) \right] e^{-xv} dv.$$

Since the function $h(t)$ is decreasing on $(-\infty, \infty)$, we obtain

$$\left(\sum_{i=1}^m a_i\right) h\left(\frac{v}{\sum_{i=1}^m a_i}\right) = \sum_{i=1}^m a_i h\left(\frac{v}{\sum_{i=1}^m a_i}\right) \geq \sum_{i=1}^m a_i h\left(\frac{v}{a_i}\right).$$

Combining this with (2.10) yields that the second derivative $[\ln \mathcal{Q}(x)]''$ is completely monotonic on $(0, \infty)$.

Complete monotonicity of $[\ln \mathcal{Q}(x)]''$ implies that the first derivative $[\ln \mathcal{Q}(x)]'$ is strictly increasing on $(0, \infty)$, therefore,

$$\begin{aligned} [\ln \mathcal{Q}(x)]' &\leq \lim_{x \rightarrow \infty} \left[\left(\sum_{i=1}^m a_i\right) \psi\left(1 + x \sum_{i=1}^m a_i\right) - \sum_{i=1}^m a_i \psi(1 + a_i x) \right] + \sum_{i=1}^m a_i \ln p_i \\ &= \lim_{x \rightarrow \infty} \left[\left(\sum_{i=1}^m a_i\right) \psi\left(x \sum_{i=1}^m a_i\right) - \sum_{i=1}^m a_i \psi(a_i x) - \frac{m-1}{x} \right] + \sum_{i=1}^m a_i \ln p_i \\ &= \lim_{x \rightarrow \infty} \left\{ \left(\sum_{i=1}^m a_i\right) \left[\psi\left(x \sum_{i=1}^m a_i\right) - \ln\left(x \sum_{i=1}^m a_i\right) \right] \right. \\ &\quad \left. - \sum_{i=1}^m a_i [\psi(a_i x) - \ln(a_i x)] + \left(\sum_{i=1}^m a_i\right) \ln\left(x \sum_{i=1}^m a_i\right) \right. \\ &\quad \left. - \sum_{i=1}^m a_i \ln(a_i x) \right\} + \sum_{i=1}^m a_i \ln p_i \end{aligned}$$

$$\begin{aligned}
&= \left(\sum_{i=1}^m a_i \right) \ln \left(\sum_{i=1}^m a_i \right) - \sum_{i=1}^m a_i \ln a_i + \sum_{i=1}^m a_i \ln p_i \\
&= \left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) \ln \left(\sum_{i=1}^m p_i \frac{a_i}{p_i} \right) - \sum_{i=1}^m p_i \frac{a_i}{p_i} \ln \frac{a_i}{p_i} \\
&\leq 0,
\end{aligned}$$

where, as did in the proof of Theorem 2.1, we used the limit (2.9) and convexity of the function $x \ln x$ on $(0, \infty)$. The proof of Theorem 2.2 is complete. \square

3. THREE INEQUALITIES FOR MULTINOMIAL COEFFICIENTS

In light of logarithmically complete monotonicity of $Q(x)$ and $\mathcal{Q}(x)$, we now offer some inequalities for multinomial coefficients.

Theorem 3.3. For $\ell, m \geq 2$, let $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$, $x_j > 0$ for $1 \leq j \leq \ell$, and $\lambda_j \in (0, 1)$ with $\sum_{j=1}^{\ell} \lambda_j = 1$. Then

$$\begin{aligned}
(3.11) \quad &\binom{\sum_{j=1}^{\ell} \lambda_j x_j \sum_{i=1}^m a_i}{a_1 \sum_{j=1}^{\ell} \lambda_j x_j, a_2 \sum_{j=1}^{\ell} \lambda_j x_j, \dots, a_m \sum_{j=1}^{\ell} \lambda_j x_j} \\
&\leq \prod_{j=1}^{\ell} \binom{x_j \sum_{i=1}^m a_i}{x_j a_1, x_j a_2, \dots, x_j a_m}^{\lambda_j}
\end{aligned}$$

and the equality in (3.11) holds if and only if $x_1 = x_2 = \dots = x_{\ell}$. In particular, when $\ell = m = 2$,

$$(3.12) \quad \binom{(a_1 + a_2)(\lambda_1 x_1 + \lambda_2 x_2)}{a_1(\lambda_1 x_1 + \lambda_2 x_2)} \leq \binom{(a_1 + a_2)x_1}{a_1 x_1}^{\lambda_1} \binom{(a_1 + a_2)x_2}{a_1 x_2}^{\lambda_2}$$

and the equality in (3.12) is valid if and only if $x_1 = x_2$.

Proof. Logarithmically complete monotonicity in Theorem 2.2 implies that the function $\mathcal{Q}(x)$ is logarithmically convex on $(0, \infty)$. Hence, we acquire

$$\mathcal{Q} \left(\sum_{j=1}^{\ell} \lambda_j x_j \right) \leq \prod_{j=1}^{\ell} \mathcal{Q}^{\lambda_j}(x_j).$$

Making use of the expression

$$\mathcal{Q}(x) = \binom{x \sum_{i=1}^m a_i}{x a_1, x a_2, \dots, x a_m} \prod_{i=1}^m p_i^{x a_i}$$

arrives at

$$\begin{aligned} & \left(a_1 \sum_{j=1}^{\ell} \lambda_j x_j, a_2 \sum_{j=1}^{\ell} \lambda_j x_j, \dots, a_m \sum_{j=1}^{\ell} \lambda_j x_j \right) \prod_{i=1}^m p_i^{a_i \sum_{j=1}^{\ell} \lambda_j x_j} \\ & \leq \prod_{j=1}^{\ell} \left[\left(x_j \sum_{i=1}^m a_i \right) \prod_{i=1}^m p_i^{a_i x_j} \right]^{\lambda_j} \end{aligned}$$

which can be rearranged as (3.11).

The inequality (3.12) can also be independently derived from logarithmically complete monotonicity of $Q(x)$. The proof of Theorem 3.3 is complete. \square

Theorem 3.4. For $\ell, m \geq 2$, let $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$ and let $x_j > 0$ for $1 \leq j \leq \ell$. Then

$$(3.13) \quad \prod_{j=1}^{\ell} \left(x_j \sum_{i=1}^m a_i \right) < \left(a_1 \sum_{j=1}^{\ell} x_j, a_2 \sum_{j=1}^{\ell} x_j, \dots, a_m \sum_{j=1}^{\ell} x_j \right).$$

In particular, when $\ell = m = 2$,

$$(3.14) \quad \begin{pmatrix} (a_1 + a_2)x_1 \\ a_1x_1 \end{pmatrix} < \begin{pmatrix} (a_1 + a_2)x_2 \\ a_1x_2 \end{pmatrix} < \begin{pmatrix} (a_1 + a_2)(x_1 + x_2) \\ a_1(x_1 + x_2) \end{pmatrix}.$$

Proof. In [2, Lemma 3], it was established that, if $g : [0, \infty) \rightarrow (0, 1]$ is differentiable and $\frac{g'}{g}$ is strictly increasing on $(0, \infty)$, then $g(x)g(y) < g(x + y)$ for $x, y \in (0, \infty)$. From this, we can inductively derive

$$\prod_{j=1}^{\ell} g(x_j) < g\left(\sum_{j=1}^{\ell} x_j\right).$$

Applying this inequality to the function $Q(x)$ yields

$$\begin{aligned} & \prod_{j=1}^{\ell} \left[\left(x_j \sum_{i=1}^m a_i \right) \prod_{i=1}^m p_i^{a_i x_j} \right] \\ & < \left(a_1 \sum_{j=1}^{\ell} x_j, a_2 \sum_{j=1}^{\ell} x_j, \dots, a_m \sum_{j=1}^{\ell} x_j \right) \prod_{i=1}^m p_i^{a_i \sum_{j=1}^{\ell} x_j} \end{aligned}$$

which can be rewritten as (3.13).

The inequality (3.14) can also be independently derived from logarithmically complete monotonicity of $Q(x)$. The proof of Theorem 3.4 is complete. \square

Theorem 3.5. For $m \geq 2$, let $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$. If $0 < a \leq c$ and $x > 0$, then

$$(3.15) \quad \begin{aligned} & \binom{(a+x)\sum_{i=1}^m a_i}{(a+x)a_1, (a+x)a_2, \dots, (a+x)a_m} \binom{c\sum_{i=1}^m a_i}{ca_1, ca_2, \dots, ca_m} \\ & \leq \binom{a\sum_{i=1}^m a_i}{aa_1, aa_2, \dots, aa_m} \binom{(c+x)\sum_{i=1}^m a_i}{(c+x)a_1, (c+x)a_2, \dots, (c+x)a_m} \end{aligned}$$

and the equality in (3.15) holds if and only if $a = c$. In particular, when $m = 2$,

$$(3.16) \quad \binom{(a+x)(a_1+a_2)}{(a+x)a_1} \binom{c(a_1+a_2)}{ca_1} \leq \binom{a(a_1+a_2)}{aa_1} \binom{(c+x)(a_1+a_2)}{(c+x)a_1}$$

and the equality in (3.16) holds if and only if $a = c$.

Proof. For $0 < a < c$, define

$$V(x) = \ln \mathcal{Q}(a+x) + \ln \mathcal{Q}(c) - \ln \mathcal{Q}(a) - \ln \mathcal{Q}(c+x).$$

Since

$$V'(x) = \frac{\mathcal{Q}'(a+x)}{\mathcal{Q}(a+x)} - \frac{\mathcal{Q}'(c+x)}{\mathcal{Q}(c+x)}$$

and logarithmically complete monotonicity of $\mathcal{Q}(x)$ implies that $\frac{\mathcal{Q}'(x)}{\mathcal{Q}(x)}$ is strictly increasing on $(0, \infty)$, we conclude that $V'(x) < 0$ and $V(x) < V(0) = 0$. Therefore,

$$\ln \mathcal{Q}(a+x) + \ln \mathcal{Q}(c) \leq \ln \mathcal{Q}(a) + \ln \mathcal{Q}(c+x),$$

which is equivalent to

$$\begin{aligned} & \ln \left[\binom{(a+x)\sum_{i=1}^m a_i}{(a+x)a_1, (a+x)a_2, \dots, (a+x)a_m} \prod_{i=1}^m p_i^{a_i(a+x)} \right] \\ & + \ln \left[\binom{c\sum_{i=1}^m a_i}{ca_1, ca_2, \dots, ca_m} \prod_{i=1}^m p_i^{a_i c} \right] \leq \ln \left[\binom{a\sum_{i=1}^m a_i}{aa_1, aa_2, \dots, aa_m} \prod_{i=1}^m p_i^{a_i a} \right] \\ & + \ln \left[\binom{(c+x)\sum_{i=1}^m a_i}{(c+x)a_1, (c+x)a_2, \dots, (c+x)a_m} \prod_{i=1}^m p_i^{a_i(c+x)} \right]. \end{aligned}$$

This can be simplified as (3.15).

The inequality (3.16) can also be independently derived from logarithmically complete monotonicity of $\mathcal{Q}(x)$. The proof of Theorem 3.5 is complete. \square

4. INEQUALITIES FOR MULTIVARIATE BETA FUNCTIONS

For $a_i > 0$ and $i \in \mathbb{N}$, the multinomial coefficient and the multivariate beta function have the relation

$$\binom{\sum_{i=1}^m a_i}{a_1, a_2, \dots, a_m} = \frac{\sum_{i=1}^m a_i}{\prod_{i=1}^m a_i} \frac{1}{B(a_1, a_2, \dots, a_m)}.$$

Therefore, from those inequalities for multinomial coefficients in Section , we can derive some inequalities for the multivariate beta function $B(a_1, a_2, \dots, a_m)$. In other words, Theorems (3.3) to (3.5) can be respectively reformulated as the following forms.

4.1 First inequality for multivariate beta function

For $\ell, m \geq 2$, let $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$, $x_j > 0$ for $1 \leq j \leq \ell$, and $\lambda_j \in (0, 1)$ with $\sum_{j=1}^{\ell} \lambda_j = 1$. Then

$$(4.17) \quad \frac{B\left(a_1 \sum_{j=1}^{\ell} \lambda_j x_j, a_2 \sum_{j=1}^{\ell} \lambda_j x_j, \dots, a_m \sum_{j=1}^{\ell} \lambda_j x_j\right)}{\prod_{j=1}^{\ell} B^{\lambda_j}(a_1 x_j, a_2 x_j, \dots, a_m x_j)} \geq \left(\frac{\prod_{j=1}^{\ell} x_j^{\lambda_j}}{\sum_{j=1}^{\ell} \lambda_j x_j}\right)^{m-1}$$

and the equality in (4.17) holds if and only if $x_1 = x_2 = \dots = x_{\ell}$. In particular, when $\ell = m = 2$,

$$(4.18) \quad \frac{B(a_1(\lambda_1 x_1 + \lambda_2 x_2), a_2(\lambda_1 x_1 + \lambda_2 x_2))}{B^{\lambda_1}(a_1 x_1, a_2 x_1) B^{\lambda_2}(a_1 x_2, a_2 x_2)} \geq \frac{x_1^{\lambda_1} x_2^{\lambda_2}}{\lambda_1 x_1 + \lambda_2 x_2}$$

and the equality in (4.18) is valid if and only if $x_1 = x_2$.

The inequality (4.17) or (4.18) implies that $x^{m-1} B(xa_1, xa_2, \dots, xa_m)$ for $m \geq 2$ is logarithmically concave with respect to $x \in (0, \infty)$. More generally, we claim that the reciprocal $\frac{1}{x^{m-1} B(xa_1, xa_2, \dots, xa_m)}$ for $m \geq 2$ is a logarithmically completely monotonic function of $x \in (0, \infty)$.

4.2 Second inequality for multivariate beta function

For $\ell, m \geq 2$, let $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$ and let $x_j > 0$ for $1 \leq j \leq \ell$. Then

$$\frac{\prod_{j=1}^{\ell} B(a_1 x_j, a_2 x_j, \dots, a_m x_j)}{B\left(a_1 \sum_{j=1}^{\ell} x_j, a_2 \sum_{j=1}^{\ell} x_j, \dots, a_m \sum_{j=1}^{\ell} x_j\right)} > \left(\frac{\sum_{i=1}^m a_i}{\prod_{i=1}^m a_i}\right)^{\ell-1} \left(\frac{\sum_{j=1}^{\ell} x_j}{\prod_{j=1}^{\ell} x_j}\right)^{m-1}.$$

In particular, when $\ell = m = 2$,

$$\frac{B(a_1 x_1, a_2 x_1) B(a_1 x_2, a_2 x_2)}{B(a_1(x_1 + x_2), a_2(x_1 + x_2))} > \left(\frac{1}{a_1} + \frac{1}{a_2}\right) \left(\frac{1}{x_1} + \frac{1}{x_2}\right).$$

4.3 Third inequality for multivariate beta function

For $m \geq 2$, let $\mathbf{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \leq i \leq m$. If $0 < a \leq c$ and $x > 0$, then

$$(4.19) \quad \left(\frac{c}{c+x}\right)^{m-1} \frac{B(ca_1, ca_2, \dots, ca_m)}{B((c+x)a_1, (c+x)a_2, \dots, (c+x)a_m)} \\ \geq \left(\frac{a}{a+x}\right)^{m-1} \frac{B(aa_1, aa_2, \dots, aa_m)}{B((a+x)a_1, (a+x)a_2, \dots, (a+x)a_m)}$$

and the equality in (4.19) holds if and only if $a = c$. In particular, when $m = 2$,

$$(4.20) \quad \frac{c}{c+x} \frac{B(a_1c, a_2c)}{B(a_1(c+x), a_2(c+x))} \geq \frac{a}{a+x} \frac{B(a_1a, a_2a)}{B(a_1(a+x), a_2(a+x))}$$

and the equality in (4.20) holds if and only if $a = c$.

The inequality (4.19) implies that the function

$$\left(\frac{t}{t+x}\right)^{m-1} \frac{B(a_1t, a_2t, \dots, a_mt)}{B(a_1(t+x), a_2(t+x), \dots, a_m(t+x))}$$

for $m \geq 2$ and $x, a_i > 0$ is strictly increasing with respect to $t \in (0, \infty)$.

Remark 4.1. For more information on inequalities for the beta function $B(x, y)$ and their applications, please refer to [3, 6, 7, 8, 9, 26, 36] and closely related references therein.

5. RECOVERING FOUR KNOWN RESULTS

From Theorems 3.3 to 3.5, we can recover inequalities and monotonicity for binomial coefficients in the papers [2, 17].

5.4 First recovery

Taking $a_1 = k \in \mathbb{N}$ and $a_2 = n - k \in \mathbb{N}$ in (3.12) results in

$$(5.21) \quad \binom{(\lambda_1 x_1 + \lambda_2 x_2)n}{(\lambda_1 x_1 + \lambda_2 x_2)k} \leq \binom{x_1 n}{x_1 k}^{\lambda_1} \binom{x_2 n}{x_2 k}^{\lambda_2}$$

and the equality in (5.21) holds if and only if $x_1 = x_2 > 0$, where $x_1, x_2, \lambda_1, \lambda_2 > 0$ and $\lambda_1 + \lambda_2 = 1$. This recovers [2, Corollary 1]. When further letting $\lambda_1 = \lambda_2 = \frac{1}{2}$ and setting $x_1 = j-1$ and $x_2 = j+1$ in (5.21), we recover a combinatorial inequality in [17, p. 4, Section 4].

5.5 Second recovery

Setting $a_1 = k \in \mathbb{N}$ and $a_2 = n - k \in \mathbb{N}$ in (3.14) gives

$$\binom{nx_1}{kx_1} \binom{nx_2}{kx_2} < \binom{n(x_1 + x_2)}{k(x_1 + x_2)}, \quad x_1, x_2 > 0$$

which is a recovery of [2, Corollary 2].

5.6 Third recovery

Letting $a_1 = k \in \mathbb{N}$ and $a_2 = n - k \in \mathbb{N}$ in (3.16) deduces

$$(5.22) \quad \binom{(a+x)n}{(a+x)k} \binom{cn}{ck} \leq \binom{an}{ak} \binom{(x+c)n}{(x+c)k}$$

for $0 < a \leq c$ and $x > 0$ and the equality is valid if and only if $a = c > 0$. The inequality (5.22) is a recovery of [2, Corollary 3].

5.7 Fourth recovery

For $a_1, a_2, x > 0$ and $0 < a < c$, the inequality (3.16) means that the function

$$F_{a,c;a_1,a_2}(x) = \binom{(a+x)(a_1+a_2)}{(a+x)a_1} / \binom{(c+x)(a_1+a_2)}{(c+x)a_1}$$

is decreasing in $x > 0$ and

$$\lim_{x \rightarrow \infty} F_{a,c;a_1,a_2}(x) = \left(\frac{a_1}{a_1+a_2} \right)^{a_1} \left(\frac{a_2}{a_1+a_2} \right)^{a_2}.$$

This generalizes [17, Lemma 2.2] which reads that the sequence $T_{n,k}(j) = \frac{\binom{(j-1)n}{(j-1)k}}{\binom{n}{jk}}$ for integers $0 \leq k \leq n$ is decreasing in $j \geq 1$ and

$$\lim_{j \rightarrow \infty} T_{n,k}(j) = \left(\frac{k}{n} \right)^k \left(\frac{n-k}{n} \right)^{n-k}.$$

Remark 5.2. Because the restrictions $a_k < 1$ and $\sum_{k=1}^n a_k = 1$ appeared in [20, Theorem 2.1] are removed off in our Theorem 2.2, the conditions in our Theorem 2.2 are more relaxed than corresponding ones in [20, Theorem 2.1]. Because logarithmically complete monotonicity is stronger than complete monotonicity, just like that logarithmic convexity is stronger than convexity, our main conclusion in Theorem 2.2 is stronger than corresponding one in [20, Theorem 2.1].

Remark 5.3. This paper is a revised version of the preprint [33] whose first version was announced almost at the same time as the preprint [21] which has been formally published as [20]. This paper is a companion of the papers [22, 23, 28, 31, 32].

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REFERENCES

1. M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 10th printing, Dover Publications, New York and Washington, 1972.
2. H. Alzer, *Complete monotonicity of a function related to the binomial probability*, J. Math. Anal. Appl. **459** (2018), no. 1, 10–15; available online at <https://doi.org/10.1016/j.jmaa.2017.10.077>.
3. H. Alzer, *Sharp inequalities for the beta function*, Indag. Math. **12** (2001), no. 1, 15–21; available online at [https://doi.org/10.1016/S0019-3577\(01\)80002-1](https://doi.org/10.1016/S0019-3577(01)80002-1).
4. R. D. Atanassov and U. V. Tsoukrovski, *Some properties of a class of logarithmically completely monotonic functions*, C. R. Acad. Bulgare Sci. **41** (1988), no. 2, 21–23.
5. C. Berg, *Integral representation of some functions related to the gamma function*, Mediterr. J. Math. **1** (2004), no. 4, 433–439; available online at <https://doi.org/10.1007/s00009-004-0022-6>.
6. P. Cerone, *Special functions: approximations and bounds*, Appl. Anal. Discrete Math. **1** (2007), no. 1, 72–91; available online at <https://doi.org/10.2298/AADM0701072C>.
7. P. Cerone, *Special Functions Approximations and Bounds via Integral Representation*. In: *Advances in Inequalities for Special Functions*. Pietro Cerone and Sever S. Dragomir eds. *Advances in Mathematical Inequalities*. Nova Science Publishers, New York, USA, pp. 1–35, 2008.
8. S. S. Dragomir, R. P. Agarwal, and N. S. Barnett, *Inequalities for beta and gamma functions via some classical and new integral inequalities*, J. Inequal. Appl. **5** (2000), no. 2, 103–165; available online at <https://doi.org/10.1155/S1025583400000084>.
9. L. Grenié and G. Molteni, *Inequalities for the beta function*, Math. Inequal. Appl. **18** (2015), no. 4, 1427–1442; available online at <https://doi.org/10.7153/mia-18-111>.
10. B.-N. Guo and F. Qi, *A completely monotonic function involving the tri-gamma function and with degree one*, Appl. Math. Comput. **218** (2012), no. 19, 9890–9897; available online at <https://doi.org/10.1016/j.amc.2012.03.075>.
11. B.-N. Guo and F. Qi, *A property of logarithmically absolutely monotonic functions and the logarithmically complete monotonicity of a power-exponential function*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **72** (2010), no. 2, 21–30.
12. B.-N. Guo and F. Qi, *On the degree of the weighted geometric mean as a complete Bernstein function*, Afr. Mat. **26** (2015), no. 7, 1253–1262; available online at <https://doi.org/10.1007/s13370-014-0279-2>.
13. B.-N. Guo and F. Qi, *Two new proofs of the complete monotonicity of a function involving the psi function*, Bull. Korean Math. Soc. **47** (2010), no. 1, 103–111; available online at <https://doi.org/10.4134/bkms.2010.47.1.103>.

14. B.-N. Guo, F. Qi, J.-L. Zhao, and Q.-M. Luo, *Sharp inequalities for polygamma functions*, *Math. Slovaca* **65** (2015), no. 1, 103–120; available online at <https://doi.org/10.1515/ms-2015-0010>.
15. N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions*, Vol. 2. Second edition. Wiley Series in Probability and Mathematical Statistics: Applied Probability and Statistics. A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1995.
16. A. Leblanc and B. C. Johnson, *A Family of Inequalities Related to Binomial Probabilities*, Tech. Report, Department of Statistics, University of Manitoba, 2006-03.
17. A. Leblanc and B. C. Johnson, *On a uniformly integrable family of polynomials defined on the unit interval*, *J. Inequal. Pure Appl. Math.* **8** (2007), no. 3, Article 67, 5 pp. Available online at <https://www.emis.de/journals/JIPAM/article878.html>.
18. D. S. Mitrinović, J. E. Pečarić, and A. M. Fink, *Classical and New Inequalities in Analysis*, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993; available online at <https://doi.org/10.1007/978-94-017-1043-5>.
19. F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (eds.), *NIST Handbook of Mathematical Functions*, Cambridge University Press, New York, 2010; available online at <http://dlmf.nist.gov/>.
20. F. Ouimet, *Complete monotonicity of multinomial probabilities and its application to Bernstein estimators on the simplex*, *J. Math. Anal. Appl.* **466** (2018), no. 2, 1609–1617; available online at <https://doi.org/10.1016/j.jmaa.2018.06.049>.
21. F. Ouimet, *Complete monotonicity of multinomial probabilities and its application to Bernstein estimators on the simplex*, arXiv preprint (2018), available online at <https://arxiv.org/abs/1804.02108>.
22. F. Qi, *A logarithmically completely monotonic function involving the q-gamma function*, HAL preprint (2018), available online at <https://hal.archives-ouvertes.fr/hal-01803352v1>.
23. F. Qi, *Complete monotonicity for a new ratio of finite many gamma functions*, HAL preprint (2020), available online at <https://hal.archives-ouvertes.fr/hal-02511909v1>.
24. F. Qi, *Limit formulas for ratios between derivatives of the gamma and digamma functions at their singularities*, *Filomat* **27** (2013), no. 4, 601–604; available online at <https://doi.org/10.2298/FIL1304601Q>.
25. F. Qi, *Properties of modified Bessel functions and completely monotonic degrees of differences between exponential and trigamma functions*, *Math. Inequal. Appl.* **18** (2015), no. 2, 493–518; available online at <https://doi.org/10.7153/mia-18-37>.
26. F. Qi and P. Cerone, *Some properties of the Fuss–Catalan numbers*, *Mathematics* **6** (2018), no. 12, Article 277, 12 pages; available online at <https://doi.org/10.3390/math6120277>.
27. F. Qi and C.-P. Chen, *A complete monotonicity property of the gamma function*, *J. Math. Anal. Appl.* **296** (2004), 603–607; available online at <https://doi.org/10.1016/j.jmaa.2004.04.026>.
28. F. Qi and B.-N. Guo, *From inequalities involving exponential functions and sums to logarithmically complete monotonicity of ratios of gamma functions*, *J. Math. Anal. Appl.* **493** (2021), no. 1, Article 124478, 19 pages; available online at <https://doi.org/10.1016/j.jmaa.2020.124478>.

29. F. Qi, B.-N. Guo, and C.-P. Chen, *Some completely monotonic functions involving the gamma and polygamma functions*, J. Aust. Math. Soc. **80** (2006), 81–88; available online at <https://doi.org/10.1017/S1446788700011393>.
30. F. Qi and W.-H. Li, *Integral representations and properties of some functions involving the logarithmic function*, Filomat **30** (2016), no. 7, 1659–1674; available online at <https://doi.org/10.2298/FIL1607659Q>.
31. F. Qi, W.-H. Li, S.-B. Yu, X.-Y. Du, and B.-N. Guo, *A ratio of many gamma functions and its properties with applications*, arXiv preprint (2019), available online at <https://arXiv.org/abs/1911.05883v1>.
32. F. Qi and D. Lim, *Monotonicity properties for a ratio of finite many gamma functions*, Adv. Difference Equ. **2020**, Paper No. 193, 9 pages; available online at <https://doi.org/10.1186/s13662-020-02655-4>.
33. F. Qi, D.-W. Niu, D. Lim, and B.-N. Guo, *Some logarithmically completely monotonic functions and inequalities for multinomial coefficients and multivariate beta functions*, HAL preprint (2018), available online at <https://hal.archives-ouvertes.fr/hal-01769288v1>.
34. F. Qi and S.-H. Wang, *Complete monotonicity, completely monotonic degree, integral representations, and an inequality related to the exponential, trigamma, and modified Bessel functions*, Glob. J. Math. Anal. **2** (2014), no. 3, 91–97; available online at <https://doi.org/10.14419/gjma.v2i3.2919>.
35. F. Qi, X.-J. Zhang, and W.-H. Li, *Lévy-Khintchine representations of the weighted geometric mean and the logarithmic mean*, Mediterr. J. Math. **11** (2014), no. 2, 315–327; available online at <https://doi.org/10.1007/s00009-013-0311-z>.
36. S.-L. Qiu and M. Vuorinen, *Some properties of the gamma and psi functions, with applications*, Math. Comp. **74** (2005), no. 250, 723–742; available online at <https://doi.org/10.1090/S0025-5718-04-01675-8>.
37. R. L. Schilling, R. Song, and Z. Vondraček, *Bernstein Functions*, 2nd ed., de Gruyter Studies in Mathematics **37**, Walter de Gruyter, Berlin, Germany, 2012; available online at <https://doi.org/10.1515/9783110269338>.
38. D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1946.
39. F. Yalcin and Y. Simsek, *A new class of symmetric beta type distributions constructed by means of symmetric Bernstein type basis functions*, Symmetry **12** (2020), no. 5, Article 779, 16 pages; available online at <https://doi.org/10.3390/sym12050779>.

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