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Non abelian 2-cohomology of semi-simple groups over rational varieties

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In memory of Jean Claude DOUAI

Abstract

The aim of this paper is to study non abelian 2-cohomology of semi-simple groups over rational varieties. The essential invariant is the $H^1(k, \text{Pic}(\bar{X}))$, where X is a rational variety over a p -adic field, or a number field, or more generally, a good field k of cohomological dimension ≤ 2 . In particular, if $ob(X) = 0$ ($ob(X)$: elementary obstruction), we show that every class in $H^2(X, \mathcal{L})$, where \mathcal{L} is an X -lien locally represented by a semi-simple simply connected group \tilde{G} , becomes neutral by the base change $\mathcal{T}'_c \rightarrow X' \rightarrow X$, where X' is the Dynkin scheme of the associated \tilde{G} form $\tilde{G}_{\mathcal{L}}$ to \mathcal{L} , and \mathcal{T}'_c a universal X' -torsor compactification. This result confirms the fact that universal torsors compactifications have more simpler arithmetic.

Subject Classification: 14F20

Keywords: Rational varieties, Universal torsors, Elementary obstruction, Brauer group, Gerbs.

1 On lower dimensions exact sequence

Let k be a perfect field and X a proper, smooth, and geometrically integral k -variety which is \bar{k} -rational (we say also rational). From Leray spectral sequence:

$$H^p(k, H^q(\bar{X}, G_m)) \Rightarrow H^{p+q}(X, G_m)$$

we get the following exact sequence in lower dimensions:

$$\begin{array}{ccccccc} 1 & \longrightarrow & \text{Pic}(X) & \longrightarrow & \text{Pic}(\bar{X})^\Gamma & \longrightarrow & \text{Br}(k) \\ & & & & & & \uparrow \\ & & & & & & \text{Br}_1(X) \\ & & & & & & \uparrow \\ & & & & & & H^1(k, \text{Pic}(\bar{X})) \\ & & & & & & \uparrow \\ & & & & & & H^3(k, G_m) \end{array} \quad (1)$$

where $\text{Br}_1(X) = \text{Br}(X)^{tr} = \text{Ker} \{ \text{Br}(X) \rightarrow \text{Br}(X \otimes_k \bar{k}) = \text{Br}(\bar{X}) \}$

Example 1.1 (cf. Prop. 1.5 of [1]): Let C be a smooth conic defined over k without k -rational point. Exact sequence (1) becomes :

$$0 \rightarrow \mathbb{Z}/2 \longrightarrow \text{Br } k \longrightarrow \text{Br}(C) \rightarrow 0 \quad (2)$$

where the generator of $\mathbb{Z}/2$ maps to the quaternions algebra class associated to C .

Indeed, $\text{Pic}(\bar{C}) = \mathbb{Z}$ implies $H^1(k, \text{Pic}(\bar{C})) = 0$.

Moreover, according to Amitsur theorem, the kernel of the restriction map:

$$r_X : \text{Br}(k) \longrightarrow \text{Br}(k(C))$$

is the cyclic group generated by the class of $[C]$ in $\text{Br}(k)$. Hence, exact sequence (2) induces the following one :

$$0 \rightarrow \mathbb{Z}/2 \longrightarrow \text{Br}(k) \longrightarrow \text{Br}(k(C)) \quad (2)'$$

where $k(C)$ is the functions field of C .

Example 1.2 If k is a p -adic field, C as in example 1.1 then $\text{Br}(k) = Q/Z$, and by Lichtenbaum's duality $\text{Br}(C) \simeq Q/Z \simeq \widehat{\text{Pic}(C)}$, so that (2) becomes :

$$0 \rightarrow \mathbb{Z}/2 \longrightarrow Q/Z \xrightarrow{\times 2} Q/Z \rightarrow 0$$

(every class in $H^2(C, \mathcal{L})$, where \mathcal{L} is a C -lien locally¹ represented by a semi-simple simply connected C -group is neutral, cf proof of theorem 2.1 later).

Theorem 1.3 Let k be a local field or a number field or more generally a field of cohomological dimension ≤ 2 , X a proper, smooth rational and geometrically integral k -variety. Then $H^1(k, \text{Pic}(\bar{X}))$ is finite² and can be inserted in the following exact sequence:

$$\text{Br}(k) \longrightarrow \text{Br}_1(X) \longrightarrow H^1(k, \text{Pic}(\bar{X})) \longrightarrow 0 \quad (3)$$

where $\text{Br}_1(X) = \text{Ker}\{\text{Br}(X) \rightarrow \text{Br}(\bar{X})\}$.

Remark 1 Under **Theor.1.3** conditions, if the elementary obstruction $ob(X)$ of X defined in [2] (Def. 2.2.1 page 413) vanishes, i.e the exact sequence :

$$1 \rightarrow \bar{k}^* \xrightarrow{\quad \sigma \quad} \bar{k}(X)^* \longrightarrow \bar{k}(X)^*/\bar{k}^* \rightarrow 1$$

¹ for étale topology ² cf [2] Corollary 2.A.2, p.461.

splits, i.e there exists a retraction $\sigma : \bar{k}(X)^* \longrightarrow \bar{k}^*$, then (3) can be inserted in the following diagram:

$$\begin{array}{ccccccc}
 & & 0 & = & H^1(k, \text{Div}(\bar{X})) & = & 0 & (D_1) \\
 & & \downarrow & & & & & \\
 \text{Br}(k) & \longrightarrow & \text{Br}_1(X) & \longrightarrow & H^1(k, \text{Pic}(\bar{X})) & \longrightarrow & 0 \\
 \parallel & & \downarrow & \longleftarrow^{t_\sigma} & \downarrow & & \\
 H^2(k, \bar{k}^*) & \longrightarrow & H^2(k, \bar{k}(X)^*) & \longrightarrow & H^2(k, \bar{k}(X)^*/\bar{k}^*) & \longrightarrow & 0 \\
 & & \downarrow & \longleftarrow^{t_\sigma} & \downarrow & & \\
 & & H^2(k, \text{Div}(\bar{X})) & = & H^2(k, \text{Div}(\bar{X})) & & \\
 & & \downarrow & & \downarrow & & \\
 & & H^3(k, G_m) & \longrightarrow & H^2(k, \text{Pic}(\bar{X})) & &
 \end{array}$$

where t_σ is induced by section σ .

Therefore $ob(X) = 0$ implies, additionally, the existence of a retraction t_σ in the exact sequence (3):

$$\text{Br}(k) \longrightarrow \text{Br}_1(X) \xrightarrow{\longleftarrow^{t_\sigma}} H^1(k, \text{Pic}(\bar{X})) \longrightarrow 0$$

and $\text{Br}_1(X) = \text{Br}(X) \oplus H^1(k, \text{Pic}(\bar{X}))$.

The group $H^1(k, \text{Pic}(\bar{X}))$ is then closely related to the Brauer group of X :

- If X is k -rational, $\text{Pic}(\bar{X})$ is a stable permutation \bar{k}/k -module, hence $H^1(k, \text{Pic}(\bar{X})) = 0$.
- If $H^1(k, \text{Pic}(\bar{X})) \neq 0$, we obtain on an obstruction to k -rationality.

2 Reduction of the cohomology of X

In this §, X is as in §1: a proper, smooth, geometrically integral and rational k -variety.

We consider cohomology with values in a lien \mathcal{L} locally ¹ represented by a semi-simple simply connected X -group \tilde{G} , which we reduce to the cohomology of X -tori and the Brauer group of an etale covering of X .

The lien \mathcal{L} is an element of $Z^1(X, \text{Autext } \tilde{G})$. We know by [5], Chap. V, §3, p. 74, that the exact sequence :

$$1 \longrightarrow \tilde{G}_{ad} \longrightarrow \text{Aut } \tilde{G} \longrightarrow \text{Autext } \tilde{G} \longrightarrow 1$$

splits and therefore \mathcal{L} defines a class $[\tilde{G}_{\mathcal{L}}]$ in $H^1(X, \text{Aut } \tilde{G})$: \mathcal{L} is represented by par $\tilde{G}_{\mathcal{L}}$.

$\tilde{G}_{\mathcal{L}}$ is semi-simple simply connected quasi-split: it admits a Killig pairing (\tilde{B}, \tilde{T}) where $\tilde{T} = \prod_{X'/X} G_{m_{X'}}$ is an induced torus of the finite etale covering X' of X , where $X' = X \otimes_k k'$, $k' = \text{Dyn}(\tilde{G}_{\mathcal{L}/k})$ refers to the Dynkin k -scheme of $\tilde{G}_{\mathcal{L}}$ (cf. Prop. 3.13-exposé XXIV of [9]). We have then the diagram:

$$\begin{array}{ccc} H^2(X, Z(\tilde{G}_{\mathcal{L}})) & \longrightarrow & H^2(X, \tilde{T}) = H^2(X', G_{m_{X'}}) = \text{Br}(X') & (D_2) \\ & \searrow \circ & \downarrow \circ & \\ & & H^2(X, \mathcal{L}) = H^2(X, \tilde{G}_{\mathcal{L}}) & \end{array}$$

where $\longrightarrow \circ$ is the Giraud's relation [8].

$H^2(X, \tilde{G}_{\mathcal{L}})$ is a principal homogeneous space under $H^2(X, Z(\tilde{G}_{\mathcal{L}}))$: let $q \in H^2(X, \tilde{G}_{\mathcal{L}})$, q can be written as $q = \alpha \cdot \varepsilon$ where $\varepsilon = [\text{Tors } \tilde{G}_{\mathcal{L}}]$ and $\alpha \in H^2(X, Z(\tilde{G}_{\mathcal{L}}))$. The class q is then related to the image of α in $H^2(X, \tilde{T}) = \text{Br}(X')$.

Theorem 2.1 *Let k be a p -adic field, or a number field, or a field of characteristic 0, of cohomological dimension ≤ 2 , such that for every finite simple central k -extension algebra, the index equals the exponent. If X is k -rational variety, \tilde{G} a semi-simple simply connected X -group, \mathcal{L} an X -lien locally 1 represented by \tilde{G} , then every class in $H^2(X, \mathcal{L})$ is neutral.*

Indeed, since X is k -rational, X' is obviously also k' -rational, where k' refers to the finite extension over which X' is defined. Hence $H^1(k', \text{Pic}(\tilde{X}')) = 0$ and $\text{Br}(k')$ maps surjectively on $\text{Br}(X')$. Diagram (D_2) leads to the following one :

$$\begin{array}{ccccc} H^2(X, Z(\tilde{G}_{\mathcal{L}})) & \longrightarrow & H^2(X, \tilde{T}) = \text{Br}(X') & \longleftarrow & \text{Br}(k') = H^2(k', G_m) = H^2\left(k, \prod_{k'/k} G_{m_{k'}}\right) \\ & \downarrow \circ & & \nearrow \circ & \downarrow \circ \\ & H^2(X, \mathcal{L}) & \longleftarrow & & H^2(k, \tilde{G}_{\mathcal{L}}) \end{array} \quad (D_3)$$

Every class in $H^2(X, \mathcal{L})$ can be pulled back to a class in $H^2(k, \tilde{G}_{\mathcal{L}})$, which is neutral by [7] or [5] results, and [4] (Prop. 5.3 and following Remark).

Applications

1. If X is a proper smooth surface on a number field, that contains a dense open subset which is a k -torsor under a torus, then $H^1(k, \text{Pic}(\bar{X})) = 0$ (cf. Prop. 6.3.2, p. 131 of [10]):

$$\begin{array}{ccccccc} \text{Br}(k) & \longrightarrow & \text{Br}(X)^{tr} & \longrightarrow & H^1(k, \text{Pic}(\bar{X})) & \longrightarrow & 0 \\ & & \wr & & \parallel & & \\ & & \text{Br}(X) & & 0 & & \end{array}$$

Isomorphism $\text{Br}(X)^{tr} \simeq \text{Br}(X)$ is due to $\text{Br}(\bar{X}) = 0$, since \bar{X} is \bar{k} -rational. X is also k -rational once it has a rational point. (here X has a generic $k(X)$ -point).

2. Let k be a p -adic field or a number field, $X \subset \mathbb{P}_k^n$ a smooth quadric. For $n \geq 4$, $\text{Br}(k) \simeq \text{Br}(X)$ (prop.1.6 of [1]). Hence, same conclusion as in **Theor 2.1**.

Example 2.2 Denote by X the cubic surface $T_0^3 + T_1^3 + T_2^3 + aT_3^3 = 0$ defined over a number field k . We know that $H^1(k, \text{Pic}(\bar{X})) \simeq (Z/3Z)^2$.

Theorem 2.3 Let k be a number field, X/k a cubic surface as above, \tilde{G} a split semi-simple simply connected X -group and $\mathcal{L} = \text{lien}(\tilde{G})$. If $|Z(\tilde{G})|$ is prime to 3, then every class in $H^2(X, \mathcal{L})$ can be pulled back to a class in $H^2(k, \mathcal{L})$ and hence, neutral by par [5].

Same result if $k = \mathbb{Q}$ and if X is a Tsfasman \mathbb{Q} -surface and $|Z(\tilde{G})|$ odd; for such surface we know indeed, that $H^1(\mathbb{Q}, \text{Pic}(\bar{X})) = Z/2$.

3 Main result

Assume that X is a proper, smooth, geometrically integral k -variety which is \bar{k} -rational and for which the elementary obstruction [2] vanishes. We know that vanishing elementary obstruction is equivalent to the existence of a universal X -torsor. (cf. prop. 2 of [2']). X admits then a universal X -torsor \mathcal{T} . Let \mathcal{T}_c be a smooth k -compactification of \mathcal{T} . We know that \mathcal{T}_c is a rational k -variety for which $\text{Pic}(\bar{\mathcal{T}}_c)$ is a stable permutation \bar{k}/k -module, hence $H^1(k, \text{Pic}(\bar{\mathcal{T}}_c)) = 0$ (cf. Theor.2.1.2 p.411 of [2], or Theor.1 of [3]). We have then the following

diagram:

$$\begin{array}{ccccccc}
 \text{Br}(k) & \xrightarrow{\quad} & \text{Br}(X) & \longrightarrow & H^1(k, \text{Pic}(\bar{X})) & \longrightarrow & H^3(k, G_m) \\
 \parallel & & \downarrow & & \downarrow & & \parallel \\
 \text{Br}(k) & \longrightarrow & \text{Br}(\mathcal{T}_c) & \longrightarrow & H^1(k, \text{Pic}(\bar{\mathcal{T}}_c)) & \longrightarrow & H^3(k, G_m) \\
 & & & & \parallel & & \\
 & & & & 0 & &
 \end{array}$$

Retraction $\text{Br}(X) \rightarrow \text{Br}(k)$ is due to $\text{ob}(X)=0$ condition (cf. large diagram (D_1) of §1.

Replace in previous diagram X by $X' = X \otimes_k k'$, where $k' = \text{Dyn}(\tilde{G}_{\mathcal{L}/k})$ is the Dynkin k -scheme of $\tilde{G}_{\mathcal{L}}$, we obtain the new diagram :

$$\begin{array}{ccc}
 \text{Br}(k') & \twoheadrightarrow & \text{Br}(\mathcal{T}'_c) \\
 \uparrow & & \uparrow \\
 \text{Br}(k) & \twoheadrightarrow & \text{Br}(\mathcal{T}_c)
 \end{array}$$

where \mathcal{T}' (resp. \mathcal{T}'_c) is a universal X' -torsor, restriction to X' of universal X -torsor \mathcal{T} (resp. \mathcal{T}_c).

Next diagram, and especially surjectivity of $\text{Br}(k') \twoheadrightarrow \text{Br}(\mathcal{T}'_c)$:

$$\begin{array}{ccccc}
 & & \text{Br}(k') & & \\
 & & \swarrow & \downarrow & \searrow \\
 H^2(X, Z(\tilde{G}_{\mathcal{L}})) & \rightarrow & H^2(X, \tilde{T}) = \text{Br}(X') & \rightarrow & \text{Br}(\mathcal{T}'_c) \\
 \swarrow & & \downarrow & & \downarrow \\
 H^1(k', \text{Pic}(\bar{X}')) & \rightarrow & H^1(k', \text{Pic}(\bar{\mathcal{T}}'_c)) & \rightarrow & H^1(k', \text{Pic}(\bar{\mathcal{T}}'_c)) = 0 \\
 \downarrow & & \downarrow \circlearrowleft & & \downarrow \\
 & & H^2(k', \tilde{G}_{\mathcal{L}}) & & \\
 \downarrow & & \swarrow & \downarrow & \downarrow \\
 H^2(X, \tilde{G}_{\mathcal{L}}) & \rightarrow & H^2(\mathcal{T}'_c, \tilde{G}_{\mathcal{L}}) & & \\
 \downarrow & & \downarrow & & \\
 H^2(X, \tilde{G}_{\mathcal{L}}) & \rightarrow & H^2(\mathcal{T}'_c, \tilde{G}_{\mathcal{L}}) & &
 \end{array} \tag{D_4}$$

show then that every class of $\text{Im}\left(H^2(X, \tilde{G}_{\mathcal{L}}) \longrightarrow H^2(\mathcal{T}'_c, \tilde{G}_{\mathcal{L}})\right)$ comes from $\text{Im}\left(H^2(k', \tilde{G}_{\mathcal{L}}) \longrightarrow H^2(\mathcal{T}'_c, \tilde{G}_{\mathcal{L}})\right)$. Hence theorem :

Theorem 3.1 *Let k be a perfect field, X a proper, smooth and geometrically integral k -variety which is \bar{k} -rational and for which elementary obstruction $ob(X) = 0$.*

Then to every X -lien \mathcal{L} locally¹ represented by a semi-simple simply connected X -group \tilde{G} , there is an associated finite etale covering $X' = X \otimes_k k'$ of X so that every class in :

$$\begin{array}{ccccc} \text{Im} \left(H^2(X, \mathcal{L}) \longrightarrow H^2(X', \mathcal{L}_{X'}) \longrightarrow H^2(\mathcal{T}'_c, \mathcal{L}_{\mathcal{T}'_c}) \right) & & & & \\ & \parallel & & \parallel & \\ & H^2(X', \tilde{G}_{\mathcal{L}}) & \longrightarrow & H^2(\mathcal{T}'_c, \tilde{G}_{\mathcal{L}}) & \end{array}$$

is in $\text{Im} \left(H^2(k', \mathcal{L}) \longrightarrow H^2(\mathcal{T}'_c, \mathcal{L}_{\mathcal{T}'_c}) \right)$.

Corollary 3.2 *Let k be a field as in theorem 2.1, X a proper, smooth and geometrically integral k -variety which is \bar{k} -rational and so that $ob(X) = 0$, then to every X -lien \mathcal{L} locally¹ represented by a semi-simple simply connected X -group \tilde{G} is associated a finite etale covering $X' = X \otimes_k k'$ of X so that every class $H^2(X, \mathcal{L})$ is neutralized by the base change from X to \mathcal{T}'_c induced by $\mathcal{T}'_c \rightarrow X' \rightarrow X$, where \mathcal{T}'_c is a smooth compactification of a universal X' -torsor.*

Indeed, if k is a number field or a p -adic field, then every class in $H^2(k, \mathcal{L})$ is neutral. ([5],[6]).

If k is of characteristic 0, of cohomological dimension ≤ 2 and so that for every simple central algebra on a finite extension of k , the index equals the exponent, cf. la prop. 5.3 de [4].

The previous theorem corroborates the fact that universal torsors compactifications have more simpler arithmetic.

Corollary 3.3 *Under theorem 4 conditions, every class in $H^2(X, \mathcal{L})/H^2(k, \mathcal{L})$ is neutralized on $H^2(\mathcal{T}'_c, \mathcal{L}_{\mathcal{T}'_c})/H^2(k', \mathcal{L})$.*

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¹ for etale topology.

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