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# A CLT FOR LINEAR SPECTRAL STATISTICS OF LARGE RANDOM INFORMATION-PLUS-NOISE MATRICES

MARWA BANNA, JAMAL NAJIM, AND JIANFENG YAO

ABSTRACT. Consider a matrix  $Y_n = \frac{\sigma}{\sqrt{n}}X_n + A_n$ , where  $\sigma > 0$  and  $X_n = (x_{ij}^n)$  is a  $N \times n$  random matrix with i.i.d. real or complex standardized entries and  $A_n$  is a  $N \times n$  deterministic matrix with bounded spectral norm. The fluctuations of the linear spectral statistics of the eigenvalues:

$$\text{Trace } f(Y_n Y_n^*) = \sum_{i=1}^N f(\lambda_i), \quad (\lambda_i \text{ eigenvalues of } Y_n Y_n^*),$$

are shown to be gaussian, in the case where  $f$  is a smooth function of class  $C^3$  with bounded support, and in the regime where both dimensions of matrix  $Y_n$  go to infinity at the same pace.

The CLT is expressed in terms of vanishing Lévy-Prohorov distance between the linear statistics' distribution and a centered Gaussian probability distribution, the variance of which depends upon  $N$  and  $n$  and may not converge. The proof combines ideas from Bai and Silverstein [3], Hachem et al. [20] and Najim and Yao [34].

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## 1. INTRODUCTION

**The model.** Consider a  $N \times n$  random matrix  $Y_n = (y_{ij}^n)$  given by:

$$Y_n = \frac{\sigma}{\sqrt{n}}X_n + A_n, \tag{1.1}$$

where  $\sigma > 0$  and  $X_n$  is a  $N \times n$  matrix whose entries  $(x_{ij}^n; i, j, n)$  are real or complex, independent and identically distributed (i.i.d.) with mean 0 and variance 1. Matrix  $A_n$  has the same dimensions and is deterministic. Matrix  $Y_n$  is sometimes coined as "Information-plus-noise" type matrix in the literature.

The purpose of this article is to study the fluctuations of linear spectral statistics of the form:

$$\text{Tr } f(Y_n Y_n^*) = \sum_{i=1}^N f(\lambda_i), \tag{1.2}$$

where  $\text{Tr}(\mathbf{M})$  refers to the trace of  $\mathbf{M}$ , the  $\lambda_i$ 's are the eigenvalues of  $Y_n Y_n^*$ , and  $f$  is a smooth function, under the regime where the dimensions  $n$  and  $N = N(n)$  go to infinity at the same pace:

$$N, n \rightarrow \infty \quad \text{and} \quad 0 < \liminf \frac{N}{n} \leq \limsup \frac{N}{n} < \infty. \tag{1.3}$$

This condition will simply be referred to as  $N, n \rightarrow \infty$  in the sequel.

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Large information-plus-noise matrices, and more generally large non-centered random matrices, have recently attracted a lot of attention. Under mild conditions over the moments of  $X_n$ 's entries and the spectral norm of matrix  $A_n$  the asymptotic behavior of the empirical distribution of  $Y_n Y_n^*$ 's eigenvalues (also called spectral distribution of  $Y_n Y_n^*$ ) defined as:

$$F^{Y_n Y_n^*}(B) = \frac{\#\{i, \lambda_i \in B\}}{N} \quad \text{for } B \text{ a Borel set in } \mathbb{R}, \quad (1.4)$$

has been studied by Girko [18, chapter 7], Dozier and Silverstein [15], Hachem et al. [23], etc. Following these results, various properties of the asymptotic spectrum were studied, see for instance [14, 31, 1, 9].

From an applied point of view, information-plus-noise matrices are versatile models in many contexts, from Rice channels in wireless communication to noisy data and small rank perturbations [33, 17, 21, 22]. From a theoretical standpoint, hermitian non-centered models of the type

$$\left( \frac{\sigma}{\sqrt{n}} X_n - z I_N \right) \left( \frac{\sigma}{\sqrt{n}} X_n - z I_N \right)^*$$

are a key device to understand the spectrum of large  $N \times N$  non-hermitian matrices  $\frac{\sigma}{\sqrt{n}} X_n$  via Girko's hermitization trick.

While fluctuations of functionals of large random covariance matrices have attracted a lot of attention, see for instance [28, 27, 8, 19, 3, 24, 20, 36, 5, 13, 32, 37, 35] and the references therein, there seems to be very few results (in fact one to the authors' knowledge) for large information-plus-noise type matrices. In the specific case of a non-centered matrix with a separable variance profile, i.e.  $\Sigma_n = \frac{1}{\sqrt{n}} D_n^{1/2} X_n \tilde{D}_n^{1/2} + A_n$ , with  $D_n, \tilde{D}_n$  deterministic diagonal matrices, the fluctuations have been described for the specific functional (known as the mutual information in wireless communications)

$$\log \det (I_n + \Sigma_n \Sigma_n^*) = \sum_{i=1}^N \log (1 + \lambda_i(\Sigma_n \Sigma_n^*)) , \quad (1.5)$$

first at a physical level of rigor by Moustakas et al. [33] for complex gaussian entries, then for general entries by Hachem et al. in [20]. This shortage of results is probably related to the fact that the addition of a deterministic component  $A_n$  to a large random matrix  $\frac{\sigma}{\sqrt{n}} X_n$  substantially increases the complexity of the computations needed to establish the CLT. The following proposition, of central use in the proof of the CLT, illustrates this fact.

**Proposition 1.1** (see Eq. (3.20) in [20]). *Let  $\mathbf{x} = (x_1, \dots, x_N)^T$  be a  $N \times 1$  vector where the  $x_i$ 's are centered i.i.d. complex random variables with unit variance and  $\sigma > 0$ . Let  $\mathbf{u}$  be a  $N \times 1$  deterministic vector and  $M = (m_{ij})$  and  $P = (p_{ij})$  be  $N \times N$  deterministic complex matrices. Denote by  $\Upsilon(M)$  the random variable*

$$\Upsilon(M) = (\mathbf{y} + \mathbf{u})^* M (\mathbf{y} + \mathbf{u}) \quad \text{where} \quad \mathbf{y} = \frac{\sigma}{\sqrt{n}} \mathbf{x},$$

and by  $\text{vdiag}(M)$  the  $N \times 1$  vector  $\text{vdiag}(M) = [M_{11}, \dots, M_{NN}]^T$ . Then  $\mathbb{E}\Upsilon(M) = \frac{\sigma^2}{n} \text{Tr}M + \mathbf{u}^* M \mathbf{u}$  and

$$\begin{aligned} & \mathbb{E}[(\Upsilon(M) - \mathbb{E}\Upsilon(M))(\Upsilon(P) - \mathbb{E}\Upsilon(P))] \\ &= \frac{\sigma^4}{n^2} \text{Tr}(MP) + \frac{\sigma^2}{n} (\mathbf{u}^* M P \mathbf{u} + \mathbf{u}^* P M \mathbf{u}) + \frac{\sigma^4 |\mathbb{E}[x_1]^2|^2}{n^2} \text{Tr}(M P^T) \\ & \quad + \frac{\sigma^2 \mathbb{E}[x_1^2]}{n} \mathbf{u}^* P M^T \bar{\mathbf{u}} + \frac{\sigma^2 \mathbb{E}[\bar{x}_1^2]}{n} \mathbf{u}^T M^T P \mathbf{u} \\ & \quad + \frac{\sigma^3 \mathbb{E}[|x_1|^2 x_1]}{n^{3/2}} (\mathbf{u}^* P \text{vdiag}(M) + \mathbf{u}^* M \text{vdiag}(P)) \\ & \quad + \frac{\sigma^3 \mathbb{E}[|x_1|^2 \bar{x}_1]}{n^{3/2}} (\text{vdiag}(P)^T M \mathbf{u} + \text{vdiag}(M)^T P \mathbf{u}) + \frac{\kappa \sigma^4}{n^2} \sum_{i=1}^N m_{ii} p_{ii}, \end{aligned} \quad (1.6)$$

where  $\kappa = \mathbb{E}|x_1|^4 - 2 - |\mathbb{E}x_1^2|^2$ .

As one may notice, the deterministic vector  $\mathbf{u}$  yields 8 extra terms in the formula above.

**Fluctuations and representation of linear spectral statistics.** We now present the main object of interest:

$$L_n(f) = \sum_{i=1}^N f(\lambda_i) - \mathbb{E} \sum_{i=1}^N f(\lambda_i)$$

as  $N, n \rightarrow \infty$ . In the case where  $f$  is a function of class  $C^{k+1}$  with compact support, denote by  $\Phi_k(f) : \mathbb{C}^+ \rightarrow \mathbb{C}$  its so-called almost analytic extension defined as

$$\Phi_k(f)(x + \mathbf{i}y) = \sum_{\ell=0}^k \frac{(\mathbf{i}y)^\ell}{\ell!} f^{(\ell)}(x) \chi(y), \quad (1.7)$$

where  $\chi : \mathbb{R} \rightarrow \mathbb{R}^+$  is a smooth, compactly supported function with value 1 in the neighbourhood of 0 and  $\bar{\partial} = \frac{1}{2}(\partial_x + \mathbf{i}\partial_y)$ . Helffer-Sjöstrand's formula yields that:

$$\sum_{i=1}^N f(\lambda_i) - \sum_{i=1}^N \mathbb{E} f(\lambda_i) = \frac{2}{\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(f)(z) \{ \operatorname{Tr} Q_n(z) - \operatorname{Tr} \mathbb{E} Q_n(z) \} \ell(dz) \quad (1.8)$$

where  $Q_n(z) = (Y_n Y_n^* - z I_N)^{-1}$  stands for the resolvent of  $Y_n Y_n^*$  and  $\ell(dz) = dx dy$  for the Lebesgue measure over  $\mathbb{C}^+$ . It is clear from (1.8) that in order to describe the fluctuations of  $L_n(f)$ , a natural approach is to study the fluctuations of the process  $(\operatorname{Tr}(Q_n(z) - \mathbb{E}Q_n(z)) ; z \in \Gamma)$  where  $\Gamma \subset \mathbb{C}$  is some given compact set. In order to proceed, we define

$$M_n(z) := \operatorname{Tr} Q_n(z) - \mathbb{E} \operatorname{Tr} Q_n(z) \quad (1.9)$$

and handle this term by martingale techniques, a strategy successfully applied in [3, 36, 24, 29, 20, 34, 5].

**Entries with non-null fourth cumulant and a family of gaussian random variables.** It is well known since the paper by Khorunzhiy et al. [30] that if the fourth moment of the entries differs from its gaussian counterpart, then other terms may appear in the variance of the trace of the resolvent, one being proportional to the fourth cumulant  $\kappa$  of the entries:

$$\kappa = \mathbb{E} |x_{11}^n|^4 - |\vartheta|^2 - 2 \quad \text{where} \quad \vartheta = \mathbb{E}(x_{11}^n)^2. \quad (1.10)$$

The same phenomenon will occur here but the convergence of these additional terms may fail to happen under usual assumptions such as the convergence of the spectral distribution  $F^{A_n A_n^*}$  of matrix  $A_n A_n^*$  to a probability measure as  $N, n \rightarrow \infty$ .

As we shall see later, the reason for this lack of convergence lies in the fact that these additional terms not only depend on the spectrum of  $A_n A_n^*$ , but also on the spectrum of  $A_n A_n^T$  and on  $A_n A_n^*$ 's eigenvectors. In order to avoid cumbersome assumptions enforcing the joint convergence of these quantities, we shall express our fluctuation results in the same way as in [35] and prove that the distribution of the linear statistics  $L_n(f)$  becomes close to a family of centered Gaussian distributions, whose variance might not converge. Namely, we shall establish that there exists a Gaussian random variable  $\mathcal{N}(0, \Theta_n(f))$  such that:

$$d_{\mathcal{L}P}(L_n(f), \mathcal{N}(0, \Theta_n(f))) \xrightarrow{N, n \rightarrow \infty} 0, \quad (1.11)$$

where  $d_{\mathcal{L}P}$  denotes the Lévy-Prohorov distance (and in particular metrizes the convergence of laws).

**A simple expression for the variance (for real  $A_n$  and real or circular  $\mathbf{x}_{ij}$ 's).** We first introduce some key quantities whose properties will be recalled and studied in Section 2. The following equations admit a unique solution  $(\delta_n, \tilde{\delta}_n)$  in the class of Stieltjes transforms of nonnegative measures with supports  $\mathbb{S}_n$  and  $\tilde{\mathbb{S}}_n$  in  $\mathbb{R}^+$  (see for instance [23, 20], see also [18, Section 7.11]).

$$\begin{cases} \delta_n(z) &= \frac{z}{n} \operatorname{Tr} \left( -z(1 + \sigma \tilde{\delta}_n(z)) I_N + \frac{A_n A_n^*}{1 + \sigma \tilde{\delta}_n(z)} \right)^{-1} \\ \tilde{\delta}_n(z) &= \frac{z}{n} \operatorname{Tr} \left( -z(1 + \sigma \delta_n(z)) I_n + \frac{A_n^* A_n}{1 + \sigma \delta_n(z)} \right)^{-1} \end{cases}, \quad z \in \mathbb{C}^+. \quad (1.12)$$

Associated to  $\delta_n$  and  $\tilde{\delta}_n$  are the  $N \times N$  and  $n \times n$  matrices:

$$\begin{cases} T_n(z) := \left( -z(1 + \sigma \tilde{\delta}_n(z)) I_N + \frac{A_n A_n^*}{1 + \sigma \tilde{\delta}_n(z)} \right)^{-1} = [t_{ij}(z)] \\ \tilde{T}_n(z) := \left( -z(1 + \sigma \delta_n(z)) I_n + \frac{A_n^* A_n}{1 + \sigma \delta_n(z)} \right)^{-1} = [\tilde{t}_{ij}(z)] \end{cases}, \quad (1.13)$$

and the quantity

$$s_n(z) := z(1 + \sigma \delta_n(z))(1 + \sigma \tilde{\delta}_n(z)). \quad (1.14)$$

With these quantities at hand, the variance  $\Theta_n(f)$  which appears in (1.11) takes a remarkably simple form, to be compared with [3, Eq. (1.17)] and [35, Eq. (4.7)], if matrix  $A_n$  is real and the  $\mathbf{x}_{ij}$ 's are real ( $\vartheta = 1$ ) or circular<sup>1</sup> ( $\vartheta = 0$ ).

$$\Theta_n(f) = \frac{1 + \vartheta}{2\pi^2} \int_{\mathbb{S}_n^2} f'(x) f'(y) \ln \left| \frac{s_n(x) - \overline{s_n(y)}}{s_n(x) - s_n(y)} \right| dx dy$$

<sup>1</sup>By circular, we mean that  $\mathbf{x}_{ij}$  has decorrelated real and imaginary part, each with the same variance 1/2, i.e.  $\mathbb{E}|\mathbf{x}_{ij}|^2 = 1$  and  $\vartheta = \mathbb{E}(\mathbf{x}_{ij})^2 = 0$ .

$$+ \frac{\sigma^4 \kappa}{\pi^2 n^2} \sum_{\substack{i=0:N \\ j=0:n}} \left( \int_{\mathbb{S}_n} f'(x) \operatorname{Im}(x t_{ii}(x) \tilde{t}_{jj}(x)) dx \right)^2. \quad (1.15)$$

The quantities  $s_n(x), t_{ii}(x), \tilde{t}_{jj}(x)$  are the limits of the corresponding quantities  $s_n, t_{ii}, \tilde{t}_{jj}$ , evaluated at  $z \in \mathbb{C}^+$ , as  $z \rightarrow x \in \mathbb{R}$ . In the case where matrix  $A_n$  is not real or  $\vartheta \notin \{0, 1\}$ , then the term proportional to  $\vartheta$  above is substantially more complicated.

While the heart of the computations needed to establish the CLT is a (substantial) variation of those performed in [20], the identification of the variance is an important contribution of this article. In particular one may notice that the quantity  $s_n$  defined in (1.14) is central to express the variance in (1.15) while it does not appear in the formula of the variance of the mutual information (1.5).

**Organization of the paper.** The main results of the paper are introduced in Section 2. Central Limit Theorems are stated in Theorem 1 for the trace of the resolvent and in Theorem 2 for general linear statistics. Simplified expressions for the variance are provided in Theorem 3. Sections 3, 4, 5 and 6 are devoted to the proofs. In Appendix A, we recall many useful estimates and in Appendix B, we provide a reminder of the most used notations all along the paper.

## 2. STATEMENT OF THE CENTRAL LIMIT THEOREM

**2.1. Notations and assumptions.** Throughout the paper,  $\mathbf{i} = \sqrt{-1}$ ,  $\mathbb{R}^+ = \{x \in \mathbb{R} : x \geq 0\}$  and  $\mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ . Denote by  $\xrightarrow{a.s.}$  the almost sure convergence, by  $\xrightarrow{\mathcal{P}}$  (resp.  $\xrightarrow{\mathcal{D}}$ ) the convergence in probability (resp. in distribution). Denote by  $\operatorname{diag}(a_i; 1 \leq i \leq k)$  the  $k \times k$  diagonal matrix whose diagonal entries are the  $a_i$ 's. Element  $(i, j)$  of the matrix  $M$  will be denoted by  $m_{ij}$  or  $[M]_{ij}$ .

For a matrix  $M$ , denote by  $M^T$  its transpose,  $M^*$  its Hermitian adjoint,  $\overline{M}$  its entry-wise conjugate,  $\det(M)$  its determinant and  $\operatorname{vdiag}(M)$  the vector whose entries are its diagonal elements  $(m_{ii})$ . When dealing with vectors and matrices,  $\|\cdot\|$  refers to the Euclidean and the spectral norm respectively.

We shall denote by  $K$  a generic constant that does not depend on  $N, n$  but whose value may change from line to line. Function  $\mathbf{1}_A$  denotes the indicator function of the set  $A$ .

Notations  $u_n = \mathcal{O}(v_n)$  and  $u_n = o(v_n)$  stand for the usual big  $\mathcal{O}$  and little  $o$  notations when  $N, n \rightarrow \infty$ . We might also use  $\mathcal{O}_z$  or  $\mathcal{O}_\varepsilon$  to underline the dependence of the constant in  $\mathcal{O}$  on  $z$  or  $\varepsilon$ . If  $X_n$  and  $Y_n$  are sequences of random variables,  $X_n = o_P(Y_n)$  stands for the fact that there exists a sequence  $Z_n$  such as  $X_n = Z_n Y_n$  and  $Z_n$  converges to zero in probability.

Denote by  $d_{LP}(P, Q)$  the Lévy-Prohorov distance between two probability measures  $P, Q$  defined as:

$$d_{LP}(P, Q) = \inf\{\varepsilon > 0 : P(A) \leq Q(A^\varepsilon) + \varepsilon \text{ for all Borel sets } A \subset \mathbb{R}^d\},$$

where  $A^\varepsilon$  is an  $\varepsilon$ -blow up of  $A$  (see [6, Chapter 1, Section 6] for more details). If  $X$  and  $Y$  are random variables with distributions  $\mathcal{L}(X)$  and  $\mathcal{L}(Y)$ , we simply write (with a slight abuse of notations)  $d_{LP}(X, Y)$  instead of  $d_{LP}(\mathcal{L}(X), \mathcal{L}(Y))$ . It is well-known that the Lévy-Prohorov distance metrizes the convergence in distribution (see for instance [16, Chapter 11]).

The set  $C_c^k(\mathbb{R})$  denotes the class of functions with  $k$  continuous derivatives and compact support.

We now state the main assumptions of the article. Recall the fact that  $N = N(n)$  and the asymptotic regime (1.3) where  $N, n \rightarrow \infty$  and denote by

$$c_n := \frac{N}{n}, \quad c_- := \liminf \frac{N}{n} \quad \text{and} \quad c_+ := \limsup \frac{N}{n}.$$

**Assumption 1.** *The random variables  $(x_{ij}^n; 1 \leq i \leq N(n), 1 \leq j \leq n, n \geq 1)$  are real or complex, independent and identically distributed (i.i.d.). They satisfy*

$$\mathbb{E}x_{ij}^n = 0, \quad \mathbb{E}|x_{ij}^n|^2 = 1 \quad \text{and} \quad \mathbb{E}|x_{ij}^n|^{16} < \infty.$$

**Remark 2.1.** *The 16th moment assumption above could be relaxed to an optimal 4th moment assumption as in [3, 35], with extra work involving the truncation of the random variables  $x_{ij}$ 's. In order to reach the optimal assumption, some of the estimates from [25] should have been improved as well. We do not pursue in this direction here.*

Associated to these moments are the quantities introduced in (1.10). We mention two important special cases: The case where  $\vartheta = 1$  corresponding to real  $x_{ij}$ 's and the case where  $\vartheta = 0$ , corresponding to complex  $x_{ij}$ 's with decorrelated real and imaginary part of equal variance.

**Assumption 2.** *The family of deterministic  $N \times n$  complex matrices  $(A_n)$  is bounded for the spectral norm:*

$$a_{\max} = \sup_{n \geq 1} \|A_n\| < \infty.$$

**2.2. Resolvent, canonical equations and deterministic equivalents.** Denote by  $Q_n(z)$  and  $\tilde{Q}_n(z)$  the resolvents of matrices  $Y_n Y_n^*$  and  $Y_n^* Y_n$ :

$$Q_n(z) = (Y_n Y_n^* - zI_N)^{-1}, \quad \tilde{Q}_n(z) = (Y_n^* Y_n - zI_n)^{-1}. \quad (2.1)$$

Their normalized traces  $\frac{1}{N} \text{Tr} Q_n(z)$  and  $\frac{1}{n} \text{Tr} \tilde{Q}_n(z)$  are respectively the Stieltjes transforms of the empirical distribution of  $Y_n Y_n^*$ 's eigenvalues and of  $Y_n^* Y_n$ 's eigenvalues.

Recall the definition of the Stieltjes transforms  $\delta_n$  and  $\tilde{\delta}_n$  as solutions of the canonical equations (1.12) and those of matrices  $T_n$  and  $\tilde{T}_n$ :

$$\begin{cases} \delta_n(z) &= \frac{\sigma}{n} \text{Tr} \left( -z(1 + \sigma \tilde{\delta}_n(z))I_N + \frac{A_n A_n^*}{1 + \sigma \tilde{\delta}_n(z)} \right)^{-1} = \frac{\sigma}{n} \text{Tr} T_n(z) \\ \tilde{\delta}_n(z) &= \frac{\sigma}{n} \text{Tr} \left( -z(1 + \sigma \delta_n(z))I_n + \frac{A_n^* A_n}{1 + \sigma \delta_n(z)} \right)^{-1} = \frac{\sigma}{n} \text{Tr} \tilde{T}_n(z) \end{cases}, \quad z \in \mathbb{C}^+.$$

The measures associated to  $\delta_n$  and  $\tilde{\delta}_n$  have respective total masses given by

$$\lim_{y \rightarrow \infty} -iy \delta_n(iy) = \frac{N}{n} \sigma \quad \text{and} \quad \lim_{y \rightarrow \infty} -iy \tilde{\delta}_n(iy) = \sigma.$$

Matrix  $T_n(z)$  defined in (1.13) is a deterministic equivalent of the resolvent  $Q_n$  in the sense that for  $z \in \mathbb{C} \setminus \mathbb{R}^+$ :

$$\frac{1}{N} \text{Tr}(Q_n(z) - T_n(z)) \xrightarrow[N, n \rightarrow \infty]{a.s.} 0 \quad \text{and} \quad u_n^* Q_n v_n - u_n^* T_n v_n \xrightarrow[N, n \rightarrow \infty]{a.s.} 0,$$

where  $u_n$  and  $v_n$  are deterministic  $N \times 1$  vectors with uniformly bounded euclidian norms (in  $n$ ), see for instance [15, 25]. A symmetric result holds for  $\tilde{Q}_n$  and  $\tilde{T}_n$ .

We will often drop the subscript  $n$  and when dealing with  $\delta_n$ ,  $\tilde{\delta}_n$ ,  $T_n$  and  $\tilde{T}_n$ , we may emphasize the  $z$ -dependence by writing  $\delta_z$ ,  $\tilde{\delta}_z$ ,  $T_z$  and  $\tilde{T}_z$  instead.

**Remark 2.2.** In the sequel, we will handle  $T_z$ ,  $\bar{T}_z$ ,  $T_z^\top$  and  $T_z^*$ . Beware that these quantities are a priori different. Definition (1.13) yields

$$T_z^\top = \left( -z(1 + \sigma \tilde{\delta}_z)I_N + \frac{\bar{A} \bar{A}^*}{1 + \sigma \tilde{\delta}_z} \right)^{-1}$$

hence the identities  $T_z^* = T_{\bar{z}}$  and  $\bar{T}_z = T_{\bar{z}}^\top$ .

**2.3. Expression of the variance and statement of the main results.** In order to express the variance, we need to introduce a number of auxiliary quantities. Let

$$\nu(z_1, z_2) = \frac{\sigma^2}{n} \frac{\text{Tr} T_{z_1} A A^* T_{z_2}}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})}, \quad \gamma(z_1, z_2) = \frac{\sigma^2}{n} \text{Tr} T_{z_1} T_{z_2}, \quad \tilde{\gamma}(z_1, z_2) = \frac{\sigma^2}{n} \text{Tr} \tilde{T}_{z_1} \tilde{T}_{z_2}. \quad (2.2)$$

The following quantity will be instrumental in the sequel.

$$\Delta_n(z_1, z_2) = (1 - \nu)^2 - z_1 z_2 \gamma \tilde{\gamma}. \quad (2.3)$$

Consider now

$$\begin{aligned} \gamma^\dagger(z_1, z_2) &= \frac{\sigma^2}{n} \text{Tr} T_{z_1} T_{z_2}^\top, & \nu^\dagger(z_1, z_2) &= \frac{\sigma^2}{n} \frac{\text{Tr} T_{z_2}^\top \bar{A} \bar{A}^* T_{z_1}}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})}, \\ \tilde{\gamma}^\dagger(z_1, z_2) &= \frac{\sigma^2}{n} \text{Tr} \tilde{T}_{z_1} \tilde{T}_{z_2}^\top, & \tilde{\nu}^\dagger(z_1, z_2) &= \frac{\sigma^2}{n} \frac{\text{Tr} T_{z_2} A A^\top T_{z_1}^\top}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})}. \end{aligned} \quad (2.4)$$

and the following counterpart to  $\Delta_n$

$$\Delta_n^\vartheta(z_1, z_2) = (1 - \vartheta \nu^\dagger) (1 - \vartheta \tilde{\nu}^\dagger) - |\vartheta|^2 z_1 z_2 \gamma^\dagger \tilde{\gamma}^\dagger. \quad (2.5)$$

**Proposition 2.3** (Properties of  $s_n$ ,  $\Delta_n$  and  $\Delta_n^\vartheta$ ). *The following properties hold:*

- (1) Function  $s_n : \mathbb{C}^+ \rightarrow \mathbb{C}^+$  is analytic and if  $z_1, z_2 \in \mathbb{C}^+$  and  $z_1 \neq z_2$  then  $s_n(z_1) \neq s_n(z_2)$ .
- (2) Function  $\Delta_n : \mathbb{C}^+ \times \mathbb{C}^+ \rightarrow \mathbb{C}$  never vanishes and the following identity holds:

$$\Delta_n(z_1, z_2) = \frac{z_1 - z_2}{s_n(z_1) - s_n(z_2)}.$$

- (3) Function  $\Delta_n^\vartheta : \mathbb{C}^+ \times \mathbb{C}^+ \rightarrow \mathbb{C}$  never vanishes.

Proposition 2.3 follows from Proposition 4.1 below.

**Remark 2.4** (Simplifications). *Simplifications may occur depending on the values of  $A$  and  $\vartheta$ :*

(1) If matrix  $A$  has real entries, then

$$\mathbf{T}_z^\top = \mathbf{T}_z, \quad \gamma = \gamma^\dagger, \quad \tilde{\gamma} = \tilde{\gamma}^\dagger.$$

Moreover

$$\nu^\dagger(z_1, z_2) = \tilde{\nu}^\dagger(z_1, z_2) = \frac{\sigma^2}{n} \frac{\text{Tr } \mathbf{T}_{z_2} \mathbf{A} \mathbf{A}^\top \mathbf{T}_{z_1}}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})}.$$

(2) In the case where  $\vartheta = 1$  (real entries  $(x_{ij})$ ) and  $A$  has real entries then

$$\Delta_n = \Delta_n^\vartheta.$$

(3) In the case where  $\vartheta = 0$  then  $\Delta_n^\vartheta = 1$ .

We are now in position to introduce the covariance function. Denote by  $\Theta_n$  the quantity:

$$\Theta_n(z_1, z_2) := \Theta_{0,n}(z_1, z_2) + \Theta_{1,n}(\vartheta, z_1, z_2) + \Theta_{2,n}(\kappa, z_1, z_2), \quad z_1, z_2 \in \mathbb{C}^+ \quad (2.6)$$

where

$$\Theta_{0,n}(z_1, z_2) := -\frac{\partial}{\partial z_2} \left( \frac{1}{\Delta_n} \frac{\partial \Delta_n}{\partial z_1} \right) = \frac{s'_n(z_1) s'_n(z_2)}{(s_n(z_1) - s_n(z_2))^2} - \frac{1}{(z_1 - z_2)^2}, \quad (2.7)$$

$$\Theta_{1,n}(\vartheta, z_1, z_2) := -\frac{\partial}{\partial z_2} \left( \frac{1}{\Delta_n^\vartheta} \frac{\partial \Delta_n^\vartheta}{\partial z_1} \right), \quad (2.8)$$

$$\Theta_{2,n}(\kappa, z_1, z_2) := \kappa \frac{\partial^2}{\partial z_1 \partial z_2} \left\{ \frac{\sigma^4 z_1 z_2}{n^2} \sum_{i=1}^N t_{ii}(z_1) t_{ii}(z_2) \sum_{j=1}^n \tilde{t}_{jj}(z_1) \tilde{t}_{jj}(z_2) \right\}. \quad (2.9)$$

Consider the following subsets of  $\mathbb{C}$ , with  $A > 0$

$$\begin{aligned} D &= [0, A] + i[0, 1], & D^\pm &= \{z \in D^+\} \cup \{\bar{z} \in D^+\}, \\ D^+ &= [0, A] + i(0, 1], & D_\varepsilon &= [0, A] + i[\varepsilon, 1], \quad (\varepsilon > 0). \end{aligned} \quad (2.10)$$

We first study the Gaussian fluctuations for the trace of the resolvent.

**Theorem 1** (CLT for the trace of the resolvent). *Recall the definition of  $M_n$  in (1.9) and let Assumptions 1 and 2 hold. Then for every  $\varepsilon > 0$ ,*

(1) *There exists  $z_0 \in \mathbb{C}^+$  such that*

$$\sup_{n \geq 1} \mathbb{E} |M_n(z_0)|^2 < \infty \quad \text{and} \quad \sup_{z_1, z_2 \in D_\varepsilon, n \geq 1} \frac{\mathbb{E} |M_n(z_1) - M_n(z_2)|^2}{|z_1 - z_2|^2} < \infty.$$

*In particular, the process  $(M_n(z), z \in D_\varepsilon)$  is tight.*

(2) *There exists a sequence  $(G_n(z), z \in D^\pm)$  of centered Gaussian processes such that for any  $z_1, z_2 \in D^\pm$ :*

$$\text{cov}(G_n(z_1), G_n(z_2)) = \Theta_n(z_1, z_2)$$

and

$$\text{cov}(G_n(z_1), \overline{G_n(z_2)}) = \text{cov}(G_n(z_1), G_n(\bar{z}_2)),$$

where  $\Theta_n$  is defined in (2.6). Moreover,  $(G_n(z), z \in D_\varepsilon)$  is tight.

(3) *For any continuous and bounded functional  $F$  from  $C(D_\varepsilon; \mathbb{C})$  to  $\mathbb{C}$ ,*

$$\mathbb{E} F(M_n) - \mathbb{E} F(G_n) \xrightarrow{N, n \rightarrow \infty} 0.$$

Theorem 1 is an extension of Bai and Silverstein's master lemma [3, Lemma 1.1] to the non-centered case. The proof is postponed to Sections 3 and 4 (computation of the covariance).

Having the CLT for the trace of the resolvent at hand, we can now extend it to non-analytic functions via Helffer-Sjöstrand's formula (1.8).

**Theorem 2** (CLT for general linear statistics). *Let Assumptions 1 and 2 hold. Let  $f_1, \dots, f_k \in C_c^3(\mathbb{R})$  and let  $L_n(\mathbf{f}) = (L_n(f_1), \dots, L_n(f_k))$  with*

$$L_n(f) = \text{Tr } f(\mathbf{Y}\mathbf{Y}^*) - \mathbb{E} \text{Tr } f(\mathbf{Y}\mathbf{Y}^*), \quad f \in \{f_1, \dots, f_k\}.$$

*Then there exists an  $\mathbb{R}^k$ -valued sequence of centered Gaussian vectors*

$$Z_n(\mathbf{f}) = (Z_n(f_1), \dots, Z_n(f_k)) \quad (2.11)$$

*with covariance given by*

$$\begin{aligned} \text{Cov}(Z_n(f), Z_n(g)) &= \frac{2}{\pi^2} \text{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial} \Phi_2(f)(z_1) \overline{\partial \Phi_2(g)(z_2)} \Theta_n(z_1, \bar{z}_2) \ell(dz_1) \ell(dz_2) \\ &\quad + \frac{2}{\pi^2} \text{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial} \Phi_2(f)(z_2) \bar{\partial} \Phi_2(g)(z_1) \Theta_n(z_1, z_2) \ell(dz_1) \ell(dz_2), \end{aligned}$$

for  $f, g \in \{f_1, \dots, f_k\}$ . Moreover, the sequence  $(Z_n(\mathbf{f}), n \geq 1)$  is tight and

$$d_{LP}(L_n(\mathbf{f}), Z_n(\mathbf{f})) \xrightarrow{N, n \rightarrow \infty} 0,$$

or equivalently for every continuous bounded function  $F : \mathbb{R}^k \rightarrow \mathbb{C}$ ,

$$\mathbb{E}F(L_n(\mathbf{f})) - \mathbb{E}F(Z_n(\mathbf{f})) \xrightarrow{N, n \rightarrow \infty} 0.$$

Proof of Theorem 2 is postponed in Section 5.

Due to the decomposition of  $\Theta_n(z_1, z_2)$  in (2.6), the covariance  $\text{Cov}(Z_n(f), Z_n(g))$  can be split into three terms

$$\text{Cov}(Z_n(f), Z_n(g)) := \Theta_{0,n}(f, g) + \Theta_{1,n}(\vartheta, f, g) + \Theta_{2,n}(\kappa, f, g)$$

where (we drop the dependence in  $\vartheta, \kappa$ ), for  $i = 0, 1, 2$ ,

$$\begin{aligned} \Theta_{i,n}(f, g) &= \frac{2}{\pi^2} \text{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial} \Phi_2(f)(z_1) \overline{\partial \Phi_2(g)(z_2)} \Theta_{i,n}(z_1, \bar{z}_2) \ell(dz_1) \ell(dz_2) \\ &\quad + \frac{2}{\pi^2} \text{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial} \Phi_2(f)(z_2) \bar{\partial} \Phi_2(g)(z_1) \Theta_{i,n}(z_1, z_2) \ell(dz_1) \ell(dz_2). \end{aligned}$$

In order to provide simplified formulas, we evaluate various quantities defined on  $\mathbb{C}^+$  along the real axis.

**Proposition 2.5** (cf. Theorem 2.1 in [14]). *Let  $x \in \mathbb{R} \setminus \{0\}$ , then the following limits exist*

$$s_n(x) := \lim_{\varepsilon \downarrow 0} s_n(x + i\varepsilon), \quad t_{ii}(x) := \lim_{\varepsilon \downarrow 0} t_{ii}(x + i\varepsilon), \quad \tilde{t}_{jj}(x) := \lim_{\varepsilon \downarrow 0} \tilde{t}_{jj}(x + i\varepsilon).$$

Recall that  $\mathcal{S}_n$  denotes the support of the measure associated to the Stieltjes transform  $\delta_n(z)$ . Alternatively,  $\mathcal{S}_n$  is the support of the probability distribution  $\mathbb{P}_n$  associated to the Stieltjes transform  $N^{-1} \text{Tr } T_n(z)$ .

We can now express simplified formulas.

**Theorem 3** (Alternative expression for the covariance formula). *Let Assumptions 1 and 2 hold and  $f, g \in C_c^3(\mathbb{R})$ . Then*

$$\Theta_{0,n}(f, g) = \frac{1}{2\pi^2} \int_{\mathcal{S}_n^2} f'(x) g'(y) \ln \left| \frac{s_n(x) - \overline{s_n(y)}}{s_n(x) - s_n(y)} \right| dx dy = \frac{1}{2\pi^2} \int_{\mathcal{S}_n^2} f'(x) g'(y) \ln \left| \frac{\Delta_n(x, y)}{\underline{\Delta}_n(x, y)} \right| dx dy,$$

where  $\Delta_n(x, y) := \lim_{\varepsilon \downarrow 0} \Delta_n(x + i\varepsilon, y + i\varepsilon)$  and  $\underline{\Delta}_n(x, y) := \lim_{\varepsilon \downarrow 0} \Delta_n(x + i\varepsilon, y - i\varepsilon)$ , and

$$\Theta_{2,n}(\kappa, f, g) = \frac{\sigma^4 \kappa}{\pi^2 n^2} \sum_{\substack{i=0:N \\ j=0:n}} \int_{\mathcal{S}_n} f'(x) \text{Im}(x t_{ii}(x) \tilde{t}_{jj}(x)) dx \int_{\mathcal{S}_n} g'(y) \text{Im}(y t_{ii}(y) \tilde{t}_{jj}(y)) dy.$$

Proof of Theorem 3 is postponed to Section 6.

**Remark 2.6** (about the term  $\Theta_{1,n}(\vartheta, f, g)$ ). *We have not succeeded so far to establish the natural formula:*

$$\Theta_{1,n}(\vartheta, f, g) \stackrel{?}{=} \frac{\vartheta}{2\pi^2} \int_{\mathcal{S}_n^2} f'(x) g'(y) \ln \left| \frac{\Delta_n^\vartheta(x, y)}{\underline{\Delta}_n^\vartheta(x, y)} \right| dx dy.$$

*We could only prove the following boundary value representation in Proposition 6.2:*

$$\Theta_{1,n}(\vartheta, f, g) = -\frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int_{\mathbb{R}^2} f(x) g(y) \Theta_{1,n}(\vartheta, x \pm_1 i\varepsilon, y \pm_2 i\varepsilon) dx dy,$$

where  $\pm_1, \pm_2 \in \{+, -\}$  and  $\pm_1 \pm_2$  is the sign resulting from the product  $\pm_1$  by  $\pm_2$ .

**Remark 2.7** (more simplifications). *The following simplifications occur:*

(1) If  $\vartheta = 0$ , then

$$\Theta_{1,n}(\vartheta, f, g) |_{\vartheta=0} = 0.$$

(2) If  $\kappa = 0$  (Gaussian moments of order 1, 2, 4) then

$$\Theta_{2,n}(\kappa, f, g) |_{\kappa=0} = 0.$$

(3) For real entries  $(x_{ij})$  (corresponding to  $\vartheta = 1$ ) and real matrix  $A$ , then

$$\Theta_{1,n}(\vartheta, f, g) |_{\vartheta=1} = \Theta_{0,n}(f, g) \quad \text{and} \quad \text{Cov}(Z_n(f), Z_n(g)) = 2\Theta_{0,n}(f, g) + \Theta_{2,n}(\kappa, f, g).$$



**Remark 2.8** (relaxing the support compactness of test functions). *Let the framework of Remark 2.7-(1) or (3) holds, so that an explicit expression of the variance as provided in Theorem 3 is available. Then combining Theorem 2 and an argument of spectrum confinement (see for instance [1], [11, Theorem 5.2]), one can obtain the following fluctuation result: let  $f \in C^3(\mathbb{R})$  (notice that  $f$  has no longer a bounded support) and let  $\mathbf{h} : \mathbb{R} \rightarrow [0, 1]$  a  $C^\infty(\mathbb{R})$  function with value 1 on  $S_n$ , then*

$$d_{LP} \left( \sum_{i=1}^N f(\lambda_i) - \sum_{i=1}^N \mathbb{E}(f\mathbf{h})(\lambda_i), Z_n(f) \right) \xrightarrow{N, n \rightarrow \infty} 0,$$

where  $Z_n(f)$  is a Gaussian random variable with variance given by  $2\Theta_{0,n}(f, f) + \Theta_{2,n}(\kappa, f, f)$ . A similar extension for a different matrix model is available in [35, Corollary 4.3].

**2.4. Remarks concerning the bias.** We have provided so far fluctuation results for quantities  $\sum_{i=1}^N f(\lambda_i) - \mathbb{E} \sum_{i=1}^N f(\lambda_i)$ . Let  $\mathbb{P}_n$  be the probability distribution associated to the Stieltjes transform  $\frac{1}{N} \text{Tr} T_n(z)$ .

The study of the biases

$$\mathbb{E} \text{Tr} Q_n(z) - \text{Tr} T_n(z) \quad \text{and} \quad \mathbb{E} \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}^+} f(\lambda) \mathbb{P}_n(d\lambda)$$

is an interesting question, computationally challenging, that we only superficially address hereafter, for the simple case of complex standard Gaussian entries.

**Proposition 2.9.** *Assume that the random variables  $(\mathbf{x}_{ij}, 1 \leq i \leq N, 1 \leq j \leq n)$  are i.i.d. complex standard Gaussian entries, that is  $\mathbf{x}_{ij} = 2^{-1}(U_{ij} + iV_{ij})$  where  $U_{ij}$  and  $V_{ij}$  are real standard Gaussian entries. Assume moreover that Assumption 2 holds. Then*

$$\mathbb{E} \text{Tr} Q_n(z) - \text{Tr} T_n(z) = \frac{1}{n} \Pi_1(|z|) \Pi_2 \left( \frac{1}{\text{Im}(z)} \right),$$

where  $\Pi_1$  and  $\Pi_2$  are polynomials with fixed degree independent from  $n$ . Denote by  $k_0$  the degree of  $\Pi_2$ . Let  $f \in C_c^{k_0+1}(\mathbb{R})$ , then

$$\mathbb{E} \sum_{i=1}^N f(\lambda_i) - N \int_{\mathbb{R}^+} f(\lambda) \mathbb{P}_n(d\lambda) = \mathcal{O} \left( \frac{1}{n} \right)$$

The first part of the proposition can be proved as in [17, Theorem 2], [12], [31, Lemma 4] and one can track down the minimal value of  $k_0$  by carefully following these proofs. The second part of the proposition is a mere application of Helffer-Sjöstrand formula.

**Remark 2.10** (relaxing the support compactness of test functions - continued). *Combining the previous proposition and Remark 2.8, one obtains the following fluctuation result for a signal plus noise matrix with standard complex Gaussian entries: let  $f \in C^{k_0+1}(\mathbb{R})$  then*

$$d_{LP} \left( \sum_{i=1}^N f(\lambda_i) - N \int_{S_n} f(\lambda) \mathbb{P}_n(d\lambda), Z_n(f) \right) \xrightarrow{N, n \rightarrow \infty} 0,$$

where  $Z_n(f)$  is a centered Gaussian random variable with variance given by  $\Theta_{0,n}(f, f)$ .

### 3. PROOF OF THEOREM 1: THE CLT FOR THE TRACE OF THE RESOLVENT

**3.1. Technical means and outline of the proof.** We first prove that under Assumptions 1 and 2  $M_n(z)$  defined in (1.9) can be written as the sum of martingale increments:

$$M_n(z) = \sum_{j=1}^n P_j(z) + o_P(1). \quad (3.1)$$

This decomposition allows to establish its Gaussian fluctuations via powerful CLTs for martingales such as [6, Th. 35.12] and [35, Lemma 5.6]. For the reader's convenience, we recall the latter.

**Lemma 3.1** ([35], Lemma 5.6). *Suppose that for each  $n$ ,  $(Y_{nj}; 1 \leq j \leq r_n)$  is a  $\mathbb{C}^d$ -valued martingale difference sequence with respect to the increasing  $\sigma$ -field  $\{\mathcal{F}_{n,j}; 1 \leq j \leq r_n\}$  having second moments. Write*

$$Y_{nj}^\top = (Y_{nj}^1, \dots, Y_{nj}^d).$$

Assume moreover that  $(\Theta_n(k, \ell))_n$  and  $(\tilde{\Theta}_n(k, \ell))_n$  are uniformly bounded sequences of complex numbers, for  $1 \leq k, \ell \leq d$ . If

$$\sum_{j=1}^{r_n} \mathbb{E} \left( Y_{nj}^k \bar{Y}_{nj}^\ell \mid \mathcal{F}_{n,j-1} \right) - \Theta_n(k, \ell) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0 \quad \text{and} \quad \sum_{j=1}^{r_n} \mathbb{E} \left( Y_{nj}^k Y_{nj}^\ell \mid \mathcal{F}_{n,j-1} \right) - \tilde{\Theta}_n(k, \ell) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0, \quad (3.2)$$

and for each  $\varepsilon > 0$ , the following Lyapunov condition holds true:

$$\sum_{j=1}^{r_n} \mathbb{E} (\|Y_{nj}\|^2 \mathbf{1}_{\|Y_{nj}\|>\varepsilon}) \xrightarrow{n \rightarrow \infty} 0. \quad (3.3)$$

Then  $d_{\mathcal{L}P} \left( \sum_{j=1}^{r_n} Y_{nj}, Z_n \right) \xrightarrow{n \rightarrow \infty} 0$ , or equivalently for any continuous bounded function  $f : \mathbb{C}^d \rightarrow \mathbb{R}$ :

$$\mathbb{E} f \left( \sum_{j=1}^{r_n} Y_{nj} \right) - \mathbb{E} f(Z_n) \xrightarrow{n \rightarrow \infty} 0,$$

where  $Z_n$  is a  $\mathbb{C}^d$ -valued centered Gaussian random vector with parameters

$$\mathbb{E} Z_n Z_n^* = (\Theta_n(k, \ell))_{k, \ell} \quad \text{and} \quad \mathbb{E} Z_n Z_n^\top = (\tilde{\Theta}_n(k, \ell))_{k, \ell}.$$

Lemma 3.1 can be strengthened with the following lemma:

**Lemma 3.2** (cf. Lemma 5.7 in [35]). *Let  $K$  be a compact set in  $\mathbb{C}$  and let  $X_1, X_2, \dots$  and  $Y_1, Y_2, \dots$  be random elements in  $C(K, \mathbb{C})$ . Assume that for all  $d \geq 1$ ,  $z_1, \dots, z_d \in K$ ,  $f \in C(\mathbb{C}^d, \mathbb{C})$  we have:*

$$\mathbb{E} f(X_n(z_1), \dots, X_n(z_d)) - \mathbb{E} f(Y_n(z_1), \dots, Y_n(z_d)) \xrightarrow{n \rightarrow \infty} 0.$$

Moreover, assume that  $(X_n)$  and  $(Y_n)$  are tight, then for every continuous and bounded functional  $F : C(K, \mathbb{C}) \rightarrow \mathbb{C}$ , we have:

$$\mathbb{E} F(X_n) - \mathbb{E} F(Y_n) \xrightarrow{n \rightarrow \infty} 0.$$

We can now provide an outline of the proof:

### Outline of the proof of Theorem 1

- (1) The martingale decomposition  $M_n(z) = \sum_{j=1}^n P_j(z) + o_P(1)$  is established in Section 3.2.
- (2) Lyapunov's condition (3.3) is established for  $Y_{nj}^\top = (P_j(z_1), \dots, P_j(z_d))$  in Section 3.2.
- (3) The tightness for  $(M_n(z), z \in D_\varepsilon)$  which is part of the assumptions of Lemma 3.2 is established in Section 3.3.
- (4) The convergences (3.2), the more demanding part of the proof, are performed in Section 4.
- (5) Finally, the tightness of the Gaussian process  $(G_n(z), z \in D_\varepsilon)$  is proved in Section 4.5.

**3.2. Sum of martingale increments and Lyapunov's condition.** We introduce now some notations. Denote by  $\mathbf{y}_j, \mathbf{a}_j, \mathbf{x}_j$  and  $\mathbf{r}_j$  the  $j^{\text{th}}$  columns of the matrices  $Y_n, A_n, X_n$  and  $\frac{\sigma}{\sqrt{n}} X_n$  respectively and let

$$Y_j Y_j^* := Y Y^* - \mathbf{y}_j \mathbf{y}_j^* = \sum_{\ell \neq j} \mathbf{y}_\ell \mathbf{y}_\ell^*.$$

Recalling that  $Q_z = (Y_n Y_n^* - z I_N)^{-1}$ , we denote by  $Q_{z,j} := Q_j(z) = (Y_j Y_j^* - z I_N)^{-1}$  and by  $\tilde{Q}_{z,j} := \tilde{Q}_j(z) = (Y_j^* Y_j - z I_n)^{-1}$  and finally note that the diagonal entries  $\tilde{q}_{z,jj} = [\tilde{Q}_z]_{jj}$  of the co-resolvent are given by

$$\tilde{q}_{jj} = \frac{-1}{z(1 + \mathbf{y}_j^* Q_{z,j} \mathbf{y}_j)}.$$

We now introduce several notations that will be used all along this paper. Denote by:

$$\begin{aligned} b_{z,j} &= \frac{-1}{z(1 + \frac{\sigma^2}{n} \text{Tr} \mathbb{E} Q_{z,j} + \mathbf{a}_j^* \mathbb{E} Q_{z,j} \mathbf{a}_j)}, \\ \tilde{b}_{z,j} &= \frac{-1}{z(1 + \frac{\sigma^2}{n} \text{Tr} Q_{z,j} + \mathbf{a}_j^* Q_{z,j} \mathbf{a}_j)}, \\ \tau_{z,j} &= \mathbf{y}_j^* Q_{z,j} \mathbf{y}_j - \frac{\sigma^2}{n} \text{Tr} \mathbb{E} Q_{z,j} - \mathbf{a}_j^* \mathbb{E} Q_{z,j} \mathbf{a}_j, \\ \hat{\tau}_{z,j} &= \mathbf{y}_j^* Q_{z,j} \mathbf{y}_j - \frac{\sigma^2}{n} \text{Tr} Q_{z,j} - \mathbf{a}_j^* Q_{z,j} \mathbf{a}_j, \\ \alpha_{z,j} &= \mathbf{y}_j^* Q_{z,j}^2 \mathbf{y}_j - \frac{\sigma^2}{n} \text{Tr} Q_{z,j}^2 - \mathbf{a}_j^* Q_{z,j}^2 \mathbf{a}_j. \end{aligned}$$

When no confusion occurs, we drop the variable  $z$  and write  $\tilde{q}_{z,jj} := \tilde{q}_{jj}(z)$ ,  $\tilde{t}_{z,jj} := \tilde{t}_{jj}(z)$ ,  $T_z := T(z)$ , etc.

Let  $\mathbb{E}_0 = \mathbb{E}$  denote the expectation and  $\mathbb{E}_j$  the conditional expectation with respect to the  $\sigma$ -field  $\mathcal{F}_{n,j}$  generated by  $\{\mathbf{x}_\ell, 1 \leq \ell \leq j\}$ . By the rank-one perturbation formula

$$Q - Q_j = z \tilde{q}_{jj} Q_j \mathbf{y}_j \mathbf{y}_j^* Q_j, \quad (3.4)$$

and the definition of  $M_n$  in (1.9), we have

$$M_n = \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \text{Tr}(\mathbb{Q} - \mathbb{Q}_j) = \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) z \tilde{q}_{jj} \mathbf{y}_j^* \mathbb{Q}_j^2 \mathbf{y}_j.$$

Note that

$$\tilde{q}_{jj} = \tilde{b}_j + z \tilde{q}_{jj} \tilde{b}_j \hat{\tau}_j = \tilde{b}_j + z \tilde{b}_j^2 \hat{\tau}_j + z^2 \tilde{b}_j^2 \hat{\tau}_j^2 \tilde{q}_{jj}, \quad (3.5)$$

and develop  $M_n(z)$  as follows

$$\begin{aligned} M_n &= z \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) (\tilde{b}_j + z \tilde{b}_j^2 \hat{\tau}_j + z^2 \tilde{b}_j^2 \hat{\tau}_j^2 \tilde{q}_{jj}) \left( \alpha_j + \frac{\sigma^2}{n} \text{Tr} \mathbb{Q}_j^2 + \mathbf{a}_j^* \mathbb{Q}_j^2 \mathbf{a}_j \right), \\ &= \sum_{j=1}^n P_j(z) + \sum_{j=1}^n P'_j(z), \end{aligned}$$

where

$$\begin{aligned} P_j(z) &:= z \mathbb{E}_j \left( \tilde{b}_j \alpha_j + z \tilde{b}_j^2 \hat{\tau}_j \left( \frac{\sigma^2}{n} \text{Tr} \mathbb{Q}_j^2 + \mathbf{a}_j^* \mathbb{Q}_j^2 \mathbf{a}_j \right) \right), \\ P'_j(z) &:= z (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ z \tilde{b}_j^2 \hat{\tau}_j \alpha_j + z^2 \tilde{b}_j^2 \hat{\tau}_j^2 \tilde{q}_{jj} \alpha_j + z^2 \tilde{b}_j^2 \hat{\tau}_j^2 \tilde{q}_{jj} \left( \frac{\sigma^2}{n} \text{Tr} \mathbb{Q}_j^2 + \mathbf{a}_j^* \mathbb{Q}_j^2 \mathbf{a}_j \right) \right]. \end{aligned}$$

Since  $\tilde{q}_{z,jj}$  and  $\tilde{b}_{z,j}$  are Stieltjes transforms of probability measures, we have

$$\max \left( |\tilde{q}_{z,jj}|, |\tilde{b}_{z,j}| \right) \leq \frac{1}{\text{Im}(z)}. \quad (3.6)$$

We decompose  $P'_j$  into three terms. By orthogonality, Cauchy-Schwarz's inequality and the estimates provided in Lemma A.2, we have

$$\begin{aligned} \mathbb{E} \left| \sum_{j=1}^n z (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ z \tilde{b}_j^2 \hat{\tau}_j \alpha_j \right] \right|^2 &= \sum_{j=1}^n \mathbb{E} \left| z (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ z \tilde{b}_j^2 \hat{\tau}_j \alpha_j \right] \right|^2, \\ &\leq \frac{4|z|^2}{\text{Im}(z)^4} \sum_{j=1}^n \left( \mathbb{E} |\hat{\tau}_j|^4 \mathbb{E} |\alpha_j|^4 \right)^{1/2} = \mathcal{O}_z \left( \frac{1}{n} \right). \end{aligned}$$

In the same way, we control the other terms and prove that  $\mathbb{E} \left| \sum_{j=1}^n P'_j(z) \right|^2 = \mathcal{O}_z(n^{-1})$ . This implies that for any  $z \in D^+ \cup \overline{D^+}$ ,  $M_n(z)$  verifies (3.1).

We now prove Lyapunov's condition (3.3). First note that  $\mathbb{E} |P_j(z)|^4 = \mathcal{O}_z(n^{-2})$  by Lemma A.2. Thus for any  $z_1, \dots, z_d \in \mathbb{C}^+$ ,

$$\sum_{j=1}^n \mathbb{E} \left[ \left( \sum_{\ell=1}^d |P_j(z_\ell)|^2 \right) \mathbf{1}_{\left\{ \sum_{\ell=1}^d |P_j(z_\ell)|^2 \geq \varepsilon^2 \right\}} \right] \leq \frac{1}{\varepsilon^2} \sum_{j=1}^n \mathbb{E} \left( \sum_{\ell=1}^d |P_j(z_\ell)|^2 \right)^2 \leq \frac{d}{\varepsilon^2} \sum_{j=1}^n \sum_{\ell=1}^d \mathbb{E} |P_j(z_\ell)|^4 \xrightarrow{N, n \rightarrow \infty} 0.$$

Lyapunov's condition is hence verified.

**3.3. Tightness of the process  $(M_n)$ .** The previous section yields the convergence in distribution of any finite collection  $(M_n(z_1), \dots, M_n(z_d))$  sampled at any points  $z_1, \dots, z_d \in D_\varepsilon$ . To extend this pointwise multidimensional convergence to the convergence of the continuous process  $M_n(\cdot) \in C(D_\varepsilon, \mathbb{C})$ , we need to verify the Arzela–Ascoli criteria. Based on [7, Theorem 7.2], it suffices to prove that there exists  $z_0 \in D_\varepsilon$  and a constant  $K$  such that  $M_n(z)$  satisfies:

$$\sup_n \mathbb{E} |M_n(z_0)|^2 < \infty \quad \text{and} \quad \sup_{n, z_1, z_2 \in D_\varepsilon} \frac{\mathbb{E} |M_n(z_1) - M_n(z_2)|^2}{|z_1 - z_2|^2} \leq K.$$

Notice that the first condition follows from (3.1), orthogonality and the fact that  $\mathbb{E} |P_j(z)|^2 = \mathcal{O}(n^{-1})$ . To prove the second condition, we follow the computations in [3, Section 3] and consider the estimates in Lemma A.3. The resolvent identity together with the fact that  $(\mathbb{E}_j - \mathbb{E}_{j-1}) \mathbb{Q}_{z_1, j} \mathbb{Q}_{z_2, j} = 0$  allow us to write:

$$\frac{M_n(z_1) - M_n(z_2)}{z_1 - z_2} = \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \text{Tr} \mathbb{Q}_{z_1} \mathbb{Q}_{z_2} = \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \text{Tr} (\mathbb{Q}_{z_1} \mathbb{Q}_{z_2} - \mathbb{Q}_{z_1, j} \mathbb{Q}_{z_2, j}).$$

Considering (3.4), we have:

$$\begin{aligned} &\text{Tr} \{ \mathbb{Q}_{z_1} \mathbb{Q}_{z_2} - \mathbb{Q}_{z_1, j} \mathbb{Q}_{z_2, j} \} \\ &= \text{Tr} \{ (\mathbb{Q}_{z_1} - \mathbb{Q}_{z_1, j}) (\mathbb{Q}_{z_2} - \mathbb{Q}_{z_2, j}) + (\mathbb{Q}_{z_1} - \mathbb{Q}_{z_1, j}) \mathbb{Q}_{z_2, j} + \mathbb{Q}_{z_1, j} (\mathbb{Q}_{z_2} - \mathbb{Q}_{z_2, j}) \}, \end{aligned}$$

$$\begin{aligned}
 &= z_1 z_2 \tilde{q}_{z_1, jj} \tilde{q}_{z_2, jj} \operatorname{Tr} \left\{ \mathbf{Q}_{z_1, j} \mathbf{Y}_j \mathbf{Y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{Y}_j \mathbf{Y}_j^* \mathbf{Q}_{z_2, j} \right\} \\
 &\quad + z_1 \tilde{q}_{z_1, jj} \operatorname{Tr} \left\{ \mathbf{Q}_{z_1, j} \mathbf{Y}_j \mathbf{Y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \right\} + z_2 \tilde{q}_{z_2, jj} \operatorname{Tr} \left\{ \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{Y}_j \mathbf{Y}_j^* \mathbf{Q}_{z_2, j} \right\}, \\
 &= z_1 z_2 \tilde{q}_{z_1, jj} \tilde{q}_{z_2, jj} (\mathbf{Y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{Y}_j)^2 \\
 &\quad + z_1 \tilde{q}_{z_1, jj} \mathbf{Y}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{Y}_j + z_2 \tilde{q}_{z_2, jj} \mathbf{Y}_j^* \mathbf{Q}_{z_2, j}^2 \mathbf{Q}_{z_1, j} \mathbf{Y}_j.
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 \frac{M_n(z_1) - M_n(z_2)}{z_1 - z_2} &= \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) (z_1 z_2 \tilde{q}_{z_1, jj} \tilde{q}_{z_2, jj} (\mathbf{Y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{Y}_j)^2) \\
 &\quad + \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) (z_1 \tilde{q}_{z_1, jj} \mathbf{Y}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{Y}_j) \\
 &\quad + \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) (z_2 \tilde{q}_{z_2, jj} \mathbf{Y}_j^* \mathbf{Q}_{z_2, j}^2 \mathbf{Q}_{z_1, j} \mathbf{Y}_j) \quad := \quad T_1 + T_2 + T_3. \quad (3.7)
 \end{aligned}$$

We show that the absolute second moment of (3.7) is uniformly bounded over  $D_\varepsilon$ . By (3.5),  $T_2$  writes:

$$\begin{aligned}
 T_2 &= z_1 \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) (\tilde{q}_{z_1, jj} \mathbf{Y}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{Y}_j), \\
 &= z_1 \sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) (b_{z_1, j} \mathbf{Y}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{Y}_j + z_1 \tilde{q}_{z_1, jj} b_{z_1, j} \mathbf{Y}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{Y}_j \tau_{z_1, j}), \\
 &= z_1 \sum_{j=1}^n b_{z_1, j} \mathbb{E}_j \left( \mathbf{Y}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{Y}_j - \frac{\sigma^2}{n} \operatorname{Tr} \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} - \mathbf{a}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{a}_j \right) \\
 &\quad + z_1^2 \sum_{j=1}^n b_{z_1, j} (\mathbb{E}_j - \mathbb{E}_{j-1}) (\tilde{q}_{z_1, jj} \mathbf{Y}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{Y}_j \tau_{z_1, j}) \quad := \quad W_1 + W_2.
 \end{aligned}$$

Using the estimates in Lemma A.3 with  $k_1 = 2, k_2 = 1$ , we obtain

$$\begin{aligned}
 \mathbb{E}|W_1|^2 &= |z_1|^2 \sum_{j=1}^n |b_{z_1, j}|^2 \mathbb{E} \left| \mathbb{E}_j \left[ \mathbf{Y}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{Y}_j - \frac{\sigma^2}{n} \operatorname{Tr} \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} - \mathbf{a}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{a}_j \right] \right|^2, \\
 &= \mathcal{O}_\varepsilon(1), \\
 \mathbb{E}|W_2|^2 &= |z_1|^2 \sum_{j=1}^n |b_{z_1, j}|^2 \mathbb{E} |(\mathbb{E}_j - \mathbb{E}_{j-1}) \tilde{q}_{z_1, jj} \mathbf{Y}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{Y}_j \tau_{z_1, j}|^2, \\
 &\leq \frac{K}{\varepsilon^2} \sum_{j=1}^n (\mathbb{E} |\mathbf{Y}_j^* \mathbf{Q}_{z_1, j}^2 \mathbf{Q}_{z_2, j} \mathbf{Y}_j|^4 \mathbb{E} |\tau_{z_1, j}|^4)^{1/2} = \mathcal{O}_{z, \varepsilon}(1).
 \end{aligned} \quad (3.8)$$

The term  $T_3$  in (3.7) can be handled similarly. We now develop the term  $T_1$  with the help of (3.5):

$$\begin{aligned}
 &\sum_{j=1}^n (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ \tilde{q}_{z_1, jj} \tilde{q}_{z_2, jj} (\mathbf{Y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{Y}_j)^2 \right] \\
 &= \sum_{j=1}^n b_{z_1, j} b_{z_2, j} (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ (\mathbf{Y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{Y}_j)^2 - \left( \frac{\sigma^2}{n} \operatorname{Tr} \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} + \mathbf{a}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{a}_j \right)^2 \right] \\
 &\quad + z_2 \sum_{j=1}^n b_{z_1, j} b_{z_2, j} (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ \tilde{q}_{z_2, jj} (\mathbf{Y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{Y}_j)^2 \tau_{z_2, j} \right] \\
 &\quad + z_1 \sum_{j=1}^n b_{z_1, j} b_{z_2, j} (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ \tilde{q}_{z_1, jj} (\mathbf{Y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{Y}_j)^2 \tau_{z_1, j} \right] \\
 &\quad + z_1 z_2 \sum_{j=1}^n b_{z_1, j} b_{z_2, j} (\mathbb{E}_j - \mathbb{E}_{j-1}) \left[ \tilde{q}_{z_1, jj} \tilde{q}_{z_2, jj} (\mathbf{Y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{Y}_j)^2 \tau_{z_1, j} \tau_{z_2, j} \right], \\
 &:= Y_1 + Y_2 + Y_3 + Y_4.
 \end{aligned}$$

The terms  $Y_2$ ,  $Y_3$  and  $Y_4$  can be handled as  $W_2$  and are of order 1. As for  $Y_1$ , we use the factorization  $a^2 - b^2 = (a + b)(a - b)$  and again the estimates in Lemma A.3 to prove that

$$\begin{aligned} \mathbb{E}|Y_1|^2 &\leq K \sum_{j=1}^n \mathbb{E} \left| \left( \mathbf{y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{y}_j \right)^2 - \left( \frac{\sigma^2}{n} \text{Tr} \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} + \mathbf{a}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{a}_j \right) \right|^2, \\ &\leq K \sum_{j=1}^n \left( \mathbb{E} \left| \mathbf{y}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{y}_j - \frac{\sigma^2}{n} \text{Tr} \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} - \mathbf{a}_j^* \mathbf{Q}_{z_1, j} \mathbf{Q}_{z_2, j} \mathbf{a}_j \right|^4 \right)^{1/2} = O_{z, \varepsilon}(1). \end{aligned}$$

We conclude that all terms in (3.7) are indeed uniformly bounded over  $D_\varepsilon$ . This completes the proof of the tightness of the process  $M_n(z)$ .

#### 4. PROOF OF THEOREM 1: COMPUTATION OF THE COVARIANCE

**4.1. General properties of  $s_n$ ,  $\Delta_n$  and  $\Delta_n^\vartheta$ .** Recall the definitions of  $s$ ,  $\gamma$ ,  $\tilde{\gamma}$ ,  $\nu$  and  $\Delta$  introduced in (1.14), (2.2)-(2.3). We provide hereafter various important properties from which Proposition 2.3 follows.

**Proposition 4.1.** *Let  $\delta$  and  $\tilde{\delta}$  be the Stieltjes transforms solution of (1.12) and recall that*

$$s_z = z(1 + \sigma\delta_z)(1 + \sigma\tilde{\delta}_z), \quad z \in \mathbb{C}^+.$$

(1) *Function  $s : \mathbb{C}^+ \rightarrow \mathbb{C}^+$  is analytic.*

(2) *Let  $z, z_1, z_2 \in \mathbb{C}^+$ , then*

$$\delta_z = \tilde{\delta}_z + \frac{\sigma(1 - c_n)}{z}.$$

*In particular, if  $s_{z_1} = s_{z_2}$  then  $z_1 = z_2$ . Similarly, if  $\delta_{z_1} = \delta_{z_2}$  then  $z_1 = z_2$ .*

(3) *Let  $z_1, z_2 \in \mathbb{C}^+$  with  $z_1 \neq z_2$ , then the following identities hold*

$$\begin{aligned} \gamma(z_1, z_2) &= \sigma \frac{\delta_{z_1} - \delta_{z_2}}{s_{z_1} - s_{z_2}}, & \tilde{\gamma}(z_1, z_2) &= \sigma \frac{\tilde{\delta}_{z_1} - \tilde{\delta}_{z_2}}{s_{z_1} - s_{z_2}}, \\ 1 - \nu(z_1, z_2) &= \frac{z_1(1 + \sigma\tilde{\delta}_{z_1}) - z_2(1 + \sigma\tilde{\delta}_{z_2})}{s_{z_1} - s_{z_2}}, & \Delta(z_1, z_2) &= \frac{z_1 - z_2}{s_{z_1} - s_{z_2}}. \end{aligned}$$

(4) *Let  $z \in \mathbb{C}^+$  then the following inequalities hold*

$$0 < \nu(z, \bar{z}) < 1, \quad \Delta(z, \bar{z}) > 0, \quad 0 < |z|^2 \gamma(z, \bar{z}) \tilde{\gamma}(z, \bar{z}) < 1. \quad (4.1)$$

(5) *Let  $z \in \mathbb{C}^+$ . If  $c_n \leq 1$  then*

$$\gamma(z, \bar{z}) \leq \frac{1}{|z|} \quad \text{and} \quad |\delta_z| \leq \frac{\sqrt{c_n}}{\sqrt{|z|}}.$$

*If  $c_n = 1$  then  $\gamma(z, \bar{z}) = \tilde{\gamma}(z, \bar{z})$  and  $\tilde{\gamma}(z, \bar{z}) \leq |z|^{-1}$ .*

*Recall the definition of  $\Delta_n^\vartheta$  in (2.5).*

(6) *Let  $z_1, z_2 \in \mathbb{C}^+$  then*

$$|z_1 z_2 \gamma(z_1, z_2) \tilde{\gamma}(z_1, z_2)| < 1, \quad |\Delta(z_1, z_2)| > 0, \quad |\Delta^\vartheta(z_1, z_2)| > 0.$$

*Proof.* Function  $s$  is obviously analytic. The mere definition of  $\delta$  and  $\tilde{\delta}$  yields

$$\frac{\delta_z}{1 + \sigma\delta_z} = \frac{\sigma}{n} \text{Tr}(-s_z + \mathbf{A}\mathbf{A}^*)^{-1} \quad \text{and} \quad \frac{\tilde{\delta}_z}{1 + \sigma\tilde{\delta}_z} = \frac{\sigma}{n} \text{Tr}(-s_z + \mathbf{A}^*\mathbf{A})^{-1} \quad (4.2)$$

from which we deduce that for all  $z \in \mathbb{C}^+$ ,  $s_z$  does not belong to the spectrum of  $\mathbf{A}\mathbf{A}^*$ . Taking the conjugate and applying the resolvent identity, we obtain

$$\frac{\text{Im}(\delta_z)}{|1 + \sigma\delta_z|^2} = \text{Im}(s_z) \frac{\sigma}{n} \text{Tr}(-s_z + \mathbf{A}\mathbf{A}^*)^{-1} (-\bar{s}_z + \mathbf{A}\mathbf{A}^*)^{-1},$$

that is  $s_z \in \mathbb{C}^+$ . Item (1) is proved.

Comparing the spectra of  $\mathbf{A}\mathbf{A}^*$  and  $\mathbf{A}^*\mathbf{A}$  we obtain

$$\frac{\sigma}{n} \text{Tr}(-s_z + \mathbf{A}^*\mathbf{A})^{-1} = \frac{\sigma}{n} \text{Tr}(-s_z + \mathbf{A}\mathbf{A}^*)^{-1} + \frac{\sigma}{s_z} (1 - c_n),$$

hence using (4.2) we get

$$\frac{\delta}{1 + \sigma\delta_z} = \frac{\tilde{\delta}}{1 + \sigma\tilde{\delta}_z} + \frac{\sigma}{s_z} (1 - c_n)$$

from which we deduce the desired identity. Applying (4.2) to  $z = z_1$  and  $z = z_2$  and subtracting yields

$$\frac{\delta_{z_1}}{1 + \sigma\delta_{z_1}} - \frac{\delta_{z_2}}{1 + \sigma\delta_{z_2}} = (s_{z_1} - s_{z_2}) \frac{\sigma}{n} \text{Tr}(-s_{z_1} + \mathbf{A}\mathbf{A}^*)^{-1}(-s_{z_2} + \mathbf{A}\mathbf{A}^*)^{-1}, \quad (4.3)$$

from which we deduce that  $s_{z_1} = s_{z_2}$  iff  $\delta_{z_1} = \delta_{z_2}$ . Assume that  $\delta_{z_1} = \delta_{z_2} = \delta^*$  then

$$s_{z_{1,2}} = z_{1,2}(1 + \sigma\delta_{z_{1,2}})(1 + \sigma\tilde{\delta}_{z_{1,2}}) = z_{1,2}(1 + \sigma\delta^*)^2 - \sigma(1 - c_n)(1 + \sigma\delta^*).$$

Hence

$$s_{z_1} - s_{z_2} = (z_1 - z_2)(1 + \sigma\delta^*)^2.$$

Necessarily,  $z_1 = z_2$ . Item (2) is proved.

The first formula of item (3) immediately follows from (4.3). The second formula can be obtained similarly. We now apply the resolvent identity to  $\delta_{z_1} - \delta_{z_2}$  and obtain, after simplification

$$\begin{aligned} \delta_{z_1} - \delta_{z_2} &= \frac{\sigma}{n} \left( z_1(1 + \sigma\tilde{\delta}_{z_1}) - z_2(1 + \sigma\tilde{\delta}_{z_2}) \right) \text{Tr} \mathbf{T}_{z_1} \mathbf{T}_{z_2} + (\delta_{z_1} - \delta_{z_2}) \frac{\sigma^2}{n} \frac{\text{Tr} \mathbf{T}_{z_1} \mathbf{A} \mathbf{A}^* \mathbf{T}_{z_2}}{(1 + \sigma\delta_{z_1})(1 + \sigma\delta_{z_2})}, \\ &= \frac{z_1(1 + \sigma\tilde{\delta}_{z_1}) - z_2(1 + \sigma\tilde{\delta}_{z_2})}{s_{z_1} - s_{z_2}} (\delta_{z_1} - \delta_{z_2}) + (\delta_{z_1} - \delta_{z_2}) \nu(z_1, z_2). \end{aligned}$$

Dividing by  $\delta_{z_1} - \delta_{z_2}$  which does not vanish if  $z_1 \neq z_2$ , we obtain the third formula.

Using the previously established formulas, we now express  $\Delta(z_1, z_2)$ .

$$\begin{aligned} \Delta(z_1, z_2) &= (1 - \nu(z_1, z_2))^2 - z_1 z_2 \gamma(z_1, z_2) \tilde{\gamma}(z_1, z_2), \\ &= \frac{(z_1(1 + \sigma\tilde{\delta}_{z_1}) - z_2(1 + \sigma\tilde{\delta}_{z_2}))^2 - \sigma^2 z_1 z_2 (\delta_{z_1} - \delta_{z_2})(\tilde{\delta}_{z_1} - \tilde{\delta}_{z_2})}{(s_{z_1} - s_{z_2})^2}. \end{aligned}$$

We focus on the numerator

$$\begin{aligned} &\left( z_1(1 + \sigma\tilde{\delta}_{z_1}) - z_2(1 + \sigma\tilde{\delta}_{z_2}) \right)^2 - \sigma^2 z_1 z_2 (\delta_{z_1} - \delta_{z_2})(\tilde{\delta}_{z_1} - \tilde{\delta}_{z_2}) \\ &= z_1^2(1 + \sigma\tilde{\delta}_{z_1})^2 + z_2^2(1 + \sigma\tilde{\delta}_{z_2})^2 - 2z_1 z_2 (1 + \sigma\tilde{\delta}_{z_1})(1 + \sigma\tilde{\delta}_{z_2}) \\ &\quad - \sigma^2 z_1 z_2 \left( \tilde{\delta}_{z_1} + (1 - c_n) \frac{\sigma}{z_1} - \tilde{\delta}_{z_2} - (1 - c_n) \frac{\sigma}{z_2} \right) (\tilde{\delta}_{z_1} - \tilde{\delta}_{z_2}), \\ &= z_1^2(1 + \sigma\tilde{\delta}_{z_1})^2 + z_2^2(1 + \sigma\tilde{\delta}_{z_2})^2 - 2z_1 z_2 (1 + \sigma\tilde{\delta}_{z_1})(1 + \sigma\tilde{\delta}_{z_2}) \\ &\quad - z_1 z_2 \left( 1 + \sigma\tilde{\delta}_{z_1} - (1 + \sigma\tilde{\delta}_{z_2}) \right)^2 + \sigma^3(1 - c_n)(z_1 - z_2)(\tilde{\delta}_{z_1} - \tilde{\delta}_{z_2}), \\ &= z_1^2(1 + \sigma\tilde{\delta}_{z_1})^2 + z_2^2(1 + \sigma\tilde{\delta}_{z_2})^2 - z_1 z_2 \left( (1 + \sigma\tilde{\delta}_{z_1})^2 + (1 + \sigma\tilde{\delta}_{z_2})^2 \right) \\ &\quad + \sigma^3(1 - c_n)(z_1 - z_2)(\tilde{\delta}_{z_1} - \tilde{\delta}_{z_2}), \\ &= (z_1 - z_2) \left( z_1(1 + \sigma\tilde{\delta}_{z_1})^2 + \sigma^2(1 - c_n)(1 + \sigma\tilde{\delta}_{z_1}) \right) \\ &\quad - (z_1 - z_2) \left( z_2(1 + \sigma\tilde{\delta}_{z_2})^2 + \sigma^2(1 - c_n)(1 + \sigma\tilde{\delta}_{z_2}) \right). \end{aligned}$$

It remains to notice that

$$s_z = z(1 + \sigma\delta_z)(1 + \sigma\tilde{\delta}_z) = z(1 + \tilde{\delta}_z)^2 + \sigma^2(1 - c_n)(1 + \sigma\tilde{\delta}_z)$$

to conclude that the numerator writes  $(z_1 - z_2)(s_{z_1} - s_{z_2})$ . The formula for  $\Delta(z_1, z_2)$  immediately follows, and item (3) is proved.

Let  $z \in \mathbb{C}^+$ , then the mere definition of  $\nu$  yields  $\nu(z, \bar{z}) > 0$ . Recall that since  $\tilde{\delta}$  is the Stieltjes transform of a measure with support in  $\mathbb{R}^+$ , then  $\text{Im}(z\tilde{\delta}_z) \geq 0$ . By the formula established in (3),

$$1 - \nu(z, \bar{z}) = \frac{\text{Im}(z) + \sigma \text{Im}(z\tilde{\delta}_z)}{\text{Im}(s_z)} > 0$$

hence  $\nu(z, \bar{z}) < 1$ . Similarly,

$$\Delta(z, \bar{z}) = \frac{z - \bar{z}}{s_z - \bar{s}_z} = \frac{\text{Im}(z)}{\text{Im}(s_z)} > 0,$$

from which we deduce that

$$0 < |z|^2 \gamma(z, \bar{z}) \tilde{\gamma}(z, \bar{z}) < (1 - \nu(z, \bar{z}))^2 < 1.$$

Proof of item (4) is completed.

Using the relation between  $\delta_z$  and  $\tilde{\delta}_z$ , we obtain

$$\tilde{\gamma}(z, \bar{z}) = \frac{\sigma}{s_z - \bar{s}_z} \left( \delta_z - \bar{\delta}_z - \frac{\sigma(1 - c_n)}{z} + \frac{\sigma(1 - c_n)}{\bar{z}} \right) = \gamma(z, \bar{z}) + \frac{\sigma^2(1 - c_n)}{|z|^2} \text{Im}(z).$$

In particular,  $\gamma(z, \bar{z}) \leq \tilde{\gamma}(z, \bar{z})$  if  $c_n \leq 1$  and  $\gamma(z, \bar{z}) = \tilde{\gamma}(z, \bar{z})$  if  $c_n = 1$ . Plugging this into the last inequality of (4.1), we get  $|z|^2 \gamma^2(z, \bar{z}) \leq 1$  which is the desired inequality. Finally, we use the elementary inequality  $|\text{Tr}(AB)| \leq \sqrt{\text{Tr} AA^*} \sqrt{\text{Tr} BB^*}$  to obtain

$$|\delta_z| = \left| \frac{\sigma}{n} \text{Tr} T_z \right| \leq \sqrt{\gamma(z, \bar{z})} \frac{\sqrt{\text{Tr} I_N}}{\sqrt{n}} \leq \frac{\sqrt{c_n}}{\sqrt{|z|}}.$$

Item (5) is proved.

Using the mere definition of  $\gamma$ , we have

$$|\gamma(z_1, z_2)| = \left| \frac{\sigma^2}{n} \text{Tr} T_{z_1} T_{z_2} \right| \leq \left( \frac{\sigma^2}{n} \text{Tr} T_{z_1} T_{z_1}^* \right)^{1/2} \left( \frac{\sigma^2}{n} \text{Tr} T_{z_2} T_{z_2}^* \right)^{1/2}.$$

Hence

$$|z_1 z_2 \gamma(z_1, z_2)| \leq \sqrt{|z_1|^2 \gamma(z_1, \bar{z}_1)} \sqrt{|z_2|^2 \gamma(z_2, \bar{z}_2)} < 1$$

by (4.1) and the first inequality of item (6) is proved. We now prove that

$$|\Delta(z_1, z_2)| \geq \sqrt{\Delta(z_1, \bar{z}_1)} \sqrt{\Delta(z_2, \bar{z}_2)} \quad (4.4)$$

where the last quantity is positive by item (4). We have

$$|1 - \nu(z_1, z_2)| > 1 - |\nu(z_1, z_2)| > 1 - \sqrt{\nu(z_1, \bar{z}_1)} \sqrt{\nu(z_2, \bar{z}_2)} > 0.$$

Hence

$$|\Delta(z_1, z_2)| > \left( 1 - \sqrt{\nu(z_1, \bar{z}_1)} \sqrt{\nu(z_2, \bar{z}_2)} \right)^2 - \sqrt{|z_1|^2 \gamma(z_1, \bar{z}_1)} \sqrt{|z_2|^2 \gamma(z_2, \bar{z}_2)} \tilde{\gamma}(z_1, \bar{z}_1) \tilde{\gamma}(z_2, \bar{z}_2). \quad (4.5)$$

We now rely on elementary inequalities (for a proof see [25, Proposition 6.1]) to conclude:

**Proposition 4.2.** (1) Let  $a_1, a_2 \geq 0$ , then

$$(1 - \sqrt{a_1 a_2})^2 > (1 - a_1)(1 - a_2).$$

(2) Assume moreover that  $b_i \geq 0$  and  $(1 - a_i)^2 - b_i > 0$  for  $i = 1, 2$ , then

$$(1 - \sqrt{a_1 a_2})^2 - \sqrt{b_1 b_2} > \sqrt{(1 - a_1)^2 - b_1} \sqrt{(1 - a_2)^2 - b_2}.$$

Using the second inequality of the previous proposition in (4.5) yields (4.4).

In order to handle  $\Delta^\vartheta$ , notice that

$$\begin{aligned} |\gamma^\dagger(z_1, z_2)| &\leq \sqrt{\gamma(z_1, \bar{z}_1)} \sqrt{\gamma(z_2, \bar{z}_2)}, & |\nu^\dagger(z_1, z_2)| &\leq \sqrt{\nu(z_1, \bar{z}_1)} \sqrt{\nu(z_2, \bar{z}_2)}, \\ |\tilde{\gamma}^\dagger(z_1, z_2)| &\leq \sqrt{\tilde{\gamma}(z_1, \bar{z}_1)} \sqrt{\tilde{\gamma}(z_2, \bar{z}_2)}, & |\tilde{\nu}^\dagger(z_1, z_2)| &\leq \sqrt{\nu(z_1, \bar{z}_1)} \sqrt{\nu(z_2, \bar{z}_2)}. \end{aligned}$$

Hence

$$\Delta^\vartheta(z_1, z_2) \geq \left( 1 - |\vartheta| \sqrt{\nu(z_1, \bar{z}_1)} \sqrt{\nu(z_2, \bar{z}_2)} \right)^2 - \sqrt{|z_1|^2 \gamma(z_1, \bar{z}_1)} \sqrt{|z_2|^2 \gamma(z_2, \bar{z}_2)} \tilde{\gamma}(z_1, \bar{z}_1) \tilde{\gamma}(z_2, \bar{z}_2). \quad (4.6)$$

Since  $|\vartheta| \leq 1$ , we obtain the same lower bound as in (4.5), from which we can conclude as previously. Item (6) is proved. Proof of Proposition 4.1 is completed.  $\square$

**4.2. Computation of the covariance: some preparation.** In order to complete the proof of Theorem 1, we shall prove that  $M_n$  satisfies (3.2) with  $\Theta_n$  defined in (2.6). Considering the decomposition of  $M_n(z)$  in (3.1), it is sufficient to prove that

$$\sum_{j=1}^n \mathbb{E}_{j-1} P_j(z_1) P_j(z_2) - \Theta_n(z_1, z_2) \xrightarrow[N, n \rightarrow \infty]{\mathcal{P}} 0 \quad \text{and} \quad \sum_{j=1}^n \mathbb{E}_{j-1} P_j(z_1) \overline{P_j(z_2)} - \Theta_n(z_1, \bar{z}_2) \xrightarrow[N, n \rightarrow \infty]{\mathcal{P}} 0.$$

Due to the expression

$$P_j(z) = \mathbb{E}_j \left\{ z \tilde{b}_{z,j} \alpha_{z,j} + z^2 \tilde{b}_{z,j}^2 \hat{\tau}_{z,j} \left( \frac{\sigma^2}{n} \text{Tr} Q_{z,j}^2 + \mathbf{a}_j^* Q_{z,j}^2 \mathbf{a}_j \right) \right\},$$

we have  $\overline{P_j(z)} = P_j(\bar{z})$ . As  $D^+ \cup \overline{D^+}$  is stable under conjugation, it suffices to prove the convergence of  $\sum_{i=1}^n \mathbb{E}_{j-1} P_j(z_1) P_j(z_2)$  for any  $z_1, z_2 \in D^+ \cup \overline{D^+}$ .

We introduce

$$\Gamma_j(z) = z \tilde{b}_{z,j} \left( \mathbf{y}_j^* Q_{z,j} \mathbf{y}_j - \frac{\sigma^2}{n} \text{Tr} Q_{z,j} - \mathbf{a}_j^* Q_{z,j} \mathbf{a}_j \right) = z \tilde{b}_{z,j} \hat{\tau}_{z,j},$$

$$A_n(z_1, z_2) = \sum_{j=1}^n \mathbb{E}_{j-1} \{ \mathbb{E}_j \Gamma_j(z_1) \mathbb{E}_j \Gamma_j(z_2) \}. \quad (4.7)$$

Since

$$\frac{\partial}{\partial z} (\hat{\tau}_{z,j}) = \alpha_{z,j} \quad \text{and} \quad \frac{\partial}{\partial z} (z \tilde{b}_{z,j}) = z^2 \tilde{b}_{z,j}^2 (\mathbf{y}_j^* \mathbf{Q}_{z,j}^2 \mathbf{y}_j - \alpha_{z,j}),$$

we can easily prove that

$$\frac{\partial^2}{\partial z_1 \partial z_2} A_n(z_1, z_2) = \sum_{j=1}^n \mathbb{E}_{j-1} P_j(z_1) P_j(z_2). \quad (4.8)$$

By the same arguments as in Bai and Silverstein [3, page 571] and [4, page 273], it is sufficient to study the convergence in probability to zero of

$$A_n(z_1, z_2) - \Upsilon_n(z_1, z_2) \quad \text{where} \quad \frac{\partial^2}{\partial z_1 \partial z_2} \Upsilon_n(z_1, z_2) = \Theta_n(z_1, z_2)$$

and the uniform boundedness (in  $n$ ) of  $\Upsilon_n$ , the latter being easy to establish by Lemma 4.8 and Lemma 4.9. We now slightly simplify the study of  $A_n(z_1, z_2)$  and prove that:

$$A_n(z_1, z_2) - \sum_{j=1}^n z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \mathbb{E}_{j-1} \{ \mathbb{E}_j \hat{\tau}_{z_1, j} \mathbb{E}_j \hat{\tau}_{z_2, j} \} \xrightarrow[n \rightarrow \infty]{\mathcal{P}} 0. \quad (4.9)$$

Indeed, by Lemma A.4,  $\mathbb{E} |\tilde{b}_j - \tilde{t}_{jj}|^2 \leq 2\mathbb{E} |\tilde{b}_j - \tilde{q}_{jj}|^2 + 2\mathbb{E} |\tilde{q}_{jj} - \tilde{t}_{jj}|^2 = \mathcal{O}(n^{-1})$ . Using Cauchy-Schwarz inequality and Lemma A.2, we obtain

$$\begin{aligned} & \left| \mathbb{E} \left[ \mathbb{E}_{j-1} \left\{ \mathbb{E}_j [\tilde{b}_{z_1, j} \hat{\tau}_{z_1, j}] \mathbb{E}_j [\tilde{b}_{z_2, j} \hat{\tau}_{z_2, j}] \right\} - \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \mathbb{E}_{j-1} \left\{ \mathbb{E}_j \hat{\tau}_{z_1, j} \mathbb{E}_j \hat{\tau}_{z_2, j} \right\} \right] \right| \\ &= \mathbb{E} \left| \mathbb{E}_{j-1} \left\{ \mathbb{E}_j [(\tilde{b}_{z_1, j} - \tilde{t}_{z_1, jj}) \hat{\tau}_{z_1, j}] \mathbb{E}_j [\tilde{b}_{z_2, j} \hat{\tau}_{z_2, j}] \right\} + \tilde{t}_{z_1, jj} \mathbb{E}_{j-1} \left\{ \mathbb{E}_j [\hat{\tau}_{z_1, j}] \mathbb{E}_j [(\tilde{b}_{z_2, j} - \tilde{t}_{z_2, jj}) \hat{\tau}_{z_2, j}] \right\} \right|, \\ &= \mathcal{O}(n^{-3/2}). \end{aligned}$$

Summing over  $j$ , we prove the convergence (4.9). Notice that  $\mathbb{E}_j(\mathbf{Q}_{z_1, j})$  is  $\mathcal{F}_{n, j-1}$  measurable. By applying Proposition 1.1 to  $M = \mathbb{E}_j(\mathbf{Q}_{z_1, j})$  and  $P = \mathbb{E}_j(\mathbf{Q}_{z_2, j})$ , we obtain

$$\sum_{j=1}^n z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \mathbb{E}_{j-1} [\mathbb{E}_j \hat{\tau}_{z_1, j} \mathbb{E}_j \hat{\tau}_{z_2, j}] := \sum_{j=1}^n (\xi_{1j} + \xi_{2j} + \xi'_{2j} + \xi_{3j} + \xi_{4j}),$$

where

$$\xi_{1j} := \frac{\kappa \sigma^4}{n^2} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \sum_{i=1}^N \mathbb{E}_j [Q_{z_1, j}]_{ii} \mathbb{E}_j [Q_{z_2, j}]_{ii}, \quad (4.10)$$

$$\begin{aligned} \xi_{2j} &:= \frac{\sigma^3}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \mathbb{E} (|x_{11}|^2 x_{11}) \\ &\quad \times \left( \frac{\mathbf{a}_j^* (\mathbb{E}_j \mathbf{Q}_{z_1, j}) \text{vdiag}(\mathbb{E}_j \mathbf{Q}_{z_2, j})}{\sqrt{n}} + \frac{\mathbf{a}_j^* (\mathbb{E}_j \mathbf{Q}_{z_2, j}) \text{vdiag}(\mathbb{E}_j \mathbf{Q}_{z_1, j})}{\sqrt{n}} \right), \end{aligned} \quad (4.11)$$

$$\begin{aligned} \xi'_{2j} &:= \frac{\sigma^3}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \mathbb{E} (|x_{11}|^2 \bar{x}_{11}) \\ &\quad \times \left( \frac{\text{vdiag}(\mathbb{E}_j \mathbf{Q}_{z_2, j})^\top (\mathbb{E}_j \mathbf{Q}_{z_1, j}) \mathbf{a}_j}{\sqrt{n}} + \frac{\text{vdiag}(\mathbb{E}_j \mathbf{Q}_{z_1, j})^\top (\mathbb{E}_j \mathbf{Q}_{z_2, j}) \mathbf{a}_j}{\sqrt{n}} \right), \end{aligned} \quad (4.12)$$

$$\begin{aligned} \xi_{3j} &:= \frac{\sigma^2}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \left( \frac{\sigma^2}{n} \text{Tr}(\mathbb{E}_j \mathbf{Q}_{z_1, j} \mathbb{E}_j \mathbf{Q}_{z_2, j}) \right. \\ &\quad \left. + \mathbf{a}_j^* \mathbb{E}_j \mathbf{Q}_{z_1, j} \mathbb{E}_j \mathbf{Q}_{z_2, j} \mathbf{a}_j + \mathbf{a}_j^* \mathbb{E}_j \mathbf{Q}_{z_2, j} \mathbb{E}_j \mathbf{Q}_{z_1, j} \mathbf{a}_j \right), \end{aligned} \quad (4.13)$$

$$\begin{aligned} \xi_{4j} &:= \frac{\sigma^2}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \left( \frac{1}{n} |\vartheta|^2 \sigma^2 \text{Tr}(\mathbb{E}_j \mathbf{Q}_{z_1, j} \mathbb{E}_j \mathbf{Q}_{z_2, j}^\top) \right. \\ &\quad \left. + \vartheta \mathbf{a}_j^* \mathbb{E}_j \mathbf{Q}_{z_2, j} \mathbb{E}_j \mathbf{Q}_{z_1, j}^\top \bar{\mathbf{a}}_j + \bar{\vartheta} \mathbf{a}_j^\top \mathbb{E}_j \mathbf{Q}_{z_1, j} \mathbb{E}_j \mathbf{Q}_{z_2, j} \mathbf{a}_j \right). \end{aligned} \quad (4.14)$$

In the following series of lemmas, we describe the behaviour of each sum. Recall the formula of the covariance in (2.6), then  $\sum_{j=1}^n \xi_{1j}$  is associated to the term  $\Theta_{2, n}$  while  $\sum_{j=1}^n \xi_{3j}$  and  $\sum_{j=1}^n \xi_{4j}$  correspond to  $\Theta_{0, n}$  and  $\Theta_{1, n}$  respectively. The terms  $\sum_{j=1}^n \xi_{2j}$  and  $\sum_{j=1}^n \xi'_{2j}$  have no contribution in the final expression.

In [20], the terms above have been studied in the case where  $z_1 = z_2 = -\rho \in (-\infty, 0)$  and many computations performed there can be established for general  $z_1, z_2 \in \mathbb{C}^+$  by mere book keeping. A technical issue however remains: the invertibility of systems of equations that appear when studying the



terms  $\sum_j \xi_{3j}$  and  $\sum_j \xi_{4j}$ . In this case, the generalization from  $z_1 = z_2 = -\rho \in (-\infty, 0)$  to general  $z_1, z_2 \in \mathbb{C}^+$  is not trivial and is carefully developed hereafter, cf. Lemma 4.8-(ii) and Lemma 4.9-(ii).

**Lemma 4.3.** *Let Assumptions 1 and 2 hold, then*

$$\sum_{j=1}^n \xi_{1j} - \frac{\kappa \sigma^4 z_1 z_2}{n^2} \sum_{i=1}^N t_{z_1, ii} t_{z_2, ii} \sum_{j=1}^n \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \xrightarrow[N, n \rightarrow \infty]{\mathbb{P}} 0.$$

Proof of Lemma 4.3 is similar to the proof of [20, Lemma 4.1] and is omitted.

**Lemma 4.4.** *Let Assumptions 1 and 2 hold, then*

$$\sum_{j=1}^n \xi_{2j} \xrightarrow[N, n \rightarrow \infty]{\mathbb{P}} 0 \quad \text{and} \quad \sum_{j=1}^n \xi'_{2j} \xrightarrow[N, n \rightarrow \infty]{\mathbb{P}} 0.$$

Proof of Lemma 4.4 is similar to the proof of [20, Lemma 4.2] and is also omitted.

**Lemma 4.5.** *Let Assumptions 1 and 2 hold, then*

$$\frac{\partial^2}{\partial z_1 \partial z_2} \sum_{j=1}^n \xi_{3j} - \frac{s'_n(z_1) s'_n(z_2)}{(s_n(z_1) - s_n(z_2))^2} + \frac{1}{(z_1 - z_2)^2} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

where  $s_n(z) = z(1 + \sigma \delta_z)(1 + \sigma \tilde{\delta}_z)$ .

**Lemma 4.6.** *Let Assumptions 1 and 2 hold, then*

$$\frac{\partial^2}{\partial z_1 \partial z_2} \sum_{j=1}^n \xi_{4j} - \Theta_{1, n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0,$$

where  $\Theta_{1, n}$  is defined in (2.6).

**4.3. Proof of Lemma 4.5.** We first recall the definition of  $\xi_{3j}$  and introduce an auxiliary quantity  $\tilde{\xi}_{3j}$ :

$$\begin{aligned} \xi_{3j} &= \frac{\sigma^2}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \left( \frac{\sigma^2}{n} \text{Tr} \{ \mathbb{E}_j \mathbf{Q}_{z_1, j} \mathbb{E}_j \mathbf{Q}_{z_2, j} \} + \mathbf{a}_j^* \mathbb{E}_j \mathbf{Q}_{z_1, j} \mathbb{E}_j \mathbf{Q}_{z_2, j} \mathbf{a}_j + \mathbf{a}_j^* \mathbb{E}_j \mathbf{Q}_{z_2, j} \mathbb{E}_j \mathbf{Q}_{z_1, j} \mathbf{a}_j \right), \\ \tilde{\xi}_{3j} &= \frac{\sigma^2}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \left( \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \{ \mathbb{E}_j \mathbf{Q}_{z_1} \mathbb{E}_j \mathbf{Q}_{z_2} \} + \mathbf{a}_j^* \mathbb{E} \{ \mathbb{E}_j \mathbf{Q}_{z_1, j} \mathbb{E}_j \mathbf{Q}_{z_2, j} \} \mathbf{a}_j + \mathbf{a}_j^* \mathbb{E} \{ \mathbb{E}_j \mathbf{Q}_{z_2, j} \mathbb{E}_j \mathbf{Q}_{z_1, j} \} \mathbf{a}_j \right). \end{aligned}$$

By rank-one perturbation and Lemma A.5, we easily prove that

$$\sum_{i=1}^n \xi_{3j} - \sum_{j=1}^n \tilde{\xi}_{3j} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} 0.$$

Consider the following notations:

$$\begin{aligned} \psi_j(z_1, z_2) &= \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \{ (\mathbb{E}_j \mathbf{Q}_{z_1}) (\mathbb{E}_j \mathbf{Q}_{z_2}) \} = \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \{ (\mathbb{E}_j \mathbf{Q}_{z_1}) \mathbf{Q}_{z_2} \}, \\ \zeta_{kj}(z_1, z_2) &= \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j \mathbf{Q}_{z_1}) (\mathbb{E}_j \mathbf{Q}_{z_2}) \mathbf{a}_k \} = \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j \mathbf{Q}_{z_1}) \mathbf{Q}_{z_2} \mathbf{a}_k \}, \\ \theta_{kj}(z_1, z_2) &= \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j \mathbf{Q}_{z_1, k}) (\mathbb{E}_j \mathbf{Q}_{z_2, k}) \mathbf{a}_k \} = \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j \mathbf{Q}_{z_1, k}) \mathbf{Q}_{z_2, k} \mathbf{a}_k \}, \\ \phi_j(z_1, z_2) &= \frac{\sigma}{n} \sum_{k=1}^j z_1 z_2 \tilde{t}_{z_1, kk} \tilde{t}_{z_2, kk} \theta_{kj}(z_1, z_2). \end{aligned} \tag{4.15}$$

If clear from the context, we simply write  $\psi_j$ ,  $\zeta_{kj}$ ,  $\theta_{kj}$  and  $\phi_j$  instead of  $\psi_j(z_1, z_2)$ ,  $\zeta_{kj}(z_1, z_2)$ ,  $\theta_{kj}(z_1, z_2)$  and  $\phi_j(z_1, z_2)$  respectively. With these notations at hand,  $\tilde{\xi}_{3j}$  writes

$$\tilde{\xi}_{3j}(z_1, z_2) = \frac{1}{n} z_1 z_2 \tilde{t}_{jj}(z_1) \tilde{t}_{jj}(z_2) (\sigma^2 \psi_j(z_1, z_2) + \sigma \theta_{jj}(z_1, z_2) + \sigma \theta_{jj}(z_2, z_1)).$$

The following part of the proof is inspired from [20]: Since it seems difficult to obtain a direct expression for the quantities  $\psi_j$ ,  $\zeta_{kj}$ ,  $\theta_{kj}$  and  $\phi_j$ , we establish in the following lemma a system of (perturbed) equations which describes the structural links between these quantities.

**Lemma 4.7.** *Let Assumptions 1 and 2 hold and recall that  $\gamma = \gamma(z_1, z_2) = \frac{\sigma^2}{n} \text{Tr} \mathbf{T}_{z_1} \mathbf{T}_{z_2}$ . Then*

$$\begin{aligned} \zeta_{kj} &= \psi_j \left( \sum_{\ell=1}^j \frac{\sigma \mathbf{a}_k^* \mathbf{T}_{z_1} \mathbf{a}_\ell \mathbf{a}_\ell^* \mathbf{T}_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})} + \sigma^3 \mathbf{a}_k^* \mathbf{T}_{z_1} \mathbf{T}_{z_2} \mathbf{a}_k \frac{1}{n} \sum_{\ell=1}^j z_1 z_2 \tilde{t}_{z_1, \ell \ell} \tilde{t}_{z_2, \ell \ell} \right) \\ &\quad + \sigma \mathbf{a}_k^* \mathbf{T}_{z_1} \mathbf{T}_{z_2} \mathbf{a}_k + \sigma \mathbf{a}_k^* \mathbf{T}_{z_1} \mathbf{T}_{z_2} \mathbf{a}_k \phi_j + \mathcal{O}_{z_1, z_2}(n^{-1/2}), \\ \zeta_{kj} &= z_1 z_2 \tilde{t}_{z_1, kk} \tilde{t}_{z_2, kk} (1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2}) \theta_{kj} + \frac{\sigma \mathbf{a}_k^* \mathbf{T}_{z_1} \mathbf{a}_k \mathbf{a}_k^* \mathbf{T}_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})} \psi_j + \mathcal{O}_{z_1, z_2}(n^{-1/2}), \end{aligned}$$

$$\psi_j = \psi_j \left( \frac{\sigma^2}{n} \sum_{\ell=1}^j \frac{\mathbf{a}_\ell^* \Gamma_{z_2} \Gamma_{z_1} \mathbf{a}_\ell}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})} + \frac{\gamma \sigma^2}{n} \sum_{\ell=1}^j z_1 z_2 \tilde{t}_{z_1, \ell \ell} \tilde{t}_{z_2, \ell \ell} \right) + \frac{\sigma^2}{n} \text{Tr} \Gamma_{z_1} \Gamma_{z_2} + \gamma \phi_j + \mathcal{O}_{z_1, z_2}(n^{-1/2}).$$

Lemma 4.7 is a generalization of computations performed in [20, Section 5] for  $z_1 = z_2 \in (-\infty, 0)$  to general  $z_1, z_2 \in \mathbb{C}^+$ . Its proof is omitted.

Combining the two first equations of Lemma 4.7, we get

$$\begin{aligned} \sigma z_1 z_2 \tilde{t}_{z_1, k k} \tilde{t}_{z_2, k k} \theta_{k j} &= \frac{\sigma^2 \mathbf{a}_k^* \Gamma_{z_1} \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})} + \frac{\sigma^2 \mathbf{a}_k^* \Gamma_{z_1} \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})} \phi_j \\ &+ \psi_j \left( \sum_{\ell=1}^j \frac{\sigma^2 \mathbf{a}_k^* \Gamma_{z_1} \mathbf{a}_\ell \mathbf{a}_\ell^* \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})^2 (1 + \sigma \delta_{z_2})^2} + \frac{\sigma^2 \mathbf{a}_k^* \Gamma_{z_1} \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})} \frac{\sigma^2}{n} \sum_{\ell=1}^j z_1 z_2 \tilde{t}_{z_1, \ell \ell} \tilde{t}_{z_2, \ell \ell} \right) \\ &- \psi_j \frac{\sigma^2 \mathbf{a}_k^* \Gamma_{z_1} \mathbf{a}_k \mathbf{a}_k^* \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})^2 (1 + \sigma \delta_{z_2})^2} + \mathcal{O}_{z_1, z_2}(n^{-1/2}). \end{aligned} \quad (4.16)$$

In order to simplify the notations, we introduce the following quantities:

$$\begin{aligned} \nu_j &:= \nu_j(z_1, z_2) = \frac{\sigma^2}{n} \sum_{k=1}^j \frac{\mathbf{a}_k^* \Gamma_{z_1} \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})}, \\ \eta_j &:= \eta_j(z_1, z_2) = \frac{\sigma^2}{n} \sum_{\ell=1}^j z_1 z_2 \tilde{t}_{z_1, \ell \ell} \tilde{t}_{z_2, \ell \ell}, \\ \omega_j &:= \omega_j(z_1, z_2) = \frac{\sigma^2}{n} \sum_{k=1}^j \sum_{\ell=1, \ell \neq k}^j \frac{\mathbf{a}_k^* \Gamma_{z_1} \mathbf{a}_\ell \mathbf{a}_\ell^* \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})^2 (1 + \sigma \delta_{z_2})^2}. \end{aligned}$$

Notice that for  $j = n$  the notation  $\nu_j$  is consistent with the definition (2.2). Eq. (4.16) yields

$$(1 - \nu_j) \phi_j - (\omega_j + \nu_j \eta_j) \psi_j = \nu_j + \mathcal{O}_{z_1, z_2}(n^{-1/2})$$

and the last equation of Lemma 4.7 writes

$$-\gamma \phi_j + (1 - \nu_j - \gamma \eta_j) \psi_j = \gamma + \mathcal{O}_{z_1, z_2}(n^{-1/2}).$$

We finally end up with a system of two perturbed linear equations for  $\phi_j$  and  $\psi_j$ :

$$\begin{cases} (1 - \nu_j) \phi_j - (\omega_j + \nu_j \eta_j) \psi_j &= \nu_j + \mathcal{O}_{z_1, z_2}(n^{-1/2}), \\ -\gamma \phi_j + (1 - \nu_j - \gamma \eta_j) \psi_j &= \gamma + \mathcal{O}_{z_1, z_2}(n^{-1/2}). \end{cases} \quad (4.17)$$

We study hereafter the properties of the determinant of the system  $\mathcal{D}_j$  given by

$$\mathcal{D}_j = (1 - \nu_j)^2 - \gamma(\eta_j + \omega_j). \quad (4.18)$$

**Lemma 4.8.** *Let Assumption 2 hold and recall the definition of  $\Delta_n$  given in (2.3). The determinant  $\mathcal{D}_j$  satisfies the following properties:*

(i) for any  $z_1, z_2 \in \mathbb{C}^+$ , we have for  $j = n$

$$\mathcal{D}_n(z_1, z_2) = \Delta_n(z_1, z_2) = \left( 1 - \frac{\sigma^2}{n} \frac{\text{Tr} \Gamma_{z_1} \mathbf{A} \mathbf{A}^* \Gamma_{z_2}}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})} \right)^2 - z_1 z_2 \gamma \tilde{\gamma},$$

(ii) for any  $z_1, z_2 \in \mathbb{C}^+$ ,

$$\liminf_n \inf_{1 \leq j \leq n} |\mathcal{D}_j(z_1, z_2)| > 0.$$

*Proof.* For  $1 \leq j \leq n$  denote by  $A_{1:j}$  the  $N \times n$  matrix defined by  $A_{1:j} := [\mathbf{a}_1, \dots, \mathbf{a}_j, \mathbf{0}, \dots, \mathbf{0}]$  and write

$$\omega_j = \frac{\sigma^2}{n} \frac{\text{Tr} A_{1:j}^* \Gamma_{z_1} A_{1:j} A_{1:j}^* \Gamma_{z_2} A_{1:j}}{(1 + \sigma \delta_{z_1})^2 (1 + \sigma \delta_{z_2})^2} - \frac{\sigma^2}{n} \sum_{k=1}^j \frac{\mathbf{a}_k^* \Gamma_{z_1} \mathbf{a}_k \mathbf{a}_k^* \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})^2 (1 + \sigma \delta_{z_2})^2}.$$

Using a standard identity [26, Section 0.7.4] applied to  $\Gamma_z$  and  $\tilde{\Gamma}_z$  yields the identity

$$\tilde{\Gamma}_z = -\frac{1}{z(1 + \sigma \delta_z)} + \frac{1}{z(1 + \sigma \delta_z)^2} \mathbf{A}^* \Gamma_z \mathbf{A} \quad (4.19)$$

from which we obtain  $(1 + \sigma \delta_z)^{-2} \mathbf{a}_k^* \Gamma_z \mathbf{a}_k = (1 + \sigma \delta_z)^{-1} + z \tilde{t}_{k k}$  and thus

$$\frac{\sigma^2}{n} \sum_{k=1}^j \frac{\mathbf{a}_k^* \Gamma_{z_1} \mathbf{a}_k \mathbf{a}_k^* \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})^2 (1 + \sigma \delta_{z_2})^2} = \frac{\sigma^2}{n} \sum_{k=1}^j z_1 z_2 \tilde{t}_{z_1, k k} \tilde{t}_{z_2, k k}$$

$$+ \frac{\sigma^2}{n} \sum_{k=1}^j \frac{z_1 \tilde{t}_{z_1, kk}}{1 + \sigma \delta_{z_2}} + \frac{\sigma^2}{n} \sum_{k=1}^j \frac{\mathbf{a}_k^* \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})^2}.$$

Introduce

$$-z \tilde{\Gamma}_{1:j}(z) := (1 + \sigma \delta_z)^{-1} \mathbf{I}_{1:j} - (1 + \sigma \delta_z)^{-2} A_{1:j}^* \mathbf{T}_z A_{1:j}$$

where  $\mathbf{I}_{1:j} := \sum_{k=1}^j e_k e_k^\top$  and  $(e_i)$  is the canonical basis of  $\mathbb{R}^n$ , one can check that

$$\eta_j + \omega_j = \frac{\sigma^2 z_1 z_2}{n} \text{Tr}(\tilde{\Gamma}_{1:j}(z_1) \tilde{\Gamma}_{1:j}(z_2)). \quad (4.20)$$

In particular  $\eta_n + \omega_n = z_1 z_2 \tilde{\gamma}$  hence the identity  $\mathcal{D}_n = (1 - \nu_n)^2 - z_1 z_2 \gamma \tilde{\gamma} = \Delta_n$  and (i) is established.

We now prove (ii) and start by showing that for any  $z \in \mathbb{C}^+$ ,

$$\liminf_n \inf_{1 \leq j \leq n} \mathcal{D}_j(z, \bar{z}) > 0. \quad (4.21)$$

It is straightforward to check that

$$0 < \nu_j(z, \bar{z}) \leq \nu_n(z, \bar{z}) \stackrel{(a)}{<} 1 \quad \text{and} \quad 0 \leq \omega_j(z, \bar{z}) + \eta_j(z, \bar{z}) \leq \omega_n(z, \bar{z}) + \eta_n(z, \bar{z}),$$

where (a) follows from Proposition 4.1-(3). Hence  $\mathcal{D}_i(z, \bar{z}) \geq \mathcal{D}_n(z, \bar{z})$ . Since by (i) we have proved that

$$\mathcal{D}_n(z, \bar{z}) = \Delta_n(z, \bar{z}) = (1 - \nu_n(z, \bar{z}))^2 - \gamma(z, \bar{z})(\eta_n(z, \bar{z}) + \omega_n(z, \bar{z})),$$

we obtain the following estimate

$$\inf_{1 \leq j \leq n} \mathcal{D}_j(z, \bar{z}) \geq \Delta_n(z, \bar{z}). \quad (4.22)$$

Recall that  $\delta_n$  and  $\tilde{\delta}_n$  are Stieltjes transforms associated to measures with respective total mass  $\sigma N n^{-1}$  and  $\sigma$  hence

$$|s_n(z)| \leq |z| \left(1 + \frac{N \sigma^2}{n \text{Im}(z)}\right) \left(1 + \frac{\sigma^2}{\text{Im}(z)}\right) \leq K_z$$

uniformly in  $n \geq 1$ . By Proposition 4.1-(3),

$$|\Delta_n(z, \bar{z})| = \frac{\text{Im}(z)}{\text{Im}(s_n(z))} \geq \frac{\text{Im}(z)}{K_z}.$$

Combining this estimate with (4.22) yields (4.21). To conclude the proof, we show that for  $z_1, z_2 \in \mathbb{C}^+$ ,

$$|\mathcal{D}_j(z_1, z_2)| \geq (\mathcal{D}_j(z_1, \bar{z}_1) \mathcal{D}_j(z_2, \bar{z}_2))^{1/2}. \quad (4.23)$$

Starting from (4.20), we have

$$\mathcal{D}_j(z_1, z_2) = (1 - \nu_j(z_1, z_2))^2 - \gamma z_1 z_2 \frac{\sigma^2}{n} \text{Tr} \tilde{\Gamma}_{1:j}(z_1) \tilde{\Gamma}_{1:j}^*(z_2).$$

From this identity, we can conclude as in the proof of Proposition 4.1-(6).  $\square$

We now go back to the proof of Lemma 4.5. By the above lemma, the system (4.17) has the solution

$$\begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} = \frac{1}{\mathcal{D}_j} \begin{bmatrix} \nu_j(1 - \nu_j - \gamma \eta_j) + \gamma(\omega_j + \nu_j \eta_j) \\ \gamma \nu_j + \gamma(1 - \nu_j) \end{bmatrix} + \mathcal{O}_{z_1, z_2}(n^{-1/2}). \quad (4.24)$$

Notice that since  $\mathbf{T}_{z_1}$  and  $\mathbf{T}_{z_2}$  commute  $\nu(z_1, z_2) = \nu(z_2, z_1)$ . Dividing by  $n$  and plugging the solution (4.24) in (4.16) yields:

$$\begin{aligned} & \frac{\sigma}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} (\theta_{jj}(z_1, z_2) + \theta_{jj}(z_2, z_1)) \\ &= 2(\nu_j - \nu_{j-1}) + \frac{\gamma}{\mathcal{D}_j} ((\omega_j - \omega_{j-1}) + 2\eta_j(\nu_j - \nu_{j-1})) \\ & \quad + \frac{1}{\mathcal{D}_j} (\nu_j - \nu_{j-1})(\nu_j(1 - \nu_j) + \gamma \omega_j) + \mathcal{O}_{z_1, z_2}(n^{-3/2}), \\ &= \frac{1}{\mathcal{D}_j} (2(\nu_j - \nu_{j-1})(1 - \nu_j) + \gamma(\omega_j - \omega_{j-1})) + \mathcal{O}_{z_1, z_2}(n^{-3/2}), \end{aligned}$$

where by convention we set  $\omega_0 = \nu_0 = \eta_0 = 0$  and  $\mathcal{D}_0 = 1$ . Therefore, we get

$$\begin{aligned} & \frac{1}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} (\sigma^2 \psi_j(z_1, z_2) + \sigma \theta_{jj}(z_1, z_2) + \sigma \theta_{jj}(z_2, z_1)) \\ &= \frac{2(\nu_j - \nu_{j-1})(1 - \nu_j) + \gamma(\eta_j - \eta_{j-1}) + \gamma(\omega_j - \omega_{j-1})}{\mathcal{D}_j} + \mathcal{O}_{z_1, z_2}(n^{-3/2}). \end{aligned}$$

Moreover, going back to the definition (4.18) of  $\mathcal{D}_j$ , we have

$$\mathcal{D}_{j-1} - \mathcal{D}_j = 2(\nu_j - \nu_{j-1})(2 - \nu_{j-1} - \nu_j) + \gamma(\eta_j - \eta_{j-1}) + \gamma(\omega_j - \omega_{j-1}),$$

$$= 2(\nu_j - \nu_{j-1})(1 - \nu_j) + \gamma(\eta_j - \eta_{j-1}) + \gamma(\omega_j - \omega_{j-1}) + \mathcal{O}_{z_1, z_2}(n^{-2}).$$

Then

$$\frac{1}{n} \sum_{j=1}^n z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} (\sigma^2 \psi_j(z_1, z_2) + \sigma \theta_{jj}(z_1, z_2) + \sigma \theta_{jj}(z_2, z_1)) = \sum_{j=1}^n \frac{\mathcal{D}_{j-1} - \mathcal{D}_j}{\mathcal{D}_j} + \mathcal{O}_{z_1, z_2}(n^{-1/2}), \quad (4.25)$$

and

$$\mathcal{D}_0 = 1 \quad \text{and} \quad |\mathcal{D}_{j-1} - \mathcal{D}_j| = \mathcal{O}_{z_1, z_2}(n^{-1}) \quad \text{for} \quad 1 \leq j \leq n. \quad (4.26)$$

For a sufficiently large fixed constant  $K$  and for any  $1 \leq j \leq n$ , we denote by  $\mathcal{B}_j := B(\mathcal{D}_j, K/n)$  the ball of center  $\mathcal{D}_j$  and radius  $K/n$  and we let  $[\mathcal{D}_j, \mathcal{D}_{j-1}] \subset \mathcal{B}_j$  be the segment joining  $\mathcal{D}_j$  and  $\mathcal{D}_{j-1}$ . We suppose that  $n$  is large enough so that  $K/n < |\mathcal{D}_j|/2$ . Thus for any  $z \in [\mathcal{D}_j, \mathcal{D}_{j-1}]$ ,

$$\frac{|\mathcal{D}_j|}{2} < |z| \leq |\mathcal{D}_j| + \frac{K}{n} \quad \text{and} \quad |z - \mathcal{D}_j| \leq \frac{K}{n}. \quad (4.27)$$

As  $z \mapsto z^{-1}$  is analytic over  $\mathcal{B} := \cup_{j=1}^n \mathcal{B}_j$ , we write

$$\frac{\mathcal{D}_{j-1} - \mathcal{D}_j}{\mathcal{D}_j} - \int_{[\mathcal{D}_j, \mathcal{D}_{j-1}]} \frac{1}{z} dz = \sum_{j=1}^n \int_{[\mathcal{D}_j, \mathcal{D}_{j-1}]} \left( \frac{1}{\mathcal{D}_j} - \frac{1}{z} \right) dz = \int_{[\mathcal{D}_j, \mathcal{D}_{j-1}]} \frac{z - \mathcal{D}_j}{z \mathcal{D}_j} dz = \mathcal{O}_{z_1, z_2}(n^{-2})$$

where the last equality follows from (4.26) and (4.27). We finally obtain

$$\left| \sum_{j=1}^n \frac{\mathcal{D}_{j-1} - \mathcal{D}_j}{\mathcal{D}_j} - \int_{\Delta_n} \frac{1}{z} dz \right| = \mathcal{O}_{z_1, z_2}(n^{-1}).$$

Using Lemma 4.8-(ii), one can prove that the r.h.s. above is uniformly bounded and apply [4, Lemma 2.14] to obtain the convergence of the derivative. Differentiating with respect to  $z_2$ , we get

$$\frac{\partial}{\partial z_2} \int_{\Delta_n} \frac{1}{z} dz = -\frac{1}{\Delta_n} \frac{\partial \Delta_n}{\partial z_2}.$$

Differentiating again with respect to  $z_1$  and relying on the identity of Proposition 4.1-(4), we conclude the proof of Lemma 4.5.

**4.4. Proof of Lemma 4.6.** As the proof of Lemma 4.6 is very close to the proof of Lemma 4.5, we only focus on the main steps. Recall the definition of  $\xi_{4j}$ ; we introduce an auxiliary quantity  $\tilde{\xi}_{4j}$ :

$$\begin{aligned} \tilde{\xi}_{4j} = \frac{\sigma^2}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} & \left( \frac{|\vartheta|^2 \sigma^2}{n} \text{Tr} \mathbb{E} \left\{ \mathbb{E}_j \mathbf{Q}_{z_1, j} \mathbb{E}_j \mathbf{Q}_{z_2, j}^\top \right\} + \vartheta \mathbf{a}_j^* \mathbb{E} \left\{ \mathbb{E}_j \mathbf{Q}_{z_2, j} \mathbb{E}_j \mathbf{Q}_{z_1, j}^\top \right\} \bar{\mathbf{a}}_j \right. \\ & \left. + \bar{\vartheta} \mathbf{a}_j^\top \mathbb{E} \left\{ \mathbb{E}_j \mathbf{Q}_{z_1, j}^\top \mathbb{E}_j \mathbf{Q}_{z_2, j} \right\} \mathbf{a}_j \right) \end{aligned}$$

and one can easily prove that  $\sum_{i=1}^n \xi_{4j} - \sum_{j=1}^n \tilde{\xi}_{4j} \xrightarrow[n \rightarrow \infty]{P} 0$ . Denote:

$$\begin{aligned} \underline{\psi}_j(z_1, z_2) &= \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \{ (\mathbb{E}_j \mathbf{Q}_{z_1}) (\mathbb{E}_j \mathbf{Q}_{z_2}^\top) \} = \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \{ (\mathbb{E}_j \mathbf{Q}_{z_1}) \mathbf{Q}_{z_2}^\top \}, \\ \underline{\zeta}_{kj}(z_1, z_2) &= \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j \mathbf{Q}_{z_1}) (\mathbb{E}_j \mathbf{Q}_{z_2}^\top) \bar{\mathbf{a}}_k \} = \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j \mathbf{Q}_{z_1}) \mathbf{Q}_{z_2}^\top \bar{\mathbf{a}}_k \}, \\ \underline{\theta}_{kj}(z_1, z_2) &= \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j \mathbf{Q}_{z_1, k}) (\mathbb{E}_j \mathbf{Q}_{z_2, k}^\top) \bar{\mathbf{a}}_k \} = \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j \mathbf{Q}_{z_1, k}) \mathbf{Q}_{z_2, k}^\top \bar{\mathbf{a}}_k \}, \\ \underline{\phi}_j &= \frac{\sigma}{n} \sum_{k=1}^j z_1 z_2 \tilde{t}_{z_1, kk} \tilde{t}_{z_2, kk} \underline{\theta}_{kj}. \end{aligned}$$

Then  $\tilde{\xi}_{4j}$  writes

$$\tilde{\xi}_{4j}(z_1, z_2) = \frac{z_1 z_2}{n} \tilde{t}_{jj}(z_1) \tilde{t}_{jj}(z_2) \left( \sigma^2 |\vartheta|^2 \underline{\psi}_j(z_1, z_2) + \sigma \vartheta \underline{\theta}_{jj}(z_1, z_2) + \sigma \bar{\vartheta} \overline{\underline{\theta}_{jj}(\bar{z}_1, \bar{z}_2)} \right). \quad (4.28)$$

Similar derivations as in the proof of Lemma 4.5 yield the perturbed system:

$$\begin{cases} (1 - \vartheta \nu_j^\dagger) \underline{\phi}_j - (\bar{\vartheta} \omega_j^\dagger + |\vartheta|^2 \nu_j^\dagger \eta_j) \underline{\psi}_j &= \nu_j^\dagger + \mathcal{O}_{z_1, z_2}(n^{-1/2}) \\ -\vartheta \gamma^\dagger \underline{\phi}_j + (1 - \bar{\vartheta} \nu_j^\dagger - |\vartheta|^2 \gamma^\dagger \eta_j) \underline{\psi}_j &= \gamma^\dagger + \mathcal{O}_{z_1, z_2}(n^{-1/2}) \end{cases}, \quad (4.29)$$

where

$$\nu_j^\dagger = \frac{\sigma^2}{n} \sum_{k=1}^j \frac{\mathbf{a}_k^* \Gamma_{z_1} \Gamma_{z_2}^\top \bar{\mathbf{a}}_k}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})},$$

$$\begin{aligned}\tilde{\nu}_j^\dagger &= \frac{\sigma^2}{n} \sum_{k=1}^j \frac{\mathbf{a}_k^\top \Gamma_{z_1}^\top \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})}, \\ \gamma^\dagger &= \frac{\sigma^2}{n} \text{Tr} \Gamma_{z_1} \Gamma_{z_2}^\top, \\ \omega_j^\dagger &= \frac{\sigma^2}{n} \sum_{k=1}^j \sum_{\ell=1, \ell \neq k}^j \frac{\mathbf{a}_k^* \Gamma_{z_1} \mathbf{a}_\ell \Gamma_{z_2}^\top \bar{\mathbf{a}}_k}{(1 + \sigma \delta_{z_1})^2 (1 + \sigma \delta_{z_2})^2},\end{aligned}$$

and the determinant  $\mathcal{D}_j^\vartheta$  of which is given by

$$\mathcal{D}_j^\vartheta = (1 - \vartheta \nu_j^\dagger)(1 - \bar{\vartheta} \tilde{\nu}_j^\dagger) - |\vartheta|^2 \gamma^\dagger (\eta_j + \omega_j^\dagger).$$

Its properties are summarized in the following lemma whose proof combines arguments from Lemma 4.8's and Proposition 4.1-(6)'s proofs.

**Lemma 4.9.** *The determinant  $\mathcal{D}_j^\vartheta$  satisfies the following:*

- (i) for  $z_1, z_2 \in \mathbb{C}^+$ , we have  $\eta_n + \omega_n^\dagger = z_1 z_2 \tilde{\gamma}^\dagger$  and thus  $\mathcal{D}_n^\vartheta$  coincides with  $\Delta_n^\vartheta$  as defined in (2.5).
- (ii) for  $z_1, z_2 \in \mathbb{C}^+$ ,

$$\liminf_n \inf_{j \in \{1, \dots, n\}} |\mathcal{D}_j^\vartheta(z_1, z_2)| > 0.$$

Solving (4.29) yields

$$\begin{bmatrix} \phi_j \\ \psi_j \end{bmatrix} = \frac{1}{\mathcal{D}_j^\vartheta} \begin{bmatrix} \nu_j^\dagger (1 - \bar{\vartheta} \tilde{\nu}_j^\dagger) + \bar{\vartheta} \gamma^\dagger \omega_j^\dagger \\ \gamma^\dagger \end{bmatrix} + \mathcal{O}_{z_1, z_2}(n^{-1/2}), \quad (4.30)$$

In order to handle the term  $\mathbf{a}_j^\top (\mathbb{E}_j \mathbf{Q}_{z_1, j}^\top) (\mathbb{E}_j \mathbf{Q}_{z_2, j}) \mathbf{a}_j$  in  $\xi_{4j}$ , we introduce the following notations:

$$\begin{aligned}\tilde{\psi}_j(z_1, z_2) &= \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \{ (\mathbb{E}_j \mathbf{Q}_{z_1}^\top) (\mathbb{E}_j \mathbf{Q}_{z_2}) \} = \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \{ (\mathbb{E}_j \mathbf{Q}_{z_1}^\top) \mathbf{Q}_{z_2} \}, \\ \tilde{\zeta}_{kj}(z_1, z_2) &= \sigma \mathbb{E} \{ \mathbf{a}_k^\top (\mathbb{E}_j \mathbf{Q}_{z_1}^\top) (\mathbb{E}_j \mathbf{Q}_{z_2}) \mathbf{a}_k \} = \sigma \mathbb{E} \{ \mathbf{a}_k^\top (\mathbb{E}_j \mathbf{Q}_{z_1}^\top) \mathbf{Q}_{z_2} \mathbf{a}_k \}, \\ \tilde{\theta}_{kj}(z_1, z_2) &= \sigma \mathbb{E} \{ \mathbf{a}_k^\top (\mathbb{E}_j \mathbf{Q}_{z_1, k}^\top) (\mathbb{E}_j \mathbf{Q}_{z_2, k}) \mathbf{a}_k \} = \sigma \mathbb{E} \{ \mathbf{a}_k^\top (\mathbb{E}_j \mathbf{Q}_{z_1, k}^\top) \mathbf{Q}_{z_2, k} \mathbf{a}_k \}, \\ \tilde{\phi}_j &= \frac{\sigma}{n} \sum_{k=1}^j z_1 z_2 \tilde{t}_{z_1, k k} \tilde{t}_{z_2, k k} \tilde{\theta}_{kj}.\end{aligned}$$

Noticing that  $\tilde{\psi}_j(z_1, z_2) = \psi_j(z_1, z_2)$  and that  $\tilde{\zeta}_{kj}(z_1, z_2) = \overline{\zeta_{kj}(\bar{z}_1, \bar{z}_2)}$  and  $\tilde{\theta}_{kj}(z_1, z_2) = \overline{\theta_{kj}(\bar{z}_1, \bar{z}_2)}$ , we get the following perturbed system

$$\begin{cases} (1 - \bar{\vartheta} \tilde{\nu}_j^\dagger) \tilde{\phi}_j - (\vartheta \omega_j^\dagger + |\vartheta|^2 \tilde{\nu}_j^\dagger \eta_j) \tilde{\psi}_j = \tilde{\nu}_j^\dagger + \mathcal{O}_{z_1, z_2}(n^{-1/2}) \\ -\bar{\vartheta} \gamma^\dagger \tilde{\phi}_j + (1 - \vartheta \nu_j^\dagger - |\vartheta|^2 \gamma^\dagger \eta_j) \tilde{\psi}_j = \gamma^\dagger + \mathcal{O}_{z_1, z_2}(n^{-1/2}). \end{cases} \quad (4.31)$$

The determinant of this system is again  $\mathcal{D}_j^\vartheta$  and the solution of the system is

$$\begin{bmatrix} \tilde{\phi}_j \\ \tilde{\psi}_j \end{bmatrix} = \frac{1}{\mathcal{D}_j^\vartheta} \begin{bmatrix} \tilde{\nu}_j^\dagger (1 - \vartheta \nu_j^\dagger) + \vartheta \gamma^\dagger \omega_j^\dagger \\ \gamma^\dagger \end{bmatrix} + \mathcal{O}(n^{-1/2}), \quad (4.32)$$

Therefore, it suffices to show that

$$\frac{\partial^2}{\partial z_1 \partial z_2} \frac{1}{n} \sum_{j=1}^n z_1 z_2 \tilde{t}_{jj}(z_1) \tilde{t}_{jj}(z_2) (\sigma^2 |\vartheta|^2 \tilde{\psi}_j + \sigma \vartheta \tilde{\theta}_{jj} + \sigma \bar{\vartheta} \tilde{\phi}_{jj}) - \Theta_{1, n} \xrightarrow{n \rightarrow \infty} 0.$$

Taking into account (4.28), similar computations as in Lemma 4.5 yield

$$\begin{aligned}\frac{1}{n} z_1 z_2 \tilde{t}_{jj}(z_1) \tilde{t}_{jj}(z_2) (\sigma^2 |\vartheta|^2 \tilde{\psi}_j + \sigma \vartheta \tilde{\theta}_{jj} + \sigma \bar{\vartheta} \tilde{\phi}_{jj}) \\ = \frac{\vartheta}{\mathcal{D}_j^\vartheta} (\nu_j^\dagger - \nu_{j-1}^\dagger) (1 - \bar{\vartheta} \tilde{\nu}_j^\dagger) + \frac{\bar{\vartheta}}{\mathcal{D}_j^\vartheta} (\tilde{\nu}_j^\dagger - \tilde{\nu}_{j-1}^\dagger) (1 - \vartheta \nu_j^\dagger) \\ + \frac{|\vartheta|^2 \gamma^\dagger}{\mathcal{D}_j^\vartheta} [(\eta_j - \eta_{j-1}) + (\omega_j - \omega_{j-1})] + \mathcal{O}(n^{-3/2}).\end{aligned}$$

Now, as

$$\begin{aligned}\mathcal{D}_{j-1}^\vartheta - \mathcal{D}_j^\vartheta &= \vartheta (\nu_j^\dagger - \nu_{j-1}^\dagger) (1 - \bar{\vartheta} \tilde{\nu}_j^\dagger) + \bar{\vartheta} (\tilde{\nu}_j^\dagger - \tilde{\nu}_{j-1}^\dagger) (1 - \vartheta \nu_j^\dagger) \\ &\quad + |\vartheta|^2 \gamma^\dagger [(\eta_j - \eta_{j-1}) + (\omega_j - \omega_{j-1})] + \mathcal{O}(n^{-2}),\end{aligned}$$

we finally get that

$$\frac{1}{n} \sum_{j=1}^n z_1 z_2 \tilde{t}_{jj}(z_1) \tilde{t}_{jj}(z_2) (\sigma^2 |\vartheta|^2 \underline{\psi}_j + \sigma \vartheta \underline{\theta}_{jj} + \sigma \bar{\vartheta} \bar{\theta}_{jj}) = \sum_{j=1}^n \frac{\mathcal{D}_{j-1}^\vartheta - \mathcal{D}_j^\vartheta}{\mathcal{D}_j^\vartheta} + \mathcal{O}(n^{-1/2}).$$

The rest of the proof is similar to the end of proof of Lemma 4.5.

**4.5. Tightness of the Gaussian process.** To complete the proof fo Theorem 2, it remains to prove that the Gaussian process  $z \mapsto G_n(z)$  is tight over  $D_\varepsilon$ . We proceed as in [35, Section 5.2.2] and prove, following Prohorov's theorem, that  $(G_n(z))_{z \in D_\varepsilon}$  is relatively compact in distribution. Following the meta model argument in [35], let  $N, n$  and  $A_n$  be given and consider the  $NM \times nM$  matrix

$$A_{n,M} = \begin{bmatrix} A_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_n \end{bmatrix}.$$

For  $M \geq 1$ ,  $\|A_{n,M}\| = \|A_n\|$ . Thus, the sequence of matrices  $(A_{n,M}; M \geq 1, n \geq 1)$  satisfies Assumption (A-2) as long as  $(A_n; n \geq 1)$  does (recall that  $N = N(n)$ ). Consider now the random matrix:

$$Y_{n,M} = \frac{\sigma}{\sqrt{nM}} X_{n,M} + A_{n,M},$$

where  $X_{n,M}$  is an  $NM \times nM$  matrix with i.i.d. random entries having the same distribution as the  $X_{ij}$ 's and satisfying Assumption (A-1) and the associated process

$$M_{n,M}(z) = \text{Tr}(Y_{n,M} Y_{n,M}^* - z I_{NM})^{-1} - \mathbb{E} \text{Tr}(Y_{n,M} Y_{n,M}^* - z I_{NM})^{-1}.$$

Notice that  $(M_{n,M}(z); z \in D_\varepsilon, M, n \geq 1)$  is tight. In fact, the arguments developed in Section 3.3 apply.

Recall the definitions of  $\delta_z, \tilde{\delta}_z, T_z$  and  $\Theta_n$  associated to  $Y_n Y_n^*$  and denote by  $\delta_{z,M}, \tilde{\delta}_{z,M}, T_{z,M}$  and  $\Theta_{n,M}$  their counterparts for the matrix  $Y_{n,M} Y_{n,M}^*$ . Then taking advantage of the block structure of  $A_{n,M}$ , it is straightforward to verify that

$$\forall M \geq 1, \quad \delta_{n,M} = \delta_z, \quad \tilde{\delta}_{n,M} = \tilde{\delta}_z, \quad T_{n,M} = \begin{bmatrix} T_n & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & T_n \end{bmatrix} \quad \text{and} \quad \Theta_n = \Theta_{n,M}.$$

As a consequence, if one fixes  $n, N$  and let  $M \rightarrow \infty$ , then the process  $M_{n,M}$  converges in distribution to a Gaussian process  $G_n$  with distribution  $\mathcal{L}(G_n)$ . Hence for all  $n \geq 1$ , the distribution  $\mathcal{L}(G_n)$  belongs to the closure of  $(\mathcal{L}(M_{n,M}), M \geq 1, n \geq 1)$  which is compact since  $(M_{n,M}(z); z \in D_\varepsilon, M \geq 1, n \geq 1)$  is tight. By [6, Chapter 2, Section 7],  $G_n$  is tight.

## 5. PROOF OF THEOREM 2: FLUCTUATIONS FOR GENERAL LINEAR STATISTICS

As mentioned before, Theorem 1 is a building block to prove the central limit theorem for non-analytic functions. For a function  $f \in C_c^{k+1}(\mathbb{R})$ , recall the definition of its almost analytic extension (1.7) and Helffer-Sjöstrand's formula (1.8). We now state the following lemma allowing the transfer of the CLT to non-analytic functions via the Helffer-Sjöstrand's formula.

**Lemma 5.1** (Lemma 6.3 in [35]). *Let  $(\varphi_n(z), z \in D^+ \cup \overline{D^+})_{n \in \mathbb{N}}$  and  $(\psi_n(z), z \in D^+ \cup \overline{D^+})_{n \in \mathbb{N}}$  be centered complex-valued continuous random processes such that  $\varphi(\bar{z}) = \overline{\varphi(z)}$  and  $\psi(\bar{z}) = \overline{\psi(z)}$ . For any  $1 \leq \ell \leq L$ , let  $g_\ell : \mathbb{R} \rightarrow \mathbb{R}$  be  $C^{k+1}$  functions having compact supports. Assume that*

(i) *the following convergence in distribution holds true: for all  $d \geq 1$  and  $(z_1, \dots, z_d) \in D^+$ ,*

$$d_{\mathcal{L}P}((\varphi_n(z_1), \dots, \varphi_n(z_d)), (\psi_n(z_1), \dots, \psi_n(z_d))) \xrightarrow[n \rightarrow \infty]{} 0,$$

(ii) *for all  $\varepsilon > 0$ ,  $\varphi_n(z)$  and  $\psi_n(z)$  are tight on  $D_\varepsilon$ ,*

(iii) *the process  $(\psi_n(z))$  is gaussian with covariance  $\kappa_n(z_1, z_2)$  for  $z_1, z_2 \in D^+ \cup \overline{D^+}$ ,*

(iv) *the following estimates hold true*

$$\forall n \in \mathbb{N}, \forall z \in D^+, \quad \text{Var } \varphi_n(z) \leq \frac{K}{(\text{Im}z)^{2k}} \quad \text{and} \quad \text{Var } \psi_n(z) \leq \frac{K}{(\text{Im}z)^{2k}}.$$

Then,

$$d_{\mathcal{L}P} \left( \frac{1}{2\pi} \text{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(\mathbf{g})(z) \varphi_n(z) \ell(dz), \frac{1}{2\pi} \text{Re} \int_{\mathbb{C}^+} \bar{\partial} \Phi_k(\mathbf{g})(z) \psi_n(z) \ell(dz) \right) \xrightarrow[n \rightarrow \infty]{} 0,$$

where  $\Phi_k(g_i)$  is the almost analytic extension of  $g_i$ ,

$$\bar{\partial}\Phi_k(g_j)(z) = \frac{1}{2}(\partial_x + i\partial_y) \sum_{\ell=0}^k \frac{(iy)^\ell}{\ell!} g_j^{(\ell)}(x) \chi(y) \quad \text{and} \quad \bar{\partial}\Phi_k(\mathbf{g}) = (\bar{\partial}\Phi_k(g_j); 1 \leq j \leq L)$$

Moreover,

$$\frac{1}{2\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial}\Phi_k(\mathbf{g})(z) \psi_n(z) \ell(dz)$$

is centered gaussian with covariance matrix:

$$\begin{aligned} \operatorname{Cov} \left( \frac{1}{2\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial}\Phi_k(g_k)(z) \varphi_n(z) \ell(dz), \frac{1}{2\pi} \operatorname{Re} \int_{\mathbb{C}^+} \bar{\partial}\Phi_k(g_\ell)(z) \psi_n(z) \ell(dz) \right) \\ = \frac{2}{\pi^2} \operatorname{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial}\Phi_k(g_k)(z_1) \overline{\bar{\partial}\Phi_k(g_\ell)(z_2)} \kappa_n(z_1, \bar{z}_2) \ell(dz_1) \ell(dz_2) \\ + \frac{2}{\pi^2} \operatorname{Re} \int_{(\mathbb{C}^+)^2} \bar{\partial}\Phi_k(g_k)(z_2) \bar{\partial}\Phi_k(g_\ell)(z_1) \kappa_n(z_1, z_2) \ell(dz_1) \ell(dz_2), \end{aligned}$$

for  $1 \leq k, \ell \leq L$ .

We apply this lemma to

$$\varphi_n(z) = \operatorname{Tr} Q_n(z) - \mathbb{E} \operatorname{Tr} Q_n(z) \quad \text{and} \quad \psi_n(z) = G_n(z),$$

where  $G_n(z)$  is the Gaussian process in Theorem 1. Notice that it remains to check first that the condition (iv) in Lemma 5.1 holds. This is the purpose of the next proposition, which is a variation of [35, Proposition 6.4].

**Proposition 5.2.** *Let Assumptions 1 and 2 hold true then for any  $z \in \mathbb{C}^+$ ,*

$$\operatorname{Var} \operatorname{Tr} Q_n(z) = \mathcal{O} \left( \frac{1}{|\operatorname{Im} z|^4} \right) \quad \text{and} \quad \operatorname{Var} \operatorname{Tr} G_n(z) = \mathcal{O} \left( \frac{1}{|\operatorname{Im} z|^4} \right).$$

Combining Lemma 5.1 with Proposition 5.2 immediatly yields Theorem 2.

## 6. PROOF OF THEOREM 3: ALTERNATIVE EXPRESSION FOR THE COVARIANCE

We first recall some properties of  $s_n$  useful in the sequel. Recall that  $\mathcal{S}_n$  is the support of the measure whose Stieltjes transform is  $\delta_n$  and denote by  $\mathcal{S}_A$  the support of the empirical distribution of the eigenvalues of  $A_n A_n^*$ .

**Proposition 6.1** (Properties of  $s_n$  near the real axis). *The following properties hold*

- (1) *The limit  $s_n(x) := \lim_{\varepsilon \downarrow 0} s_n(x + i\varepsilon)$  exists and is continuous for all  $x \in \mathbb{R} \setminus 0$ .*
- (2)  *$x \in \mathcal{S}_n^c$  implies that  $s_n(x) \in \mathcal{S}_A^c$ .*
- (3) *If  $c_n = \frac{N}{n} < 1$  then  $0 \in \mathcal{S}_n^c$ .*
- (4) *The quantity  $s_n(z)$  is bounded for  $|z| < \eta$  for some  $\eta > 0$  and  $z \neq 0$ .*

*Proof.* Items (1), (2) follow from Theorems 2.1 and 3.3 in [14]. Item (3) can be found in [10, Theorem 1.3].

To prove item (4), write

$$s_n(z) = z(1 + \sigma\delta_n(z))(1 + \sigma\bar{\delta}_n(z)) = z(1 + \sigma\delta_n(z))^2 - \sigma(1 - c_n)(1 + \sigma\delta_n(z)).$$

The first part of the r.h.s. is bounded by Proposition 4.1-(5). If  $c_n = 1$ , the second part vanishes; if  $c_n < 1$  then  $0 \notin \mathcal{S}_n$  and  $\delta_n(z)$  is analytic in a small neighbourhood of zero.  $\square$

### 6.1. A boundary value representation for the covariance.

**Proposition 6.2.** *Let  $(Z_n(f), Z_n(g))$  be the Gaussian process defined in Theorem 2, and  $\Theta_n$  the covariance defined in Theorem 1, then the covariance of  $(Z_n(f), Z_n(g))$  admits the following representation:*

$$\begin{aligned} \operatorname{cov}(Z_n(f), Z_n(g)) &= -\frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^2} f(x)g(y) (\Theta_n(x + i\varepsilon, y + i\varepsilon) + \Theta_n(x - i\varepsilon, y - i\varepsilon)) dx dy \\ &\quad + \frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^2} f(x)g(y) (\Theta_n(x + i\varepsilon, y - i\varepsilon) + \Theta_n(x - i\varepsilon, y + i\varepsilon)) dx dy, \\ &= -\frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int_{\mathbb{R}^2} f(x)g(y) \Theta_n(x \pm_1 i\varepsilon, y \pm_2 i\varepsilon) dx dy, \end{aligned}$$

where  $\pm_1, \pm_2 \in \{+, -\}$  and  $\pm_1 \pm_2$  is the sign resulting from the product  $\pm_1 1$  by  $\pm_2 1$ .

For a proof, see [35, Proposition 4.1].

**6.2. Proof of Theorem 3.** Notice that due to the symmetry of equations (1.12), we only need to consider the case where  $c \leq 1$ , which we now assume. Recall the definition of the quantity

$$\underline{\Delta}_n(x, y) = \lim_{\varepsilon \downarrow 0} \Delta_n(x + i\varepsilon, y - i\varepsilon).$$

The covariance  $\Theta_n(z_1, z_2)$  splits into three parts  $\Theta_n(z_1, z_2) = \Theta_{0,n}(z_1, z_2) + \Theta_{1,n}(z_1, z_2) + \Theta_{2,n}(z_1, z_2)$ , cf. (2.6). We first prove that

$$\begin{aligned} -\frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int_{\mathbb{R}^2} f(x)g(y) \Theta_{0,n}(x \pm_1 i\varepsilon, y \pm_2 i\varepsilon) dx dy \\ = \frac{1}{2\pi^2} \int_{\mathbb{S}_n^2} f'(x)g'(y) \ln \left| \frac{\underline{\Delta}_n(x, y)}{\Delta_n(x, y)} \right| dx dy \end{aligned} \quad (6.1)$$

Taking advantage of formula (2.7) and performing a double integration by parts yields

$$\begin{aligned} & -\frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int_{\mathbb{R}^2} f(x)g(y) \Theta_{0,n}(x \pm_1 i\varepsilon, y \pm_2 i\varepsilon) dx dy \\ &= \frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int_{\mathbb{R}^2} f(x)g(y) \frac{\partial}{\partial y} \left\{ \frac{1}{\Delta_n(x \pm_1 i\varepsilon, y \pm_2 i\varepsilon)} \frac{\partial}{\partial x} \Delta_n(x \pm_1 i\varepsilon, y \pm_2 i\varepsilon) \right\} dx dy, \\ &= -\frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int_{\mathbb{R}^2} f(x)g'(y) \frac{1}{\Delta_n(x + i\varepsilon, y + i\varepsilon)} \frac{\partial}{\partial x} \Delta_n(x + i\varepsilon, y + i\varepsilon) dx dy, \\ &\stackrel{(a)}{=} \frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int_{\mathbb{R}^2} f'(x)g'(y) \log(\Delta_n(x \pm_1 i\varepsilon, y \pm_2 i\varepsilon)) dx dy, \\ &\stackrel{(b)}{=} \frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int_{\mathbb{R}^2} f'(x)g'(y) \ln |\Delta_n(x \pm_1 i\varepsilon, y \pm_2 i\varepsilon)| dx dy, \\ &\stackrel{(c)}{=} \frac{1}{2\pi^2} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^2} f'(x)g'(y) \ln \left| \frac{\Delta_n(x + i\varepsilon, y + i\varepsilon)}{\Delta_n(x + i\varepsilon, y - i\varepsilon)} \right| dx dy, \\ &\stackrel{(d)}{=} \frac{1}{2\pi^2} \lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^2} f'(x)g'(y) \ln \left| \frac{s_n(x + i\varepsilon) - s_n(y - i\varepsilon)}{s_n(x + i\varepsilon) - s_n(y + i\varepsilon)} \right| dx dy, \end{aligned}$$

where  $\log(\cdot)$  is any branch of the complex logarithm in (a), where (b) follows from the fact that the covariance being real, the argument part of the complex logarithm necessarily vanishes and where (c) and (d) follow from the representation formula for  $\Delta_n$  (cf. Proposition 4.1-(3)) and the fact that

$$\overline{\Delta_n(z_1, z_2)} = \frac{\overline{z_1} - \overline{z_2}}{s_n(\overline{z_1}) - s_n(\overline{z_2})} = \Delta_n(\overline{z_1}, \overline{z_2}).$$

Write now

$$\begin{aligned} \ln \left| \frac{s_n(x + i\varepsilon) - s_n(y - i\varepsilon)}{s_n(x + i\varepsilon) - s_n(y + i\varepsilon)} \right| &= \frac{1}{2} \ln \left| \frac{s_n(x + i\varepsilon) - s_n(y - i\varepsilon)}{s_n(x + i\varepsilon) - s_n(y + i\varepsilon)} \right|^2, \\ &= \frac{1}{2} \ln \left( 1 + \frac{4 \operatorname{Im} s_n(x + i\varepsilon) \operatorname{Im} s_n(y + i\varepsilon)}{|s_n(x + i\varepsilon) - s_n(y + i\varepsilon)|^2} \right). \end{aligned}$$

In order to apply the dominated convergence theorem, we need to majorize the right hand side above by an integrable function of  $(x, y) \in [-K, K]^2$  where  $K$  is sufficiently large to contain the supports of functions  $f$  and  $g$ . Let  $\varepsilon_0 > 0$ . Function  $s$  being continuous on a rectangle  $[0, K] \times [0, \varepsilon_0]$ , it is bounded. In particular,  $\operatorname{Im}(s(x + i\varepsilon)) \leq |s(x + i\varepsilon)|$  is also bounded on  $[0, K] \times [0, \varepsilon_0]$ .

Let  $z_1 = x + i\varepsilon$  and  $z_2 = y + i\varepsilon$ . Then by the definition (2.3) of  $\Delta_n$ ,

$$\begin{aligned} |\Delta_n(z_1, z_2)| &\leq \left( 1 + \sqrt{\nu(z_1, \overline{z_1})} \sqrt{\nu(z_2, \overline{z_2})} \right)^2 + \sqrt{|z_1|^2 \gamma(z_1, \overline{z_1}) \tilde{\gamma}(z_1, \overline{z_1})} \sqrt{|z_2|^2 \gamma(z_2, \overline{z_2}) \tilde{\gamma}(z_2, \overline{z_2})} \\ &\leq 5 \end{aligned}$$

by Proposition 4.1-(4). By the representation of  $\Delta_n$  provided in Proposition 4.1-(3), we have

$$|\Delta_n(z_1, z_2)|^2 = \frac{|z_1 - z_2|^2}{|s(z_1) - s(z_2)|^2} \leq 25 \quad \Rightarrow \quad \frac{1}{|s(z_1) - s(z_2)|^2} \leq \frac{25}{|z_1 - z_2|^2} = \frac{25}{|x - y|^2}.$$

In the end,

$$\ln \left| \frac{s_n(x + i\varepsilon) - s_n(y - i\varepsilon)}{s_n(x + i\varepsilon) - s_n(y + i\varepsilon)} \right| \leq \ln \left( 1 + \frac{K'}{|x - y|^2} \right)$$

which is integrable. It remains to apply the dominated convergence theorem to conclude and obtain (6.1).



We now prove that

$$\begin{aligned} & -\frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int_{\mathbb{R}^2} f(x)g(y)\Theta_{2,n}(x \pm_1 \mathbf{i}\varepsilon, y \pm_2 \mathbf{i}\varepsilon) dx dy \\ & = \frac{4\sigma^4 \kappa}{\pi^2 n^2} \sum_{\substack{i=0:N \\ j=0:n}} \int_{\mathcal{S}_n} f'(x) \operatorname{Im}(x t_{ii}(x) \tilde{t}_{jj}(x)) dx \int_{\mathcal{S}_n} g'(y) \operatorname{Im}(y t_{ii}(y) \tilde{t}_{jj}(y)) dy. \end{aligned} \quad (6.2)$$

By the mere definition of  $\Theta_{2,n}$ , we have

$$\begin{aligned} & -\frac{1}{4\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \int_{\mathbb{R}^2} f(x)g(y)\Theta_{2,n}(x \pm_1 \mathbf{i}\varepsilon, y \pm_2 \mathbf{i}\varepsilon) dx dy \\ & = -\frac{\kappa\sigma^4}{4n^2\pi^2} \lim_{\varepsilon \searrow 0} \sum_{\pm_1, \pm_2} (\pm_1 \pm_2) \sum_{i,j} \int_{\mathbb{R}^2} f(x)g(y) \frac{\partial}{\partial x}(x \pm_1 \mathbf{i}\varepsilon) t_{ii}(x \pm_1 \mathbf{i}\varepsilon) \tilde{t}_{jj}(x \pm_1 \mathbf{i}\varepsilon) \\ & \quad \times \frac{\partial}{\partial y}(y \pm_2 \mathbf{i}\varepsilon) t_{ii}(y \pm_2 \mathbf{i}\varepsilon) \tilde{t}_{jj}(y \pm_2 \mathbf{i}\varepsilon) dx dy, \\ & = \frac{\kappa\sigma^4}{n^2\pi^2} \lim_{\varepsilon \searrow 0} \sum_{i,j} \int_{\mathbb{R}} f'(x) \operatorname{Im} \{ (x + \mathbf{i}\varepsilon) t_{ii}(x + \mathbf{i}\varepsilon) \tilde{t}_{jj}(x + \mathbf{i}\varepsilon) \} dx \\ & \quad \times \int_{\mathbb{R}} g'(y) \operatorname{Im} \{ (y + \mathbf{i}\varepsilon) t_{ii}(y + \mathbf{i}\varepsilon) \tilde{t}_{jj}(y + \mathbf{i}\varepsilon) \} dy. \end{aligned}$$

It remains to prove that

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}} f'(x) \operatorname{Im} \{ (x + \mathbf{i}\varepsilon) t_{ii}(x + \mathbf{i}\varepsilon) \tilde{t}_{jj}(x + \mathbf{i}\varepsilon) \} dx = \int_{\mathbb{R}} f'(x) \operatorname{Im} \{ x t_{ii}(x) \tilde{t}_{jj}(x) \} dx \quad (6.3)$$

Let  $x > 0$ , then by [14, Theorem 2.1]

$$(x + \mathbf{i}\varepsilon) t_{ii}(x + \mathbf{i}\varepsilon) \tilde{t}_{jj}(x + \mathbf{i}\varepsilon) \xrightarrow{\varepsilon \searrow 0} x t_{ii}(x) \tilde{t}_{jj}(x).$$

In order to apply the dominated convergence theorem, we handle separately the cases  $c_n = 1$  and  $c_n < 1$ .

If  $c_n = 1$ , then  $\delta_z = \tilde{\delta}_z$  (apply Prop. 4.1-(2) for instance),  $\|T_z\| = \|\tilde{T}_z\|$  and

$$\|T_z\| \leq \frac{\sqrt{n}}{\sigma\sqrt{|z|}} \quad \text{and} \quad \|\tilde{T}_z\| \leq \frac{\sqrt{n}}{\sigma\sqrt{|z|}} \quad \text{for all } z \in \mathbb{C}^+.$$

In fact,

$$\|T_z\| = \sqrt{\lambda_{\max}(T_z T_z^*)} \leq \sqrt{\operatorname{Tr} T_z T_z^*} = \sqrt{n\sigma^{-2}\gamma(z, \bar{z})} \leq \frac{\sqrt{n}}{\sigma\sqrt{|z|}},$$

where the last inequality follows from Proposition 4.1-6. Now

$$|z t_{ii}(z) \tilde{t}_{jj}(z)| \leq |z| \|T_z\| \|\tilde{T}_z\| \leq \frac{n}{\sigma^2}.$$

We therefore apply the dominated convergence theorem and prove (6.3) in the case where  $c_n = 1$ .

Assume now that  $c_n < 1$  then  $0 \notin \mathcal{S}_n$ , where  $\mathcal{S}_n$  denotes the support of the measure associated to the Stieltjes transform  $\delta_n$ . In particular, there exists  $\eta > 0$  such that  $(-\eta, \eta) \cap \mathcal{S}_n = \emptyset$ . In the sequel, we will alternatively bound  $|z t_{ii}(z) \tilde{t}_{jj}(z)|$  on the sets

$$\mathcal{D} = \{|z| \leq \eta/2\} \quad \text{and} \quad \mathcal{D}'_A = \{|z| > \eta/2\} \cap [\eta/2, A] \times [0, A],$$

where  $A > 0$  is an arbitrary constant. This will enable us to apply the dominated convergence theorem and prove (6.3).

Let  $z \in \mathcal{D}$ . One has

$$\delta_z = \frac{\sigma}{n} \sum_{i=1}^N t_{ii}(z) \quad \text{and} \quad \tilde{\delta}_z = \frac{\sigma}{n} \sum_{j=1}^n \tilde{t}_{jj}(z)$$

hence  $\operatorname{Im}(t_{ii}(z)) \leq \frac{\sigma}{n} \operatorname{Im}(\delta_z)$  and  $\operatorname{Im}(\tilde{t}_{jj}(z)) \leq \frac{\sigma}{n} \operatorname{Im}(\tilde{\delta}_z)$ . We deduce from these inequalities that the probability measure  $\mu_{ii}$  associated to the Stieltjes transform  $t_{ii}$  has a support included in  $\mathcal{S}_n$  hence

$$|t_{ii}(z)| = \left| \int_{\mathcal{S}_n} \frac{\mu_{ii}(d\lambda)}{\lambda - z} \right| \leq \frac{2}{\eta} \quad \text{for } z \in \mathcal{D}.$$

By Proposition 4.1-(2), the support of the measure associated to  $\tilde{\delta}_z$  is  $\{0\} \cup \mathcal{S}_n$ . Let  $\tilde{\mu}_{jj}$  be the probability distribution with Stieltjes transform  $\tilde{t}_{jj}$ , then  $\text{supp}(\tilde{\mu}_{jj}) \subset \{0\} \cup \mathcal{S}_n$ . Otherwise stated,  $\tilde{\mu}_{jj}$  has a Dirac component at zero and a component  $\check{\mu}_{jj}$  with support included in  $\mathcal{S}_n$  hence

$$\tilde{t}_{jj}(z) = -\frac{\alpha_j}{z} + \int_{\mathcal{S}_n} \frac{\check{\mu}_{jj}(d\lambda)}{\lambda - z} \quad \text{and} \quad |z\tilde{t}_{jj}(z)| \leq \alpha_j + \frac{2}{\eta} \quad \text{for } z \in \mathcal{D}.$$

combining the two previous estimates yields a bound for  $|zt_{ii}(z)\tilde{t}_{jj}(z)|$  for  $z \in \mathcal{D}$ .

Let  $z \in \mathcal{D}'_A$ . By Proposition 4.1-(5), we have

$$\|\mathbb{T}_z\| \leq \sqrt{n\sigma^{-2}\gamma(z, \bar{z})} \leq \sqrt{\frac{2n}{\sigma^2\eta}}.$$

Recall the identity (4.19):

$$\tilde{\mathbb{T}}_z = -\frac{1}{z(1 + \sigma\delta_z)} + \frac{1}{z(1 + \sigma\delta_z)^2} \mathbb{A}^* \mathbb{T}_z \mathbb{A}.$$

We now use the fact that for any set  $[\eta/2, A] \times [0, A]$ , the function  $z \mapsto 1 + \sigma\delta_z$  is continuous [14, Theorem 2.1] and does not vanish. In fact, if  $z \in \mathbb{C}^+$  then  $1 + \sigma\delta_z \in \mathbb{C}^+$  and if  $x \in [\eta/2, A]$  then  $\text{Re}(1 + \sigma\delta_x) > 0$  by combining Lemma 2.1 and Theorem 2.1 in [14]. Finally

$$\|z\tilde{\mathbb{T}}_z\| \leq \frac{1}{|1 + \sigma\delta_z|} + \frac{1}{|1 + \sigma\delta_z|^2} \|\mathbb{A}\|^2 \|\mathbb{T}_z\|$$

is bounded over  $\mathcal{D}'_A$ . This finally yields a bound for  $|zt_{ii}(z)\tilde{t}_{jj}(z)|$  for  $z \in \mathcal{D}'_A$ . As a consequence of these bounds and the dominated convergence theorem, we obtain (6.3). Proof of Theorem 3 is completed.

#### APPENDIX A. USEFUL IDENTITIES AND ESTIMATES

We recall a result by Bai and Silverstein [2] that allows the control of the moments of quadratic forms.

**Lemma A.1.** [2, Lemma 2.7] *Let  $\mathbf{x} = (x_1, \dots, x_n)$  be an  $n \times 1$  vector where  $x_i$  are centered i.i.d. complex random variables with unit variance. Let  $\mathbb{M}$  be an  $n \times n$  Hermitian complex matrix. Then for any  $p \geq 2$ , there exists a constant  $K_p$  depending only on  $p$  for which*

$$\mathbb{E}|\mathbf{x}^* \mathbb{M} \mathbf{x} - \text{Tr} \mathbb{M}|^p \leq K_p \left( (\mathbb{E}|x_1|^4 \text{Tr} \mathbb{M} \mathbb{M}^*)^{p/2} + \mathbb{E}|x_1|^{2p} \text{Tr}(\mathbb{M} \mathbb{M}^*)^{p/2} \right).$$

The above lemma is the key for the following estimates:

**Lemma A.2.** *Let  $\mathbf{x}$  be defined as in Lemma A.1. Let  $\mathbb{M}_n$  be an  $n \times n$  Hermitian complex matrix independent of  $\mathbf{x}_n$  and having a uniformly bounded spectral norm. Then for  $p \in [2, 8]$*

$$\max \left( \mathbb{E} \left| \frac{1}{n} \mathbf{x}_n^* \mathbb{M}_n \mathbf{x}_n - \frac{1}{n} \text{Tr} \mathbb{M}_n \right|^p, \mathbb{E} \left| \mathbf{y}_n^* \mathbb{M}_n \mathbf{y}_n - \frac{\sigma^2}{n} \text{Tr} \mathbb{M}_n - \mathbf{a}_n^* \mathbb{M}_n \mathbf{a}_n \right|^p \right) = \mathcal{O} \left( \frac{\|\mathbb{M}_n\|^p}{n^{p/2}} \right).$$

In particular,

$$\mathbb{E}|\hat{\tau}_j(z)|^p = \mathcal{O}_z(n^{-p/2}) \quad \text{and} \quad \mathbb{E}|\hat{\alpha}_j(z)|^p = \mathcal{O}_z(n^{-p/2}).$$

Recall the definition of  $D_\varepsilon = [0, A] + i[\varepsilon, 1]$ , for  $A > 0$  and  $\varepsilon \in (0, 1)$  fixed.

**Lemma A.3.** *Assume that Assumptions 1 and 2 hold. Let  $k_1, k_2 \in \mathbb{N}$  and, for any  $z_1, z_2 \in D_\varepsilon$ , define*

$$M_j := M_j(z_1, z_2) = \mathbb{Q}_j^{k_1}(z_1) \mathbb{Q}_j^{k_2}(z_2).$$

Then for any  $p \in [2, 8]$ ,

$$\begin{aligned} \sup_{z \in D_\varepsilon} \|\mathbb{Q}_j(z)\|^p &\leq \varepsilon^{-p}, \\ \sup_{z \in D_\varepsilon} \mathbb{E}|\tau_j(z)|^p &= \mathcal{O}_\varepsilon(n^{-p/2}), \\ \sup_{z \in D_\varepsilon} \mathbb{E}|\mathbf{y}_j^* M_j(z) \mathbf{y}_j|^p &= \mathcal{O}(\varepsilon^{-(k_1+k_2)p}), \\ \sup_{z_1, z_2 \in D_\varepsilon} \mathbb{E}|\mathbf{y}_j^* M_j \mathbf{y}_j - \frac{\sigma^2}{n} \text{Tr} M_j - \mathbf{a}_j^* M_j \mathbf{a}_j|^p &= \mathcal{O}_\varepsilon(n^{-p/2}). \end{aligned}$$

The estimates in Lemmas A.2 and A.3 mainly follow from Lemma A.1 and thus their proof is omitted. For more details on the estimate  $\sup_{z \in D_\varepsilon} \mathbb{E}|\tau_j(z)|^p = \mathcal{O}_\varepsilon(n^{-p/2})$ , one can check Appendix A.2 in [20].

Recalling the notations in Section 4 and by simple application of the classical identity for the inverse of a perturbed matrix:

$$(A + XRY)^{-1} = A^{-1} - A^{-1}X(R^{-1} + YA^{-1}X)^{-1}YA^{-1},$$

we obtain

$$Q = Q_j - \frac{Q_j \mathbf{y}_j \mathbf{y}_j^* Q_j}{1 + \mathbf{y}_j^* Q_j \mathbf{y}_j} \quad \text{and} \quad Q_j = Q + \frac{Q \mathbf{y}_j \mathbf{y}_j^* Q}{1 - \mathbf{y}_j^* Q \mathbf{y}_j}. \quad (\text{A.1})$$

**Lemma A.4.** [20, Theorem 3.3] *Let  $(\mathbf{u}_n)$  and  $(\mathbf{v}_n)$  be two sequences of deterministic complex  $N \times 1$  vectors bounded by*

$$\sup_{n \geq 1} \max(\|\mathbf{u}_n\|, \|\mathbf{v}_n\|) < \infty,$$

and let  $(\mathbf{U}_n)$  be a sequence of deterministic  $N \times N$  matrix with bounded spectral norm

$$\sup_{n \geq 1} \|\mathbf{U}_n\| < \infty.$$

Then, in the setting of Theorem 2:

(1) *there exists a constant  $K$  such that*

$$\sum_{j=1}^n \mathbb{E} |\mathbf{u}_n^* Q_j \mathbf{a}_j|^2 \leq K,$$

(2) *there exists a constant  $K$  such that*

$$\left| \frac{1}{n} \text{Tr} \mathbf{U} (\mathbf{T} - \mathbb{E} Q) \right| \leq \frac{K}{n},$$

(3) *there exists a constant  $K$  such that*

$$\mathbb{E} |\text{Tr} \mathbf{U} (Q - \mathbb{E} Q)|^2 \leq K,$$

(4) *for any  $p \in [1, 2]$ , there exists a constant  $K_p$  such that*

$$\max \left\{ \mathbb{E} |\mathbf{u}_n^* (Q - \mathbf{T}) \mathbf{v}_n|^{2p}, \mathbb{E} |\mathbf{u}_n^* (Q_j - \mathbf{T}_j) \mathbf{v}_n|^{2p} \right\} \leq \frac{K_p}{n^p},$$

(5) *for any  $p \in [1, 4]$ , there exists a constant  $K$  such that*

$$\max \left\{ \mathbb{E} |\tilde{q}_{jj} - \tilde{b}_j|^{2p}, \mathbb{E} |\tilde{q}_{jj} - \tilde{t}_{jj}|^{2p} \right\} \leq \frac{K}{n^p}.$$

The next result is a counterpart of [20, Lemma 5.1] and is used for the computation of the asymptotic covariance in Section 4.

**Lemma A.5.** *For any  $N \times 1$  vector  $\mathbf{a}$  with bounded Euclidean norm, we have*

$$\max_j \text{Var} \left\{ \mathbf{a}^* (\mathbb{E}_j Q_{z_1}) (\mathbb{E}_j Q_{z_2}) \mathbf{a} \right\} = \mathcal{O}_{z_1, z_2} \left( \frac{1}{n} \right) \quad \text{and} \quad \max_j \text{Var} \left\{ \text{Tr} (\mathbb{E}_j Q_{z_1}) (\mathbb{E}_j Q_{z_2}) \right\} = \mathcal{O}_{z_1, z_2} (1).$$

## APPENDIX B. REMINDER OF THE MAIN NOTATIONS

### B.1. Main notations used in Sections 1 and 2.

*Matrix model, resolvent, linear statistics.*

$$\begin{aligned} Y_n &= \frac{\sigma}{\sqrt{n}} X_n + A_n, & Q_n(z) &= (Y_n Y_n^* - z I_N)^{-1}, \\ M_n(z) &= \text{Tr} Q_n(z) - \mathbb{E} \text{Tr} Q_n(z), & L_n(f) &= \text{Tr} f(Y_n Y_n^*) - \mathbb{E} \text{Tr} f(Y_n Y_n^*). \end{aligned}$$

*Canonical equations, deterministic equivalents.*

$$\left\{ \begin{array}{l} \delta_n = \frac{\sigma}{n} \text{Tr} \left( -z(1 + \sigma \tilde{\delta}_n) I_N + \frac{A_n A_n^*}{1 + \sigma \tilde{\delta}_n} \right)^{-1} \\ \tilde{\delta}_n = \frac{\sigma}{n} \text{Tr} \left( -z(1 + \sigma \delta_n) I_n + \frac{A_n^* A_n}{1 + \sigma \delta_n} \right)^{-1} \end{array} \right\}, \quad \left\{ \begin{array}{l} \mathbf{T}_n = \left( -z(1 + \sigma \tilde{\delta}_n) I_N + \frac{A_n A_n^*}{1 + \sigma \tilde{\delta}_n} \right)^{-1} = [t_{ij}] \\ \tilde{\mathbf{T}}_n = \left( -z(1 + \sigma \delta_n) I_n + \frac{A_n^* A_n}{1 + \sigma \delta_n} \right)^{-1} = [\tilde{t}_{ij}] \end{array} \right.$$

$$s_n(z) = z(1 + \sigma \delta_n(z))(1 + \sigma \tilde{\delta}_n(z)).$$

*Definition of  $\Delta_n$ .* We have

$$\Delta_n(z_1, z_2) = (1 - \nu(z_1, z_2))^2 - z_1 z_2 \gamma(z_1, z_2) \tilde{\gamma}(z_1, z_2)$$

where

$$\nu(z_1, z_2) = \frac{\sigma^2}{n} \frac{\text{Tr} \mathbf{T}_{z_1} \mathbf{A} \mathbf{A}^* \mathbf{T}_{z_2}}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})}, \quad \gamma(z_1, z_2) = \frac{\sigma^2}{n} \text{Tr} \mathbf{T}_{z_1} \mathbf{T}_{z_2}, \quad \tilde{\gamma}(z_1, z_2) = \frac{\sigma^2}{n} \text{Tr} \tilde{\mathbf{T}}_{z_1} \tilde{\mathbf{T}}_{z_2}.$$

*Definition of  $\Delta_n^\vartheta$ .* Recall the notations  $\vartheta = \mathbb{E}(x_{11}^n)^2$ ,  $\kappa = \mathbb{E}|x_{11}^n|^4 - |\vartheta|^2 - 2$ . Then

$$\Delta_n^\vartheta(z_1, z_2) = \left(1 - \vartheta\nu^\dagger\right) \left(1 - \bar{\vartheta}\bar{\nu}^\dagger\right) - |\vartheta|^2 z_1 z_2 \gamma^\dagger \bar{\gamma}^\dagger.$$

where

$$\begin{aligned} \gamma^\dagger(z_1, z_2) &= \frac{\sigma^2}{n} \text{Tr} \mathbf{T}_{z_1} \mathbf{T}_{z_2}^\top, & \nu^\dagger(z_1, z_2) &= \frac{\sigma^2}{n} \frac{\text{Tr} \mathbf{T}_{z_2}^\top \bar{\mathbf{A}} \mathbf{A}^* \mathbf{T}_{z_1}}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})}, \\ \bar{\gamma}^\dagger(z_1, z_2) &= \frac{\sigma^2}{n} \text{Tr} \tilde{\mathbf{T}}_{z_1} \tilde{\mathbf{T}}_{z_2}^\top, & \bar{\nu}^\dagger(z_1, z_2) &= \frac{\sigma^2}{n} \frac{\text{Tr} \mathbf{T}_{z_2} \mathbf{A} \mathbf{A}^\top \tilde{\mathbf{T}}_{z_1}^\top}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})}. \end{aligned}$$

*Covariance of the trace of the resolvent.*

$$\Theta_n(z_1, z_2) = \Theta_{0,n}(z_1, z_2) + \Theta_{1,n}(\vartheta, z_1, z_2) + \Theta_{2,n}(\kappa, z_1, z_2),$$

where

$$\begin{aligned} \Theta_{0,n}(z_1, z_2) &= -\frac{\partial}{\partial z_2} \left( \frac{1}{\Delta_n} \frac{\partial \Delta_n}{\partial z_1} \right) = \frac{s'_n(z_1) s'_n(z_2)}{(s_n(z_1) - s_n(z_2))^2} - \frac{1}{(z_1 - z_2)^2}, \\ \Theta_{1,n}(\vartheta, z_1, z_2) &= -\frac{\partial}{\partial z_2} \left( \frac{1}{\Delta_n^\vartheta} \frac{\partial \Delta_n^\vartheta}{\partial z_1} \right), \\ \Theta_{2,n}(\kappa, z_1, z_2) &= \kappa \frac{\partial^2}{\partial z_1 \partial z_2} \left\{ \frac{\sigma^4 z_1 z_2}{n^2} \sum_{i=1}^N t_{ii}(z_1) t_{ii}(z_2) \sum_{j=1}^n \tilde{t}_{jj}(z_1) \tilde{t}_{jj}(z_2) \right\}. \end{aligned}$$

*Useful subsets.*

$$\begin{aligned} D &= [0, A] + \mathbf{i}[0, 1], & D^\pm &= \{z \in D^+\} \cup \{\bar{z} \in D^+\}, \\ D^+ &= [0, A] + \mathbf{i}(0, 1], & D_\varepsilon &= [0, A] + \mathbf{i}[\varepsilon, 1], \quad (\varepsilon > 0). \end{aligned}$$

*Covariance of linear statistics.*

$$\begin{aligned} \Theta_{0,n}(f, g) &= \frac{1}{2\pi^2} \int_{S_n^2} f'(x) g'(y) \ln \left| \frac{s_n(x) - \overline{s_n(y)}}{s_n(x) - s_n(y)} \right| dx dy, \\ \Theta_{2,n}(\kappa, f, g) &= \frac{\sigma^4 \kappa}{\pi^2 n^2} \sum_{\substack{i=0:N \\ j=0:n}} \int_{S_n} f'(x) \text{Im}(x t_{ii}(x) \tilde{t}_{jj}(x)) dx \int_{S_n} g'(y) \text{Im}(y t_{ii}(y) \tilde{t}_{jj}(y)) dy. \end{aligned}$$

## B.2. Main notations used in Section 3.

*Rank-one perturbations and associated quantities.* The  $j^{\text{th}}$  columns of the matrices  $\mathbf{Y}_n$ ,  $\mathbf{A}_n$ ,  $\mathbf{X}_n$  and  $\frac{\sigma}{\sqrt{n}} \mathbf{X}_n$  are denoted by  $\mathbf{y}_j$ ,  $\mathbf{a}_j$ ,  $\mathbf{x}_j$  and  $\mathbf{r}_j$  and enable to express rank-one perturbation matrices

$$\mathbf{Y}_j \mathbf{Y}_j^* = \sum_{\ell \neq j} \mathbf{y}_\ell \mathbf{y}_\ell^*, \quad \mathbf{Q}_{z,j} = \mathbf{Q}_j(z) = (\mathbf{Y}_j \mathbf{Y}_j^* - z \mathbf{I}_N)^{-1}, \quad \tilde{\mathbf{Q}}_{z,j} = \tilde{\mathbf{Q}}_j(z) = (\mathbf{Y}_j^* \mathbf{Y}_j - z \mathbf{I}_n)^{-1}.$$

The diagonal entries  $\tilde{q}_{z,jj} = [\tilde{\mathbf{Q}}_z]_{jj}$  of the co-resolvent write

$$\tilde{q}_{jj} = \frac{-1}{z(1 + \mathbf{y}_j^* \mathbf{Q}_{z,j} \mathbf{y}_j)}.$$

*Auxiliary quantities.*

$$\begin{aligned} b_{z,j} &= \frac{-1}{z(1 + \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \mathbf{Q}_{z,j} + \mathbf{a}_j^* \mathbb{E} \mathbf{Q}_{z,j} \mathbf{a}_j)}, & \tau_{z,j} &= \mathbf{y}_j^* \mathbf{Q}_{z,j} \mathbf{y}_j - \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \mathbf{Q}_{z,j} - \mathbf{a}_j^* \mathbb{E} \mathbf{Q}_{z,j} \mathbf{a}_j, \\ \tilde{b}_{z,j} &= \frac{-1}{z(1 + \frac{\sigma^2}{n} \text{Tr} \mathbf{Q}_{z,j} + \mathbf{a}_j^* \mathbf{Q}_{z,j} \mathbf{a}_j)}, & \hat{\tau}_{z,j} &= \mathbf{y}_j^* \mathbf{Q}_{z,j} \mathbf{y}_j - \frac{\sigma^2}{n} \text{Tr} \mathbf{Q}_{z,j} - \mathbf{a}_j^* \mathbf{Q}_{z,j} \mathbf{a}_j, \\ \alpha_{z,j} &= \mathbf{y}_j^* \mathbf{Q}_{z,j}^2 \mathbf{y}_j - \frac{\sigma^2}{n} \text{Tr} \mathbf{Q}_{z,j}^2 - \mathbf{a}_j^* \mathbf{Q}_{z,j}^2 \mathbf{a}_j. \end{aligned}$$

**B.3. Main notations used in Section 4.** The quantity  $P_j(z)$  already appears in Section 3.

$$\begin{aligned} P_j(z) &= \mathbb{E}_j \left\{ z \tilde{b}_{z,j} \alpha_{z,j} + z^2 \tilde{b}_{z,j}^2 \hat{\tau}_{z,j} \left( \frac{\sigma^2}{n} \text{Tr} \mathbf{Q}_{z,j}^2 + \mathbf{a}_j^* \mathbf{Q}_{z,j}^2 \mathbf{a}_j \right) \right\}, \\ \Gamma_j(z) &= z \tilde{b}_{z,j} \left( \mathbf{y}_j^* \mathbf{Q}_{z,j} \mathbf{y}_j - \frac{\sigma^2}{n} \text{Tr} \mathbf{Q}_{z,j} - \mathbf{a}_j^* \mathbf{Q}_{z,j} \mathbf{a}_j \right) = z \tilde{b}_{z,j} \hat{\tau}_{z,j}, \\ A_n(z_1, z_2) &= \sum_{j=1}^n \mathbb{E}_{j-1} \{ \mathbb{E}_j \Gamma_j(z_1) \mathbb{E}_j \Gamma_j(z_2) \}. \end{aligned}$$

Breakdown of the covariance, following Proposition 1.1.

$$\begin{aligned}
\xi_{1j} &= \frac{\kappa\sigma^4}{n^2} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \sum_{i=1}^N \mathbb{E}_j[Q_{z_1, j}]_{ii} \mathbb{E}_j[Q_{z_2, j}]_{ii}, \\
\xi_{2j} &= \frac{\sigma^3}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \mathbb{E}(|x_{11}|^2 x_{11}) \\
&\quad \times \left( \frac{\mathbf{a}_j^* (\mathbb{E}_j Q_{z_1, j}) \text{vdiag}(\mathbb{E}_j Q_{z_2, j})}{\sqrt{n}} + \frac{\mathbf{a}_j^* (\mathbb{E}_j Q_{z_2, j}) \text{vdiag}(\mathbb{E}_j Q_{z_1, j})}{\sqrt{n}} \right), \\
\xi'_{2j} &= \frac{\sigma^3}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \mathbb{E}(|x_{11}|^2 \bar{x}_{11}) \\
&\quad \times \left( \frac{\text{vdiag}(\mathbb{E}_j Q_{z_2, j})^\top (\mathbb{E}_j Q_{z_1, j}) \mathbf{a}_j}{\sqrt{n}} + \frac{\text{vdiag}(\mathbb{E}_j Q_{z_1, j})^\top (\mathbb{E}_j Q_{z_2, j}) \mathbf{a}_j}{\sqrt{n}} \right), \\
\xi_{3j} &= \frac{\sigma^2}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \left( \frac{\sigma^2}{n} \text{Tr}(\mathbb{E}_j Q_{z_1, j} \mathbb{E}_j Q_{z_2, j}) \right. \\
&\quad \left. + \mathbf{a}_j^* \mathbb{E}_j Q_{z_1, j} \mathbb{E}_j Q_{z_2, j} \mathbf{a}_j + \mathbf{a}_j^* \mathbb{E}_j Q_{z_2, j} \mathbb{E}_j Q_{z_1, j} \mathbf{a}_j \right), \\
\xi_{4j} &= \frac{\sigma^2}{n} z_1 z_2 \tilde{t}_{z_1, jj} \tilde{t}_{z_2, jj} \left( \frac{1}{n} |\vartheta|^2 \sigma^2 \text{Tr}(\mathbb{E}_j Q_{z_1, j} \mathbb{E}_j Q_{z_2, j}^\top) \right. \\
&\quad \left. + \vartheta \mathbf{a}_j^* \mathbb{E}_j Q_{z_2, j} \mathbb{E}_j Q_{z_1, j}^\top \bar{\mathbf{a}}_j + \bar{\vartheta} \mathbf{a}_j^\top \mathbb{E}_j Q_{z_1, j} \mathbb{E}_j Q_{z_2, j} \mathbf{a}_j \right).
\end{aligned}$$

Computation of  $\Theta_{0,n}$ : Auxiliary quantities satisfying a system of equations.

$$\begin{aligned}
\psi_j(z_1, z_2) &= \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \{ (\mathbb{E}_j Q_{z_1}) (\mathbb{E}_j Q_{z_2}) \} &= \frac{\sigma^2}{n} \text{Tr} \mathbb{E} \{ (\mathbb{E}_j Q_{z_1}) Q_{z_2} \}, \\
\zeta_{kj}(z_1, z_2) &= \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j Q_{z_1}) (\mathbb{E}_j Q_{z_2}) \mathbf{a}_k \} &= \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j Q_{z_1}) Q_{z_2} \mathbf{a}_k \}, \\
\theta_{kj}(z_1, z_2) &= \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j Q_{z_1, k}) (\mathbb{E}_j Q_{z_2, k}) \mathbf{a}_k \} &= \sigma \mathbb{E} \{ \mathbf{a}_k^* (\mathbb{E}_j Q_{z_1, k}) Q_{z_2, k} \mathbf{a}_k \}, \\
\phi_j(z_1, z_2) &= \frac{\sigma}{n} \sum_{k=1}^j z_1 z_2 \tilde{t}_{z_1, kk} \tilde{t}_{z_2, kk} \theta_{kj}(z_1, z_2).
\end{aligned}$$

Coefficients of the system and its determinant.

$$\begin{aligned}
\nu_j &:= \nu_j(z_1, z_2) &= \frac{\sigma^2}{n} \sum_{k=1}^j \frac{a_k^* \Gamma_{z_1} \Gamma_{z_2} a_k}{(1 + \sigma \delta_{z_1})(1 + \sigma \delta_{z_2})}, \\
\eta_j &:= \eta_j(z_1, z_2) &= \frac{\sigma^2}{n} \sum_{\ell=1}^j z_1 z_2 \tilde{t}_{z_1, \ell \ell} \tilde{t}_{z_2, \ell \ell}, \\
\omega_j &:= \omega_j(z_1, z_2) &= \frac{\sigma^2}{n} \sum_{k=1}^j \sum_{\ell=1, \ell \neq k}^j \frac{\mathbf{a}_k^* \Gamma_{z_1} \mathbf{a}_\ell \mathbf{a}_\ell^* \Gamma_{z_2} \mathbf{a}_k}{(1 + \sigma \delta_{z_1})^2 (1 + \sigma \delta_{z_2})^2}.
\end{aligned}$$

The determinant of the system is  $\mathcal{D}_j = (1 - \nu_j)^2 - \gamma(\eta_j + \omega_j)$ .

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