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► **To cite this version:**

Luc Molinet. ASYMPTOTIC STABILITY FOR SOME NON POSITIVE PERTURBATIONS OF THE CAMASSA-HOLM PEAKON WITH APPLICATION TO THE ANTIPEAKON-PEAKON PROFILE. international mathematical research notices, 2020, 21, pp.7908-7943. hal-01768579v2

HAL Id: hal-01768579

<https://hal.science/hal-01768579v2>

Submitted on 3 Jul 2018

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**ASYMPTOTIC STABILITY FOR SOME NON POSITIVE
PERTURBATIONS OF THE CAMASSA-HOLM PEAKON WITH
APPLICATION TO THE ANTIPEAKON-PEAKON PROFILE**

LUC MOLINET

ABSTRACT. We continue our investigation on the asymptotic stability of the peakon . In a first step we extend our asymptotic stability result [34] in the class of functions whose negative part of the momentum density is supported in $] - \infty, x_0]$ and the positive part in $[x_0, +\infty[$ for some $x_0 \in \mathbb{R}$. In a second step this enables us to prove the asymptotic stability of well-ordered train of antipeakons-peakons and, in particular, of the antipeakon-peakon profile. Finally, in the appendix we prove that in the case of a non negative momentum density the energy at the left of any given point decays to zero as time goes to $+\infty$. This leads to an improvement of the asymptotic stability result stated in [34].

1. INTRODUCTION

In this paper we continue our investigation of the asymptotic stability of the peakon for the Camassa-Holm equation (C-H) by studying a particular case of solutions with a non signed momentum density. This leads to an asymptotic stability result for the antipeakon-peakon profile with respect to some perturbations.

Recall that the Camassa-Holm equation reads

CHCH (1.1)
$$u_t - u_{txx} = -3uu_x + 2u_xu_{xx} + uu_{xxx}, \quad (t, x) \in \mathbb{R}^2,$$

and can be derived as a model for the propagation of unidirectional shallow water waves over a flat bottom ([7], [28]). A rigorous derivation of the Camassa-Holm equation from the full water waves problem is obtained in [1] and [13].

(C-H) is completely integrable (see [7],[8], [10] and [12]) and enjoys also a geometrical derivation (cf. [29], [30]). It possesses among others the following invariants

E (1.2)
$$M(v) = \int_{\mathbb{R}} (v - v_{xx}) dx, \quad E(v) = \int_{\mathbb{R}} v^2(x) + v_x^2(x) dx \quad \text{and} \quad F(v) = \int_{\mathbb{R}} v^3(x) + v(x)v_x^2(x) dx$$

and can be written in Hamiltonian form as

(1.3)
$$\partial_t E'(u) = -\partial_x F'(u) \quad .$$

It is also worth noticing that (1.6) can be rewritten as

CHy (1.4)
$$y_t + uy_x + 2u_x y = 0$$

that is a transport equation for the momentum density $y = u - u_{xx}$.

Date: July 3, 2018.

2010 Mathematics Subject Classification. 35Q35,35Q51, 35B40.

Key words and phrases. Camassa-Holm equation, asymptotic stability, peakon, antipeakon-peakon.

Camassa and Holm [7] exhibited peaked solitary waves solutions to (C-H) that are given by

$$u(t, x) = \varphi_c(x - ct) = c\varphi(x - ct) = ce^{-|x-ct|}, \quad c \in \mathbb{R}.$$

They are called peakon whenever $c > 0$ and antipeakon whenever $c < 0$, the profile φ_c being the unique H^1 - weak solution of the differential equation

eqpeakon

$$(1.5) \quad -c\varphi_c + c\varphi_c'' + \frac{3}{2}\varphi_c^2 = \varphi_c\varphi_c'' + \frac{1}{2}(\varphi_c')^2.$$

Note that the initial value problem associated with (C-H) has to be rewritten as

CH

$$(1.6) \quad \begin{cases} u_t + uu_x + (1 - \partial_x^2)^{-1}\partial_x(u^2 + u_x^2/2) = 0 \\ u(0) = u_0, \end{cases}$$

to give a meaning to these solutions.

Their stability seems not to enter the general framework developed for instance in [3], [22], especially because of the non smoothness of the peakon. However, Constantin and Strauss [16] succeeded in proving their orbital stability by a direct approach.

In [34], making use of the finite speed propagation of the momentum density, we proved a rigidity result for solutions with a non negative momentum density. Following the framework developed by Martel and Merle (see for instance [32], [33]) this leads to an asymptotic stability result for the peakon with respect to non negative perturbations of the momentum density.

In this paper we continue our study of the asymptotic stability of the peakons of the Camassa-Holm equation. In a first part we extend our result in [34] on the asymptotic stability result for non negative initial momentum density $y_0 \in \mathcal{M}_+$ by considering the case where $\text{supp } y_0^- \subset]-\infty, x_0]$ and $\text{supp } y_0^+ \subset [x_0, +\infty[$ for some $x_0 \in \mathbb{R}$. Here y_0^- and y_0^+ denote respectively the negative and positive part of y_0 . In a second part we use this last result to prove an asymptotic stability result for well-ordered train of antipeakons and peakons.

It is well-known (cf. [35]) that under the above hypothesis on y_0 , the solution u exists for all positive times in Y and that there exists a C^1 -function $t \mapsto x_0(t)$ such that for all non negative times, $\text{supp } y^-(t) \subset]-\infty, x_0(t)]$ and $\text{supp } y^+(t) \subset [x_0(t), +\infty[$. On the other hand, in contrast to the case of a non negative momentum density, no uniform in time bound on the total variation of y is known in this case. Actually, only an exponential estimate on $\|y(t)\|_{\mathcal{M}}$ has been derived (see (2.14)). The main novelty of this work is the proof that, also in this case, the constructed asymptotic object has a non negative momentum density and is Y -almost localized. To prove our result we separate two possible behaviors of $x_0(t)$. The first case corresponds to the case where the point $x_0(t)$ travels far to the left of the bump (at least for large times). Then locally around the bump, the momentum density is non negative and we can argue as in the case of a non negative momentum density. The boundedness of the total variation of y around the bump being proved by using an almost decay result of $E(\cdot) + M(\cdot)$ at the right of a point that travels to the right between $x_0(\cdot)$ and the bump but far away from these last ones. The second case is more involved and corresponds to the case where $x_0(t)$ stays always close to the bump. Then we prove that the total variation of y has to decay exponentially fast in time in a small interval at the left of $x_0(\cdot)$. This enables us to prove an uniform in time estimate on the total variation of $y^+(t)$ and thus of $y(t)$ by using the conservation of $M(u)$ along the trajectory. We can thus pass to

the limit on a sequence of times. The fact that the asymptotic object has a non negative momentum density follows by extending the decay of the total variation of $y(t)$ to an interval that grows with time.

At this stage we would like to emphasize that we are not able to consider antipeakon-peakon collisions. Actually, our hypothesis forces the antipeakons to be initially at the left of the peakons and this property is preserved for positive times. Recently, Bressan and Constantin ([5], [6]) succeed to construct global conservative and dissipative solutions of the (1.6) for initial data in $H^1(\mathbb{R})$ by using scalar conservation laws techniques (see also [24], [27] for a slightly different point of view). These class of solutions take into account the antipeakon-peakon collisions which are studied in more details in [25] and [26] (see also [23] and references therein).

1.1. Statement of the results. Before stating our results let us introduce the function space where our initial data will take place. Following [15], we introduce the following space of functions

$$(1.7) \quad Y = \{u \in H^1(\mathbb{R}) \text{ such that } u - u_{xx} \in \mathcal{M}(\mathbb{R})\} .$$

We denote by Y_+ the closed subset of Y defined by $Y_+ = \{u \in Y / u - u_{xx} \in \mathcal{M}_+\}$ where \mathcal{M}_+ is the set of non negative finite Radon measures on \mathbb{R} .

Let $C_b(\mathbb{R})$ be the set of bounded continuous functions on \mathbb{R} , $C_0(\mathbb{R})$ be the set of continuous functions on \mathbb{R} that tends to 0 at infinity and let $I \subset \mathbb{R}$ be an interval. A sequence $\{\nu_n\} \subset \mathcal{M}$ is said to converge tightly (resp. weakly) towards $\nu \in \mathcal{M}$ if for any $\phi \in C_b(\mathbb{R})$ (resp. $C_0(\mathbb{R})$), $\langle \nu_n, \phi \rangle \rightarrow \langle \nu, \phi \rangle$. We will then write $\nu_n \rightarrow^* \nu$ tightly in \mathcal{M} (resp. $\nu_n \rightarrow^* \nu$ in \mathcal{M}).

Throughout this paper, $y \in C_{ti}(I; \mathcal{M})$ (resp. $y \in C_w(I; \mathcal{M})$) will signify that for any $\phi \in C_b(\mathbb{R})$ (resp. $\phi \in C_0(\mathbb{R})$), $t \mapsto \langle y(t), \phi \rangle$ is continuous on I and $y_n \rightarrow^* y$ in $C_{ti}(I; \mathcal{M})$ (resp. $y_n \rightarrow^* y$ in $C_w(I; \mathcal{M})$) will signify that for any $\phi \in C_b(\mathbb{R})$ (resp. $C_0(\mathbb{R})$), $\langle y_n(\cdot), \phi \rangle \rightarrow \langle y(\cdot), \phi \rangle$ in $C(I)$.

As explained above, the aim of this paper is to extend the results in [34] under the following hypothesis.

hyp

Hypothesis 1. *We will say that $u_0 \in Y$ satisfies Hypothesis 1 if there exists $x_0 \in \mathbb{R}$ such that its momentum density $y_0 = u_0 - u_{0,xx}$ satisfies*

hyp1

$$(1.8) \quad \text{supp } y_0^- \subset]-\infty, x_0] \quad \text{and} \quad \text{supp } y_0^+ \subset [x_0, +\infty[.$$

As in [34], the key tool to prove our results is the following rigidity property for Y -almost localized solutions of (1.6) with non negative density momentum :

liouville

Theorem 1.1 ([34]). *Let $u \in C(\mathbb{R}; H^1(\mathbb{R}))$, with $u - u_{xx} \in C_w(\mathbb{R}; \mathcal{M}_+)$, be a Y -almost localized solution of (1.6) that is not identically vanishing. Then there exists $c^* > 0$ and $x_0 \in \mathbb{R}$ such that*

$$u(t) = c^* \varphi(\cdot - x_0 - c^*t), \quad \forall t \in \mathbb{R} .$$

Recall that a Y -almost localized solution is defined in the following way :

defYlocalized

Definition 1.1. We say that a solution $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ with $u - u_{xx} \in C_w(\mathbb{R}; \mathcal{M}_+)$ of (1.6) is Y -almost localized if there exist $c > 0$ and a C^1 -function $x(\cdot)$, with

$x_t \geq c > 0$, for which for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that for all $t \in \mathbb{R}$ and all $\Phi \in C(\mathbb{R})$ with $0 \leq \Phi \leq 1$ and $\text{supp } \Phi \subset [-R_\varepsilon, R_\varepsilon]^c$.

$$\boxed{\text{defloc}} \quad (1.9) \quad \int_{\mathbb{R}} (u^2(t) + u_x^2(t)) \Phi(\cdot - x(t)) dx + \left\langle \Phi(\cdot - x(t)), u(t) - u_{xx}(t) \right\rangle \leq \varepsilon.$$

Our first new result is the asymptotic stability of the peakon with respect to perturbations that satisfy Hypothesis 1 :

$\boxed{\text{asymptstab}}$ **Theorem 1.2.** *Let $c > 0$ be fixed. There exists an universal constant $0 < \eta \leq 2^{-10}$ such that for any $0 < \theta < c$ and any $u_0 \in Y$, satisfying Hypothesis 1., such that*

$$\boxed{\text{difini}} \quad (1.10) \quad \|u_0 - \varphi_c\|_{H^1} \leq \eta \left(\frac{\theta}{c}\right)^8,$$

there exists $c^* > 0$ with $|c - c^*| \ll c$ and a C^1 -function $x : \mathbb{R} \rightarrow \mathbb{R}$ with $\lim_{t \rightarrow \infty} \dot{x}(t) = c^*$ such that

$$(1.11) \quad u(t, \cdot + x(t)) \xrightarrow[t \rightarrow +\infty]{} \varphi_{c^*} \text{ in } H^1(\mathbb{R}),$$

where $u \in C(\mathbb{R}; H^1)$ is the solution emanating from u_0 . Moreover,

$$\boxed{\text{cvforte}} \quad (1.12) \quad \lim_{t \rightarrow +\infty} \|u(t) - \varphi_{c^*}(\cdot - x(t))\|_{H^1(|x| > \theta t)} = 0.$$

Remark 1.1. We emphasize that (1.12) gives a strong H^1 -convergence result at the left of the line $x = -\theta t$. This is due to the fact that, contrary to KdV type equation, there is no small linear waves travelling to the left for the Camassa-Holm equation. In the appendix we even show that in the case of a non negative momentum density, all the energy is traveling to the right.

Combining this asymptotic stability result with the one obtained in [34] and the orbital stability of well ordered trains of antipeakons-peakons proven in [21], we are able to prove the asymptotic stability of such trains that contains, as a particular case, the asymptotic stability of the antipeakon-peakon profile.

$\boxed{\text{asympt-mult-peaks}}$

Theorem 1.3. *Let be given N_- negative velocities $c_{-N_-} < \dots < c_{-2} < c_{-1} < 0$, N_+ positive velocities $0 < c_1 < c_2 < \dots < c_{N_+}$ and $0 < \theta_0 < \min(|c_{-1}|, c_1)/4$. There exist $L_0 > 0$ and $\varepsilon_0 > 0$ such that if the initial data $u_0 \in Y$ satisfies Hypothesis 1 with*

$$\boxed{\text{inini}} \quad (1.13) \quad \|u_0 - \sum_{j=1}^{N_-} \varphi_{c_{-j}}(\cdot - z_{-j}^0) - \sum_{j=1}^{N_+} \varphi_{c_j}(\cdot - z_j^0)\|_{H^1} \leq \varepsilon_0^2,$$

for some

$$z_{N_-}^0 < \dots < z_{-1}^0 < z_1^0 < \dots < z_{N_+}^0 \text{ with } |z_j^0 - z_q^0| \geq L_0 \text{ for } j \neq q,$$

then there exist $c_{-N_-}^* < \dots < c_{-2}^* < c_{-1}^* < 0 < c_1^* < \dots < c_{N_+}^*$ with $|c_j^* - c_j| \ll |c_j|$ and C^1 -functions $t \mapsto x_j(t)$, with $\dot{x}_j(t) \rightarrow c_j^*$ as $t \rightarrow +\infty$, $j \in \{[-N_-, N_+]/\{0\}\}$, such that the solution $u \in C(\mathbb{R}_+; H^1(\mathbb{R}))$ of (1.6) emanating from u_0 satisfies

$$\boxed{\text{mul1}} \quad (1.14) \quad u(\cdot + x_j(t)) \xrightarrow[t \rightarrow +\infty]{} \varphi_{c_j^*} \text{ in } H^1(\mathbb{R}), \forall j \in \{[-N_-, N_+]/\{0\}\}.$$

Moreover,

$$\boxed{\text{mul2}} \quad (1.15) \quad u - \sum_{j=1}^{N_-} \varphi_{c_{-j}^*}(\cdot - x_{-j}(t)) - \sum_{j=1}^{N_+} \varphi_{c_j^*}(\cdot - x_j(t)) \xrightarrow[t \rightarrow +\infty]{} 0 \text{ in } H^1(|x| > \theta_0 t).$$

This paper is organized as follows : in the next section we recall the well-posedness results for the class of solutions we will work with. In Section 3, we derive an almost monotonicity result that is a straightforward adaptation of the one proven in [34]. Section 4 is devoted to the proof of the Y -almost localization of the elements of the ω -limit set of the orbits associated with initial data satisfying Hypothesis 1 and being H^1 -close enough to a peakon. This is the main new contribution of this paper. In Section 5 we deduce the asymptotic stability results. Finally in the appendix we present an improvement of the asymptotic stability result given in [34] by noticing that all the energy of any solution to (1.6) with a non negative density momentum is traveling to the right. Moreover, for such a solution, the energy at the the left of any given point decays to zero as $t \rightarrow +\infty$.

2. GLOBAL WELL-POSEDNESS RESULTS

We first recall some obvious estimates that will be useful in the sequel of this paper. Noticing that $p(x) = \frac{1}{2}e^{-|x|}$ satisfies $p * y = (1 - \partial_x^2)^{-1}y$ for any $y \in H^{-1}(\mathbb{R})$ we easily get

$$\|u\|_{W^{1,1}} = \|p * (u - u_{xx})\|_{W^{1,1}} \lesssim \|u - u_{xx}\|_{\mathcal{M}}$$

and

$$\|u_{xx}\|_{\mathcal{M}} \leq \|u\|_{L^1} + \|u - u_{xx}\|_{\mathcal{M}}$$

which ensures that

$$\boxed{\text{bv}} \quad (2.1) \quad Y \hookrightarrow \{u \in W^{1,1}(\mathbb{R}) \text{ with } u_x \in \mathcal{BV}(\mathbb{R})\}.$$

It is also worth noticing that for $v \in C_0^\infty(\mathbb{R})$, satisfying Hypothesis 1,

$$\boxed{\text{formulev}} \quad (2.2) \quad v(x) = \frac{1}{2} \int_{-\infty}^x e^{x'-x} (v - v_{xx})(x') dx' + \frac{1}{2} \int_x^{+\infty} e^{x-x'} (v - v_{xx})(x') dx'$$

and

$$v_x(x) = -\frac{1}{2} \int_{-\infty}^x e^{x'-x} (v - v_{xx})(x') dx' + \frac{1}{2} \int_x^{+\infty} e^{x-x'} (v - v_{xx})(x') dx',$$

so that for $x \leq x_0$ we get

$$v_x(x) = v(x) - e^{-x} \int_{-\infty}^x e^{x'} y(x') dx' \geq v(x)$$

whereas for $x \geq x_0$ we get

$$v_x(x) = -v(x) + e^x \int_x^{+\infty} e^{-x'} y(x') dx' \geq -v(x)$$

Throughout this paper, we will denote $\{\rho_n\}_{n \geq 1}$ the mollifiers defined by

$$\boxed{\text{rho}} \quad (2.3) \quad \rho_n = \left(\int_{\mathbb{R}} \rho(\xi) d\xi \right)^{-1} n \rho(n \cdot) \text{ with } \rho(x) = \begin{cases} e^{1/(x^2-1)} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

Following [35] we approximate $v \in Y$ satisfying Hypothesis 1 by the sequence of functions

$$\boxed{\text{app}} \quad (2.4) \quad v_n = p * y_n \text{ with } y_n = -(\rho_n * y^-) \left(\cdot + \frac{1}{n} \right) + (\rho_n * y^+) \left(\cdot - \frac{1}{n} \right) \text{ and } y = v - v_{xx}$$

that belong to $Y \cap H^\infty(\mathbb{R})$ and satisfy Hypothesis 1 with the same x_0 . It is not too hard to check that

$$\boxed{\text{estyn}} \quad (2.5) \quad \|y_n\|_{L^1} \leq \|y\|_{\mathcal{M}}$$

Moreover, noticing that

$$v_n = -\left(\rho_n * (p * y^-)\right)\left(\cdot + \frac{1}{n}\right) + \left(\rho_n * (p * y^+)\right)\left(\cdot - \frac{1}{n}\right)$$

with $p * y^\mp \in H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R})$, we infer that

$$\boxed{\text{mm}} \quad (2.6) \quad v_n \rightarrow v \in H^1(\mathbb{R}) \cap W^{1,1}(\mathbb{R}).$$

that ensures that for any $v \in Y$ satisfying Hypothesis 1 it holds

$$\boxed{\text{dodo}} \quad (2.7) \quad v_x \geq v \text{ on }]-\infty, x_0[\quad \text{and} \quad v_x \geq -v \text{ on }]x_0, +\infty[.$$

$\boxed{\text{prop1}}$ **Proposition 2.1.** (Global weak solution [35])

Let $u_0 \in Y$ satisfying Hypothesis 1 for some $x_0 \in \mathbb{R}$.

1. Uniqueness and global existence : (1.6) has a unique solution

$$u \in C(\mathbb{R}_+; H^1(\mathbb{R})) \cap C^1(\mathbb{R}_+; L^2(\mathbb{R}))$$

such that $y = (1 - \partial_x^2)u \in C_{ti}(\mathbb{R}_+; \mathcal{M})$. $E(u)$, $F(u)$ and $M(u) = \langle y, 1 \rangle$ are conservation laws. Moreover, for any $t \in \mathbb{R}_+$, the density momentum $y(t)$ satisfies $\text{supp } y^-(t) \subset]-\infty, q(t, x_0)]$ and $\text{supp } y^+(t) \subset [q(t, x_0), +\infty[$ where $q(\cdot, \cdot)$ is defined by

$$\boxed{\text{defq}} \quad (2.8) \quad \begin{cases} q_t(t, x) &= u(t, q(t, x)) & , (t, x) \in \mathbb{R}^2 \\ q(0, x) &= x & , x \in \mathbb{R} \end{cases}.$$

2. Continuity with respect to the H^1 -norm: For any sequence $\{u_{0,n}\}$ bounded in Y that satisfy Hypothesis 1 and such that $u_{0,n} \rightarrow u_0$ in $H^1(\mathbb{R})$, the emanating sequence of solutions $\{u_n\} \subset C^1(\mathbb{R}_+; L^2(\mathbb{R})) \cap C(\mathbb{R}_+; H^1(\mathbb{R}))$ satisfies for any $T > 0$

$$\boxed{\text{cont1}} \quad (2.9) \quad u_n \rightarrow u \text{ in } C([0, T]; H^1(\mathbb{R})).$$

Moreover, if $\{u_{0,n}\}$ is the sequence defined by (2.4) then

$$\boxed{\text{cont2}} \quad (2.10) \quad (1 - \partial_x^2)u_n \rightharpoonup * y \text{ in } C_{ti}([0, T], \mathcal{M}).$$

3. Continuity with respect to initial data in Y equipped with its weak topology: Let $\{u_{0,n}\}$ be a bounded sequence of Y such that $u_{0,n} \rightharpoonup u_0$ in $H^1(\mathbb{R})$ and such that the emanating sequence of solution $\{u_n\}$ is bounded in $C([-T_-, T_+]; H^1) \cap L^\infty(-T_-, T_+; Y)$ for some $(T_-, T_+) \in (\mathbb{R}_+)^2$. Then the solution u of (1.6) emanating from u_0 belongs to $C([-T_-, T_+]; H^1)$ with $y = u - u_{xx} \in C_w([-T_-, T_+]; \mathcal{M})$. Moreover,

$$\boxed{\text{weakcont}} \quad (2.11) \quad u_n \xrightarrow[n \rightarrow \infty]{} u \text{ in } C_w([-T_-, T_+]; H^1(\mathbb{R})),$$

and

$$\boxed{\text{cont22}} \quad (2.12) \quad (1 - \partial_x^2)u_n \rightharpoonup * y \text{ in } C_w([-T_-, T_+], \mathcal{M}).$$

Proof. For sake of completeness let us recall that the global existence result follows from the following estimate on the density momentum y of smooth solutions with

initial data $u_0 \in Y$ that satisfies Hypothesis 1 :

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}} y_+(t, x) dx &= \frac{d}{dt} \int_{q(t, x_0)}^{+\infty} y(t, x) dx \\
&= - \int_{q(t, x_0)}^{+\infty} u_x(t, x) y(t, x) dx \\
&\leq \sup_{x \in \mathbb{R}} (-u_x(t, x)) \int_{q(t, x_0)}^{+\infty} y(t, x) dx \\
&\leq \sup_{x \in \mathbb{R}} (|u(t, x)|) \int_{\mathbb{R}} y^+(t, x) dx \\
&\leq \sqrt{E(u_0)} \int_{\mathbb{R}} y^+(t, x) dx ,
\end{aligned}$$

where one used (1.4), (2.7), the conservation of the energy and the classical Sobolev inequality :

$$\boxed{\text{sobo}} \quad (2.13) \quad \|v\|_{L^\infty} \leq \frac{1}{\sqrt{2}} \|v\|_{H^1}, \quad \forall v \in H^1(\mathbb{R}).$$

Indeed, then by Gronwall estimate and the conservation of M one gets

$$\boxed{\text{yL1}} \quad (2.14) \quad \|y\|_{L^1} \leq 2 \exp(\sqrt{E(u_0)}t) \|y_0\|_{L^1}$$

which ensures that the associated solution u can be extended for all positive times. Finally the global existence result for $u_0 \in Y$ follows by approximating u_0 as in (2.4) and proceeding as in [15].

In this way, the uniqueness and global existence results are obtained in [35] except the conservation of $M(u)$ and the fact that y belongs to $C_{ti}(\mathbb{R}_+; \mathcal{M})$. In [35], only the fact that $y \in L_{loc}^\infty(\mathbb{R}_+, \mathcal{M})$ is stated. But these properties will follow directly from (2.10) since $M(u)$ is a conservation law for any smooth solution $u \in C(\mathbb{R}_+; H^3)$ with $u - u_{xx} \in L_{loc}^\infty(\mathbb{R}_+; L^1(\mathbb{R}))$.

To prove (2.9), it suffices to notice that, according to the conservation of the H^1 -norm and (2.14), the sequence of emanating solution $\{u_n\}$ is bounded in $C(\mathbb{R}_+; H^1(\mathbb{R})) \cap L^\infty(]0, T[; W^{1,1}(\mathbb{R}))$ with $\{u_{n,x}\}$ bounded in $L^\infty(]0, T[; \mathcal{BV}(\mathbb{R}))$, for any $T > 0$. Therefore, there exists $v \in L^\infty(\mathbb{R}_+; H^1(\mathbb{R}))$ with $(1 - \partial_x^2)v \in L_{loc}^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{R}))$ such that, for any $T > 0$,

$$u_n \xrightarrow[n \rightarrow \infty]{} v \in L^\infty(]0, T[; H^1(\mathbb{R})) \text{ and } (1 - \partial_x^2)u_n \xrightarrow[n \rightarrow \infty]{} * (1 - \partial_x^2)v \text{ in } L^\infty(]-T, T[; \mathcal{M}(\mathbb{R})).$$

Moreover, in view of (1.6), $\{\partial_t u_n\}$ is bounded in $L^\infty(]0, T[; L^2(\mathbb{R}) \cap L^1(\mathbb{R}))$ and Helly's, Aubin-Lions compactness and Arzela-Ascoli theorems ensure that v is a solution to (1.6) that belongs to $C_w(]0, T[; H^1(\mathbb{R}))$ with $v(0) = u_0$ and that (2.10) holds. In particular, $v_t \in L^\infty(]0, T[; L^2(\mathbb{R}))$ and thus $v \in C(]0, T[; L^2(\mathbb{R}))$. Since $v \in L^\infty(]0, T[; H^{\frac{3}{2}-}(\mathbb{R}))$, this actually implies that $v \in C(]0, T[; H^{\frac{3}{2}-}(\mathbb{R}))$. Therefore, v belongs to the uniqueness class which ensures that $v = u$. The conservation of $E(\cdot)$ and the above weak convergence results then lead to (2.9).

To prove (2.10) we use the following $W^{1,1}$ -Lipschitz bound that is proven in [15] and [35] : Let $u_0^i \in Y$ for $i = 1, 2$ and let $u^i \in C(]0, T[; H^1) \cap L^\infty(0, T; Y)$ be the associated solution of (1.6). Setting $M = \sum_{i=1}^2 \|u^i - u_{xx}^i\|_{L^\infty(0, T; \mathcal{M})}$, it holds

$$\boxed{\text{lip}} \quad (2.15) \quad \|u^1 - u^2\|_{L^\infty(0, T; W^{1,1})} \lesssim e^{6MT} \|u_0^1 - u_0^2\|_{W^{1,1}}.$$

Let $\{u_{0,n}\}$ the sequence defined by (2.4). The sequence of emanating solutions $\{u_n\}$ is included in $C(\mathbb{R}_+; H^\infty) \cap C(\mathbb{R}_+; W^{1,1})$. Moreover in view of (2.5), (2.6) and (2.15), for any $T > 0$, $\{u_n\}$ is a Cauchy sequence in $C([0, T]; W^{1,1})$ and thus

$$\boxed{\text{convW}} \quad (2.16) \quad u_n \rightarrow u \in C(\mathbb{R}_+; W^{1,1}).$$

Now, we notice that for any $v \in BV(\mathbb{R})$ and any $\phi \in C^1(\mathbb{R})$, it holds

$$\langle v', \phi \rangle = - \int v \phi'.$$

Therefore, setting $y_n = u_n - u_{n,xx}$, (2.16) ensures that, for any $t \in \mathbb{R}_+$,

$$\int_{\mathbb{R}} y_n \phi = \int_{\mathbb{R}} (u_n \phi + u_{n,x} \phi') \rightarrow \int_{\mathbb{R}} (u \phi + u_x \phi') = \langle y, \phi \rangle$$

and thus $y_n(t) \rightharpoonup * y(t)$ tightly in \mathcal{M} . Using again that $\{\partial_t u_n\}$ is bounded in $L^\infty(0, T; L^1)$, Arzela-Ascoli theorem leads then to (2.10). Finally (2.11)-(2.12) can be proven exactly in the same way, since $u_0 \in Y$ and $\{u_n\}$ is bounded in $L^\infty(\cdot - T_-, T_+; Y)$ by hypotheses. \square

3. MONOTONICITY RESULTS

We have to prove a monotonicity result for our solutions. The novelty with respect to the monotonicity result proven in [34] is that we only require the momentum density to be non negative at the right of some curve. This is possible since, the differential equation satisfied by y being local and of order 1, we may test y with a test function Φ that vanishes on \mathbb{R}_- . Note that this is not possible to use such test function for the energy since the non local term contained in the time derivative of the energy density imposes to require a condition similar to (3.4) on the test function.

As in [33], we introduce the C^∞ -function Ψ defined on \mathbb{R} by

$$\boxed{\text{defPsi}} \quad (3.1) \quad \Psi(x) = \frac{2}{\pi} \arctan(\exp(x/6))$$

It is easy to check that $\Psi(-\cdot) = 1 - \Psi$ on \mathbb{R} , Ψ' is a positive even function and that there exists $C > 0$ such that $\forall x \leq 0$,

$$\boxed{\text{psipsi}} \quad (3.2) \quad |\Psi(x)| + |\Psi'(x)| \leq C \exp(x/6).$$

Moreover, by direct calculations, it is easy to check that

$$\boxed{\text{psi3}} \quad (3.3) \quad |\Psi'''| \leq \frac{1}{2} \Psi' \text{ on } \mathbb{R}$$

and that

$$\boxed{\text{po}} \quad (3.4) \quad \Psi'(x) \geq \Psi'(2) = \frac{1}{3\pi} \frac{e^{1/3}}{1 + e^{2/3}}, \quad \forall x \in [0, 2].$$

We also introduce the function Φ defined by

$$\boxed{\text{defPhi}} \quad (3.5) \quad \Phi(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ x/2 & \text{for } x \in [0, 2] \\ 1 & \text{for } x \geq 2 \end{cases}$$

almostdecay

Lemma 3.1. *Let $0 < \alpha < 1$ and let $u \in C(\mathbb{R}; H^1)$ with $y = (1 - \partial_x^2)u \in C_w(\mathbb{R}; \mathcal{M}^+)$ be a solution of (1.6), emanating from an initial datum $u_0 \in Y$ that satisfies Hypothesis 1, such that there exist $x : \mathbb{R}_+ \rightarrow \mathbb{R}$ of class C^1 with $\inf_{\mathbb{R}} \dot{x} \geq c_0 > 0$ and $R_0 > 0$ with*

$$\text{loc} \quad (3.6) \quad \|u(t)\|_{L^\infty(|x-x(t)|>R_0)} \leq \frac{(1-\alpha)c_0}{2^6}, \quad \forall t \in \mathbb{R}_+.$$

For $0 < \beta \leq \alpha$, $0 \leq \gamma \leq \frac{1}{3\pi(1+e^{2/3})}(1-\alpha)c_0$, $R > 0$, $t_0 \in \mathbb{R}_+$ and any C^1 -function $z : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$\text{condz} \quad (3.7) \quad (1-\alpha)\dot{x}(t) \leq \dot{z}(t) \leq (1-\beta)\dot{x}(t), \quad \forall t \in \mathbb{R}_+,$$

we set

$$\text{defI} \quad (3.8) \quad I_{t_0}^{\mp R}(t) = \left\langle u^2(t) + u_x^2(t), \Psi\left(\cdot - z_{t_0}^{\mp R}(t)\right) \right\rangle + \gamma \left\langle y(t), \Phi\left(\cdot - z_{t_0}^{\mp R}(t)\right) \right\rangle,$$

where

$$\text{condz2} \quad (3.9) \quad z_{t_0}^{\mp R}(t) = x(t_0) \mp R + z(t) - z(t_0)$$

Let also $q(\cdot, \cdot)$ be defined as in (2.8). Then if

$$\text{zo1} \quad (3.10) \quad z_{t_0}^+R(t) \geq q(t, x_0), \quad \text{for } 0 \leq t \leq t_0,$$

it holds

$$\text{mono} \quad (3.11) \quad I_{t_0}^+R(t_0) - I_{t_0}^+R(t) \leq K_0 e^{-R/6}, \quad \forall 0 \leq t \leq t_0$$

whereas if

$$\text{zo2} \quad (3.12) \quad z_{t_0}^-R(t) \geq q(t, x_0), \quad \text{for } t \geq t_0$$

it holds

$$\text{mono2} \quad (3.13) \quad I_{t_0}^-R(t) - I_{t_0}^-R(t_0) \leq K_0 e^{-R/6}, \quad \forall t \geq t_0,$$

for some constant $K_0 > 0$ that only depends on $E(u)$, c_0 , R_0 and β .

Proof. The proof is the same as in [34]. The only difference is that we replace Ψ by Φ to test the density momentum y .

We first approximate u_0 by the sequence $\{u_{0,n}\} \in Y \cap C^\infty(\mathbb{R})$ as in (2.4). Then, by Proposition 2.1, the emanating solutions u_n exist for all positive time and satisfy (2.14). In particular, for any $T_0 > 0$ fixed,

$$\|u_{n,x}\|_{L^\infty(]0, T_0[\times \mathbb{R})} \leq \sup_{t \in]0, T_0[} \|y_n(t)\|_{L^1(\mathbb{R})} \leq 2 \exp(2\sqrt{E(u_0)T_0}) \|y_0\|_{\mathcal{M}}$$

and thus for any $\varepsilon > 0$ there exists $\theta = \theta_{\varepsilon, T_0} > 0$ such that if $\|u_n - u\|_{L^\infty(]0, T_0[\times \mathbb{R})} < \theta_{\varepsilon, T_0}$ then

$$\text{qqq} \quad (3.14) \quad |q_n(t, x_0) - q(t, x_0)| < \varepsilon, \quad \forall t \in]0, T_0[.$$

Now, for any fixed $t_0, T > 0$, (2.9) ensures that there exists $n_0 = n_0(t_0 + T) \geq 0$ such that for any $n \geq n_0$,

$$\|u_n - u\|_{L^\infty(]0, t_0+T[\times \mathbb{R})} < \max\left(\frac{\alpha c_0}{2^6}, \theta_{\frac{1}{2}, t_0+T}\right).$$

which together with (3.6) and (3.14) force

$$\text{dif} \quad (3.15) \quad \sup_{t \in]0, t_0+T[} \|u_n\|_{L^\infty(|x-x(t)|>R_0)} < \frac{(1-\alpha)c_0}{2^5}$$

and

$$\boxed{\text{difq}} \quad (3.16) \quad \sup_{t \in]0, t_0 + T[} |q_n(t, x_0 - q(t, x_0))| \leq \frac{1}{2}.$$

We first prove that (3.11) holds on $[0, t_0]$ with u replaced by u_n for $n \geq n_0$. The following computations hold for u_n with $n \geq n_0$ but, to simplify the notation, we drop the index n . According to [34] we have

$$\boxed{\text{go}} \quad (3.17) \quad \frac{d}{dt} \int_{\mathbb{R}} (u^2 + u_x^2) g = \int_{\mathbb{R}} u u_x^2 g' + 2 \int_{\mathbb{R}} u h g',$$

where $h := (1 - \partial_x^2)^{-1}(u^2 + u_x^2/2)$ and

$$\boxed{\text{go2}} \quad (3.18) \quad \frac{d}{dt} \int_{\mathbb{R}} y g dx = \int_{\mathbb{R}} y u g' + \frac{1}{2} \int_{\mathbb{R}} (u^2 - u_x^2) g'.$$

Applying (3.17) with $g(t, x) = \Psi(x - z_{t_0}^R(t))$ and (3.18) with $g(t, x) = \Phi(x - z_{t_0}^R(t))$ we get

$$\begin{aligned} \frac{d}{dt} I_{t_0}^{+R}(t) &= -\dot{z}(t) \int_{\mathbb{R}} [\Psi'(u^2 + u_x^2) + \gamma \Phi' y] + \frac{\gamma}{2} \int_{\mathbb{R}} (u^2 - u_x^2) \Phi' \\ &\quad + \int_{\mathbb{R}} [\Psi' u u_x^2 + \gamma \Phi' y u] + 2 \int_{\mathbb{R}} u h \Psi' \\ \boxed{\text{go3}} \quad (3.19) \quad &\leq -\dot{z}(t) \int_{\mathbb{R}} [\Psi'(u^2 + u_x^2) + \gamma \Phi' y] + \frac{\gamma}{2} \int_{\mathbb{R}} (u^2 - u_x^2) \Phi' + J_1 + J_2. \end{aligned}$$

Now, in view of (3.4) and the conditions on γ , we have

$$\dot{z}(t) \Psi' - \frac{\gamma}{2} \Phi' \geq \frac{(1 - \alpha) c_0}{4} \Psi' \quad \text{on } \mathbb{R}$$

that leads to

$$-\dot{z}(t) \int_{\mathbb{R}} [\Psi'(u^2 + u_x^2) + \gamma \Phi' y] + \frac{\gamma}{2} \int_{\mathbb{R}} (u^2 - u_x^2) \Phi' \leq -\frac{(1 - \alpha) c_0}{4} \int_{\mathbb{R}} [\Psi'(u^2 + u_x^2) + \gamma \Phi' y]$$

where we used that, according to (3.9) and (3.16), $y \geq 0$ on the support of Φ' . Finally the terms J_1 and J_2 are treated as in [21]. For instance, to estimate J_1 we divide \mathbb{R} into two regions relating to the size of $|u|$ as follows

$$\begin{aligned} J_1(t) &= \int_{|x-x(t)| < R_0} [\Psi' u u_x^2 + \gamma \Phi' y u] + \int_{|x-x(t)| > R_0} [\Psi' u u_x^2 + \gamma \Phi' y u] \\ \boxed{\text{J0}} \quad (3.20) \quad &= J_{11} + J_{12}. \end{aligned}$$

Observe that (3.7) ensures that $\dot{x}(t) - \dot{z}(t) \geq \beta c_0$ for all $t \in \mathbb{R}$ and thus, for $|x - x(t)| < R_0$,

$$\boxed{\text{to1}} \quad (3.21) \quad x - z_{t_0}^R(t) = x - x(t) - R + (x(t) - z(t)) - (x(t_0) - z(t_0)) \leq R_0 - R - \beta c_0(t_0 - t)$$

and thus the decay properties of Ψ' and the compact support of Φ' ensure that

$$\begin{aligned} J_{11}(t) &\lesssim \left[\|u(t)\|_{L^\infty} (\|u_x(t)\|_{L^2}^2 + c_0 \|y(t)\|_{L^1}) \right] e^{R_0/6} e^{-R/6} e^{-\frac{\beta}{6} c_0(t_0 - t)} \\ \boxed{\text{J11}} \quad (3.22) \quad &\lesssim \|u_0\|_{H^1} (\|u_0\|_{H^1}^2 + c_0 \|y_0\|_{L^1}) e^{R_0/6} e^{-R/6} e^{-\frac{\beta}{6} c_0(t_0 - t)}. \end{aligned}$$

On the other hand, (3.15) ensures that for all $t \in [t_0 - T, t_0]$ it holds

$$\begin{aligned} J_{12} &\leq 4\|u\|_{L^\infty(|x-x(t)|>R_0)} \int_{|x-x(t)|>R_0} [\Psi' u_x^2 + \gamma \Phi' y] \\ \text{J12} \quad (3.23) \quad &\leq \frac{(1-\alpha)c_0}{8} \int_{|x-x(t)|>R_0} [\Psi' u_x^2 + \gamma \Phi' y] \quad . \end{aligned}$$

Gathering (3.20), (3.22), (3.23) and the estimates on J_2 that we omit, since it is exactly the same terms as in [21], we conclude that there exists $C > 0$ only depending on R_0 and $E(u)$ such that for $R \geq R_0$ and $t \in [0, t_0]$ it holds

$$\text{nini} \quad (3.24) \quad \frac{d}{dt} I_{t_0}^{+R}(t) \leq -\frac{(1-\alpha)c_0}{8} \int_{\mathbb{R}} [\Psi'(u^2 + u_x^2) + \gamma \Phi' y] + C e^{-R/6} e^{-\frac{\beta}{8}(t_0-t)} .$$

Integrating between t and t_0 we obtain (3.11) for any $t \in [0, t_0]$ and u replaced by u_n with $n \geq n_0$. Note that the constant appearing in front of the exponential now also depends on β . The convergence results (2.9)-(2.10) then ensure that (3.11) holds also for u and $t \in [0, t_0]$. Finally, (3.13) can be proven in exactly the same way by noticing that for $|x - x(t)| < R_0$ it holds

$$\begin{aligned} \text{to2} \quad (3.25) \quad &x - z_{t_0}^{-R}(t) = x - x(t) + R + (x(t) - z(t)) - (x(t_0) - z(t_0)) \geq -R_0 + R + \beta c_0(t - t_0) . \\ &\square \end{aligned}$$

Remark 3.1. It is worth noticing that the definitions of Ψ , Φ , (3.4) and (3.8) ensure that

$$(3.26) \quad I_{t_0}^{x_0}(t) \geq \frac{1}{9\pi} \left\langle u^2(t) + u_x^2(t) + y(t), \Phi(\cdot - z_{t_0}^{x_0}(t)) \right\rangle, \forall t \in \mathbb{R} .$$

4. PROPERTIES OF THE ASYMPTOTIC OBJECT

The aim of this section is to prove that for $u_0 \in Y$ satisfying Hypothesis 1 and H^1 -close enough to a peakon, one can extract from the orbit of u_0 an asymptotic object that has a non negative density momentum and gives rise to a Y -almost localized solution.

Let $u_0 \in Y$ satisfying Hypothesis 1, such that

$$\text{stab} \quad (4.1) \quad \|u_0 - c\varphi\|_{H^1} < \left(\frac{\varepsilon^2}{3c^2}\right)^4, \quad 0 < \varepsilon < c,$$

then, according to [16] and [35],

$$\text{stabo} \quad (4.2) \quad \sup_{t \in \mathbb{R}_+} \|u(t) - c\varphi(\cdot - \xi(t))\|_{H^1} < \frac{\varepsilon^2}{c},$$

where $u \in C(\mathbb{R}_+; H^1)$ is the solution emanating from u_0 and $\xi(t) \in \mathbb{R}$ is any point where the function $u(t, \cdot)$ attains its maximum. According to [34], by the implicit function theorem, we have the following lemma.

modulation **Lemma 4.1.** *There exists $0 < \varepsilon_0 < 1$, $\kappa_0 > 0$, $n_0 \in \mathbb{N}$ and $K > 1$ such that if a solution $u \in C(\mathbb{R}; Y)$ to (1.6) satisfies*

$$\text{gff} \quad (4.3) \quad \sup_{t \in \mathbb{R}_+} \|u(t) - c\varphi(\cdot - z(t))\|_{H^1} < c\varepsilon_0 ,$$

for some function $z : \mathbb{R}_+ \rightarrow \mathbb{R}$, then there exists a unique function $x : \mathbb{R}_+ \rightarrow \mathbb{R}$ such that

$$\boxed{\text{distxz}} \quad (4.4) \quad \sup_{t \in \mathbb{R}_+} |x(t) - z(t)| < \kappa_0$$

and

$$\boxed{\text{ort}} \quad (4.5) \quad \int_{\mathbb{R}} u(t)(\rho_{n_0} * \varphi')(\cdot - x(t)) = 0, \quad \forall t \in \mathbb{R}_+,$$

where $\{\rho_n\}$ is defined in (2.3) and where n_0 satisfies :

$$\boxed{\text{unic}} \quad (4.6) \quad \forall y \in [-1/2, 1/2], \quad \int_{\mathbb{R}} \varphi(\cdot - y)(\rho_{n_0} * \varphi') = 0 \Leftrightarrow y = 0.$$

Moreover, $x(\cdot) \in C^1(\mathbb{R})$ with

$$\boxed{\text{estc}} \quad (4.7) \quad \sup_{t \in \mathbb{R}_+} |\dot{x}(t) - c| \leq \frac{c}{8}$$

and if

$$\boxed{\text{gf}} \quad (4.8) \quad \sup_{t \in \mathbb{R}_+} \|u(t) - c\varphi(\cdot - z(t))\|_{H^1} < \frac{\varepsilon^2}{c} = c \left(\frac{\varepsilon}{c}\right)^2$$

for $0 < \varepsilon < c\varepsilon_0$ then

$$\boxed{\text{fg}} \quad (4.9) \quad \sup_{t \in \mathbb{R}_+} \|u(t) - c\varphi(\cdot - x(t))\|_{H^1} \leq K\varepsilon.$$

At this stage, we fix $0 < \theta < c$ and we take

$$\boxed{\text{defep}} \quad (4.10) \quad \varepsilon = \frac{1}{2^4 K} \min\left(\frac{\theta}{2^5}, c\varepsilon_0\right)$$

For $u_0 \in Y$ satisfying Hypothesis 1 and (4.1) with this ε , (4.2) ensures that (4.3) and thus (4.7) hold. Moreover, (4.9) is satisfied with $K\varepsilon \leq \min\left(\frac{\theta}{2^9}, \frac{c\varepsilon_0}{2^4}\right)$ so that

$$\boxed{\text{Kep}} \quad (4.11) \quad \sup_{t \in \mathbb{R}_+} \|u(t) - c\varphi(\cdot - x(t))\|_{H^1} \leq \frac{c\varepsilon_0}{2^4} \leq \frac{c}{2^4}.$$

It follows that

$$\boxed{\text{cc}} \quad (4.12) \quad \dot{x}(t) \geq \frac{7}{8}c, \quad \forall t \geq 0.$$

and that u satisfies the hypotheses of Lemma 3.1 for any $0 < \alpha < 1$ such that

$$\boxed{\text{okok}} \quad (4.13) \quad (1 - \alpha) \geq \frac{\theta}{4c}$$

and any $0 \leq \gamma \leq (1 - \alpha)c$. In particular, u satisfies the hypotheses of Lemma 3.1 for $\alpha = 1/3$. Note that the hypothesis (1.10) with

$$\eta_0 = \frac{1}{K^8} \min\left(\frac{1}{2^{10}}, \frac{\varepsilon_0}{2^4}\right)^8$$

implies that (4.1) holds with ε given by (4.10).

propasym **Proposition 4.2.** *Let $u_0 \in Y$ satisfying Hypotheses 1 and (4.1) with ε defined as in (4.10) and let $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ the emanating solution of (1.6). For any sequence $t_n \nearrow +\infty$ there exists a subsequence $\{t_{n_k}\} \subset \{t_n\}$ and $\tilde{u}_0 \in Y_+$ such that*

$$\boxed{\text{pp2}} \quad (4.14) \quad u(t_{n_k}, \cdot + x(t_{n_k})) \xrightarrow[n_k \rightarrow +\infty]{} \tilde{u}_0 \text{ in } H_{loc}^1(\mathbb{R})$$

where $x(\cdot)$ is a C^1 -function satisfying (4.5), (4.7) and (4.9). Moreover, the solution of (1.6) emanating from \tilde{u}_0 is Y -almost localized.

Proof. In the sequel we set $x_0(t) = q(t, x_0)$, $t \geq 0$, so that

$$\boxed{\text{yyy}} \quad (4.15) \quad \text{supp } y^-(t) \subset]-\infty, x_0(t)] \quad \text{and} \quad \text{supp } y^+(t) \subset [x_0(t), +\infty[, \quad \forall t \geq 0.$$

We separate two possible behaviors of $x_0(\cdot)$ which has to be treated in different ways. In the first case we can prove that $x(t) - x_0(t) \rightarrow +\infty$ so that for $t > 0$ large enough we are very close to the case of a non negative density momentum. On the other hand, in the second case $x(t) - x_0(t)$ is uniformly bounded and we have to use other arguments. In this case we will first prove that the total variation of y stays bounded for all positive times which enables us to pass to the limit. The non negativity of the limit follows from a decay estimate on the total variation of $y(t)$ on a growing interval at the left of $x_0(t)$.

Case 1. There exists $t_* \geq 0$ such that

$$\boxed{\text{case1}} \quad (4.16) \quad x_0(t_*) < x(t_*) - \ln(3/2).$$

In this case we notice that in view of (4.11) and (2.13), for any $t \geq 0$,

$$\boxed{\text{oo}} \quad (4.17) \quad u(t, x) < c\varphi(x - x(t)) + \frac{c}{16} < \frac{2c}{3} + \frac{c}{16} \leq \frac{3c}{4} \quad \text{for } x \leq x(t) - \ln(3/2)$$

Therefore, by a continuity argument, (2.8), (4.12), (4.17) and (4.16) lead to

$$\boxed{\text{difx0}} \quad (4.18) \quad x(t) - x_0(t) \geq \frac{c}{8}(t - t_*) + x(t_*) - x_0(t_*) > \frac{c}{8}(t - t_*) + \ln 2, \quad \forall t \geq t_*.$$

Applying the almost monotonicity result (3.13) for $t \geq t_*$ with $z(t) = x(t) - x_0(t)$ and $R = x(t_*) - x_0(t_*) > 0$ we deduce that there exists $A_0 = A_0(E(u_0), \|y(t_*)\|_{\mathcal{M}}) > 0$ such that

$$\boxed{\text{A0}} \quad (4.19) \quad \left\langle y(\cdot + x(t), \Phi(\cdot + \frac{c}{8}(t - t_*))) \right\rangle \leq A_0, \quad \forall t \geq t_*.$$

We set

$$\check{u}(t, x) = u(t, x)\Phi\left(\cdot - x(t) + \frac{c}{8}(t - t_*)\right),$$

where Φ is defined in (3.5). According to the conservation of $E(\cdot)$, (4.18) and (4.19), \check{u} is uniformly bounded in Y for positive time. Therefore for any sequence $t_n \nearrow +\infty$, there exists $\tilde{u}_0 \in Y$ and a subsequence of $\{t_{n_k}\}$ (that we still denote by t_{n_k} to simplify the notation) such that

$$\begin{aligned} \check{u}(t_{n_k}, \cdot + x(t_{n_k})) &\rightharpoonup \tilde{u}_0 \text{ in } H^1(\mathbb{R}) \\ \check{u}(t_{n_k}, \cdot + x(t_{n_k})) &\rightarrow \tilde{u}_0 \text{ in } H_{loc}^1(\mathbb{R}) \\ (\check{u} - \check{u}_{xx})(t_{n_k}, \cdot + x(t_{n_k})) &\rightharpoonup^* \tilde{y}_0 = \tilde{u}_0 - \tilde{u}_{0,xx} \text{ in } \mathcal{M}(\mathbb{R}). \end{aligned}$$

Since for $t \geq t_*$, according to (4.18) and the support property of Φ and Φ' , $(\check{u} - \check{u}_{xx})(t, \cdot + x(t)) = y(t, \cdot)$ on $] -\frac{c}{8}(t - t_*) + 2, +\infty[$ and thus is a non negative measure on $] -\frac{c}{8}(t - t_*) + 2, +\infty[$, we infer that $\tilde{u}_0 \in Y_+$. Moreover, the support properties of Φ and Φ' imply that

$$\boxed{\text{vc}} \quad (4.20) \quad u(t_{n_k}, \cdot + x(t_{n_k})) \rightarrow \tilde{u}_0 \text{ in } H_{loc}^1(\mathbb{R})$$

and, for all $A > 0$ and $\phi \in C_0(\mathbb{R})$ with $\text{supp } \phi \subset [-A, +\infty[$,

$$\boxed{\text{vcc}} \quad (4.21) \quad \left\langle y(t_{n_k}, \cdot + x(t_{n_k})), \phi \right\rangle \rightarrow \left\langle \tilde{y}_0, \phi \right\rangle.$$

Since, by (4.7), $\{x(t_n + \cdot) - x(t_n)\}$ is uniformly equi-continuous, Arzela-Ascoli theorem ensures that there exists a subsequence $\{t_{n_k}\} \subset \{t_n\}$ and $\tilde{x} \in C(\mathbb{R})$ such that for all $T > 0$,

$$\boxed{\text{cvx}} \quad (4.22) \quad x(t_{n_k} + \cdot) - x(t_{n_k}) \xrightarrow{t \rightarrow +\infty} \tilde{x} \text{ in } C([0, T]).$$

Therefore, on account of (4.20), (4.22) and part **3.** of Proposition 2.1 for any $t \in \mathbb{R}$,

$$\boxed{\text{strongcv}} \quad (4.23) \quad u(t_{n_k} + t, \cdot + x(t_{n_k} + t)) \rightarrow \tilde{u}(t, \cdot + \tilde{x}(t)) \text{ in } H_{loc}^1(\mathbb{R})$$

where $\tilde{u} \in C(\mathbb{R}_+; H^1(\mathbb{R}))$ is the solution of (1.6) emanating from $\tilde{u}_0 \in Y_+$. Moreover, for all $A > 0$ and $\phi \in C_0(\mathbb{R})$ with $\text{supp } \phi \subset [-A, +\infty[$, it holds

$$\boxed{\text{weakcv}} \quad (4.24) \quad \left\langle y(t_{n_k} + t, \cdot + x(t_{n_k} + t)), \phi \right\rangle \rightarrow \left\langle \tilde{y}(t, \cdot + \tilde{x}(t)), \phi \right\rangle,$$

where $\tilde{y} = \tilde{u} - \tilde{u}_{xx}$. Indeed, on one hand, it follows from part **3.** of Proposition 2.1 that

$$\left\langle y(t_{n_k} + t, \cdot + x(t_{n_k}) + \tilde{x}(t)), \phi \right\rangle \rightarrow \left\langle \tilde{y}(t, \cdot + \tilde{x}(t)), \phi \right\rangle$$

and on the other hand, the uniform continuity of ϕ together with (4.22) ensure that

$$\begin{aligned} & \left\langle y(t_{n_k} + t, \cdot + x(t_{n_k}) + \tilde{x}(t)) - y(t_{n_k} + t, \cdot + x(t_{n_k} + t)), \phi \right\rangle \\ &= \left\langle y(t_{n_k} + t), \phi(\cdot - x(t_{n_k})) - \tilde{x}(t) - \phi(\cdot - x(t_{n_k} + t)) \right\rangle \rightarrow 0 \end{aligned}$$

In view of (4.23) we infer that $(\tilde{u}, \tilde{x}(\cdot))$ satisfies (4.5) and (4.9) with the same ε than $(u, x(\cdot))$. Therefore, (4.10) forces $(\tilde{u}, \tilde{x}(\cdot))$ to satisfy (4.3) and the uniqueness result in Lemma 4.1 ensures that $\tilde{x}(\cdot)$ is a C^1 -function and satisfies (4.7).

Let us now prove that \tilde{u} is Y -almost localized. In the sequel, for $u \in Y$ and $\gamma \geq 0$, we defined the quantity $G(u)$ by

$$G_\gamma(u) = E(u) + \frac{c}{4} \left\langle u - u_{xx}, 1 \right\rangle.$$

We will also make use of the following functionals that measure the quantity $G(u)$ at the right and at the left of u . For $0 \leq \gamma \leq c2^{-4}$, $v \in Y$ and $R > 0$ we set

$$\boxed{\text{defJr}} \quad (4.25) \quad J_{\gamma,r}^R(v) = \left\langle v^2 + v_x^2, \Psi(\cdot - R) \right\rangle + \gamma \left\langle v - v_{xx}, \Phi(\cdot - R) \right\rangle$$

and

$$\boxed{\text{defJl}} \quad (4.26) \quad J_{\gamma,l}^R(v) = \left\langle v^2 + v_x^2, (1 - \Psi(\cdot + R)) \right\rangle + \gamma \left\langle v - v_{xx}, 1 - \Phi(\cdot + R) \right\rangle.$$

We separate $G(v)$ into two parts :

$$\begin{aligned} G_{o,\gamma}^R(v) &= \left\langle v^2 + v_x^2, 1 - \Psi(\cdot + R) + \Psi(\cdot - R) \right\rangle + \gamma \left\langle v - v_{xx}, 1 - \Phi(\cdot + R) + \Phi(\cdot - R) \right\rangle \\ &= J_{\gamma,r}^R(v) + J_{\gamma,l}^R(v), \end{aligned}$$

which almost ‘localizes’ outside the ball of radius R and

$$\begin{aligned} G_{i,\gamma}^R(v) &= \left\langle v^2 + v_x^2, \Psi(\cdot + R) - \Psi(\cdot - R) \right\rangle + \gamma \left\langle v - v_{xx}, \Phi(\cdot + R) - \Phi(\cdot - R) \right\rangle \\ &= G(v) - G_o^R(v), \end{aligned}$$

which almost ‘localizes’ inside this ball. It is however worth mentioning that, in view of (3.5), the function $\Phi(\cdot + R) - \Phi(\cdot - R)$ is indeed supported in $[-R, R + 1]$.

We first notice that the almost monotonicity result (3.11) ensures that for any $\varepsilon > 0$ there exists $R'_\varepsilon > 0$ such that

$$\boxed{\text{decR}} \quad (4.27) \quad J_{\gamma,r}^{R'_\varepsilon}(\tilde{u}(t, \cdot + \tilde{x}(t))) < \varepsilon, \quad \forall t \geq 0.$$

Indeed, let $t_0 > 0$ be fixed. Fixing $\alpha = \beta = 1/4$ and taking $z(\cdot) = (1 - \alpha)x(\cdot)$, $z(\cdot)$ clearly satisfies (3.7) and (4.9)-(4.18) ensure that it also satisfies (3.10) for $R \geq 2$. Moreover, we have $J_{\gamma,r}^R(u(t_0, \cdot + x(t_0))) = I_{t_0}^{+R}(t_0)$ where $I_{t_0}^{+R}$ is defined in (3.8). Since obviously,

$$J_{\gamma,r}^R(u(t, \cdot + x(t))) \geq I_{t_0}^{+R}(t), \quad \forall 0 \leq t \leq t_0,$$

we deduce from (3.11) that

$$\boxed{\text{monoJr}} \quad (4.28) \quad J_{\gamma,r}^R(u(t_0, \cdot + x(t_0))) \leq J_{\gamma,r}^R(u(t, \cdot + x(t))) + K_0 e^{-R/6}, \quad \forall 0 \leq t \leq t_0,$$

where K_0 is the constant appearing in (3.11). Therefore, taking $R'_\varepsilon \geq 2$ such that $K_0 e^{-R'_\varepsilon/6} < \varepsilon/2$ and $J_{\gamma,r}^{R'_\varepsilon}(u(0, \cdot + x(0))) < \varepsilon/2$ we get that for all $t \geq 0$

$$\boxed{\text{dede}} \quad (4.29) \quad J_{\gamma,r}^{R'_\varepsilon}(u(t, \cdot + x(t))) \leq J_{\gamma,r}^{R'_\varepsilon}(u(0, \cdot + x(0))) + \varepsilon/2 \leq \varepsilon$$

Passing to the weak limit, this leads to (4.27). Moreover, (4.18) ensures that for any $R > 0$ there exists $t(R) > 0$ such that

$$x_0(t) \leq x(t_R) - R + \frac{9}{10}(x(t) - x(t_R)), \quad \forall t \geq t(R).$$

Therefore, the hypotheses (3.7) and (3.12) of Lemma 3.1 are fulfilled for $z(t) = x(t_R) - R + \frac{9}{10}(x(t) - x(t_R))$ and $t \geq t_R$ and, proceeding as above, (3.13) leads to

$$\boxed{\text{monoJl}} \quad (4.30) \quad J_{\gamma,l}^R(u(t, \cdot + x(t))) \geq J_{\gamma,l}^R(u(t_R, \cdot + x(t_R))) - K_0 e^{-R/6}, \quad \forall t \geq t(R)$$

Now we notice that to prove Y -almost localization of \tilde{u} , it suffices to prove that for all $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\boxed{\text{po2}} \quad (4.31) \quad G_{o,\gamma}^{R_\varepsilon}(\tilde{u}(t, \cdot + \tilde{x}(t))) < \varepsilon, \quad \forall t \in \mathbb{R}.$$

Indeed if (4.31) is true for some $(\varepsilon, R_\varepsilon)$ then (\tilde{u}, \tilde{x}) satisfies (1.9) with $(\varepsilon/2, 2R_\varepsilon)$. As indicated above, we prove (4.31) by contradiction. Assuming that (4.31) is not true, there exists $\varepsilon_0 > 0$ such that for any $R > 0$ there exists $t_R \in \mathbb{R}$ satisfying

$$(4.32) \quad G_{o,\gamma}^R(\tilde{u}(t_R, \cdot + \tilde{x}(t_R))) \geq \varepsilon_0$$

Let $R_0 > R'_\varepsilon/10$ such that

$$(4.33) \quad G_{o,\gamma}^{R_0}(\tilde{u}(0)) \leq \frac{\varepsilon_0}{10} \quad \text{and} \quad K_0 e^{-R_0/6} < \frac{\varepsilon_0}{10}.$$

The conservation of G then forces

$$G_{i,\gamma}^{R_0}(\tilde{u}(t_{R_0}, \cdot + \tilde{x}(t_{R_0}))) \leq G_{i,\gamma}^{R_0}(\tilde{u}(0)) - \frac{9}{10}\varepsilon_0.$$

Recalling that $\Psi(\cdot + R) - \Psi(\cdot - R)$ and $\Phi(\cdot + R) - \Phi(\cdot - R)$ belong to $C_0(\mathbb{R})$, the convergence results (4.23)-(4.24) ensure that for $k \geq k_0$ with k_0 large enough,

$$G_{i,\gamma}^{R_0}(u(t_{n_k} + t_{R_0}), \cdot + x(t_{n_k} + t_{R_0})) \leq G_{i,\gamma}^{R_0}(u(t_{n_k}, \cdot + x(t_{n_k}))) - \frac{4}{5}\varepsilon_0.$$

We first assume that $t_{R_0} > 0$. By (4.27) and the conservation of G this ensures that

$$\boxed{\text{bb}} \quad (4.34) \quad J_{\gamma,l}^{R_0}(u(t_{n_k} + t_{R_0}), \cdot + x(t_{n_k} + t_{R_0})) \geq J_{\gamma,l}^{R_0}(u(t_{n_k}), \cdot + x(t_{n_k})) + \frac{7}{10}\varepsilon_0.$$

Now we take a subsequence $\{t_{n'_k}\}$ of $\{t_{n_k}\}$ such that $t_{n'_0} \geq t(R_0)$, $t_{n'_{k+1}} - t_{n'_k} \geq t_{R_0}$ and $n'_k \geq n_{k_0}$. From (4.34) and again (4.30), we get that for any $k \geq 0$,

$$J_{\gamma,l}^{R_0}(u(t_{n'_k}), \cdot + x(t_{n'_k})) \geq J_{\gamma,l}^{R_0}(u(t_{n'_0}), \cdot + x(t_{n'_0})) + \frac{3}{5}k\varepsilon_0 \xrightarrow{k \rightarrow +\infty} +\infty$$

that contradicts the conservation of G in view of (4.29) and $G_{i,\gamma}^{R_0}(u(t_{n'_k}), \cdot + x(t_{n'_k})) \rightarrow G_{i,\gamma}^{R_0}(\tilde{u}(0, \cdot + \tilde{x}_0))$. Finally, if $t_{R_0} < 0$, then for $k \geq k_0$ such that $t_{n_k} > |t_{R_0}|$ we get in the same way

$$0 \leq J_{\gamma,r}^{R_0}(u(t_{n_k}), \cdot + x(t_{n_k})) \leq J_{\gamma,r}^{R_0}(u(t_{n_k} - |t_{R_0}|), \cdot + x(t_{n_k} - |t_{R_0}|)) - \frac{7}{10}\varepsilon_0.$$

that contradicts (4.27) and $R_0 > R'_{\frac{\varepsilon}{10}}$ for $k \geq k_0$ large enough. This proves the Y -almost localization of \tilde{u} .

Case 2. For all $t \geq 0$ it holds

$$\boxed{\text{case2}} \quad (4.35) \quad x_0(t) \geq x(t) - \ln(3/2).$$

Noticing that (2.7) leads to $u_x(t, \cdot) \geq u(\cdot, \cdot)$ on $] -\infty, x_0(t)[$ and that (4.11), (2.13) and (4.35) ensure that $u(t, x_0(t) - \ln 2) \geq c\varphi(-\ln 3) - \frac{c}{16} \geq \frac{13c}{48}$, we infer that

$$\boxed{\text{cc22}} \quad (4.36) \quad u(t) \geq \frac{13}{48}c \text{ on } [x_0(t) - \ln 2, x_0(t)], \quad \forall t \geq 0.$$

We claim that

$$\boxed{\text{claimy}} \quad (4.37) \quad \|y(t)\|_{\mathcal{M}} \leq 4 \left(2 + \frac{9}{c} \sqrt{E(u_0)} \right) \|y_0\|_{\mathcal{M}}, \quad \forall t \geq 0.$$

and

$$\boxed{\text{claimy2}} \quad (4.38) \quad \|y^-(t)\|_{\mathcal{M}(]x(t) - \frac{c}{96}t, +\infty])} \xrightarrow{t \rightarrow +\infty} 0.$$

To prove this claim we approximate u_0 by $\{u_{0,n}\}$ as in (2.4) and work with the global solutions u_n emanating from $u_{0,n}$ that satisfy Hypothesis 1 with the same x_0 . We define q_n as the flow associated with u_n and we set

$$x_{0,n}(t) = q_n(t, x_0)$$

Let us recall that, in [9], it is shown that for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\boxed{\text{yy}} \quad (4.39) \quad y_n(0, x) = y_n(t, q_n(t, x))q_{n,x}(t, x)^2$$

with

$$\boxed{\text{formula1}} \quad (4.40) \quad q_{n,x}(t, x) = \exp\left(\int_0^t u_x(s, q(s, x)) ds\right).$$

Let $t_0 > 0$ be fixed. According to (2.9) and (4.36) there exists $n_0 \geq 0$ such that for all $n \geq n_0$,

$$\boxed{\text{fc2}} \quad (4.41) \quad u_n(t) \geq \frac{c}{4} \text{ on } [x_{0,n}(t) - \ln 2, x_{0,n}(t)], \quad \forall t \in [0, t_0]$$

We proceed in three steps.

Step 1. In this step, we prove that for all $t \in [0, t_0]$ and n large enough,

$$\boxed{\text{claimx0}} \quad (4.42) \quad \left| \int_{x_{0,n}(t)-\ln 2}^{x_{0,n}(t)} y_n(t, s) ds \right| \leq \exp\left(-\frac{c}{4}t\right) \|y_{0,n}\|_{L^1} .$$

Let $t \in [0, t_0]$ and $x \in [x_{0,n}(t) - \ln 2, x_{0,n}(t)]$, then one has, $\forall 0 \leq \tau \leq t$,

$$q_n(\tau, q_n^{-1}(t, x)) \leq q_n(\tau, q_n^{-1}(t, x_{0,n}(t))) = x_{0,n}(\tau)$$

where for all $t \geq 0$, $q_n^{-1}(t, \cdot)$ is the inverse mapping of $q_n(t, \cdot)$. Since, according to (2.7) and (4.41), $u_{n,x} \geq u_n \geq \frac{c}{4}$ on $[x_{0,n}(\tau) - \ln 2, x_{0,n}(\tau)]$ it holds

$$u_n(\tau, x_{0,n}(\tau)) \geq u_n\left(\tau, q_n(\tau, q_n^{-1}(t, x))\right) \geq 0$$

and thus

$$\frac{d}{d\tau} \left(x_{0,n}(\tau) - q_n(\tau, q_n^{-1}(t, x)) \right) \geq 0 \quad \text{on } [0, t] .$$

Therefore, for all $\tau \in [0, t]$ one has

$$q_n(\tau, q_n^{-1}(t, x)) \in [x_{0,n}(\tau) - \ln 2, x_{0,n}(\tau)]$$

which ensures that

$$u_{n,x}\left(\tau, q_n(\tau, q_n^{-1}(t, x))\right) \geq \frac{c}{4}$$

and (4.40) leads to

$$q_{n,x}(t, q_n^{-1}(t, x)) = \exp\left(\int_0^t u_{n,x}\left(\tau, q_n(\tau, q_n^{-1}(t, x))\right) d\tau\right) \geq \exp\left(\frac{c}{4}t\right) .$$

Therefore for all $t \in [0, t_0]$, (4.39) leads to

$$\begin{aligned} - \int_{x_{0,n}(t)-\ln 2}^{x_{0,n}(t)} y_n(t, x) dx &= - \int_{q_n^{-1}(t, x_{0,n}(t)-\ln 2)}^{x_0} y_n(t, q_n(t, \theta)) q_{n,x}(t, \theta) d\theta \\ &\leq -e^{-\frac{c}{4}t} \int_{q_n^{-1}(t, x_{0,n}(t)-\ln 2)}^{x_0} y_n(t, q_n(t, \theta)) q_{n,x}^2(t, \theta) d\theta \\ &\leq -e^{-\frac{c}{4}t} \int_{x_0-\ln 2}^{x_0} y_n(0, \theta) d\theta \end{aligned}$$

which proves (4.42).

Step 2. In this step we prove that for $t \in [0, t_0]$ and $n \geq 0$ large enough,

$$\boxed{\text{step2}} \quad (4.43) \quad \|y_n(t)\|_{L^1} \leq 2 \left(2 + \frac{9}{c} \sqrt{E(u_{0,n})} \right) \|y_{0,n}\|_{\mathcal{M}} .$$

Let $\Phi : \mathbb{R} \rightarrow \mathbb{R}_+$ defined by $\Phi \equiv 0$ on $] -\infty, -\ln 2]$, $\Phi = 1$ on \mathbb{R}_+ and $\Phi(x) = (x + \ln 2)/\ln 2$ for $x \in [-\ln 2, 0]$. Then

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}} y_n(t) \Phi(\cdot - x_{0,n}(t)) &= -\dot{x}_{0,n}(t) \int_{\mathbb{R}} y \Phi'(\cdot - x_{0,n}(t)) + \int_{\mathbb{R}} u_n y_n \Phi'(\cdot - x_{0,n}(t)) \\ &\quad + \int_{\mathbb{R}} (u_n^2 - u_{n,x}^2) \Phi'(\cdot - x_{0,n}(t)) \end{aligned} \quad \boxed{\text{gv}} \quad (4.44)$$

Since $\text{supp } \Phi'(\cdot - x_{0,n}(t)) \subset [x_{0,n}(t) - \ln 2, x_{0,n}(t)]$, (2.7) ensures that the last term of the right-hand side of (4.44) is non positive and (4.42) ensures that

$$(4.45) \quad \begin{aligned} \left| \int_{\mathbb{R}} (u_n(t) - \dot{x}_{0,n}(t)) y_n(t) \Phi'(\cdot - x_{0,n}(t)) \right| &\leq \sqrt{2E(u_{0,n})} \left| \int_{\mathbb{R}} y_n(t) \Phi'(\cdot - x_{0,n}(t)) \right| \\ &\leq \frac{\sqrt{2}}{\ln 2} \sqrt{E(u_{0,n})} \|y_{0,n}\|_{L^1} \exp(-\frac{c}{4}t). \end{aligned}$$

Therefore (4.44) leads to

$$(4.46) \quad \begin{aligned} \int_{\mathbb{R}} y_n(t) \Phi(\cdot - x_{0,n}(t)) &\leq \int_{\mathbb{R}} y_{0,n} \Phi(\cdot - x_{0,n}) + \frac{8}{c} \sqrt{E(u_{0,n})} \|y_{0,n}\|_{L^1} \\ &\leq \left(1 + \frac{4\sqrt{2}}{c \ln 2} \sqrt{E(u_{0,n})}\right) \|y_{0,n}\|_{L^1} \end{aligned}$$

Gathering this last estimate with (4.42) we get that

$$\int_{x_{0,n}(t)}^{+\infty} y_n(t, s) ds \leq \left(2 + \frac{4\sqrt{2}}{c \ln 2} \sqrt{E(u_{0,n})}\right) \|y_{0,n}\|_{\mathcal{M}}$$

that leads to (4.43) by making use of the conservation of $M(\cdot)$. (4.37) then follows by passing to the limit in n and using (2.10).

Step 3. Finally we prove that for all $t \in [0, t_0]$ and $n \geq 0$ large enough,

$$\boxed{\text{step3}} \quad (4.47) \quad \|y_n^-(t)\|_{\mathcal{M}[x_0(t) - \frac{c}{8}t, +\infty[} \leq \exp(-\frac{c}{4}t) \|y_{0,n}\|_{L^1}.$$

For this we first notice that since (2.7) leads to $u_x(t, \cdot) \geq u(t, \cdot)$ on $] -\infty, x_0(t)[$, it holds

$$u(t, x) \leq e^{x-x_0(t)} u(t, x_0(t)), \quad \forall x \leq x_0(t).$$

Therefore, (4.11) and (2.13) force

$$u(t, x) \leq \frac{1}{2} u(t, x_0(t)) \leq \frac{1}{2} \frac{17}{16} c \leq \frac{17}{32} c, \quad \forall (t, x) \in \mathbb{R}_+ \times] -\infty, x_0(t) - \ln 2].$$

On the other hand, $u_x(t) \geq u(t)$ on $] -\infty, x_0(t)[$ and (4.35) forces

$$u(t, x_0(t)) \geq \frac{2c}{3} - \frac{c}{16} = \frac{29c}{48}, \quad \forall t \geq 0.$$

Moreover, (4.11) ensures that $u \geq -2^{-4}c$ on $\mathbb{R}_+ \times \mathbb{R}$. Taking n large enough we can thus assume that

$$\boxed{\text{p11}} \quad (4.48) \quad u_n(t, x) \leq \frac{9c}{16}, \quad \forall (t, x) \in [0, t_0] \times] -\infty, x_{0,n}(t) - \ln 2],$$

$$\boxed{\text{p12}} \quad (4.49) \quad u_n(t, x_{0,n}(t)) \geq \frac{7c}{12}, \quad \forall t \in [0, t_0].$$

and

$$\boxed{\text{p13}} \quad (4.50) \quad u_n(t, x) \geq -\frac{c}{8} \quad \text{on} \quad [0, t_0] \times \mathbb{R}.$$

For any $t_1 \geq 0$ we define the function q_{n,t_1} on $\mathbb{R}_+ \times \mathbb{R}$ by

$$\boxed{\text{defqn}} \quad (4.51) \quad \begin{cases} \partial_t q_{n,t_1}(t, x) &= u_n(t, q_{n,t_1}(t, x)) & , (t, x) \in \mathbb{R}_+ \times \mathbb{R} \\ q_{n,t_1}(t_1, x) &= x & , x \in \mathbb{R} \end{cases}.$$

q_{n,t_1} is the flow associated with u_n that satisfies $q_{n,t_1}(t_1) = Id$. Obviously $q_n = q_{n,0}$ and one can easily check that for q_{n,t_1} (4.40) becomes

$$\partial_x q_{n,t_1}(t, x) = \exp\left(\int_{t_1}^t u_{n,x}(s, q_{n,t_1}(s, x)) ds\right), \quad \forall t \geq 0,$$

so that (4.50) and $u_{n,x} \geq u_n$ on $] -\infty, x_{0,n}(t)[$ ensure that

$$\boxed{\text{dw}} \quad (4.52) \quad \partial_x q_{n,t/2}(t, x) \geq \exp\left(-\frac{c}{16}t\right), \quad \forall (t, x) \in [0, t_0] \times \mathbb{R}.$$

For any $t \geq 0$, we denote by $q_{n,t_1}^{-1}(t, \cdot)$ the inverse mapping of $q_{n,t_1}(t)$. By a continuity argument, (4.48)-(4.49) and (2.8) lead to

$$x_{0,n}(t) - q_{n,t/2}(t, x_{0,n}(t/2) - \ln 2) \geq \ln 2 + \frac{c}{96}t.$$

Therefore, for any $x \in [x_{0,n}(t) - \ln 2 - \frac{c}{96}t, x_{0,n}(t)]$, we have $q_{n,t/2}^{-1}(t, x) \in [x_{0,n}(t/2) - \ln 2, x_{0,n}]$ and thus for all $t \in [0, t_0]$, a change of variables along the flow, (4.39), (4.52) and (4.42) lead to

$$\begin{aligned} - \int_{x_{0,n}(t) - \ln 2 - \frac{c}{96}t, x_{0,n}(t)}^{x_{0,n}(t)} y_n(t, x) dx &= - \int_{q_{n,t/2}^{-1}(t, x_{0,n}(t) - \ln 2 - \frac{c}{96}t)}^{x_{0,n}(t/2)} y_n(t, q_{n,t/2}(t, \theta)) \partial_x q_{n,t/2}(t, \theta) d\theta \\ &\leq -e^{-\frac{c}{16}t} \int_{x_{0,n}(t/2) - \ln 2}^{x_{0,n}(t/2)} y_n(t, t/2, q_{n,t/2}(t, \theta)) [\partial_x q_{n,t/2}(t, \theta)]^2 d\theta \\ &\leq -e^{-\frac{c}{16}t} \int_{x_{0,n}(t/2) - \ln 2}^{x_{0,n}(t/2)} y_n(t/2, \theta) d\theta \\ &\leq e^{-\frac{c}{16}t} e^{-\frac{c}{8}t} \|y_{0,n}\|_{L^1} \leq e^{-\frac{c}{16}t} \|y_{0,n}\|_{L^1}. \end{aligned}$$

This proves that

$$\|y_n^-(t)\|_{L^1([x_{0,n}(t) - \ln 2 - \frac{c}{96}t, +\infty])} \leq e^{-\frac{c}{16}t} \|y_{0,n}\|_{L^1}.$$

Now we notice that (2.10) ensures that $y_n^-(t) \rightarrow * z$ in \mathcal{M} with $z \in \mathcal{M}^+(\mathbb{R})$ and $0 \leq y^-(t) \leq z$. Therefore, passing to the limit in $n \rightarrow \infty$ and then in $t_0 \rightarrow +\infty$, we obtain that

$$\|y^-(t)\|_{\mathcal{M}([x_0(t) - \ln(3/2) - \frac{c}{96}t, +\infty])} \leq e^{-\frac{c}{16}t} \|y_0\|_{\mathcal{M}}, \quad \forall t \geq 0.$$

Finally, (4.11), (2.13) and $u_x \geq u$ on $] -\infty, x_0(t)[$ ensure that

$$\boxed{\text{f14}} \quad (4.53) \quad x_0(t) \leq x(t) + \ln(3/2), \quad \forall t \geq 0,$$

which completes the proof of (4.38).

Now, in view of (4.37), for any sequence $t_n \nearrow +\infty$, there exists $\tilde{u}_0 \in Y$ and a subsequence of $\{t_n\}$ (that we still denote by t_n to simplify the notation) such that

$$\boxed{\text{442}} \quad (4.54) \quad u(t_n, \cdot + x(t_n)) \rightarrow \tilde{u}_0 \text{ in } H_{loc}^1(\mathbb{R})$$

$$\boxed{\text{cvy}} \quad (4.55) \quad y(t_n, \cdot + x(t_n)) \rightarrow * \tilde{y}_0 = \tilde{u}_0 - \tilde{u}_{0,xx} \text{ in } \mathcal{M}(\mathbb{R}).$$

and (4.35) and (4.38) ensure that \tilde{y}_0 is a non negative bounded measure. Finally, according to part 3. of Proposition 2.1 and (4.37), the solution \tilde{u} emanating from \tilde{u}_0 belongs to $C_b(\mathbb{R}; Y_+)$. To prove the Y -almost localization of the solution \tilde{u} emanating from \tilde{u}_0 , we proceed as in the Case 1. We first obtain as in Case 1 that there exists a C^1 -function \tilde{x} such that (4.22), (4.23) hold for some subsequence

$\{t_{n_k}\}$ of $\{t_n\}$. Moreover, according to the boundedness of $\{\|y_n\|_{\mathcal{M}}\}$ we may also require that for any $\phi \in C_0(\mathbb{R})$ and any $t \in \mathbb{R}$,

$$\boxed{\text{weakcvy2}} \quad (4.56) \quad \left\langle y(t_{n_k} + t, \cdot + x(t_{n_k} + t)), \phi \right\rangle \rightarrow \left\langle \tilde{y}(t, \cdot + \tilde{x}(t)), \phi \right\rangle.$$

But then (4.35), (4.38) force

$$\boxed{\text{sup}} \quad (4.57) \quad \text{supp } \tilde{y}(t) \subset [\tilde{x}(t) - \ln 2, +\infty[, \quad \forall t \in \mathbb{R}.$$

Now, according to (4.53), $\text{supp } y^-(t) \subset]-\infty, x(t) + \ln(3/2)]$ for all $t \geq 0$. Therefore, for $R > 0$ big enough, we can thus apply Lemma 3.1 to obtain that $J_{r,\gamma}^R(u(t, \cdot + x(t)))$ is almost non increasing which ensures that (4.27) holds. Then, proceeding exactly as in the case 1, but with $\gamma = 0$, we obtain that \tilde{u} is H^1 -almost localized, i.e. for any $\varepsilon > 0$ there exists $R_\varepsilon > 0$ such that

$$G_{o,0}^{R_\varepsilon}(\tilde{u}(t, \cdot + \tilde{x}(t))) < \varepsilon, \quad \forall t \in \mathbb{R}.$$

This last estimate together with (4.27) and (4.57) prove the Y -localization of \tilde{u} . \square

5. ASYMPTOTIC STABILITY RESULTS

5.1. Asymptotic stability of a peakon. With Proposition 4.2 in hands, the proof of the asymptotic stability of a single peakon follows very closely the proof in [34] since the rest of the proof only uses Theorem 1.1 and the monotonicity results for $\gamma = 0$ (i.e. only $E(\cdot)$ is involved).

Let $u_0 \in Y$ satisfying hypothesis 1 and (4.1) with ε defined as in (4.10) and let $\{t_n\}$ be an increasing sequence that tends to $+\infty$. Combining Theorem 1.1 and Proposition 4.2 we infer that there exists a subsequence $\{t_{n_k}\} \subset \{t_n\}$, $x_0 \in \mathbb{R}$ and $c_0 > 0$ close to c such that

$$\boxed{\text{zz}} \quad (5.1) \quad u(t_{n_k}, \cdot + x(t_{n_k})) \xrightarrow[n_k \rightarrow +\infty]{} \tilde{u}_0 = \varphi_{c_0}(\cdot - x_0) \text{ in } H_{loc}^1(\mathbb{R})$$

where $x(\cdot)$ is a C^1 -function satisfying (4.5), (4.7) and (4.9). Moreover, (4.9), (4.10) and (5.1) ensure that $|x_0| \ll 1/2$.

Since by (5.1), \tilde{u}_0 satisfies the orthogonality condition (4.5), (4.6) forces $x_0 = 0$. On the other hand, (4.14) ensures that $c_0 = \lim_{n \rightarrow +\infty} \max_{\mathbb{R}} u(t'_n)$ and thus

$$u(t'_n, \cdot + x(t'_n)) - \lambda(t'_n)\varphi \xrightarrow[n \rightarrow +\infty]{} 0 \text{ in } H_{loc}^1(\mathbb{R})$$

where we set $\lambda(t) := \max_{\mathbb{R}} u(t)$, $\forall t \in \mathbb{R}$. Since this is the only possible limit, it follows that

$$\boxed{\text{pp3}} \quad (5.2) \quad u(t, \cdot + x(t)) - \lambda(t)\varphi \xrightarrow[t \rightarrow 0]{} 0 \text{ in } H_{loc}^1(\mathbb{R})$$

The convergence of the scaling parameter $\lambda(t)$ and of the derivative function \dot{x} towards c_0 at $+\infty$, as well as the strong H^1 -convergence on $] \theta t, +\infty[$ follow exactly as in ([34], Section 5) and will thus be omitted.

It remains to prove the H^1 -convergence in $] -\infty, -\theta t[$. Note that such convergence at the left was not established in [34]. In the appendix we complete the result in [34] by proving that for $u_0 \in Y_+$ the energy at the left of any given point decays to zero as time goes to $+\infty$. In the present case the desired H^1 -convergence is a direct consequence of the following monotony result : Let Ψ be the function defined in (3.1), then for any $z \in \mathbb{R}$, the functional

$$\Lambda_z : t \mapsto \int_{\mathbb{R}} \Psi(\cdot + \frac{\theta}{2}t - z)(u^2(t) + u_x^2(t))$$

is not increasing on \mathbb{R}_+ . Indeed, one has clearly $\lim_{z \rightarrow -\infty} \Lambda_z(0) = 0$ and for any given $z \in \mathbb{R}$, $-\theta t < -\frac{\theta}{2}t + z$ for $t \geq 0$ large enough. Therefore the monotony of Λ_z ensures that $\|u(t)\|_{H^1(x < \theta t)} \rightarrow 0$ as $t \rightarrow +\infty$ and the result follows since (4.12) obviously forces as well that $\|\varphi(\cdot - x(t))\|_{H^1(x < \theta t)} \rightarrow 0$ as $t \rightarrow +\infty$.

By continuity with respect to initial data in H^1 , it suffices prove this monotony result for smooth solutions. Then according to (3.17) it holds

$$\frac{d}{dt} \Lambda_z(t) = \theta \int_{\mathbb{R}} (u^2 + u_x^2) \Psi'(\cdot + \frac{\theta}{2}t - z) + \int_{\mathbb{R}} u u_x^2 \Psi'(\cdot + \frac{\theta}{2}t - z) + 2 \int_{\mathbb{R}} u h \Psi'(\cdot + \frac{\theta}{2}t - z)$$

where $h = (1 - \partial_x^2)^{-1}(u^2 + u_x^2/2) \geq 0$.

Now, we notice that by one hand, (1.10), (4.2), (2.13) and the positivity of φ yield

$$u(t) \geq -\frac{\theta^2}{4c} \geq -\frac{\theta}{8}, \quad \forall t \geq 0.$$

On the other hand, we have $\int_{\mathbb{R}} h \Psi' = \int_{\mathbb{R}} (u^2 + u_x^2/2)(1 - \partial_x^2)^{-1} \Psi'$ and that (3.3) and the positivity of the kernel associated to $(1 - \partial_x^2)^{-1}$ yield

$$(1 - \partial_x^2) \Psi' \geq \frac{1}{2} \Psi' \Rightarrow (1 - \partial_x^2)^{-1} \Psi' \leq 2 \Psi'$$

so that

$$\int_{\mathbb{R}} h \Psi' \leq 2 \int_{\mathbb{R}} (u^2 + u_x^2/2) \Psi'.$$

Gathering these estimates we eventually obtain

$$\begin{aligned} \frac{d}{dt} \Lambda_z(t) &\leq \theta \int_{\mathbb{R}} (u^2 + u_x^2) \Psi'(\cdot + \frac{\theta}{2}t - z) - \frac{\theta}{4} \int_{\mathbb{R}} u_x^2 \Psi'(\cdot + \frac{\theta}{2}t - z) - \frac{\theta}{2} \int_{\mathbb{R}} h \Psi'(\cdot + \frac{\theta}{2}t - z) \\ &\leq \theta \int_{\mathbb{R}} (u^2 + u_x^2) \Psi'(\cdot + \frac{\theta}{2}t - z) - \frac{\theta}{4} \int_{\mathbb{R}} u_x^2 \Psi'(\cdot + \frac{\theta}{2}t - z) - \theta \int_{\mathbb{R}} (u^2 + u_x^2) \Psi'(\cdot + \frac{\theta}{2}t - z) \\ &\leq 0 \end{aligned}$$

which yields the desired result.

5.2. Asymptotic stability of well ordered trains of antipeakons-peakons.

In [21] the orbital stability in $H^1(\mathbb{R})$ of well ordered trains of antipeakons-peakons is established. More precisely, the following theorem¹ is proved :

mult-peaks

Theorem 5.1 ([21]). *Let be given N_- negative velocities $c_{-N_-} < \dots < c_{-2} < c_{-1} < 0$, N_+ positive velocities $0 < c_1 < c_2 < \dots < c_{N_+}$. There exist $n_0 \in \mathbb{N}$ satisfying (4.6), $A > 0$, $L_0 > 0$ and $\varepsilon_0 > 0$ such that if $u \in C(\mathbb{R}_+; H^1)$ is the solution of (C-H) emanating from $u_0 \in Y$, satisfying Hypothesis 1 with*

huhu

$$(5.3) \quad \|u_0 - \sum_{\substack{j=-N_- \\ j \neq 0}}^{N_+} \varphi_{c_j}(\cdot - z_j^0)\|_{H^1} \leq \varepsilon_0^2$$

for some $0 < \varepsilon < \varepsilon_0$ and $z_{N_-}^0 < z_{N_-+1}^0 < \dots < z_{N_+}^0$ such that

$$z_j^0 - z_{j-1}^0 \geq L \geq L_0, \quad \forall j \in [[-N_- + 1, N_+]],$$

¹Actually, in the statement given in [21], $\partial_x \varphi_{c_i}$ appears instead of $\rho_{n_0} * \partial_x \varphi_{c_i}$ in the orthogonality condition (5.5). But it was noticed in [34] that then there is a gap in the proof of the C^1 -regularity of the functions x_i , $i = 1, \dots, N$. The proof of this version uses exactly the same arguments as developed in the appendix of [34].

then there exist $N_- + N_+$ C^1 -functions $t \mapsto x_{-N_-}(t), \dots, x_{-1}(t), x_1(t), \dots, t \mapsto x_{N_+}(t)$ uniquely determined such that

$$\boxed{\text{ini2}} \quad (5.4) \quad \sup_{t \in \mathbb{R}_+} \|u(t, \cdot) - \sum_{\substack{j=-N_- \\ j \neq 0}}^{N_+} \varphi_{c_j}(\cdot - x_j(t))\|_{H^1} \leq A\sqrt{\sqrt{\varepsilon} + L^{-1/8}}$$

and

$$\boxed{\text{mod2}} \quad (5.5) \quad \int_{\mathbb{R}} \left(u(t, \cdot) - \sum_{\substack{j=-N_- \\ j \neq 0}}^{N_+} \varphi_{c_j}(\cdot - x_j(t)) \right) (\rho_{n_0} * \partial_x \varphi_{c_i})(\cdot - x_i(t)) dx = 0, \quad i \in \{1, \dots, N\}.$$

Moreover, for $i \in [[-N_-, N_+]]/\{0\}$,

$$\boxed{\text{difdif}} \quad (5.6) \quad |\dot{x}_i - c_i| \leq A\sqrt{\sqrt{\varepsilon} + L^{-1/8}}, \quad \forall t \in \mathbb{R}_+.$$

Combining this result with the asymptotic stability of a peakon established in the preceding section, we are able to extend the asymptotic result to a train of well ordered antipeakons-peakons by following the strategy developed in [31] (see also [19]).

Proof of Theorem 1.3. We first notice that it suffices to prove the result for the part of the train travelling at the right of the line $x = \theta_0 t$, i.e. there exist $0 < c_1^* < \dots < c_{N_+}^*$ and C^1 -functions $t \mapsto x_j(t)$, with $\dot{x}_j(t) \rightarrow c_j^*$ as $t \rightarrow +\infty$, $j \in [[1, N_+]]$, such that

$$\boxed{\text{mul22}} \quad (5.7) \quad u - \sum_{j=1}^{N_+} \varphi_{c_j^*}(\cdot - x_j(t)) \xrightarrow{t \rightarrow +\infty} 0 \text{ in } H^1(x > \theta_0 t).$$

Indeed, then the result at the left of $x = -\theta_0 t$ follows by simply using the symmetry $u(t, x) \mapsto -u(t, -x)$ of the C-H equation.

As in [31] to prove the asymptotic result for the part of the train that travels to the right we proceed by induction starting from the fastest bump. More precisely, setting $z_1(t) = \theta_0 t$ and

$$z_i(t) = \frac{x_{i-1}(t) + x_i(t)}{2}, \quad \forall i \in \{2, \dots, N\},$$

we will prove by induction from $i = N$ to $i = 1$ that $x_i(t) \rightarrow c_i^*$ as $t \rightarrow +\infty$ and that

$$u(t) - \sum_{j=i}^N \varphi_{c_j^*}(\cdot - x_j(t)) \rightarrow 0 \quad \text{in } H^1(]z_i(t), +\infty[$$

Now, we notice that (5.3) and (2.7) ensure that $x_0(t) \leq x_1(t) + 2$ and thus $\text{supp } y_-(t) \subset]-\infty, z_2(t) - 1]$. Therefore applying the almost monotonicity result, we obtain that the total variation of $y(t)$ is uniformly in time bounded on $]z_2(t), +\infty[$ and thus the $N-1$ first steps of the induction can be proven exactly as in [34] (we only have to replace $y(t)$ by $\Theta(\cdot - z_2(\cdot))y$ where Θ is a continuous non decreasing function that equal 1 on $[1, +\infty[$ and vanishes on \mathbb{R}_-).

It remains to tackle the slowest bump travelling to the right. By separating the two same cases as in the preceding section and using the monotonicity result at the right of $x_1(t)$ that is proven in [[34], Lemma 6.2] (Note that we can directly apply this lemma here since $\text{supp } y_-(t) \subset]-\infty, x_1(t) + 2]$), we obtain in the same way an

asymptotic object that gives rise to an Y_+ almost localized solution of (1.6). The rest of the proof follows exactly the same line as in [34] since it does not use the non negativity of the momentum density.

5.3. Asymptotic stability of a not well-ordered train of antipeakons-peakons.

Finally we notice that we can drop the hypothesis that the set of peakons and the set of anti-peakons are each one well-ordered. Note anyway that it is crucial for us that the set of antipeakons are at the left of the set of peakons since otherwise our initial datum cannot satisfy Hypothesis 1. To do so, we make use of the fact that Camassa-Holm equation possesses special solutions called multipeakons given by

$$u(t, x) = \sum_{i=1}^N p_i(t) e^{-|x - q_i(t)|}$$

where $(p_i(t), q_i(t))$, $i = 1, \dots, N$, satisfy a differential hamiltonian system (cf. [7]). In [2] (see also [7]), the limit as $t \rightarrow \mp\infty$ of $p_i(t)$ and $\dot{q}_i(t)$, $i = 1, \dots, N$, are determined. Combining the orbital stability of well ordered train of antipeakons-peakons, the continuity with respect to initial data in $H^1(\mathbb{R})$ and the asymptotics of multipeakons, the H^1 -stability of the variety defined for $(N_-, N_+) \in (\mathbb{N}^*)^2$ by

$$\mathcal{N}_{N_-, N_+} := \left\{ v = \sum_{\substack{j=-N_- \\ j \neq 0}}^{N_+} p_j e^{-|\cdot - q_j|}, \right. \\ \left. (p_{-N_-}, \dots, p_{-1}, p_1, \dots, p_{N_+}) \in (\mathbb{R}_-^*)^{N_-} \times (\mathbb{R}_+^*)^{N_+}, q_{N_-} < q_{-1} < q_1 < \dots < q_{N_+} \right\}.$$

is proved in ([21], Corollary 1.1). Gathering this last result with the asymptotics of the multipeakons and Theorem 1.3, the following asymptotic stability result for not well ordered train of peakons can be deduced quite directly.

cor-mult-peaks

Corollary 5.1. *Let be given N_- negative real numbers $c_{-N_-} < \dots < c_{-2} < c_{-1} < 0$, N_+ positive real numbers $0 < c_1 < c_2 < \dots < c_{N_+}$, $N_- + N_+$ real numbers $q_{-N_-}^0 < \dots < q_{-1}^0 < q_1^0 < \dots < q_{N_+}^0$ and let $\lambda_{-N_-} < \dots < \lambda_{-1} < 0 < \lambda_1 < \dots < \lambda_{N_+}$ be the $N_- + N_+$ distinct eigenvalues of the matrix $(p_j^0 e^{-|q_i^0 - q_j^0|/2})_{(i,j) \in ([-N_-, N_+] \setminus \{0\})^2}$. For any $B > 0$ there exists a positive function ε with $\varepsilon(y) \rightarrow 0$ as $y \rightarrow 0$ and $\alpha_0 > 0$ such that if $u_0 \in Y$ satisfies the Hypothesis 1 with*

$$(5.8) \quad \|m_0\|_{\mathcal{M}} \leq B \quad \text{and} \quad \|u_0 - \sum_{\substack{j=-N_- \\ j \neq 0}}^{N_+} p_j^0 \exp(\cdot - q_j^0)\|_{H^1} \leq \alpha$$

for some $0 < \alpha < \alpha_0$, then there exists $c_{-N_-}^* < \dots < c_{-1}^* < 0 < c_1^* < \dots < c_{N_+}^*$ and C^1 -functions $(x_{-N_-}, \dots, x_{-1}, x_1, \dots, x_{N_+})$ with

$$|c_i^* - \lambda_i| \leq \varepsilon(\alpha) \quad \text{and} \quad \lim_{t \rightarrow +\infty} \dot{x}_i(t) = c_i^*, \quad \forall i \in ([-N_-, N_+] \setminus \{0\}),$$

such that

$$(5.9) \quad u - \sum_{\substack{j=-N_- \\ j \neq 0}}^{N_+} \varphi_{c_i^*}(\cdot - x_i(t)) \xrightarrow{t \rightarrow +\infty} 0 \quad \text{in } H^1(|x| > \min(-\lambda_{-1}, \lambda_1)/4).$$

coromul2

6. APPENDIX: AN IMPROVEMENT OF THE ASYMPTOTIC STABILITY RESULT IN THE CLASS OF SOLUTIONS WITH NON NEGATIVE MOMENTUM DENSITY

In this subsection we make a simple observation that enables to improve the asymptotic stability result given in [34]. This observation is that all the energy of the solutions of the C-H equation, that have a non negative density momentum, is traveling to the right. Moreover, as stated in the following lemma, the energy at the left of any fixed point tends to zero as $t \rightarrow +\infty$.

apen **Lemma 6.1.** *For any $u_0 \in Y_+$ and any $z \in \mathbb{R}$, denoting by $u \in C(\mathbb{R}; H^1)$ solution of (1.6) emanating from u_0 it holds*

tdt (6.1)
$$\lim_{t \rightarrow +\infty} \|u(t)\|_{H^1(]-\infty, z])} = 0 .$$

This differs from KdV like equations since for these last equations the linear waves are travelling to the left. This causes a small loss of energy to the left and the so called dispersive tail. Note that for CH-type equation, there is no linear part and if moreover the momentum density is non negative, there is also no (even small) antipeakon and thus all the energy is travelling to the right that formally leads to (6.1).

Before proving Lemma 6.1, let us state the improved asymptotic stability result :

asymptstab+ **Theorem 6.2.** *Let $c > 0$ be fixed. There exists a universal constant $0 < \eta_0 \ll 1$ such that for any $0 < \theta < c$ and any $u_0 \in Y_+$ satisfying*

difini+ (6.2)
$$\|u_0 - \varphi_c\|_{H^1} \leq \eta_0 \left(\frac{\theta}{c}\right)^8 ,$$

there exists $c^ > 0$ with $|c - c^*| \ll c$ and a C^1 -function $x : \mathbb{R} \rightarrow \mathbb{R}$ with $\lim_{t \rightarrow \infty} \dot{x} = c^*$ such that*

(6.3)
$$u(t, \cdot + x(t)) \xrightarrow{t \rightarrow +\infty} \varphi_{c^*} \text{ in } H^1(\mathbb{R}) ,$$

where $u \in C(\mathbb{R}; H^1)$ is the solution emanating from u_0 . Moreover, for any $z > 0$,

cvforter+ (6.4)
$$\lim_{t \rightarrow +\infty} \|u(t) - \varphi_{c^*}(\cdot - x(t))\|_{H^1(\mathbb{R}/[z, \theta t])} = 0 .$$

Proof of Lemma 6.1. Let $0 < \gamma < \|u_0\|_{H^1}^2$ and let $x_\gamma : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

defxg (6.5)
$$\int_{\mathbb{R}} (u^2 + u_x^2)(t) \Phi(\cdot - x_\gamma(t)) = \gamma$$

with Φ defined in (3.5). Note that $x_\gamma(\cdot)$ is well-defined since $u_0 \in Y_+$ forces $u > 0$ on \mathbb{R}^2 and thus for any fixed $t \in \mathbb{R}$, $z \mapsto \int_{\mathbb{R}} (u^2 + u_x^2)(t) \Phi(\cdot - z)$ is a decreasing continuous bijection from \mathbb{R} to $]0, \|u_0\|_{H^1}^2[$. Moreover, $u \in C(\mathbb{R}; H^1)$ ensures that $x_\gamma(\cdot)$ is a continuous function. (6.1) is clearly a direct consequence of the fact that

td3 (6.6)
$$\lim_{t \rightarrow +\infty} x_\gamma(t) = +\infty .$$

To prove (6.6) we first claim that for any $t \in \mathbb{R}$ and any $\Delta > 0$ it holds

td4 (6.7)
$$x_\gamma(t + \Delta) - x_\gamma(t) \geq \frac{1}{2} \left(\int_t^{t+\Delta} \int_{x_\gamma(t)}^{x_\gamma(t)+2} u^2(\tau, s) ds d\tau \right)^{1/2} > 0 .$$

Let us prove this claim. First we notice that by continuity with respect to initial data, it suffices to prove (6.7) for $u \in C^\infty(\mathbb{R}; H^\infty) \cap L^\infty(\mathbb{R}; Y_+)$. Then a simple

application of the implicit function theorem ensures that $t \mapsto x_\gamma(t)$ is of class C^1 . Indeed,

$$\psi : (z, v) \mapsto \int_{\mathbb{R}} (v^2 + v_x^2) \Phi(\cdot - z)$$

is of class C^1 from $\mathbb{R} \times H^1(\mathbb{R})$ into \mathbb{R} and for any $(z_0, v) \in \mathbb{R} \times Y_+ / \{0\}$, $\partial_z \psi(z_0, v) = \int_{\mathbb{R}} (v^2 + v_x^2) \Phi'(\cdot - z_0) > 0$. It then follows from (3.17) that

$$\dot{x}_\gamma(t) \int_{\mathbb{R}} (u^2 + u_x^2) \Phi'(\cdot - x_\gamma(t)) = \int_{\mathbb{R}} u u_x^2 \Phi'(\cdot - x_\gamma(t)) + 2 \int_{\mathbb{R}} u h \Phi'(\cdot - x_\gamma(t))$$

where $h = p * (u^2 + u_x^2/2)$. Now, it is worth noticing that according to [9], the following convolution estimate holds : For any $v \in H^1(\mathbb{R})$,

$$\boxed{\text{td5}} \quad (6.8) \quad p * (v^2 + v_x^2/2)(x) \geq \frac{1}{2} v^2 \quad \text{on } \mathbb{R}.$$

Combining (6.7)-(6.8) and the fact that $|v_x| \leq v$ on \mathbb{R} for any $v \in Y_+$, we eventually get

$$2\dot{x}_\gamma(t) \int_{\mathbb{R}} u^2 \Phi'(\cdot - x_\gamma(t)) \geq \int_{\mathbb{R}} u(u^2 + u_x^2) \Phi'(\cdot - x_\gamma(t)) \geq \int_{\mathbb{R}} u^3 \Phi'(\cdot - x_\gamma(t))$$

and Hölder inequality together with the fact that Φ' is a non negative function of total mass 1 lead to

$$\boxed{\text{td6}} \quad (6.9) \quad \dot{x}_\gamma(t) \geq \frac{1}{2} \left(\int_{\mathbb{R}} u^2 \Phi'(\cdot - x_\gamma(t)) \right)^{1/2}$$

Integrating this inequality between t and $t + \Delta$ yields (6.7) that obviously implies that $x_\gamma(\cdot)$ is an increasing function. In particular there exists $x_\gamma^\infty \in \mathbb{R} \cap \{+\infty\}$ such that $x_\gamma(t) \nearrow x_\gamma^\infty$ as $t \rightarrow +\infty$ and it remains to prove that $x_\gamma^\infty = +\infty$. Assuming the contrary, we first notice that since $|u_x| \leq u \leq \|u_0\|_{H^1}$ on \mathbb{R}^2 , (6.5) then yields

$$\boxed{\text{contro}} \quad (6.10) \quad \lim_{t \rightarrow +\infty} \int_{\mathbb{R}} (u^2 + u_x^2) \Phi'(\cdot - x_\gamma(t)) = \lim_{t \rightarrow +\infty} \int_{\mathbb{R}} (u^2 + u_x^2) \Phi'(\cdot - x_\gamma^\infty) = \gamma.$$

Now, taking $\Delta = 1$, (6.7) forces

$$\lim_{t \rightarrow +\infty} \int_t^{t+1} \int_{x_\gamma(t)}^{x_\gamma(t)+2} u^2(\tau, s) ds d\tau = 0$$

which, recalling that $x_\gamma(t) \rightarrow x_\gamma^\infty$, leads to

$$\lim_{t \rightarrow +\infty} \int_t^{t+1} \int_{x_\gamma^\infty}^{x_\gamma^\infty+2} u^2(\tau, s) ds d\tau = 0.$$

In particular there exists a sequence $(t_n, x_n)_{n \geq 1} \subset \mathbb{R} \times [x_\gamma^\infty, x_\gamma^\infty + 2]$ with $t_n \nearrow +\infty$ such that $u(t_n, x_n) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, making use of the fact that $|u_x| \leq u$ on \mathbb{R}^2 forces, for any $(t, x_0) \in \mathbb{R}^2$, that

$$(6.11) \quad u(t, x) \leq e^{|x_0 - x|} u(t, x_0), \quad \forall x \in \mathbb{R},$$

we infer that for any $A > 0$,

$$\boxed{\text{td7}} \quad (6.12) \quad \lim_{n \rightarrow \infty} \sup_{x \in [x_\gamma^\infty - A, x_\gamma^\infty + A]} u(t_n, x) = 0.$$

Finally, taking $A > 0$ such that $x_\gamma^\infty - A < x_{\gamma'}(0)$ with $\gamma < \gamma' < \|u_0\|_{H^1}^2$, we infer from (6.12) and the monotony of $t \mapsto x_{\gamma'}(t)$ that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} (u^2 + u_x^2)(t_n, \cdot) \Phi(\cdot - x_\gamma^\infty) = \gamma'.$$

This contradicts (6.10) and concludes the proof of the lemma. \square

Conflict of Interest : The author declares that he has no conflict of interest.

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