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Laurent Mazet, Harold Rosenberg

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MINIMAL PLANES IN ASYMPTOTICALLY FLAT THREE-MANIFOLDS

LAURENT MAZET AND HAROLD ROSENBERG

ABSTRACT. In this paper, we improve a result by Chodosh and Ketover [2]. We prove that, in an asymptotically flat 3-manifold M that contains no closed minimal surfaces, fixing $q \in M$ and a 2-plane V in $T_q M$ there is a properly embedded minimal plane Σ in M such that $q \in \Sigma$ and $T_q \Sigma = V$. We also prove that fixing three points in M there is a properly embedded minimal plane passing through these three points.

1. INTRODUCTION

This paper is inspired by the recent work of Otis Chodosh and Daniel Ketover [2]. They consider an asymptotically Euclidean 3-manifold M that contains no closed minimal surfaces. Then, for any point q in M , they prove the existence of a properly embedded minimal plane passing through the point q .

Here we obtain some more information:

- if $q \in M$ and $V \subset T_q M$ is a 2-plane, then there is a properly embedded minimal plane Σ in M with $q \in \Sigma$ and $T_q \Sigma = V$.
- if q_1, q_2, q_3 are 3 points of M , there is a properly embedded minimal plane Σ in M containing the three points.

The main ingredient is the following Theorem

Theorem. *Let $\mathbb{B}$ be the unit ball of $\mathbb{R}^3$ and let M be $\mathbb{B}$ with a Riemannian metric g satisfying*

- ∂M is strictly mean convex, and
- M contains no closed minimal surfaces.

Then we have the following statements.

- (1) *If $q \in \text{int}(M)$ and $V \subset T_q M$ is a 2-plane, there is an embedded minimal disk $D \subset M$ with $\partial D = D \cap \partial M$, $q \in D$ and $T_q D = V$.*
- (2) *If q_1, q_2, q_3 are 3 points in $\text{int}(M)$, there is an embedded minimal disk D in M with $\partial D = D \cap \partial M$ and q_1, q_2, q_3 in D .*

The introduction to the paper of Chodosh and Ketover [2] contains an excellent exposition of the problem of the existence of minimal surfaces in asymptotically flat 3-manifolds, in particular, properly embedded minimal planes. They discuss the relevance of this theory to general relativity. They also explain why they employ the theory of Brian White "moduli space

of minimal surfaces and degree theory” to prove their theorem rather than variational techniques, solving Plateau problems in geodesic balls and taking limits.

Our proof of our main Theorem above also starts with the theory of White, the moduli space of embedded minimal disks in the ball $M = (\mathbb{B}, g)$.

A natural question is what are the complete Riemannian 3-manifolds that contain properly embedded minimal planes and what are such planes. In the Euclidean $\mathbb{R}^3$, we know the only such planes are affine planes and helicoids [3]. Moreover affine planes are completely characterized by a point in it and its tangent plane at that point or by 3 non collinear points contained in it. This explains why we are interested in prescribing these constraints. We notice that in the case of the Euclidean $\mathbb{R}^3$ our proof produces affine planes and no helicoids.

Actually, in the general case, the minimal planes we construct have quadratic area growth. It would be interesting to know that these minimal planes are the only ones with a prescribed point and unit normal or 3 non collinear points (*i.e.*, not on a geodesic) and quadratic area growth as it is the case of $\mathbb{R}^3$. Actually, we believe this is true.

For other ambient spaces, the existence of properly embedded planes is still an open question. For example, when g is a complete metric of non positive curvature on $\mathbb{R}^3$ (a Hadamard manifold), we do not know if such a minimal plane exists. When the curvature is strictly pinched, by the work of Anderson they exist [1]. When S is a closed surface, there is no properly embedded minimal plane in $S \times \mathbb{R}$, with the product metric. The same problem for embedded minimal annuli $\mathbb{S}^1 \times \mathbb{R}$ is also very interesting: is it possible to construct ”catenoids” in some ambient manifolds?

The paper is organized as follow. In Section 2, we recall the main points of White’s theory. Section 3 is devoted to the proof of item 1 of the above main theorem (Theorem 3). Section 4 deals with item 2 of the main theorem (Theorem 9). In the last section, we explain how one can pass from a minimal disk in a compact ball to a properly embedded minimal plane in an asymptotically Euclidean space, here we use ideas developed by Chodosh and Ketover in [2].

2. THE SPACE OF MINIMAL DISKS

After Tomi and Tromba [6], White [8] has developed the theory of the moduli space of minimal submanifolds. Here we recall some of the aspects of this theory that we will use.

Let us denote by $\mathbb{B}$ the closed unit ball in $\mathbb{R}^3$ and $\mathbb{D}$ the closed unit disk in $\mathbb{R}^2$. Let M be $\mathbb{B}$ endowed with a Riemannian metric g .

Let $f_i : \mathbb{D} \rightarrow \mathbb{R}^3$ ($i = 1, 2$) be two $C^{k,\alpha}$ maps ($k \geq 2$ and $0 < \alpha < 1$). They are said to be equivalent if $f_1 = f_2 \circ \varphi$ for some C^1 diffeomorphism φ of $\mathbb{D}$ that is the identity on $\partial\mathbb{D} = \mathbb{S}^1$; let $\mathcal{F}$ denote the set of equivalence

classes. We also denote by $\text{Diff}_{k,\alpha}(\mathbb{D}, \partial\mathbb{D})$ the set of $C^{k,\alpha}$ diffeomorphisms of $\mathbb{D}$ that are the identity on $\partial\mathbb{D}$.

On $\mathcal{F}$, there is a boundary operator $\delta : \mathcal{F} \rightarrow C^{k,\alpha}(\mathbb{S}^1, \mathbb{R}^3)$ which is defined by $\delta([f]) = f|_{\mathbb{S}^1}$. The space of immersed minimal disks $\mathcal{M}^*$ is then the space of equivalence classes $[f]$ of $C^{k,\alpha}$ minimal immersions $f : \mathbb{D} \rightarrow M$ such that $\delta([f]) \in C^{k,\alpha}(\mathbb{S}^1, \partial M)$ and f is never tangent to ∂M (see Section 8 in [8]). We notice that if $[f_1] = [f_2]$ where f_1 and f_2 are two $C^{k,\alpha}$ immersions then $\varphi \in \text{Diff}_1(\mathbb{D}, \partial\mathbb{D})$ such that $f_1 = f_2 \circ \varphi$ is actually in $\text{Diff}_{k,\alpha}(\mathbb{D}, \partial\mathbb{D})$ (see Appendix A).

The main result of [8] is the following theorem.

Theorem 1. *The space $\mathcal{M}^*$ is a smooth Banach manifold and*

$$\delta : \mathcal{M}^* \rightarrow C^{k,\alpha}(\mathbb{S}^1, \partial M)$$

is a smooth Fredholm map of index 0.

The descriptions of the charts of $\mathcal{M}^*$ are given in [8, Theorem 3.3] (see also the proof of Lemma 6).

Actually, we are going to consider just the open subset $\mathcal{M}$ of $[f] \in \mathcal{M}^*$ where f is a minimal embedding of the disk. $\mathcal{M}$ is the space of minimal embedded disks in M with boundary in ∂M and never tangent to ∂M .

A second important result gives conditions under which the map δ is proper.

Theorem 2. *Let us assume that M satisfies the following two conditions*

- ∂M is strictly mean-convex towards M and
- M contains no closed minimal surfaces.

Then the map $\delta : \mathcal{M} \rightarrow C^{k,\alpha}(\mathbb{S}^1, \partial M)$ is proper

This result comes by combining different results by White: Theorem C in [9] gives the properness thanks to the area estimate of Theorem 2.1 in [10] (M has mean convex boundary and no closed embedded minimal surfaces) and the curvature estimate given by Theorem 0 in [7].

3. PRESCRIBING THE TANGENT PLANE TO A MINIMAL DISK IN A RIEMANNIAN 3-BALL

In this section we want to find a minimal disk in M passing through a point q with a prescribed tangent space. We still assume that M is $\mathbb{B}$ endowed with a metric g but we also assume that g can be extended to the whole $\mathbb{R}^3$. Thus the main result of this section is the following theorem

Theorem 3. *Let $M = (\mathbb{B}, g)$ be as above such that*

- (H1) ∂M is strictly mean convex towards M and
- (H2) M contains no closed minimal surfaces.

Let us consider $q \in M \setminus \partial M$ and V be a 2-dimensional subspace of $T_q M$. There exists $\sigma \in \mathcal{M}$ such that $q \in \sigma$ and $V = T_q \sigma$.

The remaining part of this section is devoted to the proof of this theorem.

Let us describe the boundary curves that we will consider. The unit tangent bundle to $\mathbb{S}^2$ is $US^2 = \{(p, v) \in \mathbb{S}^2 \times \mathbb{S}^2 \mid \langle p, v \rangle = 0\}$. For $t \in (-1, 1)$, $(p, v) \in US^2$ and $e^{i\theta} \in \mathbb{S}^1$, we define $P(p, v, t, e^{i\theta}) = tp + \sqrt{1-t^2}(\cos \theta v + \sin \theta p \wedge v)$. When $e^{i\theta}$ runs along $\mathbb{S}^1$, this describes the parallel circle $\{q \in \mathbb{S}^2 \mid \langle p, q \rangle = t\}$.

Thus we obtain a map:

$$(1) \quad \begin{array}{ccc} US^2 \times (-1, 1) & \longrightarrow & C^{k,\alpha}(\mathbb{S}^1, \mathbb{S}^2) \\ (p, v, t) & \longmapsto & \zeta \mapsto P(p, v, t, \zeta) \end{array}$$

Actually Φ gives the whole family of parallel curves on $\mathbb{S}^2$. By changing v , we only change the origin of the parametrization.

Basically, the idea of the proof of Theorem 3 consists in considering all the minimal disks in M bounded by a parallel $\Phi(p, v, t)$ and then finding a perturbation of one of them that satisfies the desired property.

3.1. Study when $|t|$ is close to 1. In this section we prove that when $|t|$ is close to 1 there is only one minimal disk in M bounded by $\Phi(p, v, t)$. More precisely we have the following statement.

Proposition 4. *There is $\bar{\varepsilon} > 0$ such that, for any $(p, v) \in US^2$ and $1 - \bar{\varepsilon} \leq |t| < 1$, there is a unique minimal disk $D_{p,v,t}$ in M bounded by $\delta(D_{p,v,t}) = \Phi(p, v, t)$.*

Moreover, when (p, v) are fixed, the $D_{p,v,t}$ form a foliation for $t \in [1 - \bar{\varepsilon}, 1)$ and $t \in (-1, -1 + \bar{\varepsilon}]$ by stable non-degenerate minimal disks. $D_{p,v,t}$ can be parametrized by

$$F(p, v, t) : (x, y) \in \mathbb{D} \mapsto \sqrt{1-t^2}(xv + yp \wedge v) + (t + u_{p,v,t}(x, y))p$$

where $(p, v, t) \mapsto u_{p,v,t}$ is a smooth family of $C^{k,\alpha}$ functions with $u_{p,v,t} = (1-t)h_{p,v} + o(1-t)$ (we have the same formula close to -1) and 0 boundary values.

Proof. First let us recall that if a Riemannian metric in $\mathbb{R}^3$ is given by symmetric matrices G and u is a function defined on a subset of $\mathbb{R}^2$ then its graph is minimal if

$$0 = \frac{1}{\sqrt{\det G}} \operatorname{div}_3 \left(\frac{\sqrt{\det G}}{\sqrt{t \nabla_3 f (G^{-1}) \nabla_3 f}} G^{-1} \nabla_3 f \right)_{|(x,y,u(x,y))}$$

where f is the function on $\mathbb{R}^3$ defined by $f(x, y, z) = u(x, y) - z$ and div_3 and ∇_3 are the Euclidean divergence and gradient operator on $\mathbb{R}^3$.

For $(p, v, t) \in US^2 \times (-1, 1)$ we are looking for a minimal disk that is a graph over the Euclidean disk bounded by $\Phi(p, v, t)$. So we are looking for a function $u_{p,v,t}$ defined on $\mathbb{D}$ with vanishing boundary values such that the surface

$$F(p, v, t) : (x, y) \in \mathbb{D} \mapsto \sqrt{1-t^2}(xv + yp \wedge v) + (t + u_{p,v,t}(x, y))p$$

is in $\mathbb{B}^3$ and is minimal for the metric g . Since t is close to 1 (the same can be done for t close to -1), we write $t = \cos \beta$ and we look for $w_{p,v,\beta} = u_{p,v,\cos \beta}$ such that

$$\tilde{F}(p, v, \beta) : (x, y) \in \mathbb{D} \mapsto \sin \beta (xv + yp \wedge v) + (\cos \beta + w_{p,v,\beta}(x, y))p$$

is in $\mathbb{B}^3$ and is minimal for the metric g . For fixed p, v, β , we define the conformal map $M_{p,v,\beta} : \mathbb{D} \times \mathbb{R} \rightarrow \mathbb{R}^3; (x, y, z) \mapsto \sin \beta (xv + yp \wedge v) + (\cos \beta + \sin \beta z)p$. So $\tilde{F}(p, v, t)$ is minimal if the graph of $\tilde{w}_{p,v,\beta} = \frac{w_{p,v,\beta}}{\sin \beta}$ in $\mathbb{D} \times \mathbb{R}$ is minimal for the metric $g_{p,v,\beta} = \frac{M_{p,v,\beta}^* g}{\sin^2 \beta}$. As explained above if G is the matrix of g in $\mathbb{R}^3$, the graph of $\tilde{w}$ in $\mathbb{D} \times \mathbb{R}$ is minimal for $g_{p,v,\beta}$ if

$$0 = \frac{1}{\sqrt{\det G_{p,v,\beta}}} \operatorname{div}_3 \left(\frac{\sqrt{\det G_{p,v,\beta}}}{\sqrt{t \nabla_3 f (G_{p,v,\beta}^{-1}) \nabla_3 f}} G_{p,v,\beta}^{-1} \nabla_3 f \right)_{|(x,y,\tilde{w}(x,y))} = H(p, v, \beta, \tilde{w})(x, y)$$

where $f(x, y, z) = \tilde{w}(x, y) - z$ and $G_{p,v,\beta} = (v, p \wedge v, p)^* (G \circ M_{p,v,\beta})(v, p \wedge v, p)$ where $(v, p \wedge v, p)$ is the matrix whose columns are $v, p \wedge v$ and p .

So one can consider the map

$$H : US^2 \times (-\varepsilon, \varepsilon) \times C_0^{k,\alpha}(\mathbb{D}) \rightarrow C^{k-2,\alpha}(\mathbb{D}); (p, v, \beta, \tilde{w}) \mapsto H(p, v, \beta, \tilde{w})$$

where $C_0^{k,\alpha}(\mathbb{D})$ is the subset of $C^{k,\alpha}$ -functions vanishing on $\partial \mathbb{D}$. We want to solve $H(p, v, \beta, \tilde{w}) = 0$.

Let us remark that for $\beta = 0$, $H(p, v, 0, \tilde{w})$ computes the mean curvature of the graph of $\tilde{w}$ in the constant metric $G_{p,v,0} = (v, p \wedge v, p)^* G(p)(v, p \wedge v, p)$. So the function $\tilde{w} \equiv 0$ is a solution, $H(p, v, 0, 0) = 0$. Besides we have

$$D_{\tilde{w}} H(p, v, 0, 0)(h) = \frac{1}{(G_{p,v,0}^{33})^{3/2}} \sum_{i,j=1}^2 (G_{p,v,0}^{ij} G_{p,v,0}^{33} - G_{p,v,0}^{i3} G_{p,v,0}^{j3}) h_{ij}$$

where the $G_{p,v,0}^{ij}$ are the coefficients of $G_{p,v,0}^{-1}$. This operator is elliptic with constant coefficients so it is invertible from $C_0^{k,\alpha}(\mathbb{D})$ to $C^{k-2,\alpha}(\mathbb{D})$ and the implicit function theorems applies. So there is a family of functions $\tilde{w}_{p,v,\beta}$ (β close to 0) which solves $H(p, v, \beta, \tilde{w}_{p,v,\beta}) = 0$ and $\tilde{w}_{p,v,0} = 0$. Moreover we have $\tilde{w}_{p,v,\beta} = \beta \tilde{h}_{p,v} + o(\beta)$.

Thus the family $(u_{p,v,t})$ is well defined for any $(p, v) \in US^2$ and $t \in (1 - \varepsilon, 1)$. Using that $t = \cos \beta$, we obtain $u_{p,v,t} = (1 - t) \tilde{h}_{p,v} + o(1 - t)$. It still remains to prove that these disks are in $\mathbb{B}$ and form local foliations. By (H1), we notice that in the metric g the Euclidean spheres $\mathbb{S}_r^2$ of radius r are mean convex for $r \in [1 - \delta, 1 + \delta]$. Since the $F(p, v, 1)(\mathbb{D}) = p$, for t close to 1, $F(p, v, t)(\mathbb{D})$ is contained in the part of $\mathbb{R}^3$ foliated by the $\mathbb{S}_r^2$ ($1 - \delta < r < 1 + \delta$). Since $\partial(F(p, v, t)(\mathbb{D})) \subset \mathbb{S}^2$, the comparison principle implies $F(p, v, t)(\mathbb{D}) \subset \mathbb{B}$ for any $(p, v, t) \in US^2 \times (1 - \varepsilon, 1)$.

If $t < t' < 1$, the boundary curve $\Phi(p, v, t')(\mathbb{S}^1)$ is between $\Phi(p, v, t)(\mathbb{S}^1)$ and p . Besides, for t' close to 1, $F(p, v, t')(\mathbb{D})$ is close to p and does not intersect $F(p, v, t)(\mathbb{D})$. So by the maximum principle $F(p, v, t')(\mathbb{D})$ does not intersect $F(p, v, t)(\mathbb{D})$ for any $1 - \varepsilon < t < t' < 1$. Thus at fixed (p, v) , the minimal disks $\{F(p, v, t)(\mathbb{D})\}_{1-\varepsilon < t < 1}$ form a foliation. As a consequence, these minimal disks are stable, non degenerate and moreover they are the unique minimal disk with boundary $\Phi(p, v, t)(\mathbb{S}^1)$ at least for $1 - \bar{\varepsilon} \leq t < 1$ ($\bar{\varepsilon} < \varepsilon$). This uniqueness statement comes from the work of White in [9] (the arguments start at the bottom of page 148, notice that here we have already constructed a foliation, see also Remark 2 below). $\square$

Remark 1. Actually the function $\tilde{h}_{p,v}$ can be computed by the equation $0 = D_\beta H(p, v, 0, 0) + D_{\tilde{w}} H(p, v, 0, 0)(\tilde{h}_{p,v})$ that can be solved since $D_{\tilde{w}} H(p, v, 0, 0)$ is invertible. Moreover, $D_\beta H(p, v, 0, 0)$ can be computed and is given by the difference of the mean curvatures at p of $\mathbb{S}^2 \subset (M, g)$ (which is positive) and $\mathbb{S}^2 \subset \mathbb{R}^3$ endowed with the constant metric $G(p)$ (which is also positive). So $D_\beta H(p, v, 0, 0)$ is a constant function on $\mathbb{D}$ and $\tilde{h}_{p,v}(x, y) = \lambda(1 - (x^2 + y^2))$ for some λ .

Remark 2. For the readers benefit, we give the arguments of White concerning uniqueness. So we fix $(\bar{p}, \bar{v}) \in U\mathbb{S}^2$. We know that for any (p, v) there is an open neighborhood $(U_{p,v})$ of p foliated by the minimal disks $\{F(p, v, t)(\mathbb{D})\}_{1-\varepsilon < t < 1} = \{D_{p,v,t}\}_{1-\varepsilon < t < 1}$. Let $(p_i, v_i, t_i) \rightarrow (\bar{p}, \bar{v}, 1)$. Let R_i be a minimal surface in M with $\partial R_i = \Phi(p_i, v_i, t_i)(\mathbb{S}^1)$ with $R_i \neq D_{p_i, v_i, t_i}$.

Let T_i be a minimal disk with boundary $\Phi(p_i, v_i, t_i)(\mathbb{S}^1)$ that minimizes the area in the connected component of $M \setminus R_i$ that contains $-p_i$. T_i is an embedded minimal disk and $\text{area}(T_i) \leq \text{area}(\partial M)$.

Unless $\text{area}(T_i) \rightarrow 0$, by Theorem 3 in [7], T_i converges to a minimal surface T with $\partial T = \bar{p}$ and T is smooth outside $\bar{p}$. By Theorem 2 in [7], T is regular even at $\bar{p}$. So this is impossible since ∂M is strictly mean convex.

So $\text{area}(T_i) \rightarrow 0$ and, by the monotonicity formula, T_i can't contain points outside of U_{p_i, v_i} so the same is true for R_i . Since the minimal disks $\{D_{p_i, v_i, t}\}_{1-\varepsilon < t < 1}$ foliate U_{p_i, v_i} this implies that $R_i = D_{p_i, v_i, t_i}$ by the maximum principle.

3.2. The family of minimal disks. In this section we are going to prove the following result.

Proposition 5. *There is a connected proper 4-dimensional submanifold $\Sigma \subset \mathcal{M}$ that contains all the minimal disks $D_{p,v,t}$ for any (p, v) and $1 - \bar{\varepsilon} \leq |t| < 1$. Moreover, for any $\sigma \in \Sigma$, $\delta(\sigma)$ is close to some $\Phi(p, v, t)$.*

Besides there is a map $F : \Sigma \rightarrow C^{2,\alpha}(\mathbb{D}, \mathbb{B})$, continuous in the C^2 topology, such that $\sigma = [F(\sigma)]$.

Actually $\delta(\sigma)$ can be chosen as close to $\Phi(p, v, t)$ as we want.

First we recall that $\delta : \mathcal{M} \rightarrow C^{2,\alpha}(\mathbb{S}^1, \mathbb{S}^2)$ is proper by Theorem 2.

The map Φ is an embedding of $U\mathbb{S}^2 \times (-1, 1)$ in $C^{2,\alpha}(\mathbb{S}^1, \mathbb{S}^2)$. Moreover, by Proposition 4, when $1 - \bar{\varepsilon} \leq |t| < 1$, there is a unique minimal disk $\sigma \in \mathcal{M}$

such that $\delta(\sigma) = \Phi(p, v, t)$ and σ is non degenerate. So when t is close to ± 1 , Φ is transverse to the boundary map δ .

Let us fix $(p_0, v_0) \in US^2$. By the Smale transversality theorem [5], there exists $\Phi' : (p_0, v_0) \times (-1, 1) \rightarrow C^{2,\alpha}(\mathbb{S}^1, \mathbb{S}^2)$, a perturbation of Φ on $(p_0, v_0) \times (-1, 1)$, which is transverse to δ . Moreover we can assume that $\Phi = \Phi'$ when $1 - \bar{\varepsilon} \leq |t| < 1$.

Thus $\delta^{-1}(\Phi'((p_0, v_0) \times (-1, 1)))$ is a submanifold of $\mathcal{M}$. Since δ is proper, $\delta^{-1}(\Phi'((p_0, v_0) \times (-1, 1)))$ is a proper 1-dimensional submanifold of $\mathcal{M}$. Let γ be its component that contains $D_{p_0, v_0, t}$ for t close to 1. Since $\delta^{-1}(\Phi'((p_0, v_0) \times (-1, 1)))$ is proper, it must contain also $D_{p_0, v_0, t}$ for t close to -1 .

Since Φ' is a perturbation of Φ on $(p_0, v_0) \times (-1, 1)$, we can extend its definition to the whole $US^2 \times (-1, 1)$ as a perturbation of Φ such that Φ' is still an embedding. We notice that a priori Φ' is not transverse to δ on $US^2 \times (-1, 1)$ but it is the case on $(p_0, v_0) \times (-1, 1)$ and $(US^2 \times ((-1, -1 + \bar{\varepsilon}] \cup [1 - \bar{\varepsilon}, 1)))$ where we choose $\Phi' = \Phi$. The Smale transversality theorem implies there is a perturbation $\tilde{\Phi}$ of Φ' on the whole $US^2 \times (-1, 1)$ which is transverse to δ and coincides with Φ' on $(p_0, v_0) \times (-1, 1)$ and $(US^2 \times ((-1, -1 + \bar{\varepsilon}] \cup [1 - \bar{\varepsilon}, 1)))$.

So $\delta^{-1}(\tilde{\Phi}(US^2 \times (-1, 1)))$ is a proper 4-dimensional submanifold of $\mathcal{M}$. Inside it, we only consider the connected component Σ that contains $D_{p, v, t}$ for any (p, v) and t close to ± 1 (Σ exists since $\gamma \subset \Sigma$).

We denote by π the map $\tilde{\Phi}^{-1} \circ \delta : \Sigma \rightarrow US^2 \times (-1, 1)$. For t close to ± 1 we know that π is a bijection. Let us define

$$\Sigma_{\pm} = \delta^{-1}(\Phi(US^2 \times [\pm(1 - \bar{\varepsilon}), \pm 1])) = \{D_{p, v, t}; (p, v, t) \in US^2 \times [\pm(1 - \bar{\varepsilon}), \pm 1]\}$$

Let us also define $\Sigma_0 = \delta^{-1}(\tilde{\Phi}(US^2 \times [-1 + \bar{\varepsilon}, 1 - \bar{\varepsilon}]))$.

So let us construct the map F that parametrizes all these minimal disks. First when $\sigma \in \Sigma_{\pm}$, we know that we can identify σ with $(p, v, t) = \pi(\sigma)$ and we define:

$$F(\sigma) = F(p, v, t) : (x, y) \in \mathbb{D} \mapsto \sqrt{1 - t^2}(xv + yp \wedge v) + (t + u_{p, v, t}(x, y))p$$

as in Proposition 4. So we need to define $F(\sigma)$ when $\sigma \in \Sigma_0$.

Lemma 6. *There is a map $F : \Sigma \rightarrow C^{2,\alpha}(\mathbb{D}, \mathbb{B})$ continuous in the C^2 topology which coincides with the above definition on $\Sigma_{\pm}$ such that $\sigma = [F(\sigma)]$. We have $F(\sigma)|_{\partial \mathbb{D}} = \tilde{\Phi}(\pi(\sigma))$.*

Proof. First we notice that F is well defined on $\partial \Sigma_0$. Let us recall what are the charts around $\bar{\sigma} \in \mathcal{M}$. By Theorem 3.3 and Section 8 in [8], if $\ker D\delta(\bar{\sigma})$ has dimension j (the set of Jacobi fields on $\bar{\sigma}$ that vanish on the boundary has dimension j), there is a neighborhood of $\bar{\sigma}$ in $\mathcal{M}$ that can be identified with a submanifold in $C^{2,\alpha}(\partial \mathbb{D}, \partial M) \times \mathbb{R}^j$ of codimension j (such that δ correspond to the projection on the first factor). Moreover there is $H : C^{2,\alpha}(\partial \mathbb{D}, \partial M) \times \mathbb{R}^j \rightarrow C^{2,\alpha}(\mathbb{D}, M)$ such that its restriction to the submanifold gives a parametrization of any minimal surface in a neighborhood of $\bar{\sigma}$.

Let $Y = Y^0 \subset \dots \subset Y^4$ be a triangulation of Σ_0 such that each 4-cell is contained in one of the above charts of $\mathcal{M}$ that are diffeomorphic to a ball. First for any σ in Y^0 , we choose a parametrization $F(\sigma)$ of σ such that, if $\sigma \in Y^0 \cap \partial\Sigma_0$, $F(\sigma)$ coincides with the preceding definition of F .

By induction, let us assume that $F : Y^p \rightarrow C^{2,\alpha}(\mathbb{D}, \mathbb{B})$ continuous in the C^2 topology is defined and consider e a $p+1$ -cell in Y^{p+1} . If e belongs to $\partial\Sigma_0$, we extend F by the preceding definition. If not, e belongs to an above chart of $\mathcal{M}$. So combining the inclusion of e in the submanifold of $C^{2,\alpha}(\partial\mathbb{D}, \partial M) \times \mathbb{R}^j$ with H , there is a continuous $X : e \rightarrow C^{2,\alpha}(\mathbb{D}, \mathbb{B})$ such that $[X(\sigma)] = \sigma$. Besides there is a continuous map $Z : \partial e \rightarrow \text{Diff}_{2,\alpha}(\mathbb{D}, \partial\mathbb{D})$ such that $F(\sigma) = X(\sigma) \circ Z(\sigma)$. Since $\text{Diff}_\infty(\mathbb{D}, \partial\mathbb{D})$ is contractible (cf. [4] and by Appendix A), Z can be extended continuously (for the C^2 topology) in $\text{Diff}_{2,\alpha}(\mathbb{D}, \partial\mathbb{D})$ to the whole e . We then extend F by $F(\sigma) = X(\sigma) \circ Z(\sigma) \in C^{2,\alpha}(\mathbb{D}, \mathbb{B})$. So we can extend F to the whole Y^{p+1} ; this finishes the proof. $\square$

The proof of Proposition 5 is finished.

3.3. Rectifying the parametrization. From now on, we will forget about the metric g on $\mathbb{B}$, so $\{F(\sigma)\}_{\sigma \in \Sigma}$ is just a family of embedded disks in $\mathbb{B}^3$ with the Euclidean metric. Actually, the parametrization of σ by $F(\sigma)$ is not good for what we are going to do next. We use the following result.

Proposition 7. *There is a continuous map $Y : \Sigma \rightarrow \text{Diff}_1(\mathbb{D}, \partial\mathbb{D})$ such that $F(\sigma) \circ Y(\sigma)$ is conformal along $\partial\mathbb{D}$. Moreover, if $\sigma \in \Sigma_\pm$, $\lim Y(\sigma) = \text{id}$ when $t \rightarrow \pm 1$.*

Proof. Actually the proof is based on the following statement: let us consider $X : \mathbb{S}^1 \rightarrow \mathbb{R}^2$ a C^1 vectorfield such that $\langle X, e_r \rangle > 0$ then there is $Y \in \text{Diff}_1(\mathbb{D}, \partial\mathbb{D})$ such that $\partial_r Y = X$ on $\partial\mathbb{D}$.

In order to construct Y , let $r_0 \in (0, 1/2)$ and $\varphi : [0, 1] \rightarrow [0, 1]$ be a non-increasing function such that $\varphi = 1$ on $[0, 1 - r_0]$, $\varphi(1) = 0$ and $\varphi'(1) = 0$ (r_0 and φ will be chosen later in order to satisfy other properties). We define

$$Y(r, \theta) = r\varphi e_r + (1 - \varphi)(e_r + (r - 1)X)$$

We also write $X = \lambda(\cos \alpha e_r + \sin \alpha e_\theta)$ with $\lambda > 0$ and $\alpha \in (-\pi/2, \pi/2)$. Thus

$$\begin{aligned} \partial_r Y &= (\varphi + r\varphi')e_r - \varphi'(e_r + (r - 1)\lambda(\cos \alpha e_r + \sin \alpha e_\theta)) \\ &\quad + (1 - \varphi)\lambda(\cos \alpha e_r + \sin \alpha e_\theta) \\ \partial_\theta Y &= r\varphi e_\theta + (1 - \varphi)(e_\theta + (r - 1)(\lambda_\theta \cos \alpha e_r + \lambda_\theta \sin \alpha e_\theta \\ &\quad - \lambda \alpha_\theta \sin \alpha e_r + \lambda \alpha_\theta \cos \alpha e_\theta + \lambda \cos \alpha e_\theta - \lambda \sin \alpha e_r)) \end{aligned}$$

Choosing r_0 such that $2r_0 \max(\lambda, \lambda_\theta, \lambda_{\alpha_\theta})$ is small and using $\varphi' = 0$ on $[0, 1 - r_0]$,

$$\begin{aligned}\partial_r Y &= (\varphi + r\varphi' - \varphi'(1 + (r-1)\lambda \cos \alpha) + (1-\varphi)\lambda \cos \alpha)e_r \\ &\quad + (-\varphi'(r-1)\lambda \sin \alpha + (1-\varphi)\lambda \sin \alpha)e_\theta \\ \partial_\theta Y &= (r\varphi + 1 - \varphi)e_\theta + \varepsilon(r, \theta)\end{aligned}$$

where $\varepsilon(r, \theta)$ can be assumed small by reducing r_0 and is vanishing if $r < 1 - r_0$. So the Euclidean Jacobian of Y is

$$J = \frac{1}{r}(\varphi + r\varphi' - \varphi'(1 + (r-1)\lambda \cos \alpha) + (1-\varphi)\lambda \cos \alpha)(r\varphi + 1 - \varphi) + \frac{1}{r}\varepsilon$$

since φ will be chosen such that $r_0\varphi'$ is bounded.

If $r \leq 1 - r_0$, we get $J = 1$. So we focus on the sign of

$$\begin{aligned}(\varphi + r\varphi' - \varphi'(1 + (r-1)\lambda \cos \alpha) + (1-\varphi)\lambda \cos \alpha) = \\ ((\varphi + (r-1)\varphi') + \lambda \cos \alpha(1 - (\varphi + (r-1)\varphi')))\end{aligned}$$

We will choose φ such that this quantity is positive. We can assume that $\lambda \leq \lambda_0$, $\lambda \cos \alpha \geq \eta$ where λ_0 and $\eta \in (0, 1)$ are some constant. Since $\varphi' \leq 0$, we have $\varphi + (r-1)\varphi' \geq \varphi \geq 0$. So if $\varphi + (r-1)\varphi' \leq 1$ and $\lambda \cos \alpha \geq 1$, one has

$$\begin{aligned}\varphi + (r-1)\varphi' + \lambda \cos \alpha(1 - \varphi - (r-1)\varphi') &\geq \lambda \cos \alpha + (\varphi + (r-1)\varphi')(1 - \lambda \cos \alpha) \\ &\geq \lambda \cos \alpha + (1 - \lambda \cos \alpha) \geq 1\end{aligned}$$

If $\varphi + (r-1)\varphi' \leq 1$ and $\lambda \cos \alpha \leq 1$,

$$\begin{aligned}\varphi + (r-1)\varphi' + \lambda \cos \alpha(1 - \varphi - (r-1)\varphi') &\geq \lambda \cos \alpha + (\varphi + (r-1)\varphi')(1 - \lambda \cos \alpha) \\ &\geq \lambda \cos \alpha \geq \eta\end{aligned}$$

If $\varphi + (r-1)\varphi' \geq 1$ we have

$$\begin{aligned}\varphi + (r-1)\varphi' + \lambda \cos \alpha(1 - \varphi - (r-1)\varphi') &\geq \varphi + (r-1)\varphi' + \lambda_0(1 - (\varphi + (r-1)\varphi')) \\ &\geq (r\varphi + (1-\varphi)(1 + \lambda_0(r-1)))'\end{aligned}$$

So we will choose φ such that $(r\varphi + (1-\varphi)(1 + \lambda_0(r-1)))' > 0$. Actually we choose $\psi : [0, 1] \rightarrow [0, 1]$ non increasing with $\psi(1) = 0$, $\psi'(1) = 0$ and $\psi(r) = 1$ on $[0, 1/2]$ and such that $(r\psi + (1-\psi)(1 + \lambda_0(r-1)))' > 0$ on $[1/2, 1]$. Then φ is defined by

$$\varphi(r) = \begin{cases} 1 & \text{if } r \leq 1 - r_0 \\ \psi(1 + \frac{r-1}{r_0}) & \text{if } r \geq 1 - r_0 \end{cases}$$

Thus the expected estimates about φ come from the ones on ψ and are independent of r_0 . Thus Y is a C^1 local diffeomorphism and then an open map. Since $Y|_{\partial \mathbb{D}} = \text{id}$, $Y(\mathbb{D}) \subset \mathbb{D}$ and Y is a global diffeomorphism. Moreover $\partial_r Y = X$.

So in order to conclude the proof of Proposition 7, we will apply the above construction to the vector fields $X(\sigma)$ given for $\zeta \in \partial\mathbb{D}$ by

$$X(\sigma)(\zeta) = D(F(\sigma))^{-1}(\zeta)(r_{-\pi/2}(\partial_\theta F(\sigma)(\zeta)))$$

where $r_{-\pi/2}$ is the rotation by angle $-\pi/2$ in $T_{F(\sigma)(\zeta)}\sigma$.

Let us remark that when $\pi(\sigma) = (p, v, t)$ with t close to 1, we have

$$F(\sigma)(x, y) = \sqrt{1-t^2}(xv + yp \wedge v) + (t + u_{p,v,t}(x, y))p$$

So $\partial_\theta F(\sigma)(e^{i\theta}) = \sqrt{1-t^2}(-\sin\theta v + \cos\theta p \wedge v) + O(1-t)$ and $\partial_r F(\sigma)(e^{i\theta}) = \sqrt{1-t^2}(\cos\theta v + \sin\theta p \wedge v) + O(1-t)$. So $r_{-\pi/2}(\partial_\theta F(\sigma)(e^{i\theta})) = \sqrt{1-t^2}(\cos\theta v + \sin\theta p \wedge v) + O(1-t) = \partial_r F(\sigma)(e^{i\theta}) + O(1-t)$. This gives us $X(\sigma)(e^{i\theta}) = e_r + O(\sqrt{1-t})$. The same is true for t near -1 .

As a consequence, all the estimates that appear in the construction can be chosen uniformly in σ . So r_0 can be chosen independently of σ and Y depends continuously on σ . The last remark is that, as $t \rightarrow \pm 1$, $Y(\sigma) \rightarrow \text{id}$. $\square$

In the sequel, we denote $\tilde{F}(\sigma) = F(\sigma) \circ Y(\sigma)$ which is conformal on the boundary.

3.4. Extending $\tilde{F}$ and the boundary behaviour. We denote by $\nu(\sigma) = \frac{\partial_x \tilde{F}(\sigma) \wedge \partial_y \tilde{F}(\sigma)}{\|\partial_x \tilde{F}(\sigma) \wedge \partial_y \tilde{F}(\sigma)\|}$ the Euclidean unit normal to σ . We also define $h(\sigma) = \frac{\partial_x \tilde{F}(\sigma)}{\|\partial_x \tilde{F}(\sigma)\|}$. Finally we define $H : \Sigma \times \mathbb{D} \rightarrow \mathbb{B} \times US^2$ by

$$H(\sigma, \zeta) = (\tilde{F}(\sigma)(\zeta), \nu(\sigma)(\zeta), h(\sigma)(\zeta)).$$

Looking at the parametrization $\tilde{F}(\sigma)$ when $\sigma \in \Sigma_\pm$, we have $H(\sigma, \zeta) \rightarrow (\pm p, p, v)$ when $\pi(\sigma) \rightarrow (p, v, \pm 1)$. This allows us to compactify $\Sigma \times \mathbb{D}$ and extend H to this compactification.

Since (π, id) is a bijection from $\Sigma_+ \times \mathbb{D} \rightarrow US^2 \times [1 - \bar{\varepsilon}, 1) \times \mathbb{D}$, one can extend $\Sigma_+ \times \mathbb{D}$ as $US^2 \times [1 - \bar{\varepsilon}, 1] \times \mathbb{D}$ then take the quotient by the relation $(p, v, 1, \zeta) \sim (p, v, 1, \zeta')$. Its like compactifying $[1 - \bar{\varepsilon}, 1) \times \mathbb{D}$ as the upper part of a 3-ball. Then H extends by continuity by $H(p, v, 1, \zeta) = (p, p, v)$. The same can be done for Σ_- .

So we get a compact manifold K where H extends to $G : K \rightarrow \mathbb{B} \times US^2$. We need to describe G on ∂K :

- if $k \in \partial K$ is a point added to $\Sigma \times \mathbb{D}$ along the compactification corresponding to $(p, v, 1)$, we have $G(k) = (p, p, v)$;
- if $k \in \partial K$ is a point added to $\Sigma \times \mathbb{D}$ along the compactification corresponding to $(p, v, -1)$, we have $G(k) = (-p, p, v)$;
- if k is not an added point and $k = (\sigma, \zeta)$ with $|\zeta| = 1$, we have $G(k) = (\delta(\sigma)(\zeta), \nu(\sigma)(\zeta), h(\sigma)(\zeta))$. If $\pi(\sigma) = (p, v, t)$, $\delta(\sigma)(\zeta) = \tilde{\Phi}(p, v, t)(\zeta)$ which is close to $\Phi(p, v, t)(\zeta)$ (and even equal on $\Sigma_\pm \times \partial\mathbb{D}$). $\nu(\sigma)(\zeta)$

is a unit normal vector to $\tilde{\Phi}(p, v, t)$. Since $\tilde{F}(\sigma)$ is conformal along $\partial\mathbb{D}$, we have:

$$\begin{aligned} h(\sigma)(e^{i\theta}) &= \frac{\cos \theta \partial_r \tilde{F}(\sigma) - \sin \theta \partial_\theta \tilde{F}(\sigma)}{\|\partial_\theta \tilde{F}(\sigma)\|}(e^{i\theta}) \\ &= \frac{\cos \theta \partial_r \tilde{F}(\sigma) - \sin \theta \partial_\theta \tilde{\Phi}(\pi(\sigma))}{\|\partial_\theta \tilde{\Phi}(\pi(\sigma))\|}(e^{i\theta}) \\ &= \cos \theta \left(\frac{\partial_\theta \tilde{\Phi}(\pi(\sigma))}{\|\partial_\theta \tilde{\Phi}(\pi(\sigma))\|} \wedge \nu \right) - \sin \theta \frac{\partial_\theta \tilde{\Phi}(\pi(\sigma))}{\|\partial_\theta \tilde{\Phi}(\pi(\sigma))\|} \\ &= \cos \theta \tilde{e}_\theta \wedge \nu - \sin \theta \tilde{e}_\theta \end{aligned}$$

$$\text{where } \tilde{e}_\theta = \frac{\partial_\theta \tilde{\Phi}(\pi(\sigma))}{\|\partial_\theta \tilde{\Phi}(\pi(\sigma))\|}.$$

Actually we have

Proposition 8. *On ∂K , choosing $\tilde{\Phi}$ close enough to Φ , G is homotopic to $\tilde{G} : \partial K \rightarrow \mathbb{S}^2 \times US^2; (\sigma, \zeta) \mapsto (\Phi(\pi(\sigma))(\zeta), \pi_1(\sigma))$ where $\pi_1(\sigma) = (p, v)$ if $\pi(\sigma) = (p, v, t)$.*

Proof. In order to construct this homotopy we can construct it for each factor $\mathbb{S}^2$ and US^2 . We focus on the second one.

To do this, we endow US^2 with a Riemannian metric such that the following identifications $\mathbb{R}P^3 \simeq SO_3 \simeq US^2$ are isometries. Let us recall that these identifications are constructed in the following way. If $(a, b, c, d) \in \mathbb{S}^3$, the unit norm quaternion $q = a + bi + cj + dk$ acts by conjugation on the unit sphere of purely imaginary quaternions by the following matrix of SO_3 :

$$\begin{pmatrix} a^2 + b^2 - c^2 - d^2 & 2(-ad + bc) & 2(ac + bd) \\ 2(ad + bc) & a^2 - b^2 + c^2 - d^2 & 2(-ab + cd) \\ 2(-ac + bd) & 2(ab + cd) & (a^2 - b^2 - c^2 + d^2) \end{pmatrix}$$

We notice that only (a, b, c, d) and $(-a, -b, -c, -d)$ are identified with the above matrix; so $\mathbb{R}P^3 \simeq SO_3$. Besides, if $(p, v) \in US^2$, we can consider the matrix $(p, v, p \wedge v) \in SO_3$. This gives the second identification $US^2 \simeq SO_3$. Finally we use them to put on US^2 the Riemannian metric of $\mathbb{R}P^3$ inherited from the one on $\mathbb{S}^3$.

As a consequence, the cut-locus of $((1, 0, 0), (0, 1, 0))$ (which is $[a : b : c : d] = [1 : 0 : 0 : 0] \in \mathbb{R}P^3$) is given by $\{a = 0\}$. So the cut-locus of I_3 is the set of matrices of trace -1 : the rotation of angle π .

In order to construct the homotopy, we are going to prove that for any (σ, ζ) the matrix $(\nu(\sigma)(\zeta), h(\sigma)(\zeta), \nu(\sigma)(\zeta) \wedge h(\sigma)(\zeta))$ is not in the cut-locus of $(p, v, p \wedge v)$ where $\pi_1(\sigma) = (p, v)$. Because of the group structure of SO_3 and the invariance of the below arguments by left multiplication, we assume that $\pi_1(\sigma) = (p_0, v_0) = ((1, 0, 0), (0, 1, 0))$. Moreover we recall that Φ is close to $\tilde{\Phi}$ and even equal when t is close to ± 1 .

At $\tilde{\Phi}(p_0, v_0, t)(e^{i\theta})$, the unit vector $\tilde{e}_\theta$ is close to $e_\theta = (0, -\sin \theta, \cos \theta)$. So we can write $\tilde{e}_\theta = (\sin \alpha, -\cos \alpha \sin \theta', \cos \alpha \cos \theta')$ with α close to 0 and θ' close to θ . We write $\nu = (\cos \varphi, \sin \varphi \cos \beta, \sin \varphi \sin \beta)$ with β close to θ and even equal if t close to ± 1 . Thus

$$\tilde{e}_\theta \wedge \nu = \begin{pmatrix} \sin \alpha \\ -\cos \alpha \sin \theta' \\ \cos \alpha \cos \theta' \end{pmatrix} \wedge \begin{pmatrix} \cos \varphi \\ \sin \varphi \cos \beta \\ \sin \varphi \sin \beta \end{pmatrix} = \begin{pmatrix} -\cos \alpha \sin \varphi \cos(\theta' - \beta) \\ -\sin \alpha \sin \varphi \sin \beta + \cos \alpha \cos \varphi \cos \theta' \\ \sin \alpha \sin \varphi \cos \beta + \cos \alpha \cos \varphi \sin \theta' \end{pmatrix}$$

Moreover $\nu \wedge h = \nu \wedge (\cos \theta \tilde{e}_\theta \wedge \nu - \sin \theta \tilde{e}_\theta) = \cos \theta \tilde{e}_\theta + \sin \theta \tilde{e}_\theta \wedge \nu$. Thus the trace of $(\nu, h, \nu \wedge h)$ takes the value

$$\begin{aligned} \text{tr}(\nu, h, \nu \wedge h) &= \cos \varphi + \cos \theta (-\sin \alpha \sin \varphi \sin \beta + \cos \alpha \cos \varphi \cos \theta') \\ &\quad + \sin \theta \cos \alpha \sin \theta' + \cos \theta \cos \alpha \cos \theta' \\ &\quad + \sin \theta (\sin \alpha \sin \varphi \cos \beta + \cos \alpha \cos \varphi \sin \theta') \\ &= \cos \varphi + \sin \alpha \sin \varphi \sin(\theta - \beta) + \cos \alpha \cos \varphi \cos(\theta - \theta') \\ &\quad + \cos \alpha \cos(\theta - \theta') \end{aligned}$$

As α is close to 0 and θ' close to θ , the trace is close to $1 + 2 \cos \varphi$.

For $|t| \geq 1 - \bar{\varepsilon}$, we know $\theta = \theta' = \beta$ and $\alpha = 0$. Moreover φ is close to 0 (its value when $t = \pm 1$). So the value of the trace is close to 3 for $|t| \geq \cos \xi$ (for some positive ξ close to 0).

For t not close to ± 1 , we know that $\tilde{e}_\theta \wedge \nu$ points to the outside of $\mathbb{B}$ at $\tilde{\Phi}(p_0, v_0, t)(e^{i\theta}) = (\cos \gamma, \sin \gamma \cos \theta'', \sin \gamma \sin \theta'')$ where $\cos \gamma$ is close to t and θ'' is close to θ since ν is never normal to $\mathbb{S}^2 = \partial \mathbb{B}$. So

$$\begin{aligned} 0 &< \langle \tilde{e}_\theta \wedge \nu, \tilde{\Phi}(p_0, v_0, t)(e^{i\theta}) \rangle \\ &= -\cos \gamma \cos \alpha \sin \varphi \cos(\theta' - \beta) \\ &\quad + \sin \gamma \cos \theta'' (-\sin \alpha \sin \varphi \sin \beta + \cos \alpha \cos \varphi \cos \theta') \\ &\quad + \sin \gamma \sin \theta'' (\sin \alpha \sin \varphi \cos \beta + \cos \alpha \cos \varphi \sin \theta') \\ &= -\cos \gamma \cos \alpha \sin \varphi \cos(\theta' - \beta) + \sin \gamma \sin \alpha \sin \varphi \sin(\theta'' - \beta) \\ &\quad + \sin \gamma \cos \alpha \cos \varphi \cos(\theta'' - \theta') \end{aligned}$$

As $\theta \simeq \theta' \simeq \theta'' \simeq \beta$ and $\alpha \simeq 0$, we get $0 < \sin(\gamma - \varphi) + \varepsilon$. So $\gamma - \pi - \eta < \varphi < \gamma + \pi + \eta$ for some small η . We notice that all the above approximations depends on how far $\tilde{\Phi}$ is from Φ . Since we can choose $\tilde{\Phi}$ as close of Φ as we want, we can make all these approximations very precise. So where we have to perturb Φ into $\tilde{\Phi}$, $\xi \leq \gamma \leq \pi - \xi$. Since we can assume $\eta < \xi$, we get

$$-\pi < -\pi + \xi - \eta < \varphi < \pi + \eta - \xi < \pi$$

So $\text{tr}(\nu, h, \nu \wedge h) \simeq 1 + 2 \cos \varphi > -1$. Thus choosing $\tilde{\Phi}$ close enough to Φ , we can be sure that, for any (σ, ζ) , $(\nu(\sigma)(\zeta), h(\sigma)(\zeta))$ is not in the cut-locus of $\pi_1(\sigma) = (p, v)$. Thus moving the points along the geodesic from $(\nu(\sigma)(\zeta), h(\sigma)(\zeta))$ to $\pi_1(\sigma)$, we get an homotopy from $(\sigma, \zeta) \mapsto (\nu(\sigma)(\zeta), h(\sigma)(\zeta))$ to $(\sigma, \zeta) \mapsto \pi_1(\sigma)$. This finishes the proof for the second factor.

For the first one, we only remark that $(\sigma, \zeta) \mapsto \tilde{\Phi}(\pi(\sigma)(\zeta))$ and $(\sigma, \zeta) \mapsto \Phi(\pi(\sigma)(\zeta))$ are close since $\tilde{\Phi}$ is a perturbation of Φ (and we have equality for points added in the compactification). So the two factors are homotopic. This finishes the proof of Proposition 8. $\square$

3.5. The final argument. We are going to prove that G is surjective on the interior of $\mathbb{B} \times US^2$. If it is true, this will tell us that for any $(q, \nu) \in \mathbb{B} \times \mathbb{S}^2$ there is a disk σ passing through q with unit normal ν : this is exactly the statement of Theorem 3.

Let us assume that this is not the case. Let $A = (a, b) \in \mathbb{B} \times US^2$ be not in the image of G . Let us notice that $\mathbb{B} \setminus \{a\}$ can be deformation retracted to $\mathbb{S}^2$ and $US^2 \setminus \{b\}$ can be deformation retracted to the equatorial $\mathbb{R}P^2$ in $\mathbb{R}P^3 \simeq US^2$ with pole at b . So $\mathbb{B} \times US^2 \setminus \{A\}$ can be deformation retracted to $\Delta = (\mathbb{S}^2 \times US^2) \cup_{\mathbb{S}^2 \times \mathbb{R}P^2} (\mathbb{B} \times \mathbb{R}P^2)$ where both terms are glued together along the common $\mathbb{S}^2 \times \mathbb{R}P^2$.

Let us consider homology with $\mathbb{Z}/2\mathbb{Z}$ coefficients since it is not clear if K is orientable. Composing G with the deformation retract defines a map $G' : K \rightarrow \Delta$ which coincides with G on ∂K . As a consequence, $[G(\partial K)] = [G'(\partial K)] = [\partial G'(K)] = 0 \in H_5(\Delta, \mathbb{Z}/2\mathbb{Z})$.

A part of the Mayer-Vietoris sequence associated to $\Delta = (\mathbb{S}^2 \times US^2) \cup_{\mathbb{S}^2 \times \mathbb{R}P^2} (\mathbb{B} \times \mathbb{R}P^2)$ is

$$H_5(\mathbb{S}^2 \times \mathbb{R}P^2) \rightarrow H_5(\mathbb{S}^2 \times US^2) \oplus H_5(\mathbb{B} \times \mathbb{R}P^2) \rightarrow H_5(\Delta)$$

Since $H_5(\mathbb{S}^2 \times \mathbb{R}P^2) = 0$ and $H_5(\mathbb{B} \times \mathbb{R}P^2) = 0$, the inclusion $i : \mathbb{S}^2 \times US^2 \hookrightarrow \Delta$ gives an injective inclusion $H_5(\mathbb{S}^2 \times US^2, \mathbb{Z}/2\mathbb{Z}) \rightarrow H_5(\Delta, \mathbb{Z}/2\mathbb{Z})$. G is homotopic to $\tilde{G} : \partial K \rightarrow \mathbb{S}^2 \times US^2 \subset \Delta$; $(\sigma, \zeta) \mapsto (\Phi(\pi(\sigma))(\zeta), \pi_1(\sigma))$ which is a degree 1 map: indeed when t is close to 1 there is exactly one antecedent. So $[G(\partial K)] = [\tilde{G}(\partial K)] = [\mathbb{S}^2 \times US^2] \neq 0 \in H_5(\Delta, \mathbb{Z}/2\mathbb{Z})$. Thus we have a contradiction and G is surjective.

4. A MINIMAL DISK CONTAINING THREE POINTS

In this section, we do similar arguments to the preceding section in order to prove that choosing three points in a Riemannian ball there is a minimal disk containing these three points.

Theorem 9. *Let $M = (\mathbb{B}, g)$ be as in Theorem 3. Let $q_1, q_2, q_3 \in M$ be three points, then there is $\sigma \in \mathcal{M}$ such that $q_i \in \sigma$ for $1 \leq i \leq 3$.*

In order to do the proof we need an equivariant version of the construction of the preceding section. Let R and S be defined as maps $C^{2,\alpha}(\mathbb{S}^1, \mathbb{S}^2) \rightarrow C^{2,\alpha}(\mathbb{S}^1, \mathbb{S}^2)$ or $C^{2,\alpha}(\mathbb{D}, \mathbb{B}) \rightarrow C^{2,\alpha}(\mathbb{D}, \mathbb{B})$ by $R(X)(z) = X(-z)$ and $S(X)(z) = X(\bar{z})$. We notice that these maps induce maps R and S on $\mathcal{M}$ such that the boundary map δ is equivariant. We also notice that R and S generate a free actions of $G = (\mathbb{Z}/2\mathbb{Z})^2$ on $\mathcal{M}$ and $C^{2,\alpha}(\mathbb{S}^1, \mathbb{S}^2)$.

On $US^2 \times (-1, 1)$ we also define $r(p, v, t) = (p, -v, t)$ and $s(p, v, t) = (-p, v, -t)$. It also generates a free action of $G = (\mathbb{Z}/2\mathbb{Z})^2$ on $US^2 \times (-1, 1)$ such that $\Phi \circ r = R \circ \Phi$ and $\Phi \circ s = S \circ \Phi$ where Φ is defined by (1).

Let us denote $\widetilde{\mathcal{M}} = \mathcal{M}/G$, $\widetilde{U} = (US^2 \times (-1, 1))/G$, $\widetilde{\Gamma} = C^{2,\alpha}(\mathbb{S}^1, \mathbb{S}^2)/G$; we denote by $\Pi : \mathcal{M} \rightarrow \widetilde{\mathcal{M}}$ the projection map. Let Ψ and $\tilde{\delta}$ be the induced map from Φ and δ . We then have a commutative diagram

$$\begin{array}{ccc} & & \widetilde{\mathcal{M}} \\ & & \downarrow \tilde{\delta} \\ \widetilde{U} & \xrightarrow{\Psi} & \widetilde{\Gamma} \end{array}$$

Since the action of G on $\mathcal{M}$ and $C^{2,\alpha}(\mathbb{S}^1, \mathbb{S}^2)$ are free and G is finite, $\widetilde{\mathcal{M}}$ and $\widetilde{\Gamma}$ are Banach manifolds and $\tilde{\delta}$ is Fredholm of index 0. Moreover $\widetilde{U}$ is a smooth manifold and Ψ is an embedding. So now the idea is to do an equivariant version of the work in Section 3.

4.1. The equivariant construction. As in the preceding section (Proposition 4), when t is close to ± 1 , there is a unique $\sigma \in \widetilde{\mathcal{M}}$ such that $\tilde{\delta}(\sigma) = \Psi(p, v, t)$ and $D\tilde{\delta}$ is invertible at σ . So we can perturb Ψ into $\tilde{\Psi} : \widetilde{U} \rightarrow \widetilde{\Gamma}$ such that $\tilde{\delta}$ and $\tilde{\Psi}$ are transverse and $\Psi = \tilde{\Psi}$ close to $t = \pm 1$.

As above $\widetilde{\Sigma} = \tilde{\delta}^{-1}(\tilde{\Psi}(\widetilde{U}))$ is a proper smooth submanifold of $\widetilde{\mathcal{M}}$. Let $\Sigma = \Pi^{-1}(\widetilde{\Sigma}) \subset \mathcal{M}$.

As above, we want to have a parametrization of all the minimal disks in Σ .

When t is close to ± 1 , we have the parametrizations

$$F(p, v, t) : (x, y) \mapsto \sqrt{1 - t^2}(xv + yp \wedge v) + (t + u_{p,v,t}(x, y))p$$

We notice that, by the implicit function theorem, we have

$$F(r(p, v, t))(z) = F(p, v, t)(-z) \text{ and } F(s(p, v, t))(z) = F(p, v, t)(\bar{z})$$

As above we define $\Sigma_{\pm} = \{D_{p,v,t}; (p, v, t) \in US^2 \times [\pm(1 - \varepsilon), \pm 1]\}$ and $\Sigma_0 = \overline{\Sigma} \setminus (\cup \Sigma_{\pm})$. So we want to extend this to the whole Σ in an equivariant way.

Lemma 10. *There is a continuous map $F : \Sigma \rightarrow C^{2,\alpha}(\mathbb{D}, \mathbb{B})$ in the C^2 topology which coincides with the above definition on $\Sigma_{\pm}$ such that $[F(\sigma)] = \sigma$ and F is equivariant:*

$$F(R(\sigma))(z) = F(\sigma)(-z) \text{ and } F(S(\sigma))(z) = F(\sigma)(\bar{z})$$

Proof. Let $\widetilde{\Sigma}_0$ be $\Pi(\Sigma_0)$. Let $\widetilde{Y}^0 \subset \dots \subset \widetilde{Y}^4$ be a triangulation of $\widetilde{\Sigma}_0$ such each 4-cell lifts to Σ_0 as 4 disjoint 4-cells. We denote by $Y^0 \subset \dots \subset Y^4$ the lift of this triangulation. So if e is a p -cell in Y^p , $R(e)$, $S(e)$ and $R \circ S(e)$ are the other p -cells above $\Pi(e)$.

As above the proof is by induction. For any 0-cell $\tilde{e}$ in $\widetilde{Y}^0$, let $\{e, R(e), S(e), R \circ S(e)\}$ be $\Pi^{-1}(\tilde{e})$. Let $F(e) \in C^{2,\alpha}(\mathbb{D}, \mathbb{B})$ be a parametrization of e and define respectively the parametrizations of $R(e)$, $S(e)$, $R \circ S(e)$ by $F(R(e))(z) =$

$F(e)(-z)$, $F(S(e))(z) = F(e)(\bar{z})$ and $F(R \circ S(e))(z) = F(e)(-\bar{z})$. Since e , $R(e)$, $S(e)$ and $R \circ S(e)$ are disjoint this is well defined and moreover F is G -equivariant.

By induction let us assume that an equivariant $F : Y^p \rightarrow C^{2,\alpha}(\mathbb{D}, \mathbb{B})$ is defined. Consider $\tilde{e}$ a $p+1$ cell in $\tilde{Y}^{p+1}$ and $\{e, R(e), S(e), R \circ S(e)\}$ be $\Pi^{-1}(\tilde{e})$. As in Lemma 6, F is defined on ∂e and can be extended to the interior of e . Then we define, for any $\sigma \in e$, $F(R(\sigma))(z) = F(\sigma)(-z)$, $F(S(\sigma))(z) = F(\sigma)(\bar{z})$ and $F(R \circ S(\sigma))(z) = F(\sigma)(-\bar{z})$. This extends the definition of F to $S(e)$, $R(e)$ and $R \circ S(e)$. So F is well defined on $\Pi^{-1}(\tilde{e})$ in an equivariant way. We then end the construction by induction. $\square$

4.2. The degree argument. Let us now define $H : \Sigma \times [-1, 1] \times \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{B} \times \mathbb{B} \times \mathbb{B}$ by $H(\sigma, x_1, z_2, z_3) = (F(\sigma)(x_1), F(\sigma)(z_2), F(\sigma)(z_3))$. Our goal is to prove that H is surjective. We notice that x_1 is real.

On $\Sigma \times [-1, 1] \times \mathbb{D}^2$, there is an action of $G = (\mathbb{Z}/2\mathbb{Z})^2$ which is defined by

$$\bar{R}(\sigma, x_1, z_2, z_3) = (R(\sigma), -x_1, -z_2, -z_3) \text{ and } \bar{S}(\sigma, x_1, z_2, z_3) = (S(\sigma), x_1, \bar{z}_2, \bar{z}_3)$$

Because of the equivariance of F , we have $H \circ \bar{R} = H$ and $H \circ \bar{S} = H$. As a consequence the map H induces a map $\tilde{H}$ on the quotient $\tilde{K} = \Sigma \times [-1, 1] \times \mathbb{D}^2/G$. It is enough to prove that $\tilde{H}$ is surjective. Let $\tilde{L}$ be the interior of $\tilde{K}$ i.e. $(\sigma, x_1, z_2, z_3) \in \tilde{L}$ if $|x_1| < 1$, $|z_2| < 1$ and $|z_3| < 1$.

Lemma 11. *The map $\tilde{H} : \tilde{L} \rightarrow \mathring{\mathbb{B}} \times \mathring{\mathbb{B}} \times \mathring{\mathbb{B}}$ has mod 2 degree equal to 1.*

Proof. It is clear that $\tilde{H} : \tilde{L} \rightarrow \mathring{\mathbb{B}} \times \mathring{\mathbb{B}} \times \mathring{\mathbb{B}}$ is proper. So it has a well defined mod 2 degree.

Let us compute this degree. Let $(p, v) \in U\mathbb{S}^2$ and t close to 1. We are interested in $H^{-1}(tp + \sqrt{1-t^2}v, tp + \sqrt{1-t^2}p \wedge v, tp - \sqrt{1-t^2}v)$. Clearly it is made of four points in K :

$$(F(p, v, t), 1, i, -1), (F(p, -v, t), -1, -i, 1), (F(-p, v, -t), 1, -i, -1), \\ (F(-p, -v, -t), -1, -i, 1)$$

We recall that x_1 is real here. As a consequence, $\tilde{H}^{-1}(tp + \sqrt{1-t^2}v, tp + \sqrt{1-t^2}p \wedge v, tp - \sqrt{1-t^2}v)$ has only one element. Let us see that it is also the case of $\tilde{H}^{-1}(q_1, q_2, q_3)$ for $(q_1, q_2, q_3) \in \mathring{\mathbb{B}} \times \mathring{\mathbb{B}} \times \mathring{\mathbb{B}}$ close to $(tp + \sqrt{1-t^2}v, tp + \sqrt{1-t^2}p \wedge v, tp - \sqrt{1-t^2}v)$. First we remark that, if

$$(F(p_n, v_n, t_n)(x_{1,n}), F(p_n, v_n, t_n)(z_{2,n}), F(p_n, v_n, t_n)(z_{3,n})) \longrightarrow \\ (tp + \sqrt{1-t^2}v, tp + \sqrt{1-t^2}p \wedge v, tp - \sqrt{1-t^2}v),$$

then $(p_n, v_n, t_n, x_{1,n}, z_{2,n}, z_{3,n}) \rightarrow (p, v, t, 1, i, -1)$. Indeed, we can assume that $(p_n, v_n, t_n, x_{1,n}, z_{2,n}, z_{3,n}) \rightarrow (\bar{p}, \bar{v}, \bar{t}, x_1, z_2, z_3)$ and we notice that $\bar{t} \neq \pm 1$ since otherwise $(F(p_n, v_n, t_n)(x_{1,n}), F(p_n, v_n, t_n)(z_{2,n}), F(p_n, v_n, t_n)(z_{3,n})) \rightarrow$

$\pm(\bar{p}, \bar{p}, \bar{p})$. Once $t_n \rightarrow \bar{t} \in (-1, 1)$, the convergence to $(p, v, t, 1, i, -1)$ is clear. So we can focus on a neighborhood of $(p, v, t, 1, i, -1)$.

Let us compute the differential of $H(p, v, t, x_1, x_2, y_2, x_3, y_3)$ at $(p, v, t, 1, 0, 1, -1, 0)$. Actually we consider the differential of

$$h(p, v, \theta, x_1, x_2, y_2, x_3, y_3) = H(p, v, \cos \theta, x_1, x_2, y_2, x_3, y_3).$$

We notice that the tangent space to (p, v) in US^2 is $\{(q, w) \in T_p \mathbb{S}^2 \times T_v \mathbb{S}^2 | (q, v) + (p, w) = 0\}$. So

$$D_{(p,v)} h(q, w) = (\cos \theta q + \sin \theta w, \cos \theta q + \sin \theta q \wedge v + \sin \theta p \wedge w, \cos \theta q - \sin \theta w)$$

We write $(q, w) = (av + bp \wedge v, -ap + cp \wedge v)$ so $q \wedge v = -bp$ and $p \wedge w = -cv$ and this derivative becomes

$$\begin{aligned} D_{(p,v)} h(q, w) = & ((-\sin \theta p + \cos \theta v)a + \cos \theta bp \wedge v + \sin \theta cp \wedge v, \\ & \cos \theta av + (\cos \theta p \wedge v - \sin \theta p)b - \sin \theta cv, \\ & (\sin \theta p + \cos \theta v)a + \cos \theta bp \wedge v - \sin \theta cp \wedge v) \end{aligned}$$

For the other derivatives we have

$$\begin{aligned} \partial_\theta h &= (-\sin \theta p + \cos \theta v, -\sin \theta v + \cos \theta p \wedge v, -\sin \theta p - \cos \theta v) \\ \partial_{x_1} h &= (\partial_x u_{p,v,\sin \theta}(1, 0)p + \sin \theta v, 0, 0) \\ \partial_{x_2} h &= (0, \partial_x u_{p,v,\sin \theta}(0, 1)p + \sin \theta v, 0) = (0, \sin \theta v, 0) \\ \partial_{y_2} h &= (0, \partial_y u_{p,v,\sin \theta}(0, 1)p + \sin \theta p \wedge v, 0) \\ \partial_{x_3} h &= (0, 0, \partial_x u_{p,v,\sin \theta}(-1, 0)p + \sin \theta v) \\ \partial_{y_3} h &= (0, 0, \partial_y u_{p,v,\sin \theta}(-1, 0)p + \sin \theta p \wedge v) = (0, 0, \sin \theta p \wedge v) \end{aligned}$$

We notice that, by Proposition 4, $\nabla u_{p,v,\cos \theta} = O(\sin^2 \theta)$. So considering the family $(D_{p,v} h(v, -p), D_{p,v} h(p \wedge v, 0), D_{p,0} h(0, p \wedge v), \partial_\theta h, \partial_{x_1} h, \partial_{x_2} h, \partial_{y_2} h, \partial_{x_3} h, \partial_{y_3} h)$ in the basis $(p, 0, 0), (p \wedge v, 0, 0), (0, p, 0), (0, 0, p), (v, 0, 0), (0, v, 0), (0, p \wedge v, 0), (0, 0, v), (0, 0, p \wedge v)$ the jacobian matrix is

$$\begin{pmatrix} -\sin \theta & 0 & 0 & -\sin \theta & \partial_x u_{p,v,\cos \theta}(1,0) & 0 & 0 & 0 & 0 \\ 0 & \cos \theta & \sin \theta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sin \theta & 0 & 0 & 0 & \partial_y u_{p,v,\cos \theta}(0,1) & 0 & 0 & 0 \\ \sin \theta & 0 & 0 & -\sin \theta & 0 & 0 & 0 & \partial_x u_{p,v,\cos \theta}(-1,0) & 0 \\ \cos \theta & 0 & 0 & \cos \theta & \sin \theta & 0 & 0 & 0 & 0 \\ \cos \theta & 0 & -\sin \theta & -\sin \theta & 0 & \sin \theta & 0 & 0 & 0 \\ 0 & -\cos \theta & 0 & \cos \theta & 0 & 0 & \sin \theta & 0 & 0 \\ \cos \theta & 0 & 0 & -\cos \theta & 0 & 0 & 0 & \sin \theta & 0 \\ 0 & \cos \theta & -\sin \theta & 0 & 0 & 0 & 0 & 0 & \sin \theta \end{pmatrix}$$

So this matrix has the form

$$\sin \theta \begin{pmatrix} -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cot \theta & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ \cot \theta & 0 & 0 & \cot \theta & 1 & 0 & 0 & 0 & 0 \\ \cot \theta & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\cot \theta & 0 & \cot \theta & 0 & 0 & 1 & 0 & 0 \\ \cot \theta & 0 & 0 & -\cot \theta & 0 & 0 & 0 & 1 & 0 \\ 0 & \cot \theta & -1 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + O(\sin^2 \theta)$$

So if θ is sufficiently close to 0 its inverse is

$$\frac{1}{\sin \theta} \begin{pmatrix} -\frac{1}{2} & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & c & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{c-1}{2} & 1 & c & \frac{1-c}{2} & 0 & 1 & 0 & 0 & 0 \\ \frac{c}{2} & 0 & -c & -\frac{c}{2} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & c & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 2c & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} + o(1)$$

where $c = \cot \theta$.

So the local inversion at the boundary applies and the map $\tilde{H}$ sends diffeomorphically a neighborhood of the point $(D_{p,v,t}, 1, 0, 1, -1, 0)$ in $\tilde{\Sigma} \times [-1, 1] \times \mathbb{D}^2$ into a neighborhood of $(tp + \sqrt{1-t^2}v, tp + \sqrt{1-t^2}p \wedge v, tp - \sqrt{1-t^2}v)$ in $\mathbb{B}^3$ when t is close to 1. This implies that when $(q_1, q_2, q_3) \in \mathbb{B}$ is close to $(tp + \sqrt{1-t^2}v, tp + \sqrt{1-t^2}p \wedge v, tp - \sqrt{1-t^2}v)$ (t close to 1) there is a unique $(\sigma, x_1, x_2, y_2, x_3, y_3) \in \tilde{\Sigma} \times (-1, 1) \times \mathbb{D}^2$ such that $\tilde{H}(\sigma, x_1, x_2, y_2, x_3, y_3) = (q_1, q_2, q_3)$ and the differential of $\tilde{H}$ at this point is invertible. So $\tilde{H}$ has mod 2 degree equal to 1. $\square$

Since $\tilde{H}$ has odd degree, it is surjective. This finishes the proof of Theorem 9.

5. MINIMAL PLANES IN ASYMPTOTICALLY FLAT 3-MANIFOLDS

In this section we prove the following result which is an extension of Theorem 1 in [2].

Theorem 12. *Let (M, g) be an asymptotically flat 3-manifold containing no closed embedded minimal surface. Let us consider either*

- $q \in M$ and $V \subset T_q M$ a 2-dimensional subspace or
- $q_1, q_2, q_3 \in M$.

Then there is a complete properly embedded minimal plane Σ in M satisfying respectively

- $q \in \Sigma$ and $T_q \Sigma = V$ or
- $q_i \in \Sigma$ for $1 \leq i \leq 3$.

First let us recall what is the asymptotic flatness hypothesis we consider. M is diffeomorphic to $\mathbb{R}^3$ and, in these coordinates, the metric can be written $g = g_0 + h$ where g_0 is the Euclidean metric and h satisfies $|h| + r|Dh| + r^2|D^2h| \rightarrow 0$ as $r \rightarrow \infty$ (r the Euclidean distance to the origin of $\mathbb{R}^3$).

Actually, once Theorems 3 and 9 are known, the proof of the above theorem follows the ideas of Chodosh and Ketover in [2]. More precisely we use a variant of [2, Proposition 7]. Before let us fix a notation. Let $\{\Gamma_R\}$ be a family of closed curves in $\mathbb{S}^2(R) = \partial\mathbb{B}(R)$ where $\mathbb{B}(R)$ is the Euclidean ball of radius R . We say that they are $C^{2,\alpha}$ almost parallel curves if, after rescaling to unit size, $\frac{1}{R}\Gamma_R \subset \mathbb{S}^2$ converges in the $C^{2,\alpha}$ sense to a parallel curve in $\mathbb{S}^2$ or to a constant map.

Proposition 13. *Let (M, g) be an asymptotically flat manifold diffeomorphic to $\mathbb{R}^3$ that contains no closed embedded minimal surfaces.*

Let $\{\Sigma_R\}$ be a family of embedded minimal disks in $\mathbb{B}(R)$ containing $q \in M$ and whose boundaries $\partial\Sigma_R \subset \partial\mathbb{B}(R)$ are $C^{2,\alpha}$ almost parallel curves. Then a subsequence of $\{\Sigma_R\}$ converges smoothly on compact subsets of M to a complete properly embedded minimal plane Σ_∞ .

The difference with [2] is that we do not assume *a priori* that $\partial\Sigma_R$ is almost equatorial.

Proof. First we notice that, for large R , the length of $\partial\Sigma_R$ is less than $2\pi(1 + o(1))R$.

Let $\lambda_R \simeq \sqrt{R}$ be such that $\Sigma_R \cap \mathbb{S}^2(\lambda_R)$ is transverse. As in [2], $\text{area}(\Sigma_R \setminus \mathbb{B}^3(\lambda_R)) \leq \pi(1 + o(1))R^2$. So $\tilde{\Sigma}_R = \frac{1}{R}(\Sigma_R \setminus \mathbb{B}^3(\lambda_R))$ is a stationary integral varifold in $(\mathbb{B}, \tilde{g}_R)$ ($\tilde{g}_R$ is the homothetic of g by $\frac{1}{R}$ and converges to the Euclidean metric as $R \rightarrow \infty$).

Since $\|\tilde{\Sigma}_R\|(\mathbb{B}) \leq \pi(1 + o(1))$, we can assume that $\tilde{\Sigma}_R$ converge as varifolds to V which is stationary in $\mathbb{B}^3 \setminus \{0\}$ endowed with the flat metric and satisfies $\|V\|(\mathbb{B} \setminus \{0\}) \leq \pi$. Actually, V extends to a stationary varifold of $\mathbb{B}$ with $\|V\|(\mathbb{B}) \leq \pi$. The origin is in the support of V since $q/R \in \Sigma_R/R$ converges to the origin.

By the monotonicity formula V is the varifold associated to a flat unit disk through the origin with multiplicity one. This implies that $\partial\Sigma_R$ is converging to an equator. Now the rest of the proof is similar to the one of [2, Proposition 7.2]. $\square$

Proof of Theorem 12. Let q and V or q_1, q_2, q_3 be as in the statement. Since the metric is asymptotically Euclidean, we consider a chart $M \simeq (\mathbb{R}^3, g)$ with the prescribed asymptotics for g . This implies that for large R , the ball $\mathbb{B}(R)$ is mean convex. So we can apply Theorem 3 or 9 and obtain a minimal disk Σ_R whose boundary is an almost parallel curve such that $(q, V) = T_q \Sigma_R$ or $q_i \in \Sigma_R$.

By Proposition 13, a subsequence of Σ_R converges to Σ_∞ a properly embedded minimal plane in M . Since the convergence is smooth $(q, V) = T_q \Sigma_\infty$ or $q_i \in \Sigma_\infty$. $\square$

Remark 3. As in [2], Theorem 12 extends to the case where M is asymptotically conical: M is $\mathbb{R}^3$ endowed outside a compact subset with a metric $g = g_\alpha + h$ where g_α is the conical metric $g_\alpha = dr^2 + r^2 \alpha^2 g_{\mathbb{S}^2}$ and h satisfies $|h| + r|Dh|_{g_\alpha} + r^2|D^2h|_{g_\alpha} \rightarrow 0$ as $r \rightarrow \infty$

APPENDIX A. ABOUT THE SMALE THEOREM

Let us explain the use of Smale's theorem in Lemma 6. *A priori* this theorem says that the group $\text{Diff}_\infty(\mathbb{D}, \partial\mathbb{D})$ is contractible. But here we are not considering C^∞ diffeomorphisms: they are $C^{k,\alpha}$.

Let us prove a first result about the regularity of the diffeomorphisms we consider. We focus on the case $k = 2$ since this is enough for us.

Lemma 14. *Let $F, G \in C^{2,\alpha}(\mathbb{D}, \mathbb{R}^3)$ be two immersions and $\varphi \in \text{Diff}_1(\mathbb{D}, \partial\mathbb{D})$ such that $F = G \circ \varphi$. Then $\varphi \in \text{Diff}_{2,\alpha}(\mathbb{D}, \partial\mathbb{D})$.*

Proof. Let σ be the surface parametrized by F and G . It is well known that φ is actually a C^2 diffeomorphism. If we look at the second differential of $F = G \circ \varphi$, we have

$$D^2F|_p = D^2G|_{\varphi(p)}(D\varphi|_p, D\varphi|_p) + DG|_{\varphi(p)}(D^2\varphi|_p)$$

Thus

$$D^2\varphi|_p = DG|_{\varphi(p)}^{-1}(D^2F|_p - D^2G|_{\varphi(p)}(D\varphi|_p, D\varphi|_p))$$

Let π_p be the orthogonal projection from $\mathbb{R}^3$ to $T_{F(p)}\sigma$ and $H_p : T_p\mathbb{D} \times T_p\mathbb{D} \rightarrow T_{F(p)}\sigma \subset \mathbb{R}^3$ defined by $H_p = D^2F|_p - D^2G|_{\varphi(p)}(D\varphi|_p, D\varphi|_p)$. Thus we have

$$\begin{aligned} D^2\varphi|_p - D^2\varphi|_q &= DG|_{\varphi(p)}^{-1}H_p - DG|_{\varphi(q)}^{-1}H_q \\ &= DG|_{\varphi(p)}^{-1}H_p - DG|_{\varphi(p)}^{-1}(\pi_p H_q) + DG|_{\varphi(p)}^{-1}(\pi_p H_q) - DG|_{\varphi(q)}^{-1}H_q \\ &= DG|_{\varphi(p)}^{-1} \circ \pi_p (H_p - H_q) + (DG|_{\varphi(p)}^{-1} \circ \pi_p - DG|_{\varphi(q)}^{-1} \circ \pi_q) H_q \end{aligned}$$

Since F and G are $C^{2,\alpha}$, $\|H_p - H_q\| \leq C|p - q|^\alpha$ and $\|DG|_{\varphi(p)}^{-1} \circ \pi_p - DG|_{\varphi(q)}^{-1} \circ \pi_q\| \leq C|p - q|$. As a consequence, we have $\|D^2\varphi|_p - D^2\varphi|_q\| \leq C|p - q|^\alpha$. So $\varphi \in C^{2,\alpha}(\mathbb{D}, \mathbb{D})$ and φ^{-1} also. $\square$

Now let us explain what is the consequence of Smale's theorem : the contractibility of $\text{Diff}_\infty(\mathbb{D}, \partial\mathbb{D})$. Let $\bar{\varphi} = (\bar{a}, \bar{b})$ be in $\text{Diff}_{k,\alpha}(\mathbb{D}, \partial\mathbb{D})$. Let a_t and b_t be the two solutions of the heat equations:

$$\begin{cases} \partial_t a_t = \Delta a & \text{on } \mathbb{R}_+^* \times \mathbb{D} \\ a_0 = \bar{a} & \text{on } \mathbb{D} \\ a_t(x, y) = x & \text{on } \mathbb{R}_+ \times \partial\mathbb{D} \end{cases} \quad \text{and} \quad \begin{cases} \partial_t b_t = \Delta b & \text{on } \mathbb{R}_+^* \times \mathbb{D} \\ b_0 = \bar{b} & \text{on } \mathbb{D} \\ b_t(x, y) = y & \text{on } \mathbb{R}_+ \times \partial\mathbb{D} \end{cases}$$

These solutions exist, are unique and $a, b \in C^k(\mathbb{R}_+ \times \mathbb{D}) \cap C^\infty(\mathbb{R}_+^* \times \mathbb{D})$. We define $\varphi_t = (a_t, b_t)$.

First we remark that $\|\varphi_t\|^2 = a_t^2 + b_t^2$ satisfies $\partial_t \|\varphi_t\|^2 \leq \Delta \|\varphi_t\|^2$. From the maximum principle, $\varphi_t(p) \in \mathbb{D}$ for all $p \in \mathbb{D}$ and $t \geq 0$. Thus for t close to 0, $\varphi_t \in \text{Diff}_\infty(\mathbb{D}, \partial\mathbb{D})$ since its Jacobian does not vanish.

Let Z be a map from a compact set to $\text{Diff}_{k,\alpha}(\mathbb{D}, \partial\mathbb{D})$ continuous in the C^k topology. By solving the heat equation as above, we construct a map $t \mapsto Z_t \in \text{Diff}_{k,\alpha}(\mathbb{D}, \partial\mathbb{D})$ for $t \in [0, \varepsilon]$ with $Z_0 = Z$ and $Z_t \in \text{Diff}_\infty(\mathbb{D}, \partial\mathbb{D})$ if $t > 0$. This map is continuous in the C^k topology. Since Z_ε is a continuous map with values in $\text{Diff}_\infty(\mathbb{D}, \partial\mathbb{D})$ which is contractible, we can deform it to the constant map.

So Z can be deformed in the C^k topology to the constant map. As a consequence, any homotopy group of $\text{Diff}_{k,\alpha}(\mathbb{D}, \partial\mathbb{D})$ is trivial. This is sufficient for the proof of Lemma 6.

REFERENCES

1. Michael T. Anderson, *Complete minimal hypersurfaces in hyperbolic n -manifolds*, Comment. Math. Helv. **58** (1983), 264–290.
2. Otis Chodosh and Daniel Ketover, *Asymptotically flat three-manifolds contain minimal planes*, preprint, arXiv:1709.09650.
3. William H. Meeks, III and Harold Rosenberg, *The uniqueness of the helicoid*, Ann. of Math. (2) **161** (2005), 727–758.
4. Stephen Smale, *Diffeomorphisms of the 2-sphere*, Proc. Amer. Math. Soc. **10** (1959), 621–626.
5. ———, *An infinite dimensional version of Sard's theorem*, Amer. J. Math. **87** (1965), 861–866.
6. Friedrich Tomi and Anthony J. Tromba, *Extreme curves bound embedded minimal surfaces of the type of the disc*, Math. Z. **158** (1978), 137–145.
7. Brian White, *Curvature estimates and compactness theorems in 3-manifolds for surfaces that are stationary for parametric elliptic functionals*, Invent. Math. **88** (1987), 243–256.
8. ———, *The space of m -dimensional surfaces that are stationary for a parametric elliptic functional*, Indiana Univ. Math. J. **36** (1987), 567–602.
9. ———, *New applications of mapping degrees to minimal surface theory*, J. Differential Geom. **29** (1989), 143–162.
10. ———, *Which ambient spaces admit isoperimetric inequalities for submanifolds?*, J. Differential Geom. **83** (2009), 213–228.

UNIVERSITÉ PARIS-EST, LAMA (UMR 8050), UPEC, UPEM, CNRS, 61, AVENUE DU GÉNÉRAL DE GAULLE, F-94010 CRÉTEIL CEDEX, FRANCE
E-mail address: laurent.mazet@math.cnrs.fr

INSTITUTO NACIONAL DE MATEMATICA PURA E APLICADA (IMPA) ESTRADA DONA CASTORINA 110, 22460-320, RIO DE JANEIRO-RJ, BRAZIL
E-mail address: rosen@impa.br