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1 Correlation dimension and phase space contraction via extreme value theory

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7 We show how to obtain theoretical and numerical estimates of correlation dimension and phase
8 space contraction by using the extreme value theory. The maxima of suitable observables sampled
9 along the trajectory of a chaotic dynamical system converge asymptotically to classical extreme
10 value laws where: (i) the inverse of the scale parameter gives the correlation dimension and (ii) the
11 extremal index is associated with the rate of phase space contraction for backward iteration, which
12 in dimension 1 and 2, is closely related to the positive Lyapunov exponent and in higher
13 dimensions is related to the metric entropy. We call it the Dynamical Extremal Index. Numerical
14 estimates are straightforward to obtain as they imply just a simple fit to a univariate distribution.
15 Numerical tests range from low dimensional maps, to generalized Henon maps and climate data.
16 The estimates of the indicators are particularly robust even with relatively short time series.

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17 **This study uses the link between extreme value laws and**
18 **dynamical systems theory to show that important dynam-**
19 **ical quantities as the correlation dimension, the entropy,**
20 **and the Lyapunov exponents can be obtained by fitting**
21 **observables computed along a trajectory of chaotic sys-**
22 **tems. All this information is contained in a newly defined**
23 **Dynamical Extreme Index. Besides being mathematically**
24 **well defined, it is almost numerically effortless to get as**
25 **(i) it does not require the specification of any additional**
26 **parameter (e.g., embedding dimension, decorrelation**
27 **time); (ii) it does not suffer from the so-called curse of**
28 **dimensionality. A numerical code for its computation is**
29 **provided.**

48 exponent and the reliability of estimates from the time series
49 of experimental phenomena is often questioned.⁶ We defer
50 the reader to the monographs^{7,8} and to the articles^{9,10} for
51 recent advancements on the various statistical tools to inves-
52 tigate the nonlinear time series.

53 The extreme value theory (EVT) has been used to char-
54 acterize the evolution of chaotic systems.^{11,12} It is possible
55 to obtain dynamical properties in phase space (fractal dimen-
56 sion or stability) by exploiting the limiting theorems of the
57 extreme value theory. The main idea is: (i) to replace the sto-
58 chastic processes used in the statistical framework with a tra-
59 jectory of a chaotic dynamical system and (ii) to study the
60 convergence of maxima of suitable observables to the class-
61 ical extreme value laws. The parameters of the EVT provide
62 estimates of dynamical properties of the system. This con-
63 nection between EVT and the dynamical properties of cha-
64 otic systems is rich not only from a theoretical but also from
65 a numerical perspective. Indeed, the estimates of local prop-
66 erties obtained with EVT do not require the introduction of
67 additional parameters and they are easy to implement numer-
68 ically. They have been used to get insights into the dynami-
69 cal behavior of atmospheric flows in Refs. 13–15. In Ref. 16,
70 it has been shown that the numerical algorithm based on
71 EVT provide reliable estimates of the dimension of high
72 dimensional systems up to phase spaces with thousands of
73 dimensions. It is therefore desirable to estimate other key
74 dynamical quantities in the EVT framework.

75 The purpose of this communication is to show that the
76 correlation dimension and the EVT are intimately related:
77 the CD arises by studying the distribution of the maxima of a
78 new suitable observable evaluated along the orbit of a
79 chaotic system. Moreover, an exponent of the limit law, the
80 extremal index, is related, for hyperbolic attractors, to
81 the positive Lyapunov exponent in dimension two and to the
82 metric entropy in higher dimensions. The idea of the rela-
83 tionship between EVT and CD comes from a previous

AQ2 31 I. INTRODUCTION

32 Since its introduction by Grassberger and Procaccia,^{1,2}
33 the correlation dimension (CD) has been used as a powerful
34 indicator for the description of the fractal structure of invari-
35 ant sets in dynamical systems. Similarly, the Lyapunov
36 exponents and the entropy^{3,4} provide an indication of the rel-
37 evant time scales associated with the dynamics and the pre-
38 dictability horizon of the system. Given the importance of
39 these quantities, there exists an increasing body of literature
40 on how to estimate CD, Lyapunov exponents, and entropy. It
41 has been shown that reliable estimates of CD can be obtained
42 with a relatively short time series.⁵ Instead, the computations
43 of Lyapunov exponents and entropy are still challenging
44 because the existing methodologies require as input addi-
45 tional parameters as the dimension of the phase space and
46 the relevant time scale of the dynamics (e.g., the decorrela-
47 tion time). Calculations are then limited to the top Lyapunov

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84 work¹⁷ where we used the extreme value theory to detect
 85 and quantify the onset of synchronization in coupled map lat-
 86 tices. The relationship between the extremal index and the
 87 Lyapunov exponent and the entropy is new and is particu-
 88 larly striking for maps with piecewise constant jacobian. In
 89 the general case, we derive a formula whose validity is con-
 90 firmed by numerical experiments. We also explain the rela-
 91 tion between our extremal index, the local dimensions, and
 92 the phase space contraction. In the rest of the paper, we will
 93 name it as the DEI, the *dynamical extremal index*. We want
 94 to point out that our DEI is a well defined quantity that can
 95 be used as a new indicator for the sensitivity associated with
 96 local hyperbolicity. We will present the theoretical results in
 97 Sec. II: some of those results can be obtained by generalizing
 98 the techniques introduced in Ref. 17; we will also address
 99 the need to develop a more appropriate theory of EVT for
 100 diffeomorphisms in higher dimensions. We will then provide
 101 several examples of classical conceptual low-dimensional
 102 dynamical systems. We will discuss the implications of our
 103 results on higher dimensional systems and the possibility to
 104 apply them to more a general time series. As an example, we
 105 will compute the indicators on climate data and explain how
 106 they provide relevant physical information on the atmo-
 107 spheric circulation over the North Atlantic.

108 **II. THEORETICAL RESULTS**

109 **A. A brief presentation of the extreme value theory and**
 110 **a new observable**

111 Let (M, μ, T) be dynamical systems given by a map T
 112 acting on the metric compact space M with distance $d(\cdot, \cdot)$
 113 and preserving the Borel measure μ . Usually, M will be a
 114 compact subset of some \mathbb{R}^n and d a distance equivalent to
 115 the standard one. Let us take the *direct product* $(M \times M,$
 116 $\mu \times \mu, T \times T)$, and denote with $(x, y) \in M \times M$, a couple of
 117 point in the Cartesian product $(M \times M)$. We then introduce
 118 the observable $\psi(x, y) = -\log d(x, y)$, and consider the pro-
 119 cess $\{\psi \circ (T^j \times T^j)\}_{j \geq 0}$, and the maximum of the sequence
 120 $\mathcal{M}_n(x, y) = \max\{\psi(x, y), \psi(Tx, Ty), \dots, \psi(T^{n-1}x, T^{n-1}y)\}$ and
 121 finally its distribution $\mathbb{P}(\mathcal{M}_n \leq u_n)$, where $\mathbb{P} = \mu \times \mu$ is the
 122 underlying probability and u_n is a suitable scaling function
 123 tending to infinity and which we are going to define.
 124 Suppose that for a given positive number τ we can find a
 125 sequence of numbers u_n such that $n\mathbb{P}(\psi \geq u_n) \rightarrow \tau, n \rightarrow \infty$.
 126 We say, that the process $\{\psi \circ (T^j \times T^j)\}_{j \geq 0}$ satisfies an
 127 extreme value law of Gumbel's type if there is a number
 128 $\theta \in (0, 1]$, the *extremal index*, such that $\mathbb{P}(\mathcal{M}_n \leq u_n) \rightarrow e^{-\theta\tau}$,
 129 $n \rightarrow \infty$. We now introduce the diagonal neighborhood S_n in
 130 the product space: $S_n = \{(x, y), d(x, y) \leq e^{-u_n}\}$. By substitut-
 131 ing the expression of ψ in $\mathbb{P}(\psi \geq u_n)$, we have

$$\mathbb{P}(\psi \geq u_n) = \mathbb{P}((x, y) \in S_n) = \int_M \mu(B(x, e^{-u_n}))d\mu(x), \quad (2.1)$$

132 where $B(x, a)$ denotes the ball of radius a centered on x .
 133 (Actually, we got the equality of the right hand side in the
 134 limit of large n when the two small corners of S_n become
 135 negligible.) The quantity $\int_M \mu(B(x, r))d\mu(x)$ scales like r^{D_2}

and the exponent D_2 is called the *correlation dimension* and
 it characterizes the fractal structure of the support of μ ; a
 more formal, from the mathematical point of view, definition
 of this fact is given in Ref. 18, Sec. 17, and references
 therein. (A precise definition consists in taking the limsup
 and liminf of the ratio of the logarithm with $\log(1/r)$.) By
 injecting successively into (2.1), we have therefore that for
 large n

$$u_n \sim \frac{-\log \tau}{D_2} + \frac{\log n}{D_2} := \frac{z}{a_n} + b_n, \quad (2.2)$$

where $\tau = e^{-z}$, $a_n = D_2$ and $b_n = \frac{\log n}{D_2}$. For numerical pur-
 poses, distribution functions like $\mathbb{P}(\mathcal{M}_n \leq z)$ are modelled,
 for n sufficiently large, by the so-called *generalized extreme*
value (GEV) distribution which is a function depending upon
 three parameters $\xi \in \mathbb{R}, \kappa \in \mathbb{R}, \sigma > 0$ and such that:
 $F_{\text{GEV}}(z; \kappa, \sigma, \xi) = \exp\left\{-\left[1 + \xi\left(\frac{z-\kappa}{\sigma}\right)\right]^{-1/\xi}\right\}$.

The parameter ξ is called the tail index; when its value
 is 0, the GEV corresponds to the Gumbel type. The param-
 eter κ is called the location parameter and σ is the scale
 parameter: for n large, the scaling constant a_n is close to σ^{-1}
 and b_n is close to κ . Therefore, if we could fit a limit law of
 Gumbel's type with suitable normalizing parameters a_n and
 b_n , we immediately get the correlation dimension. Such a
 technique was previously used with a different observable,
 and it allowed to get the so-called *information dimension*
 $D_1(x)$, another fractal dimension which provides the scaling
 of the measure of a ball around a given point x , see Ref. 19
 and references therein. Although the information dimension
 depends on the point x , its value is the same for almost all
 the choices of x with respect to the invariant measure and
 such an averaged valued, simply D_1 , is larger or equal to D_2 ,
 see Ref. 20 for an account on the different fractal dimen-
 sions. In particular, if we denote with d_H the Hausdorff
 dimension, we have $D_2 \leq D_1 \leq d_H$.

168 **B. The spectral approach with the new observable for**
 169 **conformal repellers**

Before showing our numerical simulations for the com-
 putation of the CD, let us argue how we get a Gumbel's type
 asymptotic distribution with an extremal index θ of dynami-
 cal meaning. First, we consider one-dimensional dynamical
 systems generated by uniformly expanding maps with an
 invariant set which could be a Cantor set and equipped with
 mixing Gibbs measures. These systems are better known as
conformal repellers,—see for instance²¹ for a recent contri-
 bution—whose measures are characterized by a potential φ
 of type $\varphi(x) = -\beta \log |T'(x)|$, where T' denotes the deriva-
 tive of T and $\beta \in \mathbb{R}$. If we denote them as μ_β , they are given
 by $h_\beta \nu_\beta$, where the density h_β and the *conformal* measure ν_β
 are, respectively, the eigenfunctions of the transfer operator
 (Perron-Fröbenius) and of its dual, both with eigenvalue
 $\lambda_\beta = e^{Q(\beta)}$, being $Q(\beta)$ the topological pressure. We remind
 that the transfer operator \mathcal{P}_T for the map T is defined, for an
 observable f in some suitable Banach space \mathcal{B} —for instance
 the space of Lipschitz continuous functions—by the duality
 relation: $\int \mathcal{P}_T f d\nu_\beta = \lambda_\beta \int f d\nu_\beta$. We defer to the monograph²²

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189 for an introduction to thermodynamic formalism. The conformal
 190 measure verifies the property $\nu_\beta(TA) = \lambda_\beta \int_A e^{-\phi} d\nu_\beta$,
 191 where T is one-to-one over the measurable set A . A powerful
 192 method to investigate the distribution of our process
 193 $\{\psi \circ (T^j \times T^j)\}_{j \geq 0}$ consists in perturbing the transfer opera-
 194 tor \mathcal{P} of the direct product $T \times T$. The key observation is that
 195 by repeatedly using the duality relation, we can write
 196 $\mathbb{P}(\mathcal{M}_n \leq u_n) = \lambda_\beta^{-2n} \int \int \tilde{\mathcal{P}}_n^n(h_\beta(x)h_\beta(y)) d\nu_\beta(x)d\nu_\beta(y)$, where
 197 the perturbed operator $\tilde{\mathcal{P}}_n$ is defined by acting on observables
 198 $f \in \mathcal{B}$, as $\tilde{\mathcal{P}}_n(f) = \mathcal{P}(f \mathbf{1}_{S_n^c})$, and $S_n = \{(x, y); d(x, y)$
 199 $\leq e^{-u_n}\}$. When n tends to infinity, the characteristic function
 200 of the complement of S_n , $\mathbf{1}_{S_n^c}$, goes to the identity and the
 201 operators \mathcal{P} and $\tilde{\mathcal{P}}_n$ converge to each other in \mathcal{B} . If the
 202 unperturbed operator \mathcal{P} has a spectral gap, it allows expo-
 203 nential mixing for the observables in \mathcal{B} . This compensates the
 204 lack of independence of the process $\{\psi \circ (T^j \times T^j)\}_{j \geq 0}$. The
 205 same is true for the operator $\tilde{\mathcal{P}}_n$ and the maximal, isolated,
 206 eigenvalue of \mathcal{P} , λ_β^2 , is close to that of $\tilde{\mathcal{P}}_n$, $\tilde{\lambda}_{\beta,n}^{(2)}$. More pre-
 207 cisely: $\tilde{\lambda}_{\beta,n}^{(2)} \sim \lambda_\beta^2 - (1 - \lambda_\beta^2 q_0) \mathbb{P}(S_n)$, where now $\mathbb{P} = \mu_\beta$
 208 $\times \mu_\beta$. We will define the factor q_0 in a moment. The operator
 209 $\tilde{\mathcal{P}}_n$ now decomposes as the sum of a projection along the one
 210 dimensional eigenspace associated with the eigenvalue $\tilde{\lambda}_{\beta,n}^{(2)}$
 211 and an operator with a spectral radius exponentially decreas-
 212 ing to zero and which can be neglected in the limit of large
 213 n . This allows us to write $\mathbb{P}(\mathcal{M}_n \leq u_n) \sim \lambda_\beta^{-2n} \tilde{\lambda}_{\beta,n}^{(2)n} \int \int h_\beta(x)$
 214 $h_\beta(y) d\nu_{\beta,n}(x) d\nu_{\beta,n}(y)$, where $\nu_{\beta,n}$ is the conformal
 215 measure for the perturbed operator and the double integral on
 216 the right hand side converges to 1 for $n \rightarrow \infty$. Finally, we
 217 get by approximating $\tilde{\lambda}_{\beta,n}^{(2)}$ as above: $\mathbb{P}(\mathcal{M}_n \leq u_n)$
 218 $\sim \left[1 - \frac{(1 - \lambda_\beta^2 q_0) \mathbb{P}(S_n)}{\lambda_\beta^2}\right]^n \sim \exp\left[-\frac{(1 - \lambda_\beta^2 q_0) \mathbb{P}(S_n)}{\lambda_\beta^2} n\right]$. We now
 219 remind that we are under the assumption that $n \mathbb{P}(\psi \geq u_n)$
 220 $= n \mathbb{P}(S_n) \rightarrow \tau, n \rightarrow \infty$. This leads to the Gumbel law $e^{-\theta \tau}$
 221 provided that the dynamical extremal index θ is defined as

$$\theta = \frac{1 - \lambda_\beta^{-2} q_0}{\lambda_\beta^2}. \tag{2.3}$$

222 The term q_0 is obtained by the previous perturbation theory
 223 under the assumption that the diagonal in the product space
 224 is left invariant by the direct product of the two maps. In par-
 225 ticular, we have

$$q_0 = \lim_{n \rightarrow \infty} \frac{\mathbb{P}(S_n \cap \bar{T}^{-1} S_n)}{\mathbb{P}(S_n)}, \tag{2.4}$$

226 provided that the limit exists. The technique just described
 227 was first proposed by Keller²³ as an alternative way to get
 228 EVT for systems with exponential mixing and it is based on
 229 a perturbative result by Keller and Liverani.²⁴ We defer to
 230 Ref. 23 and to our paper¹⁷ for a detailed presentation of that
 231 theory. It can be applied to conformal mixing repellers and it
 232 provides the preceding estimates, namely the asymptotic
 233 scaling for the maximal eigenvalue. We would like to point

234 out that with our choice for the observable ψ , the perturba-
 235 tive approach just sketched gives the Gumbel's law in a very
 236 direct and natural manner.

237 The computation of q_0 proceeds now as in Ref. 17 with
 238 a substantial difference: the nature of the conformal measure
 239 does not imply necessarily that the ratio $\frac{\nu_\beta(B(Tx,r))}{\nu_\beta(B(x,r))}$ is constant,
 240 which happened when the conformal measure was Lebesgue.
 241 This difficulty could be partially overcome by supposing that
 242 the potential is constant, otherwise we could bound q_0 from
 243 above and below with (close) approximations of the poten-
 244 tial. By assuming that the latter is constant and equal to $\bar{\phi}$
 245 and also that the density h_β does not vary too much, we get
 246 that q_0 is of order $e^{\bar{\phi}}$ and therefore

$$\theta \sim \frac{1 - \lambda_\beta^{-2} e^{\bar{\phi}}}{\lambda_\beta^2}. \tag{2.5}$$

247 It is worth mentioning that whenever the conformal measure
 248 is Lebesgue ($\beta = 1$), the above computation can be made rig-
 249 orous as in Proposition (5.3) in Ref. 17 and it gives

$$\theta = 1 - \frac{\int_M \frac{h^2(x)}{|T'(x)|} dx}{\int_M h^2(x) dx}, \tag{2.6}$$

250 where h is the density of the invariant measure: we defer to
 251 our paper¹⁷ for the assumptions on the system which permit
 252 to get such a result. In particular, those systems contain con-
 253 formal repellers with finitely many branches and absolutely
 254 continuous conformal measures. Notice that by introducing
 255 the invariant measure $\mu = h dm$, we could identically write

$$\theta = 1 - \frac{\int_M h(x) e^{-\log |T'(x)|} d\mu(x)}{\int_M h(x) d\mu(x)}. \tag{2.7}$$

256 If the derivative does not change too much, we get
 257 $\theta \sim 1 - e^{-\Lambda_\mu}$, where Λ_μ is the positive Lyapunov exponent
 258 of the measure μ . Alternatively, if the density h could be
 259 considered constant, we can bound (2.7) by Jensen's inequal-
 260 ity as

$$\theta \sim 1 - \int_M \frac{1}{|T'(x)|} d\mu(x) \leq 1 - e^{-\int_M \log |T'(x)| d\mu(x)} = 1 - e^{-\Lambda_\mu}.$$

261 In both cases, the DEI θ is related to the positive Lyapunov
 262 exponent: this analogy will be pursued in Sec. II C. AQ6

263 **C. Attractors and high dimensional systems**

264 For invertible maps generating attractors endowed with
 265 the SRB measure, the computation of the dynamical
 266 extremal index is less straightforward; we should stress that
 267 a spectral theory of extreme value for (invertible) uniformly
 268 hyperbolic maps is still missing. Suppose we take an hyper-
 269 bolic diffeomorphisms T preserving the ergodic SRB mea-
 270 sure \mathcal{L} . Then, the quantity q_0 in (2.4) becomes

$$q_0 = \lim_{n \rightarrow \infty} \frac{\int d\mathcal{L}(x) \int \mathbf{1}_{S_n}(x, T^{-1}y) \mathbf{1}_{S_n}(Tx, y) d\mathcal{L}(y)}{\int d\mathcal{L}(x) \int \mathbf{1}_{S_n}(x, y) d\mathcal{L}(y)}. \quad (2.8)$$

271 When we iterate backward the points $y \in B(Tx, e^{-u_n})$, we
 272 should keep only those points whose preimage is at a distance
 273 at most e^{-u_n} from x . Those preimages form a set $Q(x)$
 274 which is obtained by squeezing the ball $B(Tx, e^{-u_n})$ along the
 275 unstable manifolds. Let us suppose that the tangent expanding
 276 subspace $\Sigma_u(Tx)$ at x has dimension d . Then the measure
 277 of $Q(x)$, and therefore, by the forward invariance of the
 278 measure, of its image in $B(Tx, e^{-u_n})$ will be of order
 279 $|\det(DT(x)|_u)|^{-1} \mathcal{L}(B(Tx, e^{-u_n}))$, where $DT(x)|_u$ is the deriv-
 280 ative of T restricted to $\Sigma_u(x)$. We remember in fact that the
 281 conditional SRB measure on the unstable manifolds is
 282 smooth. This immediately gives q_0 of order

$$q_0 \sim \frac{\int d\mathcal{L}(x) |\det(DT(x)|_u)|^{-1} \mathcal{L}(B(Tx, e^{-u_n}))}{\int d\mathcal{L}(x) \mathcal{L}(B(x, e^{-u_n}))}. \quad (2.9)$$

283 We see that q_0 contains information about the dimension
 284 through the scaling of the denominator; we are now inter-
 285 ested in the contribution of the other term in the numerator.
 286 In this regard, we first remind that, for SRB measures, we
 287 can use the Pesin's formula²⁵

$$\int d\mathcal{L}(x) |\det(DT(x)|_u)| = \sum_{j=1}^d \Lambda_j^+ = h_{\mathcal{L}},$$

288 where Λ_j^+ is the positive Lyapunov exponents with multi-
 289 plicity one, and $h_{\mathcal{L}}$ is the metricentropy of the SRB measure.
 290 We now proceed under two assumptions as we did at the end
 291 of Sec. II B. Let us first assume that the derivative along the
 292 unstable subspaces does not vary too much. Then, we could
 293 estimate the DEI as

$$\theta \sim 1 - e^{-h_{\mathcal{L}}}. \quad (2.10)$$

294 For $d = 1$, we can replace the entropy with the (unique) posi-
 295 tive Lyapunov exponent $\Lambda_{\mathcal{L}}$; in the following, we will simply
 296 write it as Λ_+ .

297 The other assumption exploits the fact that for these
 298 system, and for \mathcal{L} -almost all points x we have, by Young's
 299 theorem,²⁶ that $\lim_{r \rightarrow 0} \frac{\log \mathcal{L}(B(x,r))}{\log r} = D_1$, where D_1 is the
 300 information dimension. Hence, we could guess that
 301 $\mathcal{L}(B(x, e^{-u_n})) \sim e^{-u_n D_1}$ and therefore forget about the depen-
 302 dence on the variable x . This is generally false since the multi-
 303 plicative factor in the previous scaling could depend on x .
 304 Indeed, when we integrate $\mathcal{L}(B(x, e^{-u_n}))$, we get D_2 which
 305 could be different from D_1 . If we suppose that the depen-
 306 dence on x of the prefactors is negligible, which means that
 307 we are considering a homogenous fractal invariant set with
 308 $D_1 \sim D_2$, then we have for the DEI

$$\begin{aligned} \theta &\sim 1 - \int d\mathcal{L} |\det(DT(x)|_u)|^{-1} \leq 1 - e^{-\int d\mathcal{L}(x) |\det(DT(x)|_u)|} \\ &= 1 - e^{-h_{\mathcal{L}}}, \end{aligned} \quad (2.11)$$

where the derivative is *not* supposed to be constant and
 where we have used again the Jensen's inequality to estab-
 lish the upper bound.

Those two approximations are very crude; we are in fact
 either neglecting the contributions of the prefactors in the
 local scaling of the balls in (2.9), or not taking into account
 the geometric factors when the ball $B(Tx, e^{-u_n})$ is squeezed
 at a distance e^{-u_n} from x . Moreover, the variation of the
 derivative, especially sensible in the non-uniformly hyper-
 bolic setting, could give large differences in the determina-
 tion of the DEI, as we experience for instance for the Hénon
 map, see below. The preceding relation is pretty well satisfac-
 ed for maps with one-dimensional unstable subspace and
 (piecewise) constant jacobian, like the Baker transformation,
 the Lozi map, and the solenoid. For the algebraic automor-
 phism of the torus (cat's map), a simple argument allows us
 to improve the previous rate just by taking into account the
 geometric factors. Surprisingly, relation (2.11) is pretty well
 satisfied in the example below of the generalized Hénon
 maps, where the unstable subspace has dimension larger
 than one, i.e., we have more than one positive Lyapunov
 exponent. In conclusion, our index θ traces in a satisfactory
 way the entropy. The eventual deviations are due to the vari-
 ation of the derivative and the local scaling of balls in (2.9).
 Although these effects are difficult to compute analytically,
 the DEI θ is relatively easy to compute numerically and it
 furnishes a new indicator for the local instability in chaotic
 systems.

III. NUMERICAL COMPUTATIONS

The numerical computations presented in the remaining
 of this work are performed by using the numerical algo-
 rithms and codes detailed in the [supplementary material](#). The
 stability of the results is checked against different l, n, m, s . In
 particular, we perform two sets of simulations. The first set
 of accurate simulations consist of $l = 100$ trajectories, with
 $n = 10^6$ iterations, $m = 10^3$ blocks of $s = 10^3$ length each.
 The second set of $l = 100$ simulations consists of short series
 of $n = 10^4$ iterations, with $s = m = 10^2$. This second set is
 useful to check whether the technique is reliable also for
 short time series. Except where specified, we use $\tilde{s} = 0.99$
 for the following computations. However, results are stable
 when considering different quantiles ranging from $0.97 < \tilde{s} < 0.999$.

A. Low dimensional maps

We begin the numerical computations with several
 examples on low dimensional maps. A summary of the
 results for all maps analysed is reported in Table I. For a few
 maps, we report the model equations in the [supplementary material](#) to streamline the exposition.

- Let us begin with the Bernoulli Shift map $T(x) = 3x \text{-mod } 1$. For this system, $D_2 = 1$ and $\theta = 1 - 1/3 = 2/3$. The numerical estimates (Table I) are coherent with the theoretical values for both accurate and short simulations.
- We now consider the Gauss map $T(x) = \frac{1}{x} \text{-mod } 1$ defined on the unit interval. Although, strictly speaking, this map

TABLE I. Estimates of correlation dimension D_2 and dynamical extremal index (DEI) θ obtained with $l = 100$ trajectories, consisting of $n = 10^6$ iterations or $n = 10^4$ iterations. The maxima of $\psi(x, y)$ are extracted in the block of $s = 10^3$ and $s = 10^2$ length, for a total of $m = 10^3$ or $m = 10^2$ blocks. The quantile for the estimate of the DEI is $\tilde{s} = 0.99$. For the Arnold Cat's map, the convergence to theoretical value is lower and the estimates are provided only for $\tilde{s} = 0.99999$ and $n = 10^7$.

Map	D_2 (classical)	$D_2 (n = 10^6)$	$D_2 (n = 10^4)$	θ (from Lyapunov)	$\theta (n = 10^6)$	$\theta (n = 10^4)$
Bernoulli's shifts	1	1.00 ± 0.02	1.01 ± 0.14	0.667	0.668 ± 0.004	0.69 ± 0.04
Gauss map	1	1.00 ± 0.03	0.96 ± 0.16	0.773	0.773 ± 0.005	0.78 ± 0.04
Cantor IFS	0.667	0.64 ± 0.01	0.59 ± 0.13	0.5	0.502 ± 0.005	0.50 ± 0.05
Baker map	1.41	1.46 ± 0.02	1.42 ± 0.25	0.47	0.49 ± 0.02	0.50 ± 0.04
Lozi map	1.38	1.39 ± 0.11	1.29 ± 0.25	0.37	0.37 ± 0.01	0.37 ± 0.05
Henon map	1.22	1.24 ± 0.03	1.13 ± 0.25	0.34	0.43 ± 0.01	0.43 ± 0.06
Solenoid $a = 1/3$	1.6309	1.64 ± 0.04	1.55 ± 0.17	0.5	0.51 ± 0.01	0.59 ± 0.03
Solenoid $a = 1/4$	1.5	1.52 ± 0.03	1.57 ± 0.20	0.5	0.51 ± 0.01	0.53 ± 0.03
Arnold Cat's map	1.987	2.00 ± 0.06	...	0.51	0.53 ± 0.06	...

364 does not fit the assumptions in Ref. 17 since in the latter
 365 paper, we consider maps with finitely many branches, we
 366 still try formula (2.6). For the Gauss map, the density is
 367 explicit and reads $h(x) = \frac{1}{\log 2} \frac{1}{1+x}$. The integral in (2.6) can
 368 be easily computed and gives $\theta = 4 \log(2) - 2 \sim 0.77$,
 369 whereas D_2 is expected to be 1. The numerical estimates
 370 are coherent with the theoretical values (Table I).

371 • Returning to a map with constant slope 3, we now look at
 372 the transformation generating the classical ternary Cantor
 373 set. In order to compute numerically the GEV function,
 374 one should access the invariant Cantor set, which is of
 375 zero Lebesgue measure. We need therefore to use the
 376 backward iterates of the map (otherwise almost all the for-
 377 ward orbits will fall into the holes), and the measures
 378 allowing us to compute the time averages are the so-called
 379 *balanced measures*, given suitable weights to the pre-
 380 images of the map: see our article, Ref. 27 Sec. 3.2.2 for a
 381 description of such measures. For the ternary Cantor set
 382 and choosing equal weights 1/2 for the two preimages, it
 383 is easy to check that such a balanced measure coincides
 384 with the Gibbs measure with $\beta = \log 2 / \log 3$ which is the
 385 Hausdorff dimension of the invariant set. The measure μ_{d_H}
 386 is called *uniform*, see Ref. 20, Sec. 3. The potential φ will
 387 be equal to $-\log 2$ and $\lambda = 1$, since by Bowen's formula
 388 $Q(d_H) = 0$. Therefore, for the ternary Cantor set, we get a
 389 DEI equal to 0.5 which is perfectly confirmed by the
 390 numerical simulations (Table I).

391 • For the Lozi map: $x_{n+1} = a|x_n| + y_n + 1$, $y_{n+1} = bx_n$,
 392 $a = 1.7$, $b = 0.5$, Λ_+ is of order 0, 47,²⁸ which gives, with
 393 our approximation, a DEI of order $\theta = 0.37$. Previous
 394 numerical computations for D_2 gave $D_2 \sim 1.38$.²⁹ Our
 395 computations (Table I) are coherent with the theoretical
 396 values.

397 • For the Hénon map $x_{n+1} = ax_n^2 + y_n + 1$, $y_{n+1} = bx_n$,
 398 $a = 1.4$, $b = 0.3$, Λ_+ is of order 0, 42,²⁸ which gives, with
 399 our approximation, a DEI of order $\theta = 0.34$. Previous
 400 numerical computations for D_2 gave $D_2 \sim 1.22$.²⁹ The
 401 GEV computations give $D_2 = 1.24 \pm 0.11$ but $\theta = 0.43$
 402 ± 0.01 for $n = 10^6$ (See Table I for the results with
 403 $n = 10^4$ iterations). The discrepancy of the DEI estimate
 404 does not get any better with the increase of \tilde{s} or n . As said
 405 before, we do not expect θ to coincide with the estimate
 406 0.34 due to the variation of the derivative and the non-
 407 uniform hyperbolicity of the map.

• Let us consider the cat's map with the associated matrix 408

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \text{ The stable and unstable manifolds for such a } 409$$

map are orthogonal, so we could suppose that the pre- 410
 image of the ball $B(Tx, e^{-u_n})$ will intersect the ball 411
 $B(x, e^{-u_n})$ in a rectangle $R(x)$ centered at x and with the 412
 shortest side of length $(\lambda_+)^{-1} e^{-u_n}$, where $\lambda_+ = \frac{3+\sqrt{5}}{2}$ is the 413
 eigenvalue larger than 1 corresponding to the unstable 414
 direction. An elementary calculation shows immediately 415
 that $q_0 \sim \mathcal{L}(R(x)) / \mathcal{L}(B(x, e^{-u_n}))$ is approximately given by 416
 $\frac{4}{\pi} (\lambda_+)^{-1}$ which gives an extremal index as 0.51. Previous 417
 numerical computations for D_2 gave $D_2 \sim 1.987$.²⁹ The 418
 numerical computation with the GEV fitting gives 419
 $D_2 = 2.00 \pm 0.06$ and $\theta = 0.552 \pm 0.005$ for $n = 10^6$. 420
 In order to investigate the discrepancy with our theoretical 421
 estimate, we raised the quantile from $\tilde{s} = 0.99$ to 422
 $\tilde{s} = 0.999$, i.e., we select more extreme clusters. The esti- 423
 mates for this case are $\theta = 0.54 \pm 0.02$, more compatible 424
 with the theoretical one. Finally, if we consider longer tra- 425
 jectories ($n = 10^7$ iterates) with an even higher quantile 426
 ($\tilde{s} = 0.9999$), we get $\theta = 0.53 \pm 0.06$, which is even closer 427
 to the theoretical guess. 428

• We now consider the baker's map (see supplementary 429
 material); it depends on three parameters α , γ_a , and γ_b . The 430
 positive Lyapunov exponent is given by Ref. 20, Eq. 431
 (5.14) 432

$$\Lambda_+ = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha}.$$

With the value $\alpha = 1/3$, $\gamma_a = 1/5$, $\gamma_b = 1/4$, we get $\Lambda_+ \sim 0$, 433
 64 which gives, with our approximation, an extremal 434
 index of order 0, 47. In the paper, Ref. 20 Eq. (5.18), we 435
 gave an implicit formula expressing D_2 as a function of α 436
 and with respect to the SRB measure. For $\alpha = 1/3$, this 437
 estimate reads $D_2 \simeq 1.41$. The GEV estimates are given in 438
 Table I and are consistent with the theory. 439

• We next consider an attractor embedded in \mathbb{R}^3 , the so- 440
 called solenoid, see supplementary material; it depends 441
 upon the parameter $a \in (0, 0.5)$. The attractor is foliated 442
 by one-dimensional unstable manifolds, while each merid- 443
 ional disk is a two-dimensional stable manifold each of 444
 which intersecting the attractor over a Cantor set. The 445
 Lyapunov exponents are 446

$$\Lambda_- = \log a < 0, \quad \Lambda_+ = \log 2,$$

447 while the Hausdorff dimension d_H is given by the formula³⁰

$$d_H = 1 + \frac{\log 2}{-\log a}.$$

448 The numerical computations for the solenoid provide a fur-
449 ther test of the validity of the numerical algorithm and are
450 provided in Table I.

451 **B. High dimensional generalized Hénon maps**

452 We now analyze the generalized Hénon maps defined in
453 Ref. 31 and further analyzed in Ref. 32. They are defined as

$$x_{n+1}(1) = ax_n(d-1)^2 - bx_n(d) \quad x_{n+1}(i) = x_n(i-1). \quad (3.1)$$

454 When the parameter $a=1.76$, the number of positive
455 Lyapunov exponents is $d-1$; we could therefore test our
456 relation (2.11) by computing the entropy $h_{\mathcal{L}}$ as the sum of
457 positive Lyapunov exponents (see Table II in Ref. 32) for a
458 given d . We also perform the computation of the dimension
459 D_2 and compare it to the Kaplan-Yorke dimension D_{KY} given
460 in Ref. 32; we used such a dimension because we did not
461 find an explicit computation of D_2 in the literature. The good
462 agreement between our numerical results (Fig. 1) confirm
463 the validity of Eq. (2.10) with the caveat that an exact corre-
464 spondence cannot be derived for the geometric factor that
465 stretch balls in phase space in different dimensions: the ori-
466 gin of this discrepancy has been discussed in detail at the
467 end of Sec. IIC.

468 **C. Application to atmospheric data**

469 We now consider an application to atmospheric data.
470 The purpose of this application is to show that the applicabil-
471 ity of the technique on real data provides results that have a

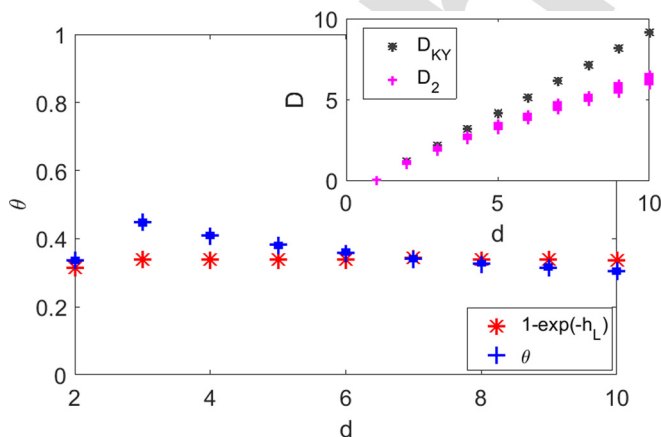


FIG. 1. Estimates of the dynamical extremal index θ and correlation dimension D_2 (inset) obtained for the Generalized Henon maps [Eq. (3.1)] in different dimensions d . The values represent the estimates obtained taking 30 couples of trajectories, iterated for $n=10^6$ iterations. Each couple is displayed using a single marker, but the uncertainty is so small that the difference between couples is hardly recognizable. The quantile used for the estimation is $\bar{s} = 0.98$. The results are compared to those obtained using the Kaplan-Yorke dimension D_{KY} and the entropy $h_{\mathcal{L}}$. This map has $d-1$ positive Lyapunov exponents.

coherent interpretation in terms of the underlying physics of
the systems. In order to provide evidence of the robustness
of our results, we will study several trajectories of a climate
models which incorporate observations of the past 110 years,
and compute θ and D_2 for several sub-periods showing that
the results are numerically stable. We study the atmospheric
circulation over the North Atlantic and focus on a single field
that represents its major features: the sea-level pressure
(SLP).^{33,34} Indeed, it has been shown that SLP fields can be
used to study teleconnection patterns as well as storm track
activity and atmospheric blocking.^{35,36} The trajectories of
our dynamical systems are successions of SLP fields
extracted with daily frequency from the ERA-20 CM reanal-
ysis project over the period 1900–2010.³⁷ The ERA-20 CM
consists of 10 members ensemble of a (climate) model
whose task is to reconstruct at best the 1900–2010 atmo-
spheric dynamics by constraining the model to include the
information from available surface observations. Each mem-
ber of the ERA 20 CM is therefore a slightly perturbed
reconstruction of the atmospheric dynamics in the past
110 years. The choice of the North Atlantic domain (80° W
 \leq Long. $\leq 50^\circ$ E, 22.5° N \leq Lat. $\leq 70^\circ$ N) is motivated by
the better observational coverage over the region in the first
part of the analysis period compared to other regions of the
globe.³⁸ Before presenting the results for D_2 and θ , we would
like to stress that (i) our analysis will only be representative
of the North-Atlantic domain and D_2 will be a proxy of the
active degrees of freedom of the atmospheric circulation in
this area. Therefore, our results cannot be used to estimate
the dimension of the full atmospheric climate attractor. (ii)
Previous results^{15,39,40} have shown that the estimates
obtained for the daily dimensions are robust with respect to
the changes in the datasets, resolution of the climate models,
and are linearly insensitive to the size of the domain. This
gives us confidence on the applicability of the numerical
algorithm described in this paper for climate data since it is
largely based on those used in Refs. 15, 39, and 40.

The results for D_2 and θ on the SLP fields of the ERA-
20 CM ensemble are presented in Fig. 2. For each estimate,
we fix the reference trajectory x as the first member (M1) of
the ERA-20 CM ensemble because this is always considered
as the reference simulation, while y is alternatively set as the
 M_i th member with $i=2, 3, \dots, 10$. The dependence of the
results on the reference member are tested in the [supplemen-](#)
[tary material](#) Fig. S1. To test the robustness of the results, we
provide four estimates of D_2 and θ : (i) using the full data in
the period 1900–2010, (ii) using 1900–1955 data, (iii) using
1900–1928 data, and (iv) considering only the first 14 years
(1900–1914) of data. For each member, the results are
reported in Fig. 2. The ensemble averages of D_2 and θ for
the different periods are instead reported in Table II.
Estimates are consistent for different periods and the value
of $D_2 \simeq 9$ found on average, is slightly lower than the esti-
mates of d_H found in Ref. 15 (we remind that $D_2 < d_H$). The
value of D_2 roughly corresponds to the number of spatial
degrees of freedom active in a North-Atlantic SLP field as
explained in Ref. 15. Indeed, the domain used for this analy-
sis can host about 9 large spatial structures reparted between
3 and 4 extratropical cyclones at time and the same number

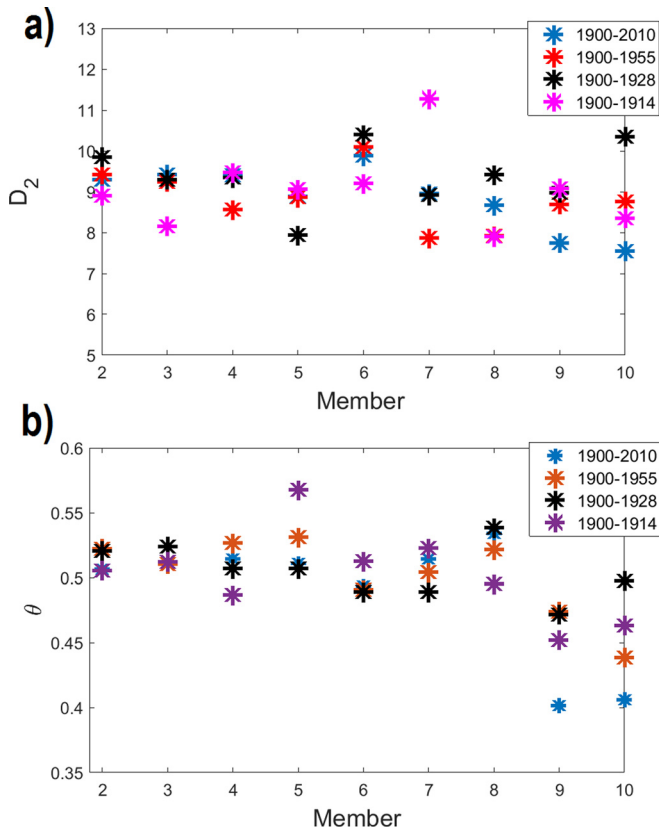


FIG. 2. Estimates of correlation dimension D_2 (a) and extremal index θ (b) obtained for daily sea-level pressure maps for four different periods in the ERA-20CM reanalysis. The values represent the estimates obtained taking as reference trajectory x the member M1 and as y , the remaining 9 ensemble members.

the indicators among the members. In particular, M9 and M10 have a minimum of θ around 1960. This could be due to the different boundary conditions applied to the members and detailed in Ref. 37.

D. Additive noise

In our previous papers,^{17,42,43} we have analyzed the effect of additive noise on the parameters of the extreme value laws. It consists in defining a family of maps $T_\xi = T + \varepsilon\xi$ with ξ a random variable sampled from some distribution \mathbb{G} (we will take here the uniform distribution on some small ball of radius ε around 0). The iteration of the single map T will be now replaced by the concatenation $T_{\xi_n} \circ T_{\xi_{n-1}} \cdots \circ T_{\xi_1}$ and the evaluation of an observable computed along this orbit will be given by the probability measure \mathbb{P} which is the product of \mathbb{G}^N with the so-called stationary measure μ_S , verifying, for any real measurable bounded function f : $\int f d\mu_s = \int f \circ T_\xi d\mu_s$; see Ref. 19 Chap. 7, for a general introduction to the matter. In the aforementioned papers, Refs. 42 and 43, we have shown analytically that for dynamical systems perturbed additively, the extremal index $\theta = 1$, no matter what the intensity of the noise is. The proof was supported by numerical experiments, using also different noise types. The extremal index is a parameter that quantifies the amount of clustering, the stickiness of the trajectory in phase space. In our setting, clustering happens in the presence of invariant sets, which are periodic points in Ref. 42. By looking at formula (2.4), we see that we estimate the proportion of the neighborhood of the invariant set returning to itself; as we argued above, that estimate gives information on the rate of backward volume contraction in the unstable direction. Since the noise generally destructs these invariants sets, we expect the extremal index be equal to 1 or quickly approaching 1 when the noise increases. This is confirmed by the numerical experiments reported in Fig. 3 where the value of θ is plotted against the intensity of the noise ε for three maps: $3x \bmod 1$ map, the Baker map, and the Lozi map. In all cases, indeed $\theta \rightarrow 1$ for large enough noise. However, with respect to the observables discussed in Ref. 42, we find some remarkable differences on the intensity of the noise needed to observe changes of the extremal index from the deterministic values: whereas in Ref. 42, we observed significant deviation from the deterministic behavior for very small noise intensities ($\varepsilon \geq 10^{-4}$), here we need $\varepsilon \geq 10^{-2}$, i.e., only large noise amplitudes perturb the estimates of D_2 and θ . This difference can be easily explained: in Ref. 42, the extremal index was used to explore the local stability at periodic fixed points, where the dynamics is deeply affected even by a small noise. Here, instead, the extremal index tracks a global property that it is stable with respect to small stochastic perturbations. We underline that, for the Lozi map, we cannot obtain estimates of θ for noise larger than 0.1 because the dynamics fall out the basin of attraction.

IV. DISCUSSION AND CONCLUSIONS

Using the extreme value theory, we have introduced a new and efficient way to compute the correlation dimension

of anticyclones (see the textbook of Holton,⁴¹ for estimates of the typical size of these objects). θ is, in fact, the inverse of the average time the two trajectories x and y cluster together. The value of the DEI $\theta = 0.5$ corresponds therefore to a contraction of the phase space associated with a time-scale between 2 and 3 days. This is the typical decay rate of baroclinic eddies associated with the low pressure systems observed in SLP fields (see again the textbook by Holton⁴¹ for the decay rates). We finally notice that our formula (2.11) gives for the entropy the value $\log 2$. In Fig. S2, we show a moving window computation of D_2 and θ . No clear trend emerges that could be attributed to anthropogenic forcing. This result is consistent with those found for d_H in Ref. 39. We remark however some differences in the variability of

TABLE II. Estimates of correlation dimension D_2 and extremal index θ obtained for daily sea-level pressure maps for four different periods of the ERA-20CM reanalysis. The values represent average over the 9 ensemble members and uncertainty is expressed as the standard deviation of the ensemble mean.

Period	D_2	θ
1900-2010	8.9 ± 0.8	0.48 ± 0.05
1900-1955	8.8 ± 0.7	0.50 ± 0.03
1900-1928	9.4 ± 0.8	0.50 ± 0.02
1900-1914	9.0 ± 1.0	0.50 ± 0.03

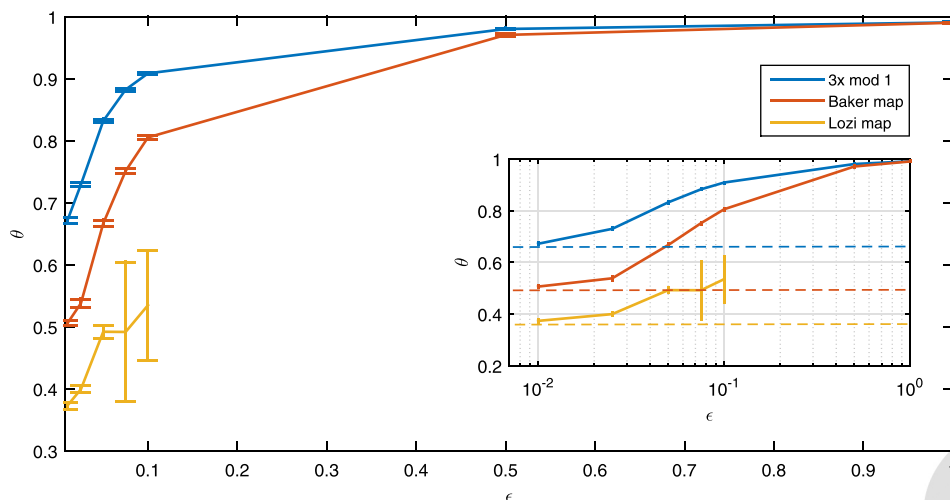


FIG. 3. Dynamical extremal index θ vs intensity of the additive noise ε for three different maps: $3 \times \text{mod } 1$ (blue), Baker map (red), and Lozi map (orange). The error bar indicates the standard deviation of the sample of $l = 100$ trajectories, each consisting of $n = 10^6$ iterations. The quantile for the estimate of the extremal index is $\hat{s} = 0.99$. The inset shows the same data in the semilog scale, with the deterministic values represented by the dotted lines.

601 D_2 . Moreover, for higher dimensional maps, we introduced
 602 the quantity q_0 , related to the expectation of the inverse of
 603 the determinant of the derivative along the expanding sub-
 604 space. Therefore, the extremal index $\theta = 1 - q_0$ is a *measure*
 605 *of the average rate of phase space contraction for backward*
 606 *iteration*. Although this quantity slightly differs from the
 607 entropy or from the positive Lyapunov exponent when the
 608 expanding subspace has dimension one, it provides an
 609 important piece of information on the dynamics of the sys-
 610 tem. In fact it can be linked to the global predictability and
 611 therefore *considered as a new indicator of the local instabil-*
 612 *ity in chaotic systems*. We would like also to emphasize that
 613 both θ and D_2 can be computed simultaneously just by look-
 614 ing at the GEV function and this makes our method quite
 615 rapid and economically efficient from a numerical point of
 616 view. We have shown that even for a short time series of
 617 only of 10^4 iterations, the estimates are robust and consistent
 618 with the theoretical expectations. We have also presented a
 619 first application of these indicators to climate data proving
 620 that the indicators are useful to infer the spatial number of
 621 degrees of freedom and the typical time scales of the atm-
 622 ospheric dynamics on the North Atlantic region. Finally, we
 623 have observed their sensitivity to the different boundary con-
 624 ditions imposed for the climate simulations analyzed. This
 625 implies that the indicators could be useful in characterizing
 626 and comparing also different climate datasets as those ana-
 627 lyzed in international campaigns.

628 Our interpretation of θ together with that on the correla-
 629 tion dimension D_2 could be useful also to analyze the times
 630 series arising from the evolution of chaotic systems. Indeed,
 631 these quantities are particularly straightforward to obtain
 632 from numerical computations. Moreover, the results obtained
 633 can also be used to detect the embedding dimension, namely
 634 by replacing the sample of data with delay vectors of vari-
 635 able lengths; we stress that computing the GEV with those
 636 delay vectors will allow us to get exactly the embedding
 637 dimension. We mean to develop further this approach in a
 638 future paper.

639 Finally, the computation of the DEI could be helpful to
 640 distinguish purely stochastic sequences for which the
 641 extremal index should approach 1, see Sec. III D, from
 642 dynamical systems with an underlying chaotic behavior even

in the presence of small stochastic perturbations. Again, 643
 these further applications of our approach with EVT will be 644
 the objects of forthcoming investigations. 645

SUPPLEMENTARY MATERIAL 646

See [supplementary material](#) for: (i) the algorithm for the 647
 estimation of the correlation dimension D_2 and the 648
 Dynamical Extremal Index (DEI) θ , (ii) a commented 649
 numerical MATLAB code for such estimation, (iii) the 650
 model equations for the maps used, and (iv) the supplement- 651
 ary figures. 652

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¹P. Grassberger and I. Procaccia, *The Theory of Chaotic Attractors* 664
 (Springer, 2004), pp. 170–189. 665

²P. Grassberger and I. Procaccia, *Phys. Rev. Lett.* **50**, 346 (1983). 666

³A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, *Physica D* **16**, 285 667
 (1985). 668

⁴M. T. Rosenstein, J. J. Collins, and C. J. De Luca, *Physica D* **65**, 117 669
 (1993). 670

⁵J. Theiler, S. Eubank, A. Longtin, B. Galdrikian, and J. D. Farmer, *Physica* 671
D **58**, 77 (1992). 672

⁶J.-P. Eckmann and D. Ruelle, *Physica D* **56**, 185 (1992). 673

⁷H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis* (Cambridge 674
 University Press, 2004), Vol. 7. 675

⁸A. Pikovsky and A. Politi, *Lyapunov Exponents: A Tool to Explore* 676
Complex Dynamics (Cambridge University Press, 2016). 677

⁹H. Kantz, G. Radons, and H. Yang, *J. Phys. A: Math. Theor.* **46**, 254009 678
 (2013). 679

¹⁰A. Politi, *Phys. Rev. Lett.* **118**, 144101 (2017). 680

¹¹A. C. M. Freitas, J. M. Freitas, and M. Todd, *Probab. Theory Relat. Fields* 681
147, 675 (2010). 682

¹²D. Faranda, V. Lucarini, G. Turchetti, and S. Vaienti, *J. Stat. Phys.* **145**, 683
 1156 (2011). 684

¹³D. Faranda and S. Vaienti, *Geophys. Res. Lett.* **40**, 5782, <https://doi.org/10.1002/2013GL057811> (2013). 685
 686

- 687 ¹⁴D. Faranda, M. C. Alvarez-Castro, and P. Yiou, *Clim. Dyn.* **47**, 3803
688 (2016). 710
- 689 ¹⁵D. Faranda, G. Messori, and P. Yiou, *Sci. Rep.* **7**, 41278 (2017). 711
- AQ8 690 ¹⁶F. M. E. Pons, G. Messori, M. C. Alvarez-Castro, and D. Faranda, preprint
691 arXiv:hal-01650250 (2017). 712
- AQ9 692 ¹⁷D. Faranda, H. Ghoudi, P. Guiraud, and S. Vaienti, “■,” Nonlinearity (to
693 be published); , preprint arXiv:1708.00191. 713
- 694 ¹⁸Y. B. Pesin, *Dimension Theory in Dynamical Systems: Contemporary*
695 *Views and Applications* (University of Chicago Press, 2008). 714
- 696 ¹⁹V. Lucarini, D. Faranda, J. M. Freitas, M. Holland, T. Kuna, M. Nicol, M.
697 Todd, S. Vaienti *et al.*, *Extremes and Recurrence in Dynamical Systems*
698 (John Wiley & Sons, 2016). 715
- 699 ²⁰D. Bessis, G. Paladin, G. Turchetti, and S. Vaienti, *J. Stat. Phys.* **51**, 109 (1988). 716
- 700 ²¹A. Ferguson and M. Pollicott, *Ergodic Theory Dyn. Syst.* **32**, 961 (2012). 717
- 701 ²²G. Keller, *Equilibrium States in Ergodic Theory* (Cambridge University
702 Press, 1998), Vol. 42. 718
- 703 ²³G. Keller, *Dyn. Syst.* **27**, 11 (2012). 719
- 704 ²⁴G. Keller and C. Liverani, *J. Stat. Phys.* **135**, 519 (2009). 720
- 705 ²⁵M. Viana, *Lectures on Lyapunov Exponents* (Cambridge University Press,
706 2014), Vol. 145. 721
- 707 ²⁶L.-S. Young, *Ergodic Theory Dyn. Syst.* **2**, 109 (1982). 722
- 708 ²⁷V. Lucarini, D. Faranda, G. Turchetti, and S. Vaienti, *Chaos* **22**, 023135
709 (2012). 723
- ²⁸G. Paladin and S. Vaienti, *J. Stat. Phys.* **57**, 289 (1989). 710
- ²⁹J. C. Sprott and G. Rowlands, *Int. J. Bifurcation Chaos* **11**, 1865
(2001). 711
- ³⁰K. Simon, *Proc. Am. Math. Soc.* **125**, 1221 (1997). 712
- ³¹G. Baier and M. Klein, *Phys. Lett. A* **151**, 281 (1990). 713
- ³²H. Richter, *Int. J. Bifurcation Chaos* **12**, 1371 (2002). 714
- ³³J. W. Hurrell, *Science* **269**, 676 (1995); ISSN 0036-8075. 715
- ³⁴G. Moore, I. A. Renfrew, and R. S. Pickart, *J. Clim.* **26**, 2453 (2013). 716
- ³⁵J. C. Rogers, *J. Clim.* **10**, 1635 (1997). 717
- ³⁶L. Comas-Bru and F. McDermott, *Q. J. R. Meteorol. Soc.* **140**, 354
(2014). 718
- ³⁷H. Hersbach, C. Peubey, A. Simmons, P. Berrisford, P. Poli, and D. Dee,
Q. J. R. Meteorol. Soc. **141**, 2350 (2015). 719
- ³⁸O. Krueger, F. Schenk, F. Feser, and R. Weisse, *J. Clim.* **26**, 868 (2013). 720
- ³⁹D. Rodrigues, M. C. Alvarez-Castro, G. Messori, P. Yiou, Y. Robin, and
D. Faranda, ■ (2017). 721
- ⁴⁰D. Faranda, G. Messori, M. C. Alvarez-Castro, and P. Yiou, *Nonlinear*
Processes Geophys. **24**, 713 (2017). 722
- ⁴¹J. R. Holton, *Am. J. Phys.* **41**, 752 (1973). 723
- ⁴²D. Faranda, J. M. Freitas, V. Lucarini, G. Turchetti, and S. Vaienti,
Nonlinearity **26**, 2597 (2013). 724
- ⁴³H. Aytac, J. Freitas, and S. Vaienti, *Trans. Am. Math. Soc.* **367**, 8229
725 (2015). 726
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