



**HAL**  
open science

# Correlation dimension and phase space contraction via extreme value theory

Davide Faranda, Sandro Vaienti

► **To cite this version:**

Davide Faranda, Sandro Vaienti. Correlation dimension and phase space contraction via extreme value theory. *Chaos: An Interdisciplinary Journal of Nonlinear Science*, 2018, 28 (4), pp.041103. 10.1063/1.5027386 . hal-01768181

**HAL Id: hal-01768181**


**<https://hal.science/hal-01768181>**

Submitted on 24 Apr 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## AUTHOR QUERY FORM

	<p><b>Journal:</b> Chaos</p> <p><b>Article Number:</b> 008894CHA</p>	<p>Please provide your responses and any corrections by annotating this PDF and uploading it to AIP's eProof website as detailed in the Welcome email.</p>
---	---	--

Dear Author,

Below are the queries associated with your article; please answer all of these queries before sending the proof back to AIP.

**Article checklist:** In order to ensure greater accuracy, please check the following and make all necessary corrections before returning your proof.

1. Is the title of your article accurate and spelled correctly?
2. Please check affiliations including spelling, completeness, and correct linking to authors.
3. Did you remember to include acknowledgment of funding, if required, and is it accurate?

Location in article	Query / Remark: click on the Q link to navigate to the appropriate spot in the proof. There, insert your comments as a PDF annotation.
<a href="#">AQ1</a>	Please check that the author names are in the proper order and spelled correctly. Also, please ensure that each author's given and surnames have been correctly identified (given names are highlighted in red and surnames appear in blue).
<a href="#">AQ2</a>	Please check the hierarchy of section headings and their citations.
<a href="#">AQ3</a>	In the sentence beginning "We will present..." please confirm that "next section" refers to Sec. II.
<a href="#">AQ4</a>	As per standard AIP journal style, Eqs. (3.1)–(4.12) have been renumbered as Eqs. (2.1)–(3.1) and all citations in the text have been updated accordingly. Please check all renumbering carefully throughout.
<a href="#">AQ5</a>	Footnotes in the article are not allowed, hence, we have moved the footnotes in the text. Kindly check.
<a href="#">AQ6</a>	In the sentence beginning "In both cases..." please confirm that "next section" refers to Sec. II C.
<a href="#">AQ7</a>	In the sentence beginning "We now..." please confirm that "previous section" refers to Sec. II B.
<a href="#">AQ8</a>	Please provide published information in Ref. 16.
<a href="#">AQ9</a>	If preprint Ref. 17 has subsequently been published elsewhere, please provide updated reference information (article title, volume number, page number, and year).
<a href="#">AQ10</a>	Please provide journal title, volume, and page number in Ref. 39.

Thank you for your assistance.

# 1 Correlation dimension and phase space contraction via extreme value theory

2 Davide Faranda<sup>1,a)</sup> and Sandro Vaienti<sup>2,b)</sup>

3 <sup>1</sup>LSCE-IPSL, CEA Saclay l'Orme des Merisiers, CNRS UMR 8212 CEA-CNRS-UVSQ,  
4 Université Paris-Saclay, 91191 Gif-sur-Yvette, France

5 <sup>2</sup>Aix Marseille Univ., Université de Toulon, CNRS, CPT, 13009 Marseille, France

6 (Received 1 March 2018; accepted 10 April 2018; published online xx xx xxxx)

7 We show how to obtain theoretical and numerical estimates of correlation dimension and phase  
8 space contraction by using the extreme value theory. The maxima of suitable observables sampled  
9 along the trajectory of a chaotic dynamical system converge asymptotically to classical extreme  
10 value laws where: (i) the inverse of the scale parameter gives the correlation dimension and (ii) the  
11 extremal index is associated with the rate of phase space contraction for backward iteration, which  
12 in dimension 1 and 2, is closely related to the positive Lyapunov exponent and in higher  
13 dimensions is related to the metric entropy. We call it the Dynamical Extremal Index. Numerical  
14 estimates are straightforward to obtain as they imply just a simple fit to a univariate distribution.  
15 Numerical tests range from low dimensional maps, to generalized Henon maps and climate data.  
16 The estimates of the indicators are particularly robust even with relatively short time series.

Published by AIP Publishing. <https://doi.org/10.1063/1.5027386>

17 **This study uses the link between extreme value laws and**  
18 **dynamical systems theory to show that important dynam-**  
19 **ical quantities as the correlation dimension, the entropy,**  
20 **and the Lyapunov exponents can be obtained by fitting**  
21 **observables computed along a trajectory of chaotic sys-**  
22 **tems. All this information is contained in a newly defined**  
23 **Dynamical Extreme Index. Besides being mathematically**  
24 **well defined, it is almost numerically effortless to get as**  
25 **(i) it does not require the specification of any additional**  
26 **parameter (e.g., embedding dimension, decorrelation**  
27 **time); (ii) it does not suffer from the so-called curse of**  
28 **dimensionality. A numerical code for its computation is**  
29 **provided.**

48 exponent and the reliability of estimates from the time series  
49 of experimental phenomena is often questioned.<sup>6</sup> We defer  
50 the reader to the monographs<sup>7,8</sup> and to the articles<sup>9,10</sup> for  
51 recent advancements on the various statistical tools to inves-  
52 tigate the nonlinear time series.

53 The extreme value theory (EVT) has been used to char-  
54 acterize the evolution of chaotic systems.<sup>11,12</sup> It is possible  
55 to obtain dynamical properties in phase space (fractal dimen-  
56 sion or stability) by exploiting the limiting theorems of the  
57 extreme value theory. The main idea is: (i) to replace the sto-  
58 chastic processes used in the statistical framework with a tra-  
59 jectory of a chaotic dynamical system and (ii) to study the  
60 convergence of maxima of suitable observables to the class-  
61 ical extreme value laws. The parameters of the EVT provide  
62 estimates of dynamical properties of the system. This con-  
63 nection between EVT and the dynamical properties of cha-  
64 otic systems is rich not only from a theoretical but also from  
65 a numerical perspective. Indeed, the estimates of local prop-  
66 erties obtained with EVT do not require the introduction of  
67 additional parameters and they are easy to implement numer-  
68 ically. They have been used to get insights into the dynami-  
69 cal behavior of atmospheric flows in Refs. 13–15. In Ref. 16,  
70 it has been shown that the numerical algorithm based on  
71 EVT provide reliable estimates of the dimension of high  
72 dimensional systems up to phase spaces with thousands of  
73 dimensions. It is therefore desirable to estimate other key  
74 dynamical quantities in the EVT framework.

75 The purpose of this communication is to show that the  
76 correlation dimension and the EVT are intimately related:  
77 the CD arises by studying the distribution of the maxima of a  
78 new suitable observable evaluated along the orbit of a  
79 chaotic system. Moreover, an exponent of the limit law, the  
80 extremal index, is related, for hyperbolic attractors, to  
81 the positive Lyapunov exponent in dimension two and to the  
82 metric entropy in higher dimensions. The idea of the rela-  
83 tionship between EVT and CD comes from a previous

## AQ2 31 I. INTRODUCTION

32 Since its introduction by Grassberger and Procaccia,<sup>1,2</sup>  
33 the correlation dimension (CD) has been used as a powerful  
34 indicator for the description of the fractal structure of invari-  
35 ant sets in dynamical systems. Similarly, the Lyapunov  
36 exponents and the entropy<sup>3,4</sup> provide an indication of the rel-  
37 evant time scales associated with the dynamics and the pre-  
38 dictability horizon of the system. Given the importance of  
39 these quantities, there exists an increasing body of literature  
40 on how to estimate CD, Lyapunov exponents, and entropy. It  
41 has been shown that reliable estimates of CD can be obtained  
42 with a relatively short time series.<sup>5</sup> Instead, the computations  
43 of Lyapunov exponents and entropy are still challenging  
44 because the existing methodologies require as input addi-  
45 tional parameters as the dimension of the phase space and  
46 the relevant time scale of the dynamics (e.g., the decorrela-  
47 tion time). Calculations are then limited to the top Lyapunov

<sup>a)</sup>Also at London Mathematical Laboratory, 14 Buckingham Street, London  
WC2N 6DF, United Kingdom. Electronic mail: [davide.faranda@cea.fr](mailto:davide.faranda@cea.fr)

<sup>b)</sup>Electronic mail: [vaienti@cpt.univ-mrs.fr](mailto:vaienti@cpt.univ-mrs.fr)

84 work<sup>17</sup> where we used the extreme value theory to detect  
 85 and quantify the onset of synchronization in coupled map lat-  
 86 tices. The relationship between the extremal index and the  
 87 Lyapunov exponent and the entropy is new and is particu-  
 88 larly striking for maps with piecewise constant jacobian. In  
 89 the general case, we derive a formula whose validity is con-  
 90 firmed by numerical experiments. We also explain the rela-  
 91 tion between our extremal index, the local dimensions, and  
 92 the phase space contraction. In the rest of the paper, we will  
 93 name it as the DEI, the *dynamical extremal index*. We want  
 94 to point out that our DEI is a well defined quantity that can  
 95 be used as a new indicator for the sensitivity associated with  
 96 local hyperbolicity. We will present the theoretical results in  
 97 Sec. II: some of those results can be obtained by generalizing  
 98 the techniques introduced in Ref. 17; we will also address  
 99 the need to develop a more appropriate theory of EVT for  
 100 diffeomorphisms in higher dimensions. We will then provide  
 101 several examples of classical conceptual low-dimensional  
 102 dynamical systems. We will discuss the implications of our  
 103 results on higher dimensional systems and the possibility to  
 104 apply them to more a general time series. As an example, we  
 105 will compute the indicators on climate data and explain how  
 106 they provide relevant physical information on the atmo-  
 107 spheric circulation over the North Atlantic.

108 **II. THEORETICAL RESULTS**

109 **A. A brief presentation of the extreme value theory and**  
 110 **a new observable**

111 Let  $(M, \mu, T)$  be dynamical systems given by a map  $T$   
 112 acting on the metric compact space  $M$  with distance  $d(\cdot, \cdot)$   
 113 and preserving the Borel measure  $\mu$ . Usually,  $M$  will be a  
 114 compact subset of some  $\mathbb{R}^n$  and  $d$  a distance equivalent to  
 115 the standard one. Let us take the *direct product*  $(M \times M,$   
 116  $\mu \times \mu, T \times T)$ , and denote with  $(x, y) \in M \times M$ , a couple of  
 117 point in the Cartesian product  $(M \times M)$ . We then introduce  
 118 the observable  $\psi(x, y) = -\log d(x, y)$ , and consider the pro-  
 119 cess  $\{\psi \circ (T^j \times T^j)\}_{j \geq 0}$ , and the maximum of the sequence  
 120  $\mathcal{M}_n(x, y) = \max\{\psi(x, y), \psi(Tx, Ty), \dots, \psi(T^{n-1}x, T^{n-1}y)\}$  and  
 121 finally its distribution  $\mathbb{P}(\mathcal{M}_n \leq u_n)$ , where  $\mathbb{P} = \mu \times \mu$  is the  
 122 underlying probability and  $u_n$  is a suitable scaling function  
 123 tending to infinity and which we are going to define.  
 124 Suppose that for a given positive number  $\tau$  we can find a  
 125 sequence of numbers  $u_n$  such that  $n\mathbb{P}(\psi \geq u_n) \rightarrow \tau, n \rightarrow \infty$ .  
 126 We say, that the process  $\{\psi \circ (T^j \times T^j)\}_{j \geq 0}$  satisfies an  
 127 extreme value law of Gumbel's type if there is a number  
 128  $\theta \in (0, 1]$ , the *extremal index*, such that  $\mathbb{P}(\mathcal{M}_n \leq u_n) \rightarrow e^{-\theta\tau}$ ,  
 129  $n \rightarrow \infty$ . We now introduce the diagonal neighborhood  $S_n$  in  
 130 the product space:  $S_n = \{(x, y), d(x, y) \leq e^{-u_n}\}$ . By substitut-  
 131 ing the expression of  $\psi$  in  $\mathbb{P}(\psi \geq u_n)$ , we have

$$\mathbb{P}(\psi \geq u_n) = \mathbb{P}((x, y) \in S_n) = \int_M \mu(B(x, e^{-u_n}))d\mu(x), \quad (2.1)$$

132 where  $B(x, a)$  denotes the ball of radius  $a$  centered on  $x$ .  
 133 (Actually, we got the equality of the right hand side in the  
 134 limit of large  $n$  when the two small corners of  $S_n$  become  
 135 negligible.) The quantity  $\int_M \mu(B(x, r))d\mu(x)$  scales like  $r^{D_2}$

and the exponent  $D_2$  is called the *correlation dimension* and  
 it characterizes the fractal structure of the support of  $\mu$ ; a  
 more formal, from the mathematical point of view, definition  
 of this fact is given in Ref. 18, Sec. 17, and references  
 therein. (A precise definition consists in taking the limsup  
 and liminf of the ratio of the logarithm with  $\log(1/r)$ .) By  
 injecting successively into (2.1), we have therefore that for  
 large  $n$

$$u_n \sim \frac{-\log \tau}{D_2} + \frac{\log n}{D_2} := \frac{z}{a_n} + b_n, \quad (2.2)$$

where  $\tau = e^{-z}$ ,  $a_n = D_2$  and  $b_n = \frac{\log n}{D_2}$ . For numerical pur-  
 poses, distribution functions like  $\mathbb{P}(\mathcal{M}_n \leq z)$  are modelled,  
 for  $n$  sufficiently large, by the so-called *generalized extreme*  
*value (GEV)* distribution which is a function depending upon  
 three parameters  $\xi \in \mathbb{R}, \kappa \in \mathbb{R}, \sigma > 0$  and such that:  
 $F_{\text{GEV}}(z; \kappa, \sigma, \xi) = \exp\left\{-\left[1 + \xi\left(\frac{z-\kappa}{\sigma}\right)\right]^{-1/\xi}\right\}$ .

The parameter  $\xi$  is called the tail index; when its value  
 is 0, the GEV corresponds to the Gumbel type. The param-  
 eter  $\kappa$  is called the location parameter and  $\sigma$  is the scale  
 parameter: for  $n$  large, the scaling constant  $a_n$  is close to  $\sigma^{-1}$   
 and  $b_n$  is close to  $\kappa$ . Therefore, if we could fit a limit law of  
 Gumbel's type with suitable normalizing parameters  $a_n$  and  
 $b_n$ , we immediately get the correlation dimension. Such a  
 technique was previously used with a different observable,  
 and it allowed to get the so-called *information dimension*  
 $D_1(x)$ , another fractal dimension which provides the scaling  
 of the measure of a ball around a given point  $x$ , see Ref. 19  
 and references therein. Although the information dimension  
 depends on the point  $x$ , its value is the same for almost all  
 the choices of  $x$  with respect to the invariant measure and  
 such an averaged valued, simply  $D_1$ , is larger or equal to  $D_2$ ,  
 see Ref. 20 for an account on the different fractal dimen-  
 sions. In particular, if we denote with  $d_H$  the Hausdorff  
 dimension, we have  $D_2 \leq D_1 \leq d_H$ .

168 **B. The spectral approach with the new observable for**  
 169 **conformal repellers**

Before showing our numerical simulations for the com-  
 putation of the CD, let us argue how we get a Gumbel's type  
 asymptotic distribution with an extremal index  $\theta$  of dynami-  
 cal meaning. First, we consider one-dimensional dynamical  
 systems generated by uniformly expanding maps with an  
 invariant set which could be a Cantor set and equipped with  
 mixing Gibbs measures. These systems are better known as  
*conformal repellers*,—see for instance<sup>21</sup> for a recent contri-  
 bution—whose measures are characterized by a potential  $\varphi$   
 of type  $\varphi(x) = -\beta \log |T'(x)|$ , where  $T'$  denotes the deriva-  
 tive of  $T$  and  $\beta \in \mathbb{R}$ . If we denote them as  $\mu_\beta$ , they are given  
 by  $h_\beta \nu_\beta$ , where the density  $h_\beta$  and the *conformal* measure  $\nu_\beta$   
 are, respectively, the eigenfunctions of the transfer operator  
 (Perron-Fröbenius) and of its dual, both with eigenvalue  
 $\lambda_\beta = e^{Q(\beta)}$ , being  $Q(\beta)$  the topological pressure. We remind  
 that the transfer operator  $\mathcal{P}_T$  for the map  $T$  is defined, for an  
 observable  $f$  in some suitable Banach space  $\mathcal{B}$ —for instance  
 the space of Lipschitz continuous functions—by the duality  
 relation:  $\int \mathcal{P}_T f d\nu_\beta = \lambda_\beta \int f d\nu_\beta$ . We defer to the monograph<sup>22</sup>

AQ3

AQ4

AQ5

189 for an introduction to thermodynamic formalism. The conformal  
 190 measure verifies the property  $\nu_\beta(TA) = \lambda_\beta \int_A e^{-\phi} d\nu_\beta$ ,  
 191 where  $T$  is one-to-one over the measurable set  $A$ . A powerful  
 192 method to investigate the distribution of our process  
 193  $\{\psi \circ (T^j \times T^j)\}_{j \geq 0}$  consists in perturbing the transfer opera-  
 194 tor  $\mathcal{P}$  of the direct product  $T \times T$ . The key observation is that  
 195 by repeatedly using the duality relation, we can write  
 196  $\mathbb{P}(\mathcal{M}_n \leq u_n) = \lambda_\beta^{-2n} \int \int \tilde{\mathcal{P}}_n^n(h_\beta(x)h_\beta(y)) d\nu_\beta(x) d\nu_\beta(y)$ , where  
 197 the perturbed operator  $\tilde{\mathcal{P}}_n$  is defined by acting on observables  
 198  $f \in \mathcal{B}$ , as  $\tilde{\mathcal{P}}_n(f) = \mathcal{P}(f \mathbf{1}_{S_n^c})$ , and  $S_n = \{(x, y); d(x, y)$   
 199  $\leq e^{-u_n}\}$ . When  $n$  tends to infinity, the characteristic function  
 200 of the complement of  $S_n$ ,  $\mathbf{1}_{S_n^c}$ , goes to the identity and the  
 201 operators  $\mathcal{P}$  and  $\tilde{\mathcal{P}}_n$  converge to each other in  $\mathcal{B}$ . If the  
 202 unperturbed operator  $\mathcal{P}$  has a spectral gap, it allows expo-  
 203 nential mixing for the observables in  $\mathcal{B}$ . This compensates the  
 204 lack of independence of the process  $\{\psi \circ (T^j \times T^j)\}_{j \geq 0}$ . The  
 205 same is true for the operator  $\tilde{\mathcal{P}}_n$  and the maximal, isolated,  
 206 eigenvalue of  $\mathcal{P}$ ,  $\lambda_\beta^2$ , is close to that of  $\tilde{\mathcal{P}}_n$ ,  $\tilde{\lambda}_{\beta,n}^{(2)}$ . More pre-  
 207 cisely:  $\tilde{\lambda}_{\beta,n}^{(2)} \sim \lambda_\beta^2 - (1 - \lambda_\beta^2 q_0) \mathbb{P}(S_n)$ , where now  $\mathbb{P} = \mu_\beta$   
 208  $\times \mu_\beta$ . We will define the factor  $q_0$  in a moment. The operator  
 209  $\tilde{\mathcal{P}}_n$  now decomposes as the sum of a projection along the one  
 210 dimensional eigenspace associated with the eigenvalue  $\tilde{\lambda}_{\beta,n}^{(2)}$   
 211 and an operator with a spectral radius exponentially decreas-  
 212 ing to zero and which can be neglected in the limit of large  
 213  $n$ . This allows us to write  $\mathbb{P}(\mathcal{M}_n \leq u_n) \sim \lambda_\beta^{-2n} \tilde{\lambda}_{\beta,n}^{(2)n} \int \int h_\beta(x)$   
 214  $h_\beta(y) d\nu_{\beta,n}(x) d\nu_{\beta,n}(y)$ , where  $\nu_{\beta,n}$  is the conformal  
 215 measure for the perturbed operator and the double integral on  
 216 the right hand side converges to 1 for  $n \rightarrow \infty$ . Finally, we  
 217 get by approximating  $\tilde{\lambda}_{\beta,n}^{(2)}$  as above:  $\mathbb{P}(\mathcal{M}_n \leq u_n)$   
 218  $\sim \left[1 - \frac{(1 - \lambda_\beta^2 q_0) \mathbb{P}(S_n)}{\lambda_\beta^2}\right]^n \sim \exp\left[-\frac{(1 - \lambda_\beta^2 q_0) \mathbb{P}(S_n)}{\lambda_\beta^2} n\right]$ . We now  
 219 remind that we are under the assumption that  $n \mathbb{P}(\psi \geq u_n)$   
 220  $= n \mathbb{P}(S_n) \rightarrow \tau, n \rightarrow \infty$ . This leads to the Gumbel law  $e^{-\theta \tau}$   
 221 provided that the dynamical extremal index  $\theta$  is defined as

$$\theta = \frac{1 - \lambda_\beta^{-2} q_0}{\lambda_\beta^2}. \tag{2.3}$$

222 The term  $q_0$  is obtained by the previous perturbation theory  
 223 under the assumption that the diagonal in the product space  
 224 is left invariant by the direct product of the two maps. In par-  
 225 ticular, we have

$$q_0 = \lim_{n \rightarrow \infty} \frac{\mathbb{P}(S_n \cap \bar{T}^{-1} S_n)}{\mathbb{P}(S_n)}, \tag{2.4}$$

226 provided that the limit exists. The technique just described  
 227 was first proposed by Keller<sup>23</sup> as an alternative way to get  
 228 EVT for systems with exponential mixing and it is based on  
 229 a perturbative result by Keller and Liverani.<sup>24</sup> We defer to  
 230 Ref. 23 and to our paper<sup>17</sup> for a detailed presentation of that  
 231 theory. It can be applied to conformal mixing repellers and it  
 232 provides the preceding estimates, namely the asymptotic  
 233 scaling for the maximal eigenvalue. We would like to point

234 out that with our choice for the observable  $\psi$ , the perturba-  
 235 tive approach just sketched gives the Gumbel's law in a very  
 236 direct and natural manner.

237 The computation of  $q_0$  proceeds now as in Ref. 17 with  
 238 a substantial difference: the nature of the conformal measure  
 239 does not imply necessarily that the ratio  $\frac{\nu_\beta(B(Tx,r))}{\nu_\beta(B(x,r))}$  is constant,  
 240 which happened when the conformal measure was Lebesgue.  
 241 This difficulty could be partially overcome by supposing that  
 242 the potential is constant, otherwise we could bound  $q_0$  from  
 243 above and below with (close) approximations of the poten-  
 244 tial. By assuming that the latter is constant and equal to  $\bar{\phi}$   
 245 and also that the density  $h_\beta$  does not vary too much, we get  
 246 that  $q_0$  is of order  $e^{\bar{\phi}}$  and therefore

$$\theta \sim \frac{1 - \lambda_\beta^{-2} e^{\bar{\phi}}}{\lambda_\beta^2}. \tag{2.5}$$

247 It is worth mentioning that whenever the conformal measure  
 248 is Lebesgue ( $\beta = 1$ ), the above computation can be made rig-  
 249 orous as in Proposition (5.3) in Ref. 17 and it gives

$$\theta = 1 - \frac{\int_M \frac{h^2(x)}{|T'(x)|} dx}{\int_M h^2(x) dx}, \tag{2.6}$$

250 where  $h$  is the density of the invariant measure: we defer to  
 251 our paper<sup>17</sup> for the assumptions on the system which permit  
 252 to get such a result. In particular, those systems contain con-  
 253 formal repellers with finitely many branches and absolutely  
 254 continuous conformal measures. Notice that by introducing  
 255 the invariant measure  $\mu = h dm$ , we could identically write

$$\theta = 1 - \frac{\int_M h(x) e^{-\log |T'(x)|} d\mu(x)}{\int_M h(x) d\mu(x)}. \tag{2.7}$$

256 If the derivative does not change too much, we get  
 257  $\theta \sim 1 - e^{-\Lambda_\mu}$ , where  $\Lambda_\mu$  is the positive Lyapunov exponent  
 258 of the measure  $\mu$ . Alternatively, if the density  $h$  could be  
 259 considered constant, we can bound (2.7) by Jensen's inequal-  
 260 ity as

$$\theta \sim 1 - \int_M \frac{1}{|T'(x)|} d\mu(x) \leq 1 - e^{-\int_M \log |T'(x)| d\mu(x)} = 1 - e^{-\Lambda_\mu}.$$

261 In both cases, the DEI  $\theta$  is related to the positive Lyapunov  
 262 exponent: this analogy will be pursued in Sec. II C. AQ6

**C. Attractors and high dimensional systems** 263

264 For invertible maps generating attractors endowed with  
 265 the SRB measure, the computation of the dynamical  
 266 extremal index is less straightforward; we should stress that  
 267 a spectral theory of extreme value for (invertible) uniformly  
 268 hyperbolic maps is still missing. Suppose we take an hyper-  
 269 bolic diffeomorphisms  $T$  preserving the ergodic SRB mea-  
 270 sure  $\mathcal{L}$ . Then, the quantity  $q_0$  in (2.4) becomes

$$q_0 = \lim_{n \rightarrow \infty} \frac{\int d\mathcal{L}(x) \int \mathbf{1}_{S_n}(x, T^{-1}y) \mathbf{1}_{S_n}(Tx, y) d\mathcal{L}(y)}{\int d\mathcal{L}(x) \int \mathbf{1}_{S_n}(x, y) d\mathcal{L}(y)}. \quad (2.8)$$

271 When we iterate backward the points  $y \in B(Tx, e^{-u_n})$ , we  
 272 should keep only those points whose preimage is at a distance  
 273 at most  $e^{-u_n}$  from  $x$ . Those preimages form a set  $Q(x)$   
 274 which is obtained by squeezing the ball  $B(Tx, e^{-u_n})$  along the  
 275 unstable manifolds. Let us suppose that the tangent expanding  
 276 subspace  $\Sigma_u(Tx)$  at  $x$  has dimension  $d$ . Then the measure  
 277 of  $Q(x)$ , and therefore, by the forward invariance of the  
 278 measure, of its image in  $B(Tx, e^{-u_n})$  will be of order  
 279  $|\det(DT(x)|_u)|^{-1} \mathcal{L}(B(Tx, e^{-u_n}))$ , where  $DT(x)|_u$  is the deriv-  
 280 ative of  $T$  restricted to  $\Sigma_u(x)$ . We remember in fact that the  
 281 conditional SRB measure on the unstable manifolds is  
 282 smooth. This immediately gives  $q_0$  of order

$$q_0 \sim \frac{\int d\mathcal{L}(x) |\det(DT(x)|_u)|^{-1} \mathcal{L}(B(Tx, e^{-u_n}))}{\int d\mathcal{L}(x) \mathcal{L}(B(x, e^{-u_n}))}. \quad (2.9)$$

283 We see that  $q_0$  contains information about the dimension  
 284 through the scaling of the denominator; we are now inter-  
 285 ested in the contribution of the other term in the numerator.  
 286 In this regard, we first remind that, for SRB measures, we  
 287 can use the Pesin's formula<sup>25</sup>

$$\int d\mathcal{L}(x) |\det(DT(x)|_u)| = \sum_{j=1}^d \Lambda_j^+ = h_{\mathcal{L}},$$

288 where  $\Lambda_j^+$  is the positive Lyapunov exponents with multi-  
 289 plicity one, and  $h_{\mathcal{L}}$  is the metricentropy of the SRB measure.  
 290 We now proceed under two assumptions as we did at the end  
 291 of Sec. II B. Let us first assume that the derivative along the  
 292 unstable subspaces does not vary too much. Then, we could  
 293 estimate the DEI as

$$\theta \sim 1 - e^{-h_{\mathcal{L}}}. \quad (2.10)$$

294 For  $d = 1$ , we can replace the entropy with the (unique) posi-  
 295 tive Lyapunov exponent  $\Lambda_{\mathcal{L}}$ ; in the following, we will simply  
 296 write it as  $\Lambda_+$ .

297 The other assumption exploits the fact that for these  
 298 system, and for  $\mathcal{L}$ -almost all points  $x$  we have, by Young's  
 299 theorem,<sup>26</sup> that  $\lim_{r \rightarrow 0} \frac{\log \mathcal{L}(B(x,r))}{\log r} = D_1$ , where  $D_1$  is the  
 300 information dimension. Hence, we could guess that  
 301  $\mathcal{L}(B(x, e^{-u_n})) \sim e^{-u_n D_1}$  and therefore forget about the depen-  
 302 dence on the variable  $x$ . This is generally false since the multi-  
 303 plicative factor in the previous scaling could depend on  $x$ .  
 304 Indeed, when we integrate  $\mathcal{L}(B(x, e^{-u_n}))$ , we get  $D_2$  which  
 305 could be different from  $D_1$ . If we suppose that the depen-  
 306 dence on  $x$  of the prefactors is negligible, which means that  
 307 we are considering a homogenous fractal invariant set with  
 308  $D_1 \sim D_2$ , then we have for the DEI

$$\begin{aligned} \theta &\sim 1 - \int d\mathcal{L} |\det(DT(x)|_u)|^{-1} \leq 1 - e^{-\int d\mathcal{L}(x) |\det(DT(x)|_u)|} \\ &= 1 - e^{-h_{\mathcal{L}}}, \end{aligned} \quad (2.11)$$

where the derivative is *not* supposed to be constant and  
 where we have used again the Jensen's inequality to estab-  
 lish the upper bound.

Those two approximations are very crude; we are in fact  
 either neglecting the contributions of the prefactors in the  
 local scaling of the balls in (2.9), or not taking into account  
 the geometric factors when the ball  $B(Tx, e^{-u_n})$  is squeezed  
 at a distance  $e^{-u_n}$  from  $x$ . Moreover, the variation of the  
 derivative, especially sensible in the non-uniformly hyper-  
 bolic setting, could give large differences in the determina-  
 tion of the DEI, as we experience for instance for the Hénon  
 map, see below. The preceding relation is pretty well satisfac-  
 ed for maps with one-dimensional unstable subspace and  
 (piecewise) constant jacobian, like the Baker transformation,  
 the Lozi map, and the solenoid. For the algebraic automor-  
 phism of the torus (cat's map), a simple argument allows us  
 to improve the previous rate just by taking into account the  
 geometric factors. Surprisingly, relation (2.11) is pretty well  
 satisfied in the example below of the generalized Hénon  
 maps, where the unstable subspace has dimension larger  
 than one, i.e., we have more than one positive Lyapunov  
 exponent. In conclusion, our index  $\theta$  traces in a satisfactory  
 way the entropy. The eventual deviations are due to the vari-  
 ation of the derivative and the local scaling of balls in (2.9).  
 Although these effects are difficult to compute analytically,  
 the DEI  $\theta$  is relatively easy to compute numerically and it  
 furnishes a new indicator for the local instability in chaotic  
 systems.

### III. NUMERICAL COMPUTATIONS

The numerical computations presented in the remaining  
 of this work are performed by using the numerical algo-  
 rithms and codes detailed in the [supplementary material](#). The  
 stability of the results is checked against different  $l, n, m, s$ . In  
 particular, we perform two sets of simulations. The first set  
 of accurate simulations consist of  $l = 100$  trajectories, with  
 $n = 10^6$  iterations,  $m = 10^3$  blocks of  $s = 10^3$  length each.  
 The second set of  $l = 100$  simulations consists of short series  
 of  $n = 10^4$  iterations, with  $s = m = 10^2$ . This second set is  
 useful to check whether the technique is reliable also for  
 short time series. Except where specified, we use  $\tilde{s} = 0.99$   
 for the following computations. However, results are stable  
 when considering different quantiles ranging from  $0.97 < \tilde{s} < 0.999$ .

#### A. Low dimensional maps

We begin the numerical computations with several  
 examples on low dimensional maps. A summary of the  
 results for all maps analysed is reported in Table I. For a few  
 maps, we report the model equations in the [supplementary material](#) to streamline the exposition.

- Let us begin with the Bernoulli Shift map  $T(x) = 3x \text{-mod } 1$ . For this system,  $D_2 = 1$  and  $\theta = 1 - 1/3 = 2/3$ . The numerical estimates (Table I) are coherent with the theoretical values for both accurate and short simulations.
- We now consider the Gauss map  $T(x) = \frac{1}{x} \text{-mod } 1$  defined on the unit interval. Although, strictly speaking, this map

TABLE I. Estimates of correlation dimension  $D_2$  and dynamical extremal index (DEI)  $\theta$  obtained with  $l = 100$  trajectories, consisting of  $n = 10^6$  iterations or  $n = 10^4$  iterations. The maxima of  $\psi(x, y)$  are extracted in the block of  $s = 10^3$  and  $s = 10^2$  length, for a total of  $m = 10^3$  or  $m = 10^2$  blocks. The quantile for the estimate of the DEI is  $\tilde{s} = 0.99$ . For the Arnold Cat's map, the convergence to theoretical value is lower and the estimates are provided only for  $\tilde{s} = 0.99999$  and  $n = 10^7$ .

Map	$D_2$ (classical)	$D_2 (n = 10^6)$	$D_2 (n = 10^4)$	$\theta$ (from Lyapunov)	$\theta (n = 10^6)$	$\theta (n = 10^4)$
Bernoulli's shifts	1	$1.00 \pm 0.02$	$1.01 \pm 0.14$	0.667	$0.668 \pm 0.004$	$0.69 \pm 0.04$
Gauss map	1	$1.00 \pm 0.03$	$0.96 \pm 0.16$	0.773	$0.773 \pm 0.005$	$0.78 \pm 0.04$
Cantor IFS	0.667	$0.64 \pm 0.01$	$0.59 \pm 0.13$	0.5	$0.502 \pm 0.005$	$0.50 \pm 0.05$
Baker map	1.41	$1.46 \pm 0.02$	$1.42 \pm 0.25$	0.47	$0.49 \pm 0.02$	$0.50 \pm 0.04$
Lozi map	1.38	$1.39 \pm 0.11$	$1.29 \pm 0.25$	0.37	$0.37 \pm 0.01$	$0.37 \pm 0.05$
Henon map	1.22	$1.24 \pm 0.03$	$1.13 \pm 0.25$	0.34	$0.43 \pm 0.01$	$0.43 \pm 0.06$
Solenoid $a = 1/3$	1.6309	$1.64 \pm 0.04$	$1.55 \pm 0.17$	0.5	$0.51 \pm 0.01$	$0.59 \pm 0.03$
Solenoid $a = 1/4$	1.5	$1.52 \pm 0.03$	$1.57 \pm 0.20$	0.5	$0.51 \pm 0.01$	$0.53 \pm 0.03$
Arnold Cat's map	1.987	$2.00 \pm 0.06$	...	0.51	$0.53 \pm 0.06$	...

364 does not fit the assumptions in Ref. 17 since in the latter  
 365 paper, we consider maps with finitely many branches, we  
 366 still try formula (2.6). For the Gauss map, the density is  
 367 explicit and reads  $h(x) = \frac{1}{\log 2} \frac{1}{1+x}$ . The integral in (2.6) can  
 368 be easily computed and gives  $\theta = 4 \log(2) - 2 \sim 0.77$ ,  
 369 whereas  $D_2$  is expected to be 1. The numerical estimates  
 370 are coherent with the theoretical values (Table I).

371 • Returning to a map with constant slope 3, we now look at  
 372 the transformation generating the classical ternary Cantor  
 373 set. In order to compute numerically the GEV function,  
 374 one should access the invariant Cantor set, which is of  
 375 zero Lebesgue measure. We need therefore to use the  
 376 backward iterates of the map (otherwise almost all the for-  
 377 ward orbits will fall into the holes), and the measures  
 378 allowing us to compute the time averages are the so-called  
 379 *balanced measures*, given suitable weights to the pre-  
 380 images of the map: see our article, Ref. 27 Sec. 3.2.2 for a  
 381 description of such measures. For the ternary Cantor set  
 382 and choosing equal weights 1/2 for the two preimages, it  
 383 is easy to check that such a balanced measure coincides  
 384 with the Gibbs measure with  $\beta = \log 2 / \log 3$  which is the  
 385 Hausdorff dimension of the invariant set. The measure  $\mu_{d_H}$   
 386 is called *uniform*, see Ref. 20, Sec. 3. The potential  $\varphi$  will  
 387 be equal to  $-\log 2$  and  $\lambda = 1$ , since by Bowen's formula  
 388  $Q(d_H) = 0$ . Therefore, for the ternary Cantor set, we get a  
 389 DEI equal to 0.5 which is perfectly confirmed by the  
 390 numerical simulations (Table I).

391 • For the Lozi map:  $x_{n+1} = a|x_n| + y_n + 1$ ,  $y_{n+1} = bx_n$ ,  
 392  $a = 1.7$ ,  $b = 0.5$ ,  $\Lambda_+$  is of order 0, 47,<sup>28</sup> which gives, with  
 393 our approximation, a DEI of order  $\theta = 0.37$ . Previous  
 394 numerical computations for  $D_2$  gave  $D_2 \sim 1.38$ .<sup>29</sup> Our  
 395 computations (Table I) are coherent with the theoretical  
 396 values.

397 • For the Hénon map  $x_{n+1} = ax_n^2 + y_n + 1$ ,  $y_{n+1} = bx_n$ ,  
 398  $a = 1.4$ ,  $b = 0.3$ ,  $\Lambda_+$  is of order 0, 42,<sup>28</sup> which gives, with  
 399 our approximation, a DEI of order  $\theta = 0.34$ . Previous  
 400 numerical computations for  $D_2$  gave  $D_2 \sim 1.22$ .<sup>29</sup> The  
 401 GEV computations give  $D_2 = 1.24 \pm 0.11$  but  $\theta = 0.43$   
 402  $\pm 0.01$  for  $n = 10^6$  (See Table I for the results with  
 403  $n = 10^4$  iterations). The discrepancy of the DEI estimate  
 404 does not get any better with the increase of  $\tilde{s}$  or  $n$ . As said  
 405 before, we do not expect  $\theta$  to coincide with the estimate  
 406 0.34 due to the variation of the derivative and the non-  
 407 uniform hyperbolicity of the map.

• Let us consider the cat's map with the associated matrix 408

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \text{ The stable and unstable manifolds for such a } 409$$

map are orthogonal, so we could suppose that the pre- 410  
 image of the ball  $B(Tx, e^{-u_n})$  will intersect the ball 411  
 $B(x, e^{-u_n})$  in a rectangle  $R(x)$  centered at  $x$  and with the 412  
 shortest side of length  $(\lambda_+)^{-1} e^{-u_n}$ , where  $\lambda_+ = \frac{3+\sqrt{5}}{2}$  is the 413  
 eigenvalue larger than 1 corresponding to the unstable 414  
 direction. An elementary calculation shows immediately 415  
 that  $q_0 \sim \mathcal{L}(R(x)) / \mathcal{L}(B(x, e^{-u_n}))$  is approximately given by 416  
 $\frac{4}{\pi} (\lambda_+)^{-1}$  which gives an extremal index as 0.51. Previous 417  
 numerical computations for  $D_2$  gave  $D_2 \sim 1.987$ .<sup>29</sup> The 418  
 numerical computation with the GEV fitting gives 419  
 $D_2 = 2.00 \pm 0.06$  and  $\theta = 0.552 \pm 0.005$  for  $n = 10^6$ . 420  
 In order to investigate the discrepancy with our theoretical 421  
 estimate, we raised the quantile from  $\tilde{s} = 0.99$  to 422  
 $\tilde{s} = 0.999$ , i.e., we select more extreme clusters. The esti- 423  
 mates for this case are  $\theta = 0.54 \pm 0.02$ , more compatible 424  
 with the theoretical one. Finally, if we consider longer tra- 425  
 jectories ( $n = 10^7$  iterates) with an even higher quantile 426  
 ( $\tilde{s} = 0.9999$ ), we get  $\theta = 0.53 \pm 0.06$ , which is even closer 427  
 to the theoretical guess. 428

429 • We now consider the baker's map (see supplementary 429  
 material); it depends on three parameters  $\alpha$ ,  $\gamma_a$ , and  $\gamma_b$ . The 430  
 positive Lyapunov exponent is given by Ref. 20, Eq. 431  
 (5.14) 432

$$\Lambda_+ = \alpha \log \frac{1}{\alpha} + (1 - \alpha) \log \frac{1}{1 - \alpha}.$$

With the value  $\alpha = 1/3$ ,  $\gamma_a = 1/5$ ,  $\gamma_b = 1/4$ , we get  $\Lambda_+ \sim 0$ , 433  
 64 which gives, with our approximation, an extremal 434  
 index of order 0, 47. In the paper, Ref. 20 Eq. (5.18), we 435  
 gave an implicit formula expressing  $D_2$  as a function of  $\alpha$  436  
 and with respect to the SRB measure. For  $\alpha = 1/3$ , this 437  
 estimate reads  $D_2 \simeq 1.41$ . The GEV estimates are given in 438  
 Table I and are consistent with the theory. 439

440 • We next consider an attractor embedded in  $\mathbb{R}^3$ , the so- 440  
 called solenoid, see supplementary material; it depends 441  
 upon the parameter  $a \in (0, 0.5)$ . The attractor is foliated 442  
 by one-dimensional unstable manifolds, while each merid- 443  
 ional disk is a two-dimensional stable manifold each of 444  
 which intersecting the attractor over a Cantor set. The 445  
 Lyapunov exponents are 446

$$\Lambda_- = \log a < 0, \quad \Lambda_+ = \log 2,$$

447 while the Hausdorff dimension  $d_H$  is given by the formula<sup>30</sup>

$$d_H = 1 + \frac{\log 2}{-\log a}.$$

448 The numerical computations for the solenoid provide a fur-  
449 ther test of the validity of the numerical algorithm and are  
450 provided in Table I.

451 **B. High dimensional generalized Hénon maps**

452 We now analyze the generalized Hénon maps defined in  
453 Ref. 31 and further analyzed in Ref. 32. They are defined as

$$x_{n+1}(1) = ax_n(d-1)^2 - bx_n(d) \quad x_{n+1}(i) = x_n(i-1). \quad (3.1)$$

454 When the parameter  $a=1.76$ , the number of positive  
455 Lyapunov exponents is  $d-1$ ; we could therefore test our  
456 relation (2.11) by computing the entropy  $h_{\mathcal{L}}$  as the sum of  
457 positive Lyapunov exponents (see Table II in Ref. 32) for a  
458 given  $d$ . We also perform the computation of the dimension  
459  $D_2$  and compare it to the Kaplan-Yorke dimension  $D_{KY}$  given  
460 in Ref. 32; we used such a dimension because we did not  
461 find an explicit computation of  $D_2$  in the literature. The good  
462 agreement between our numerical results (Fig. 1) confirm  
463 the validity of Eq. (2.10) with the caveat that an exact corre-  
464 spondence cannot be derived for the geometric factor that  
465 stretch balls in phase space in different dimensions: the ori-  
466 gin of this discrepancy has been discussed in detail at the  
467 end of Sec. IIC.

468 **C. Application to atmospheric data**

469 We now consider an application to atmospheric data.  
470 The purpose of this application is to show that the applicabil-  
471 ity of the technique on real data provides results that have a

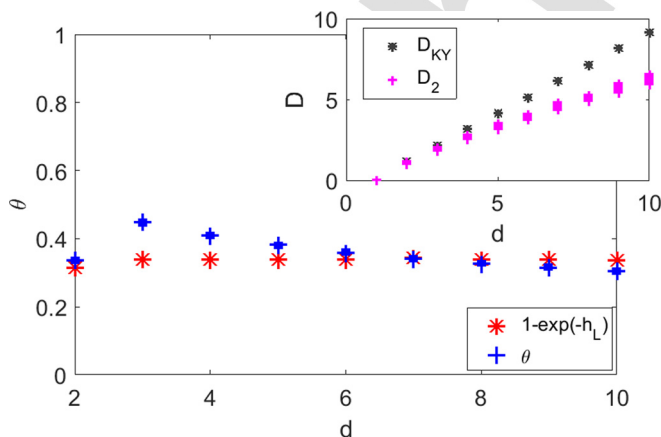


FIG. 1. Estimates of the dynamical extremal index  $\theta$  and correlation dimension  $D_2$  (inset) obtained for the Generalized Henon maps [Eq. (3.1)] in different dimensions  $d$ . The values represent the estimates obtained taking 30 couples of trajectories, iterated for  $n=10^6$  iterations. Each couple is displayed using a single marker, but the uncertainty is so small that the difference between couples is hardly recognizable. The quantile used for the estimation is  $\bar{s} = 0.98$ . The results are compared to those obtained using the Kaplan-Yorke dimension  $D_{KY}$  and the entropy  $h_{\mathcal{L}}$ . This map has  $d-1$  positive Lyapunov exponents.

coherent interpretation in terms of the underlying physics of 472  
the systems. In order to provide evidence of the robustness 473  
of our results, we will study several trajectories of a climate 474  
models which incorporate observations of the past 110 years, 475  
and compute  $\theta$  and  $D_2$  for several sub-periods showing that 476  
the results are numerically stable. We study the atmospheric 477  
circulation over the North Atlantic and focus on a single field 478  
that represents its major features: the sea-level pressure 479  
(SLP).<sup>33,34</sup> Indeed, it has been shown that SLP fields can be 480  
used to study teleconnection patterns as well as storm track 481  
activity and atmospheric blocking.<sup>35,36</sup> The trajectories of 482  
our dynamical systems are successions of SLP fields 483  
extracted with daily frequency from the ERA-20 CM reanal- 484  
ysis project over the period 1900–2010.<sup>37</sup> The ERA-20 CM 485  
consists of 10 members ensemble of a (climate) model 486  
whose task is to reconstruct at best the 1900–2010 atmo- 487  
spheric dynamics by constraining the model to include the 488  
information from available surface observations. Each mem- 489  
ber of the ERA 20 CM is therefore a slightly perturbed 490  
reconstruction of the atmospheric dynamics in the past 491  
110 years. The choice of the North Atlantic domain ( $80^\circ$  W 492  
 $\leq$  Long.  $\leq 50^\circ$  E,  $22.5^\circ$  N  $\leq$  Lat.  $\leq 70^\circ$  N) is motivated by 493  
the better observational coverage over the region in the first 494  
part of the analysis period compared to other regions of the 495  
globe.<sup>38</sup> Before presenting the results for  $D_2$  and  $\theta$ , we would 496  
like to stress that (i) our analysis will only be representative 497  
of the North-Atlantic domain and  $D_2$  will be a proxy of the 498  
active degrees of freedom of the atmospheric circulation in 499  
this area. Therefore, our results cannot be used to estimate 500  
the dimension of the full atmospheric climate attractor. (ii) 501  
Previous results<sup>15,39,40</sup> have shown that the estimates 502  
obtained for the daily dimensions are robust with respect to 503  
the changes in the datasets, resolution of the climate models, 504  
and are linearly insensitive to the size of the domain. This 505  
gives us confidence on the applicability of the numerical 506  
algorithm described in this paper for climate data since it is 507  
largely based on those used in Refs. 15, 39, and 40. 508

The results for  $D_2$  and  $\theta$  on the SLP fields of the ERA- 509  
20 CM ensemble are presented in Fig. 2. For each estimate, 510  
we fix the reference trajectory  $x$  as the first member (M1) of 511  
the ERA-20 CM ensemble because this is always considered 512  
as the reference simulation, while  $y$  is alternatively set as the 513  
 $M$ th member with  $i=2, 3, \dots, 10$ . The dependence of the 514  
results on the reference member are tested in the [supplemen-](#) 515  
[tary material](#) Fig. S1. To test the robustness of the results, we 516  
provide four estimates of  $D_2$  and  $\theta$ : (i) using the full data in 517  
the period 1900–2010, (ii) using 1900–1955 data, (iii) using 518  
1900–1928 data, and (iv) considering only the first 14 years 519  
(1900–1914) of data. For each member, the results are 520  
reported in Fig. 2. The ensemble averages of  $D_2$  and  $\theta$  for 521  
the different periods are instead reported in Table II. 522  
Estimates are consistent for different periods and the value 523  
of  $D_2 \simeq 9$  found on average, is slightly lower than the 524  
estimates of  $d_H$  found in Ref. 15 (we remind that  $D_2 < d_H$ ). 525  
The value of  $D_2$  roughly corresponds to the number of spatial 526  
degrees of freedom active in a North-Atlantic SLP field as 527  
explained in Ref. 15. Indeed, the domain used for this analy- 528  
sis can host about 9 large spatial structures reparted between 529  
3 and 4 extratropical cyclones at time and the same number 530



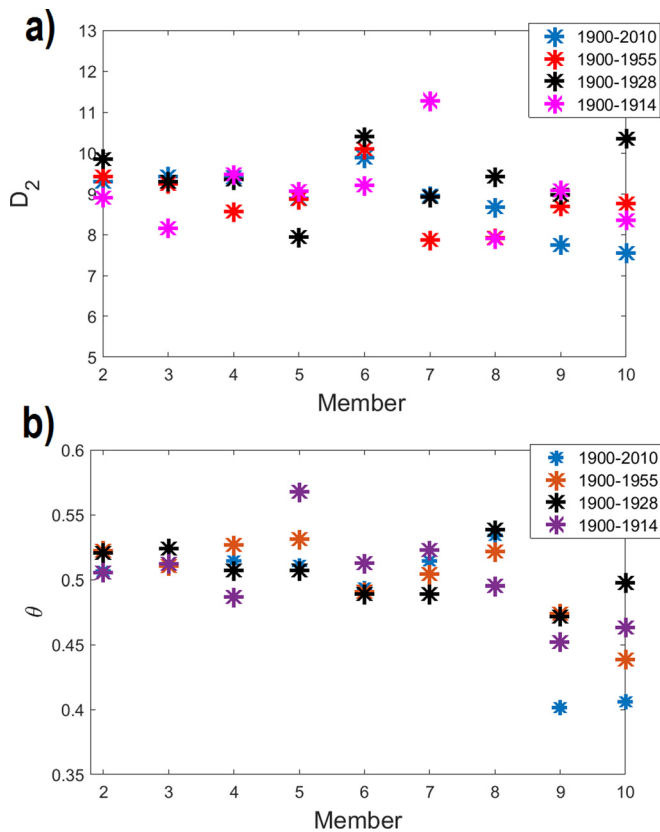


FIG. 2. Estimates of correlation dimension  $D_2$  (a) and extremal index  $\theta$  (b) obtained for daily sea-level pressure maps for four different periods in the ERA-20CM reanalysis. The values represent the estimates obtained taking as reference trajectory  $x$  the member M1 and as  $y$ , the remaining 9 ensemble members.

531 of anticyclones (see the textbook of Holton,<sup>41</sup> for estimates  
 532 of the typical size of these objects).  $\theta$  is, in fact, the inverse  
 533 of the average time the two trajectories  $x$  and  $y$  cluster  
 534 together. The value of the DEI  $\theta = 0.5$  corresponds therefore  
 535 to a contraction of the phase space associated with a time-  
 536 scale between 2 and 3 days. This is the typical decay rate of  
 537 baroclinic eddies associated with the low pressure systems  
 538 observed in SLP fields (see again the textbook by Holton<sup>41</sup>  
 539 for the decay rates). We finally notice that our formula (2.11)  
 540 gives for the entropy the value  $\log 2$ . In Fig. S2, we show a  
 541 moving window computation of  $D_2$  and  $\theta$ . No clear trend  
 542 emerges that could be attributed to anthropogenic forcing.  
 543 This result is consistent with those found for  $d_H$  in Ref. 39.  
 544 We remark however some differences in the variability of

TABLE II. Estimates of correlation dimension  $D_2$  and extremal index  $\theta$  obtained for daily sea-level pressure maps for four different periods of the ERA-20CM reanalysis. The values represent average over the 9 ensemble members and uncertainty is expressed as the standard deviation of the ensemble mean.

Period	$D_2$	$\theta$
1900-2010	$8.9 \pm 0.8$	$0.48 \pm 0.05$
1900-1955	$8.8 \pm 0.7$	$0.50 \pm 0.03$
1900-1928	$9.4 \pm 0.8$	$0.50 \pm 0.02$
1900-1914	$9.0 \pm 1.0$	$0.50 \pm 0.03$

the indicators among the members. In particular, M9 and M10 have a minimum of  $\theta$  around 1960. This could be due to the different boundary conditions applied to the members and detailed in Ref. 37.

**D. Additive noise**

In our previous papers,<sup>17,42,43</sup> we have analyzed the effect of additive noise on the parameters of the extreme value laws. It consists in defining a family of maps  $T_\xi = T + \varepsilon\xi$  with  $\xi$  a random variable sampled from some distribution  $\mathbb{G}$  (we will take here the uniform distribution on some small ball of radius  $\varepsilon$  around 0). The iteration of the single map  $T$  will be now replaced by the concatenation  $T_{\xi_n} \circ T_{\xi_{n-1}} \cdots \circ T_{\xi_1}$  and the evaluation of an observable computed along this orbit will be given by the probability measure  $\mathbb{P}$  which is the product of  $\mathbb{G}^N$  with the so-called stationary measure  $\mu_S$ , verifying, for any real measurable bounded function  $f$ :  $\int f d\mu_S = \int f \circ T_\xi d\mu_S$ ; see Ref. 19 Chap. 7, for a general introduction to the matter. In the aforementioned papers, Refs. 42 and 43, we have shown analytically that for dynamical systems perturbed additively, the extremal index  $\theta = 1$ , no matter what the intensity of the noise is. The proof was supported by numerical experiments, using also different noise types. The extremal index is a parameter that quantifies the amount of clustering, the stickiness of the trajectory in phase space. In our setting, clustering happens in the presence of invariant sets, which are periodic points in Ref. 42. By looking at formula (2.4), we see that we estimate the proportion of the neighborhood of the invariant set returning to itself; as we argued above, that estimate gives information on the rate of backward volume contraction in the unstable direction. Since the noise generally destructs these invariants sets, we expect the extremal index be equal to 1 or quickly approaching 1 when the noise increases. This is confirmed by the numerical experiments reported in Fig. 3 where the value of  $\theta$  is plotted against the intensity of the noise  $\varepsilon$  for three maps:  $3x \bmod 1$  map, the Baker map, and the Lozi map. In all cases, indeed  $\theta \rightarrow 1$  for large enough noise. However, with respect to the observables discussed in Ref. 42, we find some remarkable differences on the intensity of the noise needed to observe changes of the extremal index from the deterministic values: whereas in Ref. 42, we observed significant deviation from the deterministic behavior for very small noise intensities ( $\varepsilon \geq 10^{-4}$ ), here we need  $\varepsilon \geq 10^{-2}$ , i.e., only large noise amplitudes perturb the estimates of  $D_2$  and  $\theta$ . This difference can be easily explained: in Ref. 42, the extremal index was used to explore the local stability at periodic fixed points, where the dynamics is deeply affected even by a small noise. Here, instead, the extremal index tracks a global property that it is stable with respect to small stochastic perturbations. We underline that, for the Lozi map, we cannot obtain estimates of  $\theta$  for noise larger than 0.1 because the dynamics fall out the basin of attraction.

**IV. DISCUSSION AND CONCLUSIONS**

Using the extreme value theory, we have introduced a new and efficient way to compute the correlation dimension

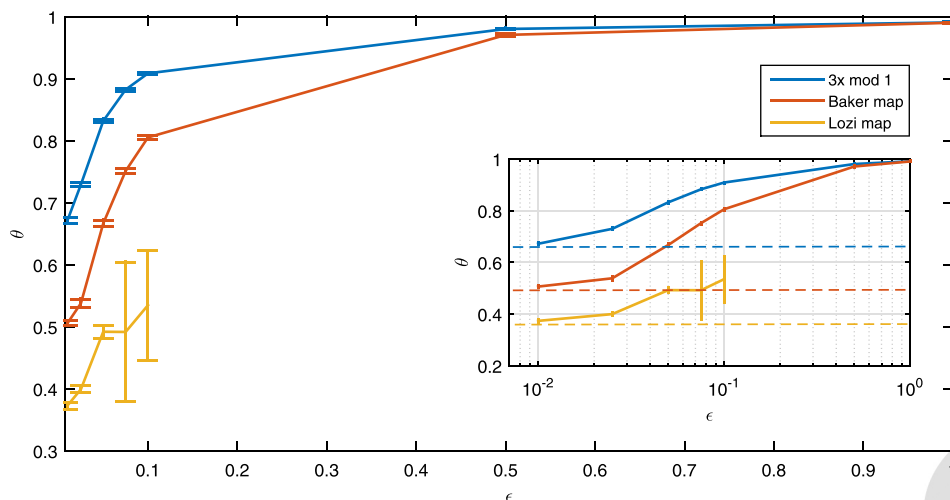


FIG. 3. Dynamical extremal index  $\theta$  vs intensity of the additive noise  $\varepsilon$  for three different maps:  $3 \times \text{mod } 1$  (blue), Baker map (red), and Lozi map (orange). The error bar indicates the standard deviation of the sample of  $l = 100$  trajectories, each consisting of  $n = 10^6$  iterations. The quantile for the estimate of the extremal index is  $\hat{s} = 0.99$ . The inset shows the same data in the semilog scale, with the deterministic values represented by the dotted lines.

601  $D_2$ . Moreover, for higher dimensional maps, we introduced  
 602 the quantity  $q_0$ , related to the expectation of the inverse of  
 603 the determinant of the derivative along the expanding sub-  
 604 space. Therefore, the extremal index  $\theta = 1 - q_0$  is a *measure*  
 605 *of the average rate of phase space contraction for backward*  
 606 *iteration*. Although this quantity slightly differs from the  
 607 entropy or from the positive Lyapunov exponent when the  
 608 expanding subspace has dimension one, it provides an  
 609 important piece of information on the dynamics of the sys-  
 610 tem. In fact it can be linked to the global predictability and  
 611 therefore *considered as a new indicator of the local instabil-*  
 612 *ity in chaotic systems*. We would like also to emphasize that  
 613 both  $\theta$  and  $D_2$  can be computed simultaneously just by look-  
 614 ing at the GEV function and this makes our method quite  
 615 rapid and economically efficient from a numerical point of  
 616 view. We have shown that even for a short time series of  
 617 only of  $10^4$  iterations, the estimates are robust and consistent  
 618 with the theoretical expectations. We have also presented a  
 619 first application of these indicators to climate data proving  
 620 that the indicators are useful to infer the spatial number of  
 621 degrees of freedom and the typical time scales of the atm-  
 622 ospheric dynamics on the North Atlantic region. Finally, we  
 623 have observed their sensitivity to the different boundary con-  
 624 ditions imposed for the climate simulations analyzed. This  
 625 implies that the indicators could be useful in characterizing  
 626 and comparing also different climate datasets as those ana-  
 627 lyzed in international campaigns.

628 Our interpretation of  $\theta$  together with that on the correla-  
 629 tion dimension  $D_2$  could be useful also to analyze the times  
 630 series arising from the evolution of chaotic systems. Indeed,  
 631 these quantities are particularly straightforward to obtain  
 632 from numerical computations. Moreover, the results obtained  
 633 can also be used to detect the embedding dimension, namely  
 634 by replacing the sample of data with delay vectors of vari-  
 635 able lengths; we stress that computing the GEV with those  
 636 delay vectors will allow us to get exactly the embedding  
 637 dimension. We mean to develop further this approach in a  
 638 future paper.

639 Finally, the computation of the DEI could be helpful to  
 640 distinguish purely stochastic sequences for which the  
 641 extremal index should approach 1, see Sec. III D, from  
 642 dynamical systems with an underlying chaotic behavior even

in the presence of small stochastic perturbations. Again, 643  
 these further applications of our approach with EVT will be 644  
 the objects of forthcoming investigations. 645

## SUPPLEMENTARY MATERIAL 646

See [supplementary material](#) for: (i) the algorithm for the 647  
 estimation of the correlation dimension  $D_2$  and the 648  
 Dynamical Extremal Index (DEI)  $\theta$ , (ii) a commented 649  
 numerical MATLAB code for such estimation, (iii) the 650  
 model equations for the maps used, and (iv) the supplement- 651  
 ary figures. 652

## ACKNOWLEDGMENTS 653

S.V. was supported by the MATH AM-Sud Project 655  
 Physeco, by the Leverhulme Trust thorough the Network 656  
 Grant No. IN-2014-021, and by the project APEX Systèmes 657  
 dynamiques: Probabilités et Approximation Diophantienne 658  
 PAD funded by the Région PACA (France). D.F. was 659  
 partially supported by the ERC Grant A2C2 (No. 338965). 660  
 The authors warmly thank the referee whose comments and 661  
 advices helped them to improve the paper. 662

<sup>1</sup>P. Grassberger and I. Procaccia, *The Theory of Chaotic Attractors* 664  
 (Springer, 2004), pp. 170–189. 665

<sup>2</sup>P. Grassberger and I. Procaccia, *Phys. Rev. Lett.* **50**, 346 (1983). 666

<sup>3</sup>A. Wolf, J. B. Swift, H. L. Swinney, and J. A. Vastano, *Physica D* **16**, 285 667  
 (1985). 668

<sup>4</sup>M. T. Rosenstein, J. J. Collins, and C. J. De Luca, *Physica D* **65**, 117 669  
 (1993). 670

<sup>5</sup>J. Theiler, S. Eubank, A. Longtin, B. Galdrikian, and J. D. Farmer, *Physica* 671  
*D* **58**, 77 (1992). 672

<sup>6</sup>J.-P. Eckmann and D. Ruelle, *Physica D* **56**, 185 (1992). 673

<sup>7</sup>H. Kantz and T. Schreiber, *Nonlinear Time Series Analysis* (Cambridge 674  
 University Press, 2004), Vol. 7. 675

<sup>8</sup>A. Pikovsky and A. Politi, *Lyapunov Exponents: A Tool to Explore* 676  
*Complex Dynamics* (Cambridge University Press, 2016). 677

<sup>9</sup>H. Kantz, G. Radons, and H. Yang, *J. Phys. A: Math. Theor.* **46**, 254009 678  
 (2013). 679

<sup>10</sup>A. Politi, *Phys. Rev. Lett.* **118**, 144101 (2017). 680

<sup>11</sup>A. C. M. Freitas, J. M. Freitas, and M. Todd, *Probab. Theory Relat. Fields* 681  
**147**, 675 (2010). 682

<sup>12</sup>D. Faranda, V. Lucarini, G. Turchetti, and S. Vaienti, *J. Stat. Phys.* **145**, 683  
 1156 (2011). 684

<sup>13</sup>D. Faranda and S. Vaienti, *Geophys. Res. Lett.* **40**, 5782, <https://doi.org/10.1002/2013GL057811> (2013). 685  
 686

- 687 <sup>14</sup>D. Faranda, M. C. Alvarez-Castro, and P. Yiou, *Clim. Dyn.* **47**, 3803  
688 (2016). 710
- 689 <sup>15</sup>D. Faranda, G. Messori, and P. Yiou, *Sci. Rep.* **7**, 41278 (2017). 711
- AQ8 690 <sup>16</sup>F. M. E. Pons, G. Messori, M. C. Alvarez-Castro, and D. Faranda, preprint  
691 [arXiv:hal-01650250](https://arxiv.org/abs/hal-01650250) (2017). 712
- AQ9 692 <sup>17</sup>D. Faranda, H. Ghoudi, P. Guiraud, and S. Vaienti, “■,” Nonlinearity (to  
693 be published); , preprint [arXiv:1708.00191](https://arxiv.org/abs/1708.00191). 713
- 694 <sup>18</sup>Y. B. Pesin, *Dimension Theory in Dynamical Systems: Contemporary*  
695 *Views and Applications* (University of Chicago Press, 2008). 714
- 696 <sup>19</sup>V. Lucarini, D. Faranda, J. M. Freitas, M. Holland, T. Kuna, M. Nicol, M.  
697 Todd, S. Vaienti *et al.*, *Extremes and Recurrence in Dynamical Systems*  
698 (John Wiley & Sons, 2016). 715
- 699 <sup>20</sup>D. Bessis, G. Paladin, G. Turchetti, and S. Vaienti, *J. Stat. Phys.* **51**, 109 (1988). 716
- 700 <sup>21</sup>A. Ferguson and M. Pollicott, *Ergodic Theory Dyn. Syst.* **32**, 961 (2012). 717
- 701 <sup>22</sup>G. Keller, *Equilibrium States in Ergodic Theory* (Cambridge University  
702 Press, 1998), Vol. 42. 718
- 703 <sup>23</sup>G. Keller, *Dyn. Syst.* **27**, 11 (2012). 719
- 704 <sup>24</sup>G. Keller and C. Liverani, *J. Stat. Phys.* **135**, 519 (2009). 720
- 705 <sup>25</sup>M. Viana, *Lectures on Lyapunov Exponents* (Cambridge University Press,  
706 2014), Vol. 145. 721
- 707 <sup>26</sup>L.-S. Young, *Ergodic Theory Dyn. Syst.* **2**, 109 (1982). 722
- 708 <sup>27</sup>V. Lucarini, D. Faranda, G. Turchetti, and S. Vaienti, *Chaos* **22**, 023135  
709 (2012). 723
- <sup>28</sup>G. Paladin and S. Vaienti, *J. Stat. Phys.* **57**, 289 (1989). 710
- <sup>29</sup>J. C. Sprott and G. Rowlands, *Int. J. Bifurcation Chaos* **11**, 1865  
(2001). 711
- <sup>30</sup>K. Simon, *Proc. Am. Math. Soc.* **125**, 1221 (1997). 712
- <sup>31</sup>G. Baier and M. Klein, *Phys. Lett. A* **151**, 281 (1990). 713
- <sup>32</sup>H. Richter, *Int. J. Bifurcation Chaos* **12**, 1371 (2002). 714
- <sup>33</sup>J. W. Hurrell, *Science* **269**, 676 (1995); ISSN 0036-8075. 715
- <sup>34</sup>G. Moore, I. A. Renfrew, and R. S. Pickart, *J. Clim.* **26**, 2453 (2013). 716
- <sup>35</sup>J. C. Rogers, *J. Clim.* **10**, 1635 (1997). 717
- <sup>36</sup>L. Comas-Bru and F. McDermott, *Q. J. R. Meteorol. Soc.* **140**, 354  
(2014). 718
- <sup>37</sup>H. Hersbach, C. Peubey, A. Simmons, P. Berrisford, P. Poli, and D. Dee,  
*Q. J. R. Meteorol. Soc.* **141**, 2350 (2015). 719
- <sup>38</sup>O. Krueger, F. Schenk, F. Feser, and R. Weisse, *J. Clim.* **26**, 868 (2013). 720
- <sup>39</sup>D. Rodrigues, M. C. Alvarez-Castro, G. Messori, P. Yiou, Y. Robin, and  
D. Faranda, ■ (2017). 721
- <sup>40</sup>D. Faranda, G. Messori, M. C. Alvarez-Castro, and P. Yiou, *Nonlinear*  
*Processes Geophys.* **24**, 713 (2017). 722
- <sup>41</sup>J. R. Holton, *Am. J. Phys.* **41**, 752 (1973). 723
- <sup>42</sup>D. Faranda, J. M. Freitas, V. Lucarini, G. Turchetti, and S. Vaienti,  
*Nonlinearity* **26**, 2597 (2013). 724
- <sup>43</sup>H. Aytac, J. Freitas, and S. Vaienti, *Trans. Am. Math. Soc.* **367**, 8229  
725 (2015). 726
- AQ10