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Stabilization of Continuous-Time Linear Systems subject to Input Quantization ⋆

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Abstract

This paper deals with the stabilization of continuous-time linear time-invariant systems subject to uniform input quantization. Specifically, the right-hand side of the closed-loop system is rewritten as a linear system subject to a discontinuous perturbation due to the quantization error. Then, the controller design is performed to achieve finite-time convergence of the closed-loop trajectories towards a compact invariant set surrounding the origin. Furthermore, a computationally tractable design procedure for the proposed controller based on linear matrix inequalities, and some insights on the simulation of the closed-loop system are presented. In addition, the effectiveness of the proposed control design procedure is shown in a numerical example.

Key words: Linear systems, Quantizers, Convex optimization, Discontinuous control, Numerical simulations, Feedback stabilization.

1 Introduction

Recent technology enhancement has enabled the control of dynamical systems via digital controllers. When a continuous-time plant is controlled through a digital controller, side effect as time-delays, asynchronism, quantization, (see [14] and references therein), can turn into an excessive performance degradation like the appearance of limit cycles, chaotic phenomena or even instability of the closed-loop system, [7,13]. Concerning the effect of quantization in control systems, such a topic has been extensively addressed by researchers over the last years; see, *e.g.*, [5,7,12,13,17,22,26,27] just to cite a few.

This paper pertains to the stabilization of continuous-time linear time-invariant plants with uniformly-quantized input via static state feedback control. Specifically, pursuing the general approach introduced in [13], we model the uniform quantizer as a discontinuous static isolated nonlinearity entering into the dynamics of the closed-loop system. At this stage, since the resulting closed-loop system is described by a discontinuous right-hand side differential equation, the existence of solutions to the closed-loop system is not guaranteed; see [16]. Therefore, to tackle the problem under consideration, we adopt, for the closed-loop system, the notion of solution due to Krasovskii; see [11]. Then, by the use of the sector conditions for the uniform quantizer presented in [15], coupled through S-procedure (see [3]) to a quadratic Lyapunov-like function, we propose a condition to guarantee the finite-time convergence of the closed-loop trajectories towards a compact invariant set surrounding the origin, (asymptotic stability is usually impossible to prove due to the deadzone effect induced by uniform quantization; see, *e.g.*, [8,15,27]). Afterwards, via the use of the projection lemma (see [25]), such a condition is turned into a design procedure based on the solutions of a convex optimization problem that in one shot provides: the controller gain, the invariant set wherein the closed-loop trajectories ultimately converge, while minimizing the size of such a set. Moreover, some insights on the simulation of the considered closed-loop system are discussed.

It is worthwhile to notice that, although the approach introduced in [13], (consisting in modeling quantizers as isolated nonlinearities), has enabled to build constructive design tools for quantized control systems, as for the case of other types of isolated nonlinearities such as, saturation, backlash etc., surprisingly no much work has been done in that direction, except for [12,17,27], though the results in [12,17] relate to discrete-time systems subjected to a logarithmic quantizer (deadzone-free). Specifically, the majority of the results available in the literature (see, *e.g.*, [7,22] and the references therein), has dealt with the stability analysis of quantized closed-loop systems involving a controller de-

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signed while ignoring the presence of the quantizer. In this sense, this paper wants to fill this gap by proposing a design technique tailored to quantized systems. The contribution of this paper with respect to [27] is twofold. On one hand, the novel sector conditions for the uniform quantizer presented in [15] are for the first time exploited to design a static state feedback controller in the presence of uniform input quantization. On the other hand, the use of the projection lemma is considered to potentially improve the proposed design technique. Notice also that even if the spirit of the pursued approach is similar to [7,20], the results presented in this paper allows to deal with multi-inputs systems, derive constructive conditions for the design of a state feedback controller and explicitly characterize the set wherein the closed-loop trajectories ultimately converge, directly from the knowledge of a Lyapunov-like function. Therefore, our results can be considered as complementary with respect to those in [7,20].

The paper is organized as follows. Section 2 presents the system under consideration and the problem we solve. Section 3 is dedicated to the main results. Section 4 is devoted to numerical issues about the controller design. Moreover, some aspects on the simulation of the closed-loop system are discussed. Finally, Section 5 shows the effectiveness of the presented results in a numerical multi-inputs example.

Notation: The set $\mathscr{B}(x,\delta))$ denotes the δ radius closed Euclidean ball centered at x. $\mathbf{I}_n$ denotes the identity matrix and $\mathbf{0}$ denotes the null matrix (equivalently the null vector) of appropriate dimensions. For a matrix $A \in \mathbb{R}^{n\times m}$, A', $A_{(i)}$ denotes its $i-th$ row, and $trace(A)$ denote its transpose and its trace, and $\mathrm{He}(A) = A + A'$. The matrix $\mathrm{diag}\{A_1, A_2, \ldots, A_n\}$ is the block-diagonal matrix having $A_1, A_2, \ldots, A_n$ as diagonal blocks and in symmetric matrices $\bullet$ stands for symmetric blocks. For a vector $x \in \mathbb{R}^n$, $x_{(i)}$ denotes its $i-th$ component, x' denotes its transpose, $|x|$ stands for the componentwise absolute value operator, $\mathrm{sign}(x)$ is the componentwise sign function, with $\mathrm{sign}(0) = 0$, and $\lfloor x \rfloor$ the componentwise floor operator. The set $\Delta\mathbb{Z}^p$ is the set of the p-tuples of integers multiple of Δ. The symbol $\langle\cdot,\cdot\rangle$ denotes the standard Euclidean inner product and $\times$ stands for the standard Cartesian product. For a set U, $\mathbf{int}(U)$ denotes the interior of U. The double arrows notation $F\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ indicates that F is a set-valued mapping with $F(x) \subset \mathbb{R}^m$. Throughout the paper, a.a. stands for *almost all* in a Lebesgue-measure sense. For a function $f\colon A \to B$, $\mathrm{rge}\, f := \{y \in B\colon \exists x \in A \text{ such that } y = f(x)\}$.

Preliminary definitions: In this paper we deal with differential inclusions in the form

$$\dot{x} \in F(x) \tag{1}$$

where $F(x)\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. Consider the following definitions given mainly in [18,23].

Definition 1 *Let $\mathbb{I} \subset \mathbb{R}_{\geqslant 0}$ be an interval. Given $x_0 \in \mathbb{R}^n$ an absolutely continuous function $\phi(t)\colon \mathbb{I} \to \mathbb{R}^n$ is said to be a solution to* (1) *from x_0, if $\phi(0) = x_0$, and $\dot{\phi}(t) \in F(\phi(t))$ for a.a. $t \in \mathbb{I}$.*

Definition 2 *A solution $\phi\colon \mathbb{I} \to \mathbb{R}^n$ to* (1) *from x_0 is said to be maximal, if there does not exist any other solution $\overline{\phi}\colon \overline{\mathbb{I}} \to \mathbb{R}^n$, with $\mathbb{I} \subset \overline{\mathbb{I}}$ and such that $\phi(t) = \overline{\phi}(t)$ for every $t \in \mathbb{I}$. Moreover, ϕ is said to be complete, if $\mathbb{I} = \mathbb{R}_{\geqslant 0}$.*

Definition 3 *A set $\mathscr{A} \subset \mathbb{R}^n$ is strongly forward invariant for* (1)*, if every maximal solution ϕ to* (1) *is complete, and $\phi(0) \in \mathscr{A}$ implies* $\mathrm{rge}\,\phi \subset \mathscr{A}$.

2 Problem Statement

Consider the following continuous-time linear system with quantized input

$$\begin{cases} \dot{x} = Ax + B\mathrm{q}(u) \\ x(0) = x_0 \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$, $x_0 \in \mathbb{R}^n$ are respectively the state, the input of the system and the initial state. A, B are real matrices of suitable dimensions, and $\mathrm{q}(\cdot)$ is the uniform quantizer, which is described by the static nonlinear functions defined as

$$\mathrm{q}(u) := \Delta\,\mathrm{sign}(u) \left\lfloor \frac{|u|}{\Delta} \right\rfloor \tag{3}$$

where Δ is a positive given real scalar characterizing the quantization error bound. Assuming that the state x is fully accessible, we want to stabilize system (2) via the following control law $u = Kx$. Therefore, by defining the function $\Psi(u) := \mathrm{q}(u) - u$, the closed-loop system reads as

$$\begin{cases} \dot{x} = (A + BK)x + B\Psi(Kx) \\ x(0) = x_0. \end{cases} \tag{4}$$

Notice that, due to the presence of the uniform quantizer, the right-hand side of (4) is a discontinuous function of the state, then there is no guarantees about the existence of solutions when intended in a classical sense; see [11]. To this end, as in [7], in this paper we focus on Krasovskii solutions to system (4), that is the solutions to the following differential inclusion

$$\dot{x} \in \mathscr{K}\left((A + BK)x + B\Psi(Kx)\right) \tag{5}$$

where the Krasovskii operator $\mathscr{K}$ is defined by [1] $\mathscr{K}(f(x)) := \bigcap_{\delta>0} \overline{\mathrm{co}} f(\mathscr{B}(x,\delta))$. The existence of such solutions is guaranteed under the very mild requirement of local boundedness of the right-hand side of (4), obviously verified in our case. Moreover, as pointed out in [18,19], Krasovskii solutions are arbitrarily close to the solutions to (4) obtained by perturbing the state x with arbitrarily small perturbations; see [19]. This fact provides a further reason to consider Krasovskii solutions, beyond the merely issue concerning the existence of solutions.

Remark 1 *As pointed out in [22,27], the presence of the uniform quantizer defined in* (3)*, due to its deadzone effect, can represent a real obstacle to the asymptotic stabilization of the closed-loop system. Namely, one should be aware that if the matrix A is not Hurwitz, then the asymptotic stability of the origin for the closed-loop system* (4) *cannot be achieved via any choice of the gain K.*

Concerning closed-loop system (4), notice that according to [24] every solution to (4) is a Krasovskii solution to [2] [3]

$$\dot{x} \in (A + BK)x + B \bigtimes_{i=1}^{p} \mathscr{K}\left(\Psi(K_{(i)}x)\right). \tag{6}$$

In the sequel, for notational simplicity, we denote the right-hand side of (6) as $\mathscr{W}(x)$, so that (6) reads as $\dot{x} \in \mathscr{W}(x)$. Now we are in position to state the problem we solve.

[1] Missing $\overline{\mathrm{co}}$ in the published version.

[2] In the published version there is a missing B in (6).

[3] Systems (4) and (6) are swapped in the published version. The error does not propagate and the (correct) argument used in the paper is that solutions to (6) are Krasovskii solutions to (4) and not the other way around.

Problem 1 *Given the matrices A, B of adequate dimensions and a positive real quantization error bound Δ, determine a gain K and a compact set $\mathscr{S}_u \subset \mathbb{R}^n$ containing the origin, such that*

- *$\mathscr{S}_u$ is strongly forward invariant for* (6)*;*
- *For every initial condition $x_0 \in \mathbb{R}^n \backslash \mathscr{S}_u$, the resulting maximal solutions to* (6) *are bounded, complete and converge in a finite time into $\mathscr{S}_u$.*

3 Main results

To solve Problem 1, we want to exploit the sector conditions for the uniform quantizer presented in [15], which are recalled below.

Lemma 1 *For every $u \in \mathbb{R}^p$ and every $v \in \times_{i=1}^{p} \mathscr{K}(\Psi(u_{(i)}))$, the following relations are verified*

$$v' S_1 v - \mathrm{trace}(S_1)\Delta^2 \leqslant 0 \tag{7}$$

$$v' S_2 (v+u) \leqslant 0 \tag{8}$$

for any diagonal positive definite matrices $S_1, S_2 \in \mathbb{R}^{p\times p}$.

The next result gives a first condition to solve Problem 1.

Proposition 1 *If there exist a symmetric positive definite matrix $P \in \mathbb{R}^{n\times n}$, two diagonal positive definite matrices $S_1, S_2 \in \mathbb{R}^{p\times p}$, a matrix $K \in \mathbb{R}^{p\times n}$ and a positive scalar τ such that*

$$\mathscr{N} = \begin{bmatrix} \mathrm{He}(P(A+BK)) + \tau P & PB - K'S_2 \\ \bullet & -S_1 - 2S_2 \end{bmatrix} < \mathbf{0} \tag{9}$$

$$\mathrm{trace}(S_1)\Delta^2 - \tau \leqslant 0 \tag{10}$$

then K, and

$$\mathscr{S}_u = \{x \in \mathbb{R}^n : x'Px \leqslant 1\} \tag{11}$$

are solution to Problem 1.

PROOF. Consider the following quadratic function $V(x) = x'Px$, we want to prove that there exists a positive real scalar β such that

$$\langle \nabla V(x), w\rangle \leqslant -\beta x'x \qquad \forall x \in \mathbb{R}^n \backslash \mathbf{int}(\mathscr{S}_u), w \in \mathscr{W}(x). \tag{12}$$

By S-procedure arguments, this can be obtained by showing that for every $x \in \mathbb{R}^n$, there exists a positive real scalar τ such that

$$\langle \nabla V(x), w\rangle - \tau(1 - x'Px) \leqslant -\beta x'x \qquad \forall w \in \mathscr{W}(x). \tag{13}$$

On the other hand, due to (6), for every $w \in \mathscr{W}(x)$ there exists a $v \in \times_{i=1}^{p} \mathscr{K}(\Psi(K_{(i)}x))$ such that $w = (A+BK)x + Bv$, thus still by S-procedure arguments, by continuity of the function Kx (see [15]), thanks to Lemma 1, (13) is ensured by proving that for each $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^p$,

$$\begin{aligned} &\langle \nabla V(x), (A+BK)x + Bv\rangle - \tau(1 - x'Px) - v'S_1 v + \\ &\mathrm{trace}(S_1)\Delta^2 - 2v'S_2(v + Kx) \leqslant -\beta x'x. \end{aligned} \tag{14}$$

By straightforward calculations the left-hand side of the above relation can be rewritten as follows

$$\begin{bmatrix} x \\ v \end{bmatrix}' \mathscr{N} \begin{bmatrix} x \\ v \end{bmatrix} + \mathrm{trace}(S_1)\Delta^2 - \tau. \tag{15}$$

Thus in view of (9) and (10), it follows that there exists a small enough positive scalar β such that for every $x \in \mathbb{R}^n, w \in \mathscr{W}(x)$, one has $\langle \nabla V(x), w\rangle - \tau(1 - x'Px) \leqslant -\beta(x'x + v'v)$, which in turn gives (13).

To conclude, by following standard arguments (see, *e.g.*, [2,21]), from (12) it follows that, $\mathscr{S}_u$ is strongly forward invariant for (6), every solution to (6) is bounded (then complete; see [18, Proposition 6.10.]) and converges to $\mathscr{S}_u$ at most in[4] $\frac{\lambda_{\max}(P)}{\beta} \ln(V(x_0))$ units of time. □

As it appears in Proposition 1, condition (9) is nonlinear in the decision variables, thus from a numerical standpoint the solution to Problem 1 directly via Proposition 1 appears unlikely; see [3]. For this, the next result allows to make a first step toward a convex design procedure based on the result given in Proposition 1.

Corollary 1 *If there exist a symmetric positive definite matrix $J \in \mathbb{R}^{n\times n}$, a matrix $Y \in \mathbb{R}^{p\times n}$, a matrix $F \in \mathbb{R}^{n\times n}$, two diagonal positive definite matrices $S_1, S_2 \in \mathbb{R}^{p\times p}$ and a positive scalar τ such that* (10) *is verified and*

$$\begin{bmatrix} -\mathrm{He}(F) & J + AF + BY - F' & B \\ \bullet & \tau J + \mathrm{He}(AF+BY) & -Y'S_2 + B \\ \bullet & \bullet & -S_1 - 2S_2 \end{bmatrix} < \mathbf{0} \tag{16}$$

then $K = YF^{-1}$ and the set $\mathscr{S}_u = \{x \in \mathbb{R}^n : x'Px \leqslant 1\}$, with $P = F'^{-1}JF^{-1}$, are solution to Problem 1.

PROOF. The proof is inspired by [25]. From Proposition 1, notice that $\mathscr{N} = W'\mathscr{Q}W$, where

$$W = \begin{bmatrix} A+BK & B \\ \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}, \mathscr{Q} = \begin{bmatrix} \mathbf{0} & P & \mathbf{0} \\ \bullet & \tau P & -K'S_2 \\ \bullet & \bullet & -S_1 - 2S_2 \end{bmatrix}.$$

Thus, (9) can be rewritten equivalently as $W'\mathscr{Q}W < \mathbf{0}$. Moreover, being S_1 and S_2 positive definite, $U'\mathscr{Q}U < \mathbf{0}$, with $U' = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$, is obviously satisfied. Thus, by the projection lemma; see [25], the satisfaction of (9), whenever S_1 and S_2 are required to be positive definite, is equivalent to find a matrix $\mathscr{X}$ such that[5]

$$\mathscr{Q} + (W_r^{\perp})'\mathscr{X}U_r^{\perp} + (U_r^{\perp})'\mathscr{X}'W_r^{\perp} < \mathbf{0} \tag{17}$$

where, $U_r^{\perp}$ and $W_r^{\perp}$ are some matrices having as rows a basis of the row-null space, respectively of U and W. Now,

[4] Typo in the published version, $\mathbb{P}$ should be P.

[5] Typo in (17), $W'^{\perp}_r$ should be $(W_r^{\perp})'$, same for $U'^{\perp}_r$.

by selecting $U_r^{\perp} = \begin{bmatrix} \mathbf{I}_{2n} & \mathbf{0}_{2n\times p} \end{bmatrix}$ and $W_r^{\perp} = \begin{bmatrix} -\mathbf{I} & A+BK & B \end{bmatrix}$, and by partitioning $\mathscr{X} = \begin{bmatrix} X_1 & X_2 \end{bmatrix}$, where $X_1, X_2 \in \mathbb{R}^{n\times n}$, from (17) one gets

$$\begin{bmatrix} -\mathrm{He}(X_1) & P - X_2 + X_1'(A+BK) & X_1'B \\ \bullet & \mathrm{He}(X_2'(A+BK)) + \tau P & X_2'B - K'S_2 \\ \bullet & \bullet & -S_1 - 2S_2 \end{bmatrix} < \mathbf{0}. \tag{18}$$

At this stage, by setting in the above expression $X_1 = X_2 = X$, then by pre-and-post multiplying the left-hand side of the resulting matrix by $\mathrm{diag}\{X'^{-1}, X'^{-1}, \mathbf{I}\}$ and $\mathrm{diag}\{X^{-1}, X^{-1}, \mathbf{I}\}$ and finally by setting $X^{-1} = F$, $J = F'PF$ and $Y = KF$ yields the left-hand side of (16). Then, since the satisfaction of (16) implies the satisfaction of (9), thanks to Proposition 1 the assertion is proven. □

4 Numerical Issues

4.1 Controller design

The implicit objective in solving Problem 1 is to shrink the size of the set $\mathscr{S}_u$ as much as possible, so as to keep the closed-loop trajectories sufficiently close to the origin. Thus, as shown in [27], one can select among all the feasible solutions to condition (16), that one providing the smallest size of the set $\mathscr{S}_u$. To achieve this objective, one need firstly to choose a suitable measure for the set $\mathscr{S}_u$. Now, being the ultimate set an ellipsoid, several size criteria can be adopted. A typical choice adopted in the literature consists in minimizing the volume of the set $\mathscr{S}_u$ which is proportional to $\sqrt{\det(P^{-1})}$; see [3]. However, since the variable P does not directly appear in (16), the adoption of the latter criterion is not handy and it can even lead to nonconvex problems. To overcome this issue, one can implicitly minimize the volume of $\mathscr{S}_u$ by minimizing $\mathrm{trace}(P^{-1})$. Then, to link the matrix P to the other decision variables the following further constraint is added

$$\begin{bmatrix} N & F \\ \bullet & J \end{bmatrix} \geqslant \mathbf{0} \tag{19}$$

which, for every symmetric positive definite matrix N, is equivalent to $P^{-1} \leqslant N$. Essentially, the latter constraint guarantees that $\mathrm{trace}(P^{-1}) \leqslant \mathrm{trace}(N)$, so as the minimization $\mathrm{trace}(P^{-1})$ can be performed via the minimization of $\mathrm{trace}(N)$, and N is a new decision variable. At this stage, it is important to note that (16) is still nonlinear in the decision variables. On the other hand, relation (16) becomes linear with respect to the decision variables once τ and S_2 are fixed. The selection of τ and S_2 is not a difficult task and it can be performed through a grid search. Therefore, Problem 1 can be solved through the solution of the following convex optimization problem, along with a grid search stage aimed at selecting the values of τ and S_2 providing the best value for the objective function.

$$\begin{aligned} &\underset{F,Y,S_1,N,J}{\text{minimize}} && \mathrm{trace}(N) \\ &\text{subject to} && N > \mathbf{0}, S_1 > \mathbf{0}, J > \mathbf{0}, (16), (10), (19). \end{aligned} \tag{20}$$

Nevertheless, due to the dead-zone effect induced by the quantizer, by solving the above optimization problem, one may run into solutions characterized by a high norm control gain, situation that need to be ruled out to envision the physical construction of the proposed controller. To this end, we implicitly limit the control gain by constraining the eigenvalues of the matrix $A+BK$ to lay in suitable subsets of the open-left half complex plane. This kind of additional constraints can be easily expressed in a linear matrix inequality form; see [9]. A typical choice is to consider as region the closed circle centered in $(-\omega, 0)$ with radius $r > 0$, where ω is a positive real scalar, *i.e.*, $\{z \in \mathbb{C} \colon |z+\omega| \leqslant r\}$; such a condition is guaranteed by adding the following constraint; see [28],

$$\begin{bmatrix} -rQ & \omega F + AF + BY \\ \bullet & -r(F' + F - Q) \end{bmatrix} < \mathbf{0} \tag{21}$$

where Q is a symmetric matrix with adequate dimensions.

Remark 2 *The introduction of the slack variable F allows to decouple the matrix P defining the set $\mathscr{S}_u$ from the gain K. This in turn allows to introduce an extra variable, i.e., Q, in* (21)*. On the one hand, this in general leads to a less conservative design technique when further constraints like* (21) *are considered; see [6]. On the other hand, this approach potentially enables to obtain better result in term of size of the set $\mathscr{S}_u$, being the variable J, (and then indirectly P), not subject to any additional constraint.*

Notice that, as long as the gain K is known, condition (9) is linear in the decision variables, modulo the variable τ. Therefore, as a second step, by combining Proposition 1 along with a griding procedure for the scalar τ, by solving a finite number of linear matrix inequalities, one may attempt to tighten the set $\mathscr{S}_u$. This is a typical approach pursued in the literature; see, *e.g.*, [28]. In particular, throughout this further stage one may also account for different measures for the set $\mathscr{S}_u$ than the trace criterion considered during the design stage.

4.2 Numerical integration

To overcome the issues on the existence of solutions to the closed-loop system, we addressed the study of such a system by mean of the notion of Krasovskii solution, and resting on this notion we shown some stability and stabilization results for the closed-loop system. Thus, at this stage, the question that naturally arises is how to simulate the closed-loop system while taking into account the notion of solution adopted throughout the paper. For this purpose, we need to introduce the notion of approximated polygonal arc and Euler solution, which are both given in [10].

Definition 4 (Polygonal approximation) *Given $x_0 \in \mathbb{R}^n$ and $T > 0$, consider the following construction*

- *Fix an arbitrary partition of the interval $[0,T]$, $0 < t_1 < t_2 < \cdots < t_N$, with $t_N = T$ and $\max_{k\in\{0,\ldots,N-1\}} \{t_{k+1} - t_k\} \leqslant \varepsilon$.*
- *Compute $x_{k+1} = x_k + (t_{k+1} - t_k)(Ax_k + B\mathrm{q}(Kx_k))$, for $k = 0,\ldots,N$ and $x(0) = x_0$.*
- *Build the affine function such that $x_\varepsilon(t_k) = x_k$ for $k = 0,\ldots,N$.*

The function $x_\varepsilon(t)$ is said to be an ε-polygonal approximation for closed-loop system (4)*.*

Definition 5 *A function $\varphi_{\mathscr{E}}(t)$ is said to be an Euler solution to* (4) *if it is the uniform limit for $\varepsilon \to 0$ of a polygonal approximation $x_\varepsilon(t)$ obtained by some partition of the interval* $[0,T]$.

The interest in considering Euler solutions stems from the fact that, as proven in [4], Euler solutions are Krasovskii solutions. In particular, notice that, among all the possible polygonal approximations one can consider, the simplest and straightforwardly attainable through a numerical procedure arises from selecting a uniform partitioning of the time interval. Namely, let N be an arbitrarily positive integer, then fix $\delta = \frac{T}{N}$, and for $k = 1,2,\ldots,N-1$, $t_{k+1} = t_k + \frac{T}{N}$. Thus, the sequence of polygonal approximations $\{x_{\frac{T}{N}}\}_{N=1}^{\infty}$, if converges uniformly, it converges uniformly to a Krasovskii solution to (4). Therefore, for N sufficiently large, the function $x_{\frac{T}{N}}$ can represent a good approximation of a Krasovskii solution to the considered system. However, often given $x_0 \in \mathbb{R}^n$, there may exist multiple Krasovskii solutions from x_0, and some of them may not be Euler solutions. Moreover, establishing whether the considered sequence uniformly converges whenever N approaches infinity could be nontrivial. Therefore, this aspect is still worth of further investigations.

5 Numerical example

Consider the following example derived from [1] for which $A = \begin{bmatrix} -0.5 & 1.5 & 4 \\ 4.3 & 6 & 5 \\ 3.2 & 6.8 & 7.2 \end{bmatrix}, B = \begin{bmatrix} -0.7 & -1.3 \\ 0 & -4.3 \\ 0.8 & -1.5 \end{bmatrix}$ and assume $\Delta = 0.5$. As a first step, by considering $\tau = 1.8, S_2 = \mathrm{diag}\{1.4 \cdot 10^{-6}, 4.3 \cdot 10^{-5}\}$, selected via a grid search, and by considering the further constraint in (21) with $\omega = 10, r = 8.5$, the solution to the optimization problem (20) yields $K = \begin{bmatrix} -0.71 & 1.9 & -27 \\ 4.3 & 4.1 & 4.3 \end{bmatrix}$.

As a second step, we perform an analysis stage via Proposition 1, while considering as measure for the set $\mathscr{S}_u$ its volume (the volume of the set $\mathscr{S}_u$ can be minimized by considering as convex objective $-\log\det(P)$; see [3]). An iterative research for the scalar τ allows to select the more convenient value for such a parameter, that is $\tau = 7.2$, which leads to $P = \begin{bmatrix} 29.33 & 13.25 & -14.07 \\ 13.25 & 65 & -136.7 \\ -14.07 & -136.7 & 404.8 \end{bmatrix}$. In Fig. 2, the evolution of the closed-loop system from different initial conditions is reported. Simulations show that trajectories converge towards two equilibria, (obviously also the origin is an equilibrium point for the closed-loop system, though unstable). It is interesting to notice that these two equilibrium points belong respectively to the two surfaces $Kx = [\Delta \ -\Delta]'$ and $Kx = [-\Delta \ \Delta]'$ wherein the function $\mathrm{q}(Kx)$ is discontinuous. It can be shown that the two mentioned equilibria are actually Krasovskii equilibria, and they can be computed as illustrated in [8]. Fig 1 reports a closed-loop trajectory converging towards an equilibrium. Specifically, in approaching the equilibrium point, the state slides onto different quantization boundaries making the control inputs switch repeatedly among the two adjacent quantization levels (chattering). Simulations also show the accuracy provided by Euler integration in capturing the peculiar behaviors due to the discontinuity introduced by the quantizer, as the two Krasovskii equilibria.

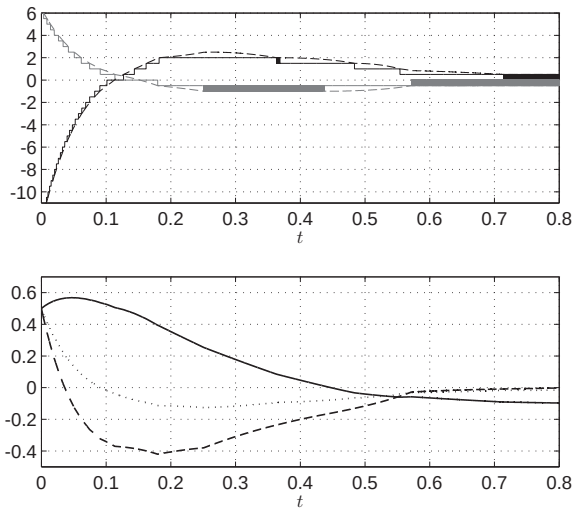

Fig. 1. The evolution of the closed-loop system from $x_0 = (0.5, 0.5, 0.5)$: Above the control inputs ($\mathrm{q}(u_1)$ (solid-black), $\mathrm{q}(u_2)$ (solid-gray), and the two quantization-free inputs ($K_{(1)}x(t)$ (dashed-black), $K_{(2)}x(t)$ (dashed-gray). Below the closed-loop states: x_1 (solid), x_2 (dashed), x_3 (dotted).

6 Conclusion

In this paper we tackled the stabilization problem for linear systems subject to input quantization, with respect to Krasovskii solutions. Specifically, via the use of the novel sector conditions for the uniform quantizer presented in [15], and the adoption of the projection lemma, a design procedure based on the solution of a convex optimization problem was provided. In addition some hints on the numerical simulations of the closed-loop system are discussed. Finally, the effectiveness of the proposed methodology is displayed in a multi-inputs example.

Future research directions include the extension of this methodology to the case of the logarithmic quantizer, and to the case of static state feedback control via quantized state.

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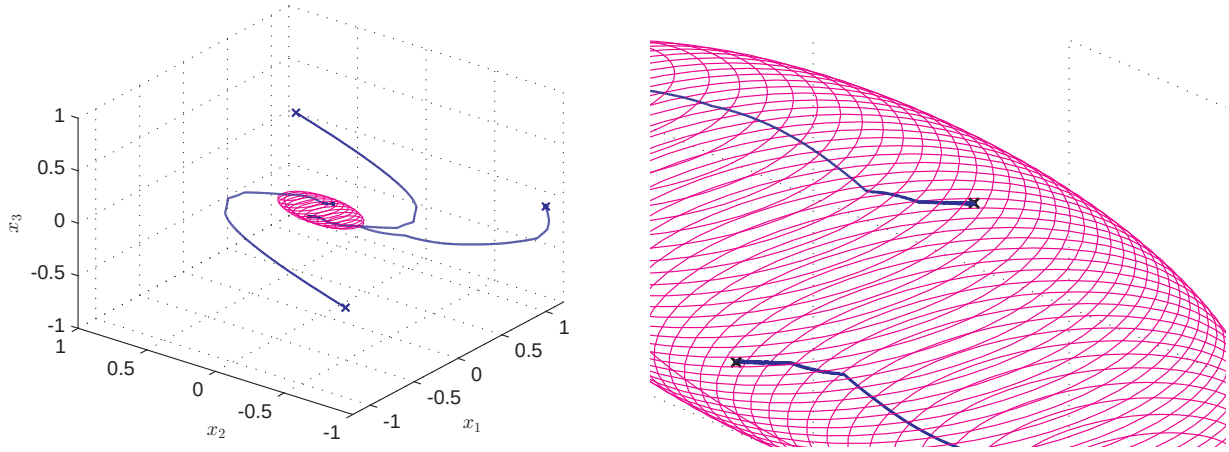

(a)
Some closed-loop trajectories.

(b)
A close-up showing the two equilibrium points (x).

Fig. 2. The evolution of the closed-loop system from different initial conditions. The trajectory is obtained by integrating the closed-loop model with an Euler first order method with step size 10^{-3}.

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