Quadric Conformal Geometric Algebra of $R^{9,6}$

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Abstract. Geometric Algebra can be understood as a set of tools to represent, construct and transform geometric objects. Some Geometric Algebras like the well-defined Conformal Geometric Algebra constructs lines, circles, planes, and spheres from control points just by using the outer product. There exist some Geometric Algebras to handle more complex objects such as quadric surfaces; however in this case, none of them is known to build quadric surfaces from control points. This paper presents a novel Geometric Algebra framework, the Geometric Algebra of $\mathbb{R}^{9,6}$, to deal with quadric surfaces where an arbitrary quadric surface is constructed by the mere wedge of nine points. The proposed framework enables us not only to intuitively represent quadric surfaces but also to construct objects using Conformal Geometric Algebra. Our proposed framework also provides the computation of the intersection of quadric surfaces, the normal vector, and the tangent plane of a quadric surface.

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1. Introduction

Geometric Algebra provides useful and, more importantly, intuitively understandable tools to represent, construct and manipulate geometric objects. Intensively explored by physicists, Geometric Algebra has been applied in quantum mechanics and electromagnetism [3]. Geometric Algebra has also found some interesting applications in data manipulation for Geographic Information Systems (GIS) [13]. More recently, it turns out that Geometric Algebra can be applied even in computer graphics, either to basic geometric primitive manipulations [17] or to more complex illumination processes as in [14] where spherical harmonics are substituted by Geometric Algebra entities.

This paper presents a Geometric Algebra framework to handle quadric surfaces which can be applied to detect collision in computer graphics, and to calibrate omnidirectional cameras, usually embedding a mirror with a quadric surface, in computer vision. Handling quadric surfaces in Geometric Algebra requires a high dimensional space to work as seen in subsequent sections. Nevertheless, no low-dimensional Geometric Algebra framework is yet introduced that handles general orientation quadric surfaces and their construction using contact points.

1.1. High dimensional Geometric Algebras

Following Conformal Geometric Algebra (CGA) and its well-defined properties [5], the Geometric Algebra community recently started to explore new frameworks that work in higher dimensional spaces. The motivation for this direction is to increase the dimension of the relevant Euclidean space ($\mathbb{R}^n$ with $n > 3$) and/or to investigate more complex geometric objects.

The Geometric Algebra of $\mathbb{R}^{p,q}$ is denoted by $\mathbb{G}_{p,q}$ where p is the number of basis vectors squared to $+1$ and q is that of basis vectors squared to -1. Then, the CGA of $\mathbb{R}^3$ is denoted by $\mathbb{G}_{4,1}$. Extending from dimension 5 to 6 leads to $\mathbb{G}_{3,3}$ defining either 3D projective geometry (see Dorst [4]) or line geometry (see Klawitter [12]). Conics in $\mathbb{R}^2$ are represented by the conic space of Perwass [16] with $\mathbb{G}_{5,3}$. Conics in $\mathbb{R}^2$ are also defined by the Double Conformal Geometric Algebra (DCGA) with $\mathbb{G}_{6,2}$ introduced by Easter and Hitzer [8]. DCGA is extended to handle cubic curves (and some other even higher order curves) in the Triple Conformal Geometric Algebra with $\mathbb{G}_{9,3}$ [9] and in the Double Conformal Space-Time Algebra with $\mathbb{G}_{4,8}$ [7]. We note that the dimension of the algebras generated by any n-dimensional vector spaces ($n = p + q$) grows exponentially as they have 2^n basis elements. Although most multivectors are extremely sparse, very few implementations exist that can handle high dimensional Geometric Algebras. This problem is discussed further in Section 7.2.

1.2. Geometric Algebra and quadric surfaces

A framework to handle quadric surfaces was introduced by Zamora [18] for the first time. Though this framework constructs a quadric surface from control points, it supports only axis-aligned quadric surfaces.

There exist two main Geometric Algebra frameworks to manipulate general quadric surfaces.

On one hand, DCGA with $\mathbb{G}_{8,2}$, defined by Easter and Hitzer [8], constructs quadric and more general surfaces from their implicit equation coefficients specified by the user. A quadric (torus, Dupin- or Darboux cyclide) is represented by a bivector containing 15 coefficients that are required to construct the implicit equation of the surface. This framework preserves many properties of CGA and thus supports not only object transformations using versors but also differential operators. However, it is incapable of handling the intersection between two general quadrics and, to our best knowledge, cannot construct general quadric surfaces from control points.

On the other hand, quadric surfaces are also represented in a framework of $\mathbb{G}_{4,4}$ as first introduced by Parkin [15] and developed further by Du et al. [6]. Since this framework is based on a duplication of the projective geometry of $\mathbb{R}^3$, it is referred to as Double Perspective Geometric Algebra (DPGA) hereafter. DPGA represents quadric surfaces by bivector entities. The quadric expression, however, comes from a so-called "sandwiching" duplication of the product. DPGA handles quadric intersection and conics. It also handles versors transformations. However, to our best knowledge, it cannot construct general quadric surfaces from control points. This incapability seems true because, for example, wedging 9 control points together in this space results in 0 due to its vector space dimension.

1.3. Contributions

Our proposed framework, referred to as Quadric Conformal Geometric Algebra (QCGA) hereafter, is a new type of CGA, specifically dedicated to quadric surfaces. Through generalizing the conic construction in $\mathbb{R}^2$ by Perwass [16], QCGA is capable of constructing quadric surfaces using either control points or implicit equations. Moreover, QCGA can compute the intersection of quadric surfaces, the surface tangent, and normal vectors for a quadric surface point.

1.4. Notation

We use the following notation throughout the paper. Lower-case bold letters denote basis blades and multivectors (multivector $\mathbf{a}$). Italic lower-case letters refer to multivector components $(a_1, x, y^2, \cdots)$. For example, a_i is the i^{th} coordinate of the multivector $\mathbf{a}$. Constant scalars are denoted using lower-case default text font (constant radius r). The superscript star used in $\mathbf{x}^*$ represents the dualization of the multivector $\mathbf{x}$. Finally, subscript ϵ on $\mathbf{x}_\epsilon$ refers to the Euclidean vector associated with the point $\mathbf{x}$ of QCGA.

Note that in geometric algebra, the inner product, contractions and outer product have priority over the full geometric product. For instance, $a \wedge bI = (a \wedge b)I$.

2. QCGA definition

This section introduces QCGA. We specify its basis vectors and give the definition of a point.

2.1. QCGA basis and metric

The QCGA $\mathbb{G}_{9,6}$ is defined over a 15-dimensional vector space. The base vectors of the space $\mathbb{R}^{9,6}$ are basically divided into three groups: $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ (corresponding to the Euclidean vectors in $\mathbb{R}^3$), $\{\mathbf{e}_{o1}, \mathbf{e}_{o2}, \mathbf{e}_{o3}, \mathbf{e}_{o4}, \mathbf{e}_{o5}, \mathbf{e}_{o6}\}$, and $\{\mathbf{e}_{\infty1}, \mathbf{e}_{\infty2}, \mathbf{e}_{\infty3}, \mathbf{e}_{\infty4}, \mathbf{e}_{\infty5}, \mathbf{e}_{\infty6}\}$. The inner products between them are as defined in Table 1.

For some computation constraints, a diagonal metric matrix may be required. The orthonormal vector basis of $\mathbb{R}^{9,6}$ with the Euclidean basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and 6 basis vectors $\{\mathbf{e}_{+1}, \mathbf{e}_{+2}, \mathbf{e}_{+3}, \mathbf{e}_{+4}, \mathbf{e}_{+5}, \mathbf{e}_{+6}\}$ each of which

TABLE 1. Inner product between QCGA basis vectors.

	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_{o1}$	$\mathbf{e}_{\infty 1}$	$\mathbf{e}_{o2}$	$\mathbf{e}_{\infty 2}$	$\mathbf{e}_{o3}$	$\mathbf{e}_{\infty 3}$	$\mathbf{e}_{o4}$	$\mathbf{e}_{\infty 4}$	$\mathbf{e}_{o5}$	$\mathbf{e}_{\infty 5}$	$\mathbf{e}_{o6}$	$\mathbf{e}_{\infty 6}$
$\mathbf{e}_1$	1	0	0	·	·	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_2$	0	1	0	·	·	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_3$	0	0	1	·	·	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{o1}$	·	·	·	0	−1	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{\infty 1}$	·	·	·	−1	0	·	·	·	·	·	·	·	·	·	·
$\mathbf{e}_{o2}$	·	·	·	·	·	0	−1	·	·	·	·	·	·	·	·
$\mathbf{e}_{\infty 2}$	·	·	·	·	·	−1	0	·	·	·	·	·	·	·	·
$\mathbf{e}_{o3}$	·	·	·	·	·	·	·	0	−1	·	·	·	·	·	·
$\mathbf{e}_{\infty 3}$	·	·	·	·	·	·	·	−1	0	·	·	·	·	·	·
$\mathbf{e}_{o4}$	·	·	·	·	·	·	·	·	·	0	−1	·	·	·	·
$\mathbf{e}_{\infty 4}$	·	·	·	·	·	·	·	·	·	−1	0	·	·	·	·
$\mathbf{e}_{o5}$	·	·	·	·	·	·	·	·	·	·	·	0	−1	·	·
$\mathbf{e}_{\infty 5}$	·	·	·	·	·	·	·	·	·	·	·	−1	0	·	·
$\mathbf{e}_{o6}$	·	·	·	·	·	·	·	·	·	·	·	·	·	0	−1
$\mathbf{e}_{\infty 6}$	·	·	·	·	·	·	·	·	·	·	·	·	·	−1	0

squares to $+1$ along with six other basis vectors $\{\mathbf{e}_{-1}, \mathbf{e}_{-2}, \mathbf{e}_{-1}, \mathbf{e}_{-4}, \mathbf{e}_{-5}, \mathbf{e}_{-6}\}$ each of which squares to -1 corresponds to a diagonal metric matrix. The transformation from the original basis to this new basis (with diagonal metric) can be defined as follows:

$$\mathbf{e}_{\infty i} = \mathbf{e}_{+i} + \mathbf{e}_{-i}, \quad \mathbf{e}_{oi} = \frac{1}{2}(\mathbf{e}_{-i} - \mathbf{e}_{+i}), \quad i \in \{1, \cdots, 6\}. \tag{2.1}$$

For clarity, we also define the 6-blades

$$\begin{aligned} \mathbf{I}_\infty &= \mathbf{e}_{\infty 1} \wedge \mathbf{e}_{\infty 2} \wedge \mathbf{e}_{\infty 3} \wedge \mathbf{e}_{\infty 4} \wedge \mathbf{e}_{\infty 5} \wedge \mathbf{e}_{\infty 6}, \\ \mathbf{I}_o &= \mathbf{e}_{o1} \wedge \mathbf{e}_{o2} \wedge \mathbf{e}_{o3} \wedge \mathbf{e}_{o4} \wedge \mathbf{e}_{o5} \wedge \mathbf{e}_{o6}, \end{aligned} \tag{2.2}$$

the 5-blades

$$\begin{aligned} \mathbf{I}_\infty^\triangleright &= (\mathbf{e}_{\infty 1} - \mathbf{e}_{\infty 2}) \wedge (\mathbf{e}_{\infty 2} - \mathbf{e}_{\infty 3}) \wedge \mathbf{e}_{\infty 4} \wedge \mathbf{e}_{\infty 5} \wedge \mathbf{e}_{\infty 6}, \\ \mathbf{I}_o^\triangleright &= (\mathbf{e}_{o1} - \mathbf{e}_{o2}) \wedge (\mathbf{e}_{o2} - \mathbf{e}_{o3}) \wedge \mathbf{e}_{o4} \wedge \mathbf{e}_{o5} \wedge \mathbf{e}_{o6}, \end{aligned} \tag{2.3}$$

the pseudo-scalar of $\mathbb{R}^3$

$$\mathbf{I}_\epsilon = \mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3, \tag{2.4}$$

and the pseudo-scalar

$$\mathbf{I} = \mathbf{I}_\epsilon \wedge \mathbf{I}_\infty \wedge \mathbf{I}_o. \tag{2.5}$$

The inverse of the pseudo-scalar results in

$$\mathbf{I}^{-1} = -\mathbf{I}. \tag{2.6}$$

The dual of a multivector indicates division by the pseudo-scalar, e.g., $\mathbf{a}^* = -\mathbf{aI}$, $\mathbf{a} = \mathbf{a}^*\mathbf{I}$. From eq. (1.19) in [10], we have the useful duality between outer and inner products of non-scalar blades $\mathbf{a}$ and $\mathbf{b}$ in Geometric Algebra:

$$(\mathbf{a} \wedge \mathbf{b})^* = \mathbf{a} \cdot \mathbf{b}^*, \quad \mathbf{a} \wedge (\mathbf{b}^*) = (\mathbf{a} \cdot \mathbf{b})^*, \quad \mathbf{a} \wedge (\mathbf{bI}) = (\mathbf{a} \cdot \mathbf{b})\mathbf{I}, \tag{2.7}$$

which indicates that

$$\mathbf{a} \wedge \mathbf{b} = 0 \;\Leftrightarrow\; \mathbf{a} \cdot \mathbf{b}^* = 0, \quad \mathbf{a} \cdot \mathbf{b} = 0 \;\Leftrightarrow\; \mathbf{a} \wedge \mathbf{b}^* = 0. \tag{2.8}$$

2.2. Point in QCGA

The point $\mathbf{x}$ of QCGA corresponding to the Euclidean point $\mathbf{x}_\epsilon = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \in \mathbb{R}^3$ is defined as

$$\mathbf{x} = \mathbf{x}_\epsilon + \tfrac{1}{2}(x^2\mathbf{e}_{\infty 1} + y^2\mathbf{e}_{\infty 2} + z^2\mathbf{e}_{\infty 3}) + xy\mathbf{e}_{\infty 4} + xz\mathbf{e}_{\infty 5} + yz\mathbf{e}_{\infty 6} + \mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3}. \tag{2.9}$$

Note that the null vectors $\mathbf{e}_{o4}, \mathbf{e}_{o5}, \mathbf{e}_{o6}$ are not present in the definition of the point. This is merely to keep the convenient properties of CGA points, namely, the inner product between two points is identical with the squared distance between them. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be two points, their inner product is

$$\begin{aligned}\mathbf{x}_1 \cdot \mathbf{x}_2 = &\Big(x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + z_1\mathbf{e}_3 + \frac{1}{2}x_1^2\mathbf{e}_{\infty 1} + \frac{1}{2}y_1^2\mathbf{e}_{\infty 2} + \frac{1}{2}z_1^2\mathbf{e}_{\infty 3} \\ &+ x_1y_1\mathbf{e}_{\infty 4} + x_1z_1\mathbf{e}_{\infty 5} + y_1z_1\mathbf{e}_{\infty 6} + \mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3}\Big) \cdot \\ &\Big(x_2\mathbf{e}_1 + y_2\mathbf{e}_2 + z_2\mathbf{e}_3 + \frac{1}{2}x_2^2\mathbf{e}_{\infty 1} + \frac{1}{2}y_2^2\mathbf{e}_{\infty 2} + \frac{1}{2}z_2^2\mathbf{e}_{\infty 3} \\ &+ x_2y_2\mathbf{e}_{\infty 4} + x_2z_2\mathbf{e}_{\infty 5} + y_2z_2\mathbf{e}_{\infty 6} + \mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3}\Big).\end{aligned} \tag{2.10}$$

from which together with Table 1, it follows that

$$\begin{aligned}\mathbf{x}_1 \cdot \mathbf{x}_2 &= \Big(x_1x_2 + y_1y_2 + z_1z_2 - \frac{1}{2}x_1^2 - \frac{1}{2}x_2^2 - \frac{1}{2}y_1^2 - \frac{1}{2}y_2^2 - \frac{1}{2}z_1^2 - \frac{1}{2}z_2^2\Big) \\ &= -\frac{1}{2}\left\|\mathbf{x}_{1\epsilon} - \mathbf{x}_{2\epsilon}\right\|^2.\end{aligned} \tag{2.11}$$

We see that the inner product is equivalent to minus half the squared Euclidean distance between $\mathbf{x}_1$ and $\mathbf{x}_2$.

3. QCGA objects

QCGA is an extension of CGA, thus the objects defined in CGA are also defined in QCGA. The following sections explore the plane, the line, and the sphere to show their definitions in QCGA, and similarity between these objects in CGA and their counterparts in QCGA.

3.1. Plane

3.1.1. Primal plane. As in CGA, a plane $\boldsymbol{\pi}$ in QCGA is computed using the wedge of three linearly independent points $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ on the plane:

$$\boldsymbol{\pi} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{I}_\infty \wedge \mathbf{I}_o^\triangleright. \tag{3.1}$$

The multivector $\boldsymbol{\pi}$ corresponds to the primal form of a plane in QCGA, with grade 14, composed of six components. The $\mathbf{e}_{o2o3}, \mathbf{e}_{o1o3}, \mathbf{e}_{o1o2}$ components have the same coefficient and can thus be factorized, resulting in a form defined with only four coefficients x_n, y_n, z_n and h:

$$\begin{aligned}\boldsymbol{\pi} = &\Big(\mathrm{x}_n\mathbf{e}_{23} - \mathrm{y}_n\mathbf{e}_{13} + \mathrm{z}_n\mathbf{e}_{12}\Big)\mathbf{I}_\infty \wedge \mathbf{I}_o \\ &+ \frac{\mathrm{h}}{3}\mathbf{e}_{123}\mathbf{I}_\infty \wedge \Big(\mathbf{e}_{o2o3} - \mathbf{e}_{o1o3} + \mathbf{e}_{o1o2}\Big) \wedge \mathbf{e}_{o4o5o6}.\end{aligned} \tag{3.2}$$

Proposition 3.1. *A point* $\mathbf{x}$ *with Euclidean coordinates* $\mathbf{x}_\epsilon$ *lies on the plane* $\boldsymbol{\pi}$ *iff* $\mathbf{x} \wedge \boldsymbol{\pi} = 0$.

Proof.

$$\begin{aligned}
\mathbf{x} \wedge \boldsymbol{\pi} =& \Big(x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 + \frac{1}{2}x^2\mathbf{e}_{\infty 1} + \frac{1}{2}y^2\mathbf{e}_{\infty 2} + \frac{1}{2}z^2\mathbf{e}_{\infty 3} \\
& + xy\mathbf{e}_{\infty 4} + xz\mathbf{e}_{\infty 5} + yz\mathbf{e}_{\infty 6} + \mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3} \Big) \\
& \wedge \Big((\mathrm{x}_n\mathbf{e}_{23} - \mathrm{y}_n\mathbf{e}_{13} + \mathrm{z}_n\mathbf{e}_{12})\mathbf{I}_\infty \wedge \mathbf{I}_o \\
& + \frac{\mathrm{h}}{3}\mathbf{e}_{123}\mathbf{I}_\infty \wedge (\mathbf{e}_{o2o3} - \mathbf{e}_{o1o3} + \mathbf{e}_{o1o2}) \wedge \mathbf{e}_{o4o5o6} \Big).
\end{aligned} \tag{3.3}$$

Using distributivity and anticommutativity of the outer product, we obtain

$$\begin{aligned}
\mathbf{x} \wedge \boldsymbol{\pi} &= \Big(x\mathrm{x}_n + y\mathrm{y}_n + z\mathrm{z}_n - \frac{1}{3}\mathrm{h}(1+1+1) \Big) \mathbf{I} \\
&= \big(x\mathrm{x}_n + y\mathrm{y}_n + z\mathrm{z}_n - \mathrm{h} \big) \mathbf{I} \\
&= (\mathbf{x}_\epsilon \cdot \mathbf{n}_\epsilon - \mathrm{h})\, \mathbf{I},
\end{aligned} \tag{3.4}$$

which corresponds to the Hessian form of the plane with Euclidean normal $\mathbf{n}_\epsilon = \mathrm{x}_n\mathbf{e}_1 + \mathrm{y}_n\mathbf{e}_2 + \mathrm{z}_n\mathbf{e}_3$ and with orthogonal distance h from the origin. □

3.1.2. Dual plane. The dualization of the primal form of the plane is

$$\boldsymbol{\pi}^* = \mathbf{n}_\epsilon + \frac{1}{3}\mathrm{h}(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 3}). \tag{3.5}$$

Proposition 3.2. *A point* $\mathbf{x}$ *with Euclidean coordinates* $\mathbf{x}_\epsilon$ *lies on the dual plane* $\boldsymbol{\pi}^*$ *iff* $\mathbf{x} \cdot \boldsymbol{\pi}^* = 0$.

Proof. Consequence of (2.8). □

Because of (2.11), a plane can also be obtained as the bisection plane of two points $\mathbf{x}_1$ and $\mathbf{x}_2$ in a similar way as in CGA.

Proposition 3.3. *The dual plane* $\boldsymbol{\pi}^* = \mathbf{x}_1 - \mathbf{x}_2$ *is the dual orthogonal bisecting plane between the points* $\mathbf{x}_1$ *and* $\mathbf{x}_2$.

Proof. From Proposition 3.2, every point $\mathbf{x}$ on $\boldsymbol{\pi}^*$ satisfies $\mathbf{x} \cdot \boldsymbol{\pi}^* = 0$,

$$\mathbf{x} \cdot (\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{x} \cdot \mathbf{x}_1 - \mathbf{x} \cdot \mathbf{x}_2 = 0. \tag{3.6}$$

As seen in (2.11), the inner product between two points results in the squared Euclidean distance between the two points. We thus have

$$\mathbf{x} \cdot (\mathbf{x}_1 - \mathbf{x}_2) = 0 \quad \Leftrightarrow \quad \|\mathbf{x}_\epsilon - \mathbf{x}_{1\epsilon}\|^2 = \|\mathbf{x}_\epsilon - \mathbf{x}_{2\epsilon}\|^2 . \tag{3.7}$$

This corresponds to the equation of the orthogonal bisecting dual plane between $\mathbf{x}_{1\epsilon}$ and $\mathbf{x}_{2\epsilon}$. □

3.2. Line

3.2.1. Primal line. A primal line $\mathbf{l}$ is a 13-vector constructed from two linearly independent points $\mathbf{x}_1$ and $\mathbf{x}_2$ as follows:

$$\mathbf{l} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{I}_\infty \wedge \mathbf{I}_o^{\triangleright}. \tag{3.8}$$

The outer product between the 6−vector $\mathbf{I}_\infty$ and the two points $\mathbf{x}_1$ and $\mathbf{x}_2$ removes all their $\mathbf{e}_{\infty i}$ components ($i \in \{1, \cdots, 6\}$). Accordingly, they can be reduced to $\mathbf{x}_1 = (\mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3} + \mathbf{x}_{\epsilon 1})$ and $\mathbf{x}_2 = (\mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3} + \mathbf{x}_{\epsilon 2})$ respectively. For clarity, (3.8) is simplified "in advance" as

$$\begin{aligned}
\mathbf{l} &= \mathbf{x}_1 \wedge (\mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3} + \mathbf{x}_{\epsilon 2}) \wedge \mathbf{I}_\infty \wedge (\mathbf{e}_{o1} - \mathbf{e}_{o2}) \wedge (\mathbf{e}_{o2} - \mathbf{e}_{o3}) \wedge \mathbf{e}_{o4o5o6} \\
&= \mathbf{x}_1 \wedge (\mathbf{x}_{\epsilon 2} \wedge (\mathbf{e}_{o1} - \mathbf{e}_{o2}) \wedge (\mathbf{e}_{o2} - \mathbf{e}_{o3}) + 3\mathbf{e}_{o1} \wedge \mathbf{e}_{o2} \wedge \mathbf{e}_{o3}) \wedge \mathbf{I}_\infty \wedge \mathbf{e}_{o4o5o6} \\
&= \big(3\mathbf{e}_{o1} \wedge \mathbf{e}_{o2} \wedge \mathbf{e}_{o3} \wedge (\mathbf{x}_{\epsilon 2} - \mathbf{x}_{\epsilon 1}) + \mathbf{x}_{\epsilon 1} \wedge \mathbf{x}_{\epsilon 2} \wedge (\mathbf{e}_{o1} - \mathbf{e}_{o2}) \wedge (\mathbf{e}_{o2} - \mathbf{e}_{o3})\big) \\
&\quad \wedge \mathbf{I}_\infty \wedge \mathbf{e}_{o4o5o6}.
\end{aligned} \tag{3.9}$$

Setting $\mathbf{u}_\epsilon = \mathbf{x}_{\epsilon 2} - \mathbf{x}_{\epsilon 1}$ and $\mathbf{v}_\epsilon = \mathbf{x}_{\epsilon 1} \wedge \mathbf{x}_{\epsilon 2}$ gives

$$\begin{aligned}
\mathbf{l} &= \big(3\mathbf{e}_{o1} \wedge \mathbf{e}_{o2} \wedge \mathbf{e}_{o3} \wedge \mathbf{u}_\epsilon + \mathbf{v}_\epsilon \wedge (\mathbf{e}_{o1o2} - \mathbf{e}_{o1o3} + \mathbf{e}_{o2o3})\big) \wedge \mathbf{I}_\infty \wedge \mathbf{e}_{o4o5o6} \\
&= -3\,\mathbf{u}_\epsilon\,\mathbf{I}_\infty \wedge \mathbf{I}_o + \mathbf{v}_\epsilon\,\mathbf{I}_\infty \wedge \mathbf{I}_o^{\triangleright}.
\end{aligned} \tag{3.10}$$

Note that $\mathbf{u}_\epsilon$ and $\mathbf{v}_\epsilon$ correspond to the 6 Plücker coefficients of a line in $\mathbb{R}^3$. More precisely, $\mathbf{u}_\epsilon$ is the support vector of the line and $\mathbf{v}_\epsilon$ is its moment.

Proposition 3.4. *A point $\mathbf{x}$ with Euclidean coordinates $\mathbf{x}_\epsilon$ lies on the line $\mathbf{l}$ iff $\mathbf{x} \wedge \mathbf{l} = 0$.*

Proof.

$$\begin{aligned}
\mathbf{x} \wedge \mathbf{l} &= (\mathbf{x}_\epsilon + \mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3}) \wedge (-3\,\mathbf{u}_\epsilon\,\mathbf{I}_\infty \wedge \mathbf{I}_o + \mathbf{v}_\epsilon\,\mathbf{I}_\infty \wedge \mathbf{I}_o^{\triangleright}) \\
&= -3\mathbf{x}_\epsilon \wedge \mathbf{u}_\epsilon\,\mathbf{I}_\infty \wedge \mathbf{I}_o + \mathbf{x}_\epsilon \wedge \mathbf{v}_\epsilon\,\mathbf{I}_\infty \wedge \mathbf{I}_o^{\triangleright} + \mathbf{v}_\epsilon\,\mathbf{I}_\infty \wedge \big((\mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3}) \wedge \mathbf{I}_o^{\triangleright}\big) \\
&= -3(\mathbf{x}_\epsilon \wedge \mathbf{u}_\epsilon - \mathbf{v}_\epsilon)\,\mathbf{I}_\infty \wedge \mathbf{I}_o + \mathbf{x}_\epsilon \wedge \mathbf{v}_\epsilon\,\mathbf{I}_\infty \wedge \mathbf{I}_o^{\triangleright}.
\end{aligned} \tag{3.11}$$

The 6-blade $\mathbf{I}_\infty \wedge \mathbf{I}_o$ and the 5-blade $\mathbf{I}_\infty \wedge \mathbf{I}_o^{\triangleright}$ are linearly independent. Therefore, $\mathbf{x} \wedge \mathbf{l} = 0$ yields

$$\mathbf{x} \wedge \mathbf{l} = 0 \Leftrightarrow \begin{cases} \mathbf{x}_\epsilon \wedge \mathbf{u}_\epsilon = \mathbf{v}_\epsilon, \\ \mathbf{x}_\epsilon \wedge \mathbf{v}_\epsilon = 0. \end{cases} \tag{3.12}$$

As $\mathbf{x}_\epsilon, \mathbf{u}_\epsilon$ and $\mathbf{v}_\epsilon$ are Euclidean entities, (3.12) corresponds to the Plücker equations of a line [11]. □

3.2.2. Dual line. Dualizing the entity $\mathbf{l}$ consists in computing with duals:

$$\begin{aligned}
\mathbf{l}^* &= (-3\,\mathbf{u}_\epsilon\,\mathbf{I}_\infty \wedge \mathbf{I}_o + \mathbf{v}_\epsilon\,\mathbf{I}_\infty \wedge \mathbf{I}_o^{\triangleright})(-\mathbf{I}) \\
&= 3\,\mathbf{u}_\epsilon\,\mathbf{I}_\epsilon + (\mathbf{e}_{\infty 3} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 1}) \wedge \mathbf{v}_\epsilon\,\mathbf{I}_\epsilon.
\end{aligned} \tag{3.13}$$

Proposition 3.5. *A point $\mathbf{x}$ lies on the dual line $\mathbf{l}^*$ iff $\mathbf{x} \cdot \mathbf{l}^* = 0$.*

Proof. Consequence of (2.8). □

Note that a dual line $\mathbf{l}^*$ can also be constructed from the intersection of two dual planes as follows:

$$\mathbf{l}^* = \boldsymbol{\pi}_1^* \wedge \boldsymbol{\pi}_2^*. \tag{3.14}$$

3.3. Sphere

3.3.1. Primal sphere. We define a sphere $\mathbf{s}$ using four points as the 14-blade

$$\mathbf{s} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \mathbf{x}_4 \wedge \mathbf{I}_\infty^\triangleright \wedge \mathbf{I}_o^\triangleright. \tag{3.15}$$

The outer product of the points with $\mathbf{I}_\infty^\triangleright$ removes all $\mathbf{e}_{\infty 4}, \mathbf{e}_{\infty 5}, \mathbf{e}_{\infty 6}$ components of these points, i.e., the cross terms (xy, xz, and yz). The same remark holds for $\mathbf{I}_o^\triangleright$ and $\mathbf{e}_{o4}, \mathbf{e}_{o5}, \mathbf{e}_{o6}$. For clarity, we omit these terms below. We thus have

$$\mathbf{s} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \mathbf{x}_3 \wedge \Big(\frac{1}{2}(x_4^2 \mathbf{e}_{\infty 1} + y_4^2 \mathbf{e}_{\infty 2} + z_4^2 \mathbf{e}_{\infty 3}) \mathbf{I}_\infty^\triangleright \wedge \mathbf{I}_o^\triangleright - 3\mathbf{I}_\infty^\triangleright \wedge \mathbf{I}_o + \mathbf{x}_{4\epsilon} \mathbf{I}_\infty^\triangleright \wedge \mathbf{I}_o^\triangleright \Big). \tag{3.16}$$

Note the similarities with a CGA point $\mathbf{x}_{4\epsilon} + \mathbf{e}_o + \frac{1}{2}\|\mathbf{x}_{4\epsilon}\|^2 \mathbf{e}_\infty$. Then, the explicit outer product with $\mathbf{x}_3$ gives:

$$\begin{aligned} \mathbf{s} =& \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge (\mathbf{x}_{3\epsilon} \wedge \mathbf{x}_{4\epsilon} \mathbf{I}_o^\triangleright \wedge \mathbf{I}_\infty^\triangleright + 3(\mathbf{x}_{3\epsilon} - \mathbf{x}_{4\epsilon}) \mathbf{I}_o \wedge \mathbf{I}_\infty^\triangleright \\ &+ \frac{1}{2}\|\mathbf{x}_{4\epsilon}\|^2 \mathbf{x}_{3\epsilon} \mathbf{I}_\infty \wedge \mathbf{I}_o^\triangleright - \frac{1}{2}\|\mathbf{x}_{3\epsilon}\|^2 \mathbf{x}_{4\epsilon} \mathbf{I}_\infty \wedge \mathbf{I}_o^\triangleright \\ &+ \frac{3}{2}(\|\mathbf{x}_{4\epsilon}\|^2 - \|\mathbf{x}_{3\epsilon}\|^2) \mathbf{I}_o \wedge \mathbf{I}_\infty. \end{aligned} \tag{3.17}$$

Again we remark that the resulting entity has striking similarities with a point pair of CGA. More precisely, let $\mathbf{c}_\epsilon$ be the Euclidean midpoint between the two entities $\mathbf{x}_3$ and $\mathbf{x}_4$, $\mathbf{d}_\epsilon$ be the unit vector from $\mathbf{x}_3$ to $\mathbf{x}_4$, and r be half of the Euclidean distance between the two points in exactly the same way as Hitzer et al in [10], namely

$$2\mathrm{r} = |\mathbf{x}_{3\epsilon} - \mathbf{x}_{4\epsilon}|, \qquad \mathbf{d}_\epsilon = \frac{\mathbf{x}_{3\epsilon} - \mathbf{x}_{4\epsilon}}{2\mathrm{r}}, \qquad \mathbf{c}_\epsilon = \frac{\mathbf{x}_{3\epsilon} + \mathbf{x}_{4\epsilon}}{2}. \tag{3.18}$$

Then, (3.17) can be rewritten by

$$\begin{aligned} \mathbf{s} =& \mathbf{x}_1 \wedge \mathbf{x}_2 \\ & \wedge 2\mathrm{r}\Big(\mathbf{d}_\epsilon \wedge \mathbf{c}_\epsilon \, \mathbf{I}_o^\triangleright \wedge \mathbf{I}_\infty^\triangleright \\ & + 3\mathbf{d}_\epsilon \, \mathbf{I}_o \wedge \mathbf{I}_\infty^\triangleright + \frac{1}{2}\big((\mathbf{c}_\epsilon^2 + \mathrm{r}^2)\mathbf{d}_\epsilon - 2\mathbf{c}_\epsilon \mathbf{c}_\epsilon \cdot \mathbf{d}_\epsilon\big) \mathbf{I}_\infty \wedge \mathbf{I}_o^\triangleright \Big). \end{aligned} \tag{3.19}$$

The bottom part corresponds to a point pair, as defined in [10], that belongs to the round object family. Applying the same development to the two points $\mathbf{x}_1$ and $\mathbf{x}_2$ again results in round objects:

$$\mathbf{s} = -\frac{1}{6}\Big(\|\mathbf{x}_{c\epsilon}\|^2 - \mathrm{r}^2 \Big) \mathbf{I}_\epsilon \wedge \mathbf{I}_\infty \wedge \mathbf{I}_o^\triangleright + \mathbf{e}_{123} \wedge \mathbf{I}_\infty^\triangleright \wedge \mathbf{I}_o + (\mathbf{x}_{c\epsilon} \mathbf{I}_\epsilon) \wedge \mathbf{I}_\infty \wedge \mathbf{I}_o. \tag{3.20}$$

Note that $\mathbf{x}_{c\epsilon}$ corresponds to the center point of the sphere and r to its radius. It can be further simplified into

$$\mathbf{s} = \big(\mathbf{x}'_c - \frac{1}{6}\mathrm{r}^2(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 3}) \big) \mathbf{I}, \tag{3.21}$$

which is dualized to

$$\mathbf{s}^* = \mathbf{x}'_c - \frac{1}{6}\mathrm{r}^2(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 3}), \tag{3.22}$$

where $\mathbf{x}'_c$ corresponds to $\mathbf{x}_c$ without the cross terms xy, xz, yz. Since a QCGA point has no $\mathbf{e}_{o4}, \mathbf{e}_{o5}, \mathbf{e}_{o6}$ components, building a sphere with these cross terms is also valid. However, inserting these cross terms (that actually do not appear in the primal form) raises some issues in computing intersections with other objects.

Proposition 3.6. *A point* $\mathbf{x}$ *lies on the sphere* $\mathbf{s}$ *iff* $\mathbf{x} \wedge \mathbf{s} = 0$*.*

Proof. Since the components $\mathbf{e}_{\infty 4}, \mathbf{e}_{\infty 5}$ and $\mathbf{e}_{\infty 6}$ of $\mathbf{x}$ are removed by the outer product with $\mathbf{s}$ of (3.17), we ignore them to obtain

$$\begin{aligned} \mathbf{x} \wedge \mathbf{s} &= \mathbf{x} \wedge (\mathbf{s}^* \mathbf{I}) = \mathbf{x} \cdot \mathbf{s}^* \, \mathbf{I} && (3.23) \\ &= \left(\mathbf{x}_\epsilon + \mathbf{e}_{o1} + \mathbf{e}_{o2} + \mathbf{e}_{o3} + \frac{1}{2}x^2 \mathbf{e}_{\infty 1} + \frac{1}{2}y^2 \mathbf{e}_{\infty 2} + \frac{1}{2}z^2 \mathbf{e}_{\infty 3}\right) && (3.24) \\ &\quad \cdot \left(\mathbf{x'}_c - \frac{1}{6}\mathrm{r}^2(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 3})\right)\mathbf{I}, \end{aligned}$$

which can be rewritten by

$$\begin{aligned} \mathbf{x} \wedge \mathbf{s} = \Bigg(& x\mathrm{x}_c + y\mathrm{y}_c + z\mathrm{z}_c - \left(\frac{1}{2}\mathrm{x}_c^2 - \frac{1}{6}\mathrm{r}^2\right) - \left(\frac{1}{2}\mathrm{y}_c^2 - \frac{1}{6}\mathrm{r}^2\right) - \left(\frac{1}{2}\mathrm{z}_c^2 - \frac{1}{6}\mathrm{r}^2\right) \\ & - \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 \Bigg) \mathbf{I} = 0. \qquad (3.25) \end{aligned}$$

This can take a more compact form defining a sphere

$$(x - \mathrm{x}_c)^2 + (y - \mathrm{y}_c)^2 + (z - \mathrm{z}_c)^2 = \mathrm{r}^2. \qquad (3.26)$$

□

3.3.2. Dual sphere. The dualization of the primal sphere $\mathbf{s}$ gives:

$$\mathbf{s}^* = \mathbf{x'}_c - \frac{1}{6}\mathrm{r}^2(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 3}). \qquad (3.27)$$

Proposition 3.7. *A point* $\mathbf{x}$ *lies on the dual sphere* $\mathbf{s}^*$ *iff* $\mathbf{x} \cdot \mathbf{s}^* = 0$*.*

Proof. Consequence of (2.8). □

4. Quadric surfaces

This section describes how QCGA handles quadric surfaces. All QCGA objects defined in Section 3 become thus part of a more general framework.

4.1. Primal quadric surfaces

The implicit formula of a quadric surface in $\mathbb{R}^3$ is

$$F(x,y,z) = \mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + \mathrm{d}xy + \mathrm{e}xz + \mathrm{f}yz + \mathrm{g}x + \mathrm{h}y + \mathrm{i}z + \mathrm{j} = 0. \quad (4.1)$$

A quadric surface is constructed by wedging 9 points together with 5 null basis vectors as follows

$$\mathbf{q} = \mathbf{x}_1 \wedge \mathbf{x}_2 \wedge \cdots \wedge \mathbf{x}_9 \wedge \mathbf{I}_o^{\triangleright}. \qquad (4.2)$$

The multivector $\mathbf{q}$ corresponds to the primal form of a quadric surface with grade 14 and 12 components. Again 3 of these components have the same coefficient and can be combined together into the form defined by 10 coefficients a, b, ..., j, as in

$$\begin{aligned}\mathbf{q} = \mathbf{e}_{123}\Big(\big(2\mathrm{a}\mathbf{e}_{o1} + 2\mathrm{b}\mathbf{e}_{o2} + 2\mathrm{c}\mathbf{e}_{o3} + \mathrm{d}\mathbf{e}_{o4} + \mathrm{e}\mathbf{e}_{o5} + \mathrm{f}\mathbf{e}_{o6}\big) \cdot \mathbf{I}_\infty \Big) \wedge \mathbf{I}_o \\ + \big(\mathrm{g}\mathbf{e}_1 + \mathrm{h}\mathbf{e}_2 + \mathrm{i}\mathbf{e}_3\big)\mathbf{e}_{123}\,\mathbf{I}_\infty \wedge \mathbf{I}_o + \frac{\mathrm{j}}{3}\mathbf{e}_{123}\,\mathbf{I}_\infty \wedge \big((\mathbf{e}_{\infty1} + \mathbf{e}_{\infty2} + \mathbf{e}_{\infty3}) \cdot \mathbf{I}_o\big) \\ = \Big(- \big(2\mathrm{a}\mathbf{e}_{o1} + 2\mathrm{b}\mathbf{e}_{o2} + 2\mathrm{c}\mathbf{e}_{o3} + \mathrm{d}\mathbf{e}_{o4} + \mathrm{e}\mathbf{e}_{o5} + \mathrm{f}\mathbf{e}_{o6}\big) \\ + \big(\mathrm{g}\mathbf{e}_1 + \mathrm{h}\mathbf{e}_2 + \mathrm{i}\mathbf{e}_3\big) - \frac{\mathrm{j}}{3}\big(\mathbf{e}_{\infty1} + \mathbf{e}_{\infty2} + \mathbf{e}_{\infty3}\big)\Big)\,\mathbf{I} = \mathbf{q}^*\mathbf{I},\end{aligned} \tag{4.3}$$

where in the second equality we used the duality property. The expression for the dual quadric vector is therefore

$$\begin{aligned}\mathbf{q}^* = -\big(2\mathrm{a}\mathbf{e}_{o1} + 2\mathrm{b}\mathbf{e}_{o2} + 2\mathrm{c}\mathbf{e}_{o3} + \mathrm{d}\mathbf{e}_{o4} + \mathrm{e}\mathbf{e}_{o5} + \mathrm{f}\mathbf{e}_{o6}\big) \\ + \big(\mathrm{g}\mathbf{e}_1 + \mathrm{h}\mathbf{e}_2 + \mathrm{i}\mathbf{e}_3\big) - \frac{\mathrm{j}}{3}(\mathbf{e}_{\infty1} + \mathbf{e}_{\infty2} + \mathbf{e}_{\infty3}).\end{aligned} \tag{4.4}$$

Proposition 4.1. *A point* $\mathbf{x}$ *lies on the quadric surface* $\mathbf{q}$ *iff* $\mathbf{x} \wedge \mathbf{q} = 0$.

Proof.

$$\begin{aligned}\mathbf{x} \wedge \mathbf{q} &= \mathbf{x} \wedge (\mathbf{q}^*\mathbf{I}) = (\mathbf{x} \cdot \mathbf{q}^*)\mathbf{I} \\ &= \Big(\mathbf{x} \cdot \Big(- \big(2\mathrm{a}\mathbf{e}_{o1} + 2\mathrm{b}\mathbf{e}_{o2} + 2\mathrm{c}\mathbf{e}_{o3} + \mathrm{d}\mathbf{e}_{o4} + \mathrm{e}\mathbf{e}_{o5} + \mathrm{f}\mathbf{e}_{o6}\big) \\ &\quad + \big(\mathrm{g}\mathbf{e}_1 + \mathrm{h}\mathbf{e}_2 + \mathrm{i}\mathbf{e}_3\big) - \frac{\mathrm{j}}{3}(\mathbf{e}_{\infty1} + \mathbf{e}_{\infty2} + \mathbf{e}_{\infty3})\Big)\Big)\mathbf{I} \\ &= \big(\mathrm{a}x^2 + \mathrm{b}y^2 + \mathrm{c}z^2 + \mathrm{d}xy + \mathrm{e}xz + \mathrm{f}yz + \mathrm{g}x + \mathrm{h}y + \mathrm{i}z + \mathrm{j}\big)\mathbf{I}.\end{aligned} \tag{4.5}$$

This corresponds to the formula (4.1) representing a general quadric surface. □

4.2. Dual quadric surfaces

The dualization of a primal quadric surface leads to the 1-vector dual quadric surface $\mathbf{q}^*$ of (4.4). We have the following proposition whose proof is a consequence of (2.8).

Proposition 4.2. *A point* $\mathbf{x}$ *lies on the dual quadric surface* $\mathbf{q}^*$ *iff* $\mathbf{x} \cdot \mathbf{q}^* = 0$.

5. Normals and tangents

This section presents the computation of the normal Euclidean vector $\mathbf{n}_\epsilon$ and the tangent plane $\boldsymbol{\pi}^*$ of a point $\mathbf{x}$ (associated to the Euclidean point $\mathbf{x}_\epsilon = x\mathbf{e}_1 + y\mathbf{e}_2 + y\mathbf{e}_3$) on a dual quadric surface $\mathbf{q}^*$. The implicit formula of the dual quadric surface is considered as the following scalar field

$$F(x, y, z) = \mathbf{x} \cdot \mathbf{q}^*. \tag{5.1}$$

The normal vector $\mathbf{n}_\epsilon$ of a point $\mathbf{x}$ is computed as the gradient of the implicit surface (scalar field) at $\mathbf{x}$:

$$\mathbf{n}_\epsilon = \nabla F(x,y,z) = \frac{\partial F(x,y,z)}{\partial x}\mathbf{e}_1 + \frac{\partial F(x,y,z)}{\partial y}\mathbf{e}_2 + \frac{\partial F(x,y,z)}{\partial z}\mathbf{e}_3. \tag{5.2}$$

Since the partial derivative with respect to the x component is defined by

$$\frac{\partial F(x,y,z)}{\partial x} = \lim_{h\mapsto 0} \frac{F(x+h,y,z) - F(x,y,z)}{h}, \tag{5.3}$$

we have

$$\frac{\partial F(x,y,z)}{\partial x} = \lim_{h\mapsto 0} \frac{\mathbf{x}_2 \cdot \mathbf{q}^* - \mathbf{x}\cdot\mathbf{q}^*}{h} = \left(\lim_{h\mapsto 0} \frac{\mathbf{x}_2 - \mathbf{x}}{h}\right)\cdot \mathbf{q}^*, \tag{5.4}$$

where $\mathbf{x}_2$ is the point obtained by translating $\mathbf{x}$ along the x-axis by the value h. Note that $\mathbf{x}_2 - \mathbf{x}$ represents the dual orthogonal bisecting plane spanned by $\mathbf{x}_2$ and $\mathbf{x}$ (see Proposition 3.3). Accordingly, we have

$$\begin{aligned}\lim_{h\mapsto 0} \frac{\mathbf{x}_2 - \mathbf{x}}{h} &= x\mathbf{e}_{\infty 1} + y\mathbf{e}_{\infty 4} + z\mathbf{e}_{\infty 5} + \mathbf{e}_1 \\ &= (\mathbf{x}\cdot\mathbf{e}_1)\mathbf{e}_{\infty 1} + (\mathbf{x}\cdot\mathbf{e}_2)\mathbf{e}_{\infty 4} + (\mathbf{x}\cdot\mathbf{e}_3)\mathbf{e}_{\infty 5} + \mathbf{e}_1.\end{aligned} \tag{5.5}$$

This argument can also be applied to the partial derivative with respect to the y and z components. Therefore, we obtain

$$\begin{aligned}\mathbf{n}_\epsilon = &\Big(\big((\mathbf{x}\cdot\mathbf{e}_1)\mathbf{e}_{\infty 1} + (\mathbf{x}\cdot\mathbf{e}_2)\mathbf{e}_{\infty 4} + (\mathbf{x}\cdot\mathbf{e}_3)\mathbf{e}_{\infty 5} + \mathbf{e}_1\big)\cdot\mathbf{q}^*\Big)\mathbf{e}_1 \\ &+ \Big(\big((\mathbf{x}\cdot\mathbf{e}_2)\mathbf{e}_{\infty 2} + (\mathbf{x}\cdot\mathbf{e}_1)\mathbf{e}_{\infty 4} + (\mathbf{x}\cdot\mathbf{e}_3)\mathbf{e}_{\infty 6} + \mathbf{e}_2\big)\cdot\mathbf{q}^*\Big)\mathbf{e}_2 \\ &+ \Big(\big((\mathbf{x}\cdot\mathbf{e}_3)\mathbf{e}_{\infty 3} + (\mathbf{x}\cdot\mathbf{e}_1)\mathbf{e}_{\infty 5} + (\mathbf{x}\cdot\mathbf{e}_2)\mathbf{e}_{\infty 6} + \mathbf{e}_3\big)\cdot\mathbf{q}^*\Big)\mathbf{e}_3.\end{aligned} \tag{5.6}$$

On the other hand, the tangent plane at a surface point $\mathbf{x}$ can be computed from the Euclidean normal vector $\mathbf{n}_\epsilon$ and the point $\mathbf{x}$. Since the plane orthogonal distance from the origin is $\sqrt{-2(\mathbf{e}_{o1} + \mathbf{e}_{o3} + \mathbf{e}_{o3})\cdot\mathbf{x}}$, the tangent plane $\boldsymbol{\pi}^*$ is obtained as

$$\boldsymbol{\pi}^* = \mathbf{n}_\epsilon + \frac{1}{3}(\mathbf{e}_{\infty 1} + \mathbf{e}_{\infty 2} + \mathbf{e}_{\infty 3})\sqrt{-2(\mathbf{e}_{o1} + \mathbf{e}_{o3} + \mathbf{e}_{o3})\cdot\mathbf{x}}. \tag{5.7}$$

6. Intersections

Let us consider two geometric objects corresponding to dual quadrics[1] $\mathbf{a}^*$ and $\mathbf{b}^*$. Assuming that the two objects are linearly independent, i.e., $\mathbf{a}^*$ and $\mathbf{b}^*$ are linearly independent, we consider the outer product $\mathbf{c}^*$ of these two objects

$$\mathbf{c}^* = \mathbf{a}^* \wedge \mathbf{b}^*. \tag{6.1}$$

If a point $\mathbf{x}$ lies on $\mathbf{c}^*$, then

$$\mathbf{x}\cdot\mathbf{c}^* = \mathbf{x}\cdot(\mathbf{a}^*\wedge\mathbf{b}^*) = 0. \tag{6.2}$$

The inner product computation of (6.2) leads to

$$\mathbf{x}\cdot\mathbf{c}^* = (\mathbf{x}\cdot\mathbf{a}^*)\mathbf{b}^* - (\mathbf{x}\cdot\mathbf{b}^*)\mathbf{a}^* = 0. \tag{6.3}$$

[1]The term "quadric" (without being followed by surface) encompasses quadric surfaces and conic sections.

Our assumption of linear independence between $\mathbf{a}^*$ and $\mathbf{b}^*$ indicates that (6.3) holds if and only if $\mathbf{x}\cdot\mathbf{a}^* = 0$ and $\mathbf{x}\cdot\mathbf{b}^* = 0$, i.e. the point $\mathbf{x}$ lies on both quadrics. Thus, $\mathbf{c}^* = \mathbf{a}^*\wedge\mathbf{b}^*$ represents the intersection of the linearly independent quadrics $\mathbf{a}^*$ and $\mathbf{b}^*$, and a point $\mathbf{x}$ lies on this intersection if and only if $\mathbf{x}\cdot\mathbf{c}^* = 0$.

6.1. Quadric-Line intersection

For example, in computer graphics, making a Geometric Algebra compatible with a raytracer requires only to be able to compute a surface normal and a line−object intersection. This section defines the line−quadric intersection.

Similarly to (6.1), the intersection $\mathbf{x}_\pm$ between a dual line $\mathbf{l}^*$ and a dual quadric $\mathbf{q}^*$ is computed by $\mathbf{l}^*\wedge\mathbf{q}^*$. Any point $\mathbf{x}$ lying on the line $\mathbf{l}$ defined by two points $\mathbf{x}_1$ and $\mathbf{x}_2$ can be represented by the parametric formula $\mathbf{x}_\epsilon = \alpha(\mathbf{x}_{1\epsilon} - \mathbf{x}_{2\epsilon}) + \mathbf{x}_{2\epsilon} = \alpha\mathbf{u}_\epsilon + \mathbf{x}_{2\epsilon}$. Note that $\mathbf{u}_\epsilon$ could also be computed directly from the dual line $\mathbf{l}^*$ (see (3.13)). Any point $\mathbf{x}_2 \in \mathbf{l}$ can be used, especially the closest point of $\mathbf{l}$ from the origin, i.e. $\mathbf{x}_{2\epsilon} = \mathbf{v}_\epsilon\cdot\mathbf{u}_\epsilon^{-1}$. Accordingly, computing the intersection between the dual line $\mathbf{l}^*$ and the dual quadric $\mathbf{q}^*$ becomes equivalent to finding α such that $\mathbf{x}$ lies on the dual quadric, i.e., $\mathbf{x}\cdot\mathbf{q}^* = 0$, leading to a second degree equation in α, as shown in (4.1). In this situation, the problem is reduced to computing the roots of this equation. However, we have to consider four cases: the case where the line is tangent to the quadric, the case where the intersection is empty, the case where the line intersects the quadric into two points, and the case where one of the two points exists at infinity. To identify each case, we use the discriminant δ defined as:

$$\delta = \beta^2 - 4(\mathbf{x}_{2\epsilon}\cdot\mathbf{q}^*)\sum_{i=1}^{6}(\mathbf{u}\cdot\mathbf{e}_{oi})(\mathbf{q}^*\cdot\mathbf{e}_{\infty i}), \tag{6.4}$$

where

$$\begin{aligned}\beta = {}& 2\mathbf{u}\cdot\big(\mathrm{a}(x_{2\epsilon}\cdot\mathbf{e}_1)\mathbf{e}_1 + \mathrm{b}(x_{2\epsilon}\cdot\mathbf{e}_2)\mathbf{e}_2 + \mathrm{c}(x_{2\epsilon}\cdot\mathbf{e}_3)\mathbf{e}_3\big)+\\ & \mathrm{d}\big((\mathbf{u}\wedge\mathbf{e}_1)\cdot(\mathbf{x}_{2\epsilon}\wedge\mathbf{e}_2) + (\mathbf{x}_{2\epsilon}\wedge\mathbf{e}_1)\cdot(\mathbf{u}\wedge\mathbf{e}_2)\big)+\\ & \mathrm{e}\big((\mathbf{u}\wedge\mathbf{e}_1)\cdot(\mathbf{x}_{2\epsilon}\wedge\mathbf{e}_3) + (\mathbf{x}_{2\epsilon}\wedge\mathbf{e}_1)\cdot(\mathbf{u}\wedge\mathbf{e}_3)\big)+\\ & \mathrm{f}\big((\mathbf{u}\wedge\mathbf{e}_2)\cdot(\mathbf{x}_{2\epsilon}\wedge\mathbf{e}_3) + (\mathbf{x}_{2\epsilon}\wedge\mathbf{e}_2)\cdot(\mathbf{u}\wedge\mathbf{e}_3)\big) + \mathbf{q}^*\cdot\mathbf{u}_\epsilon.\end{aligned} \tag{6.5}$$

If $\delta < 0$, the line does not intersect the quadric (the solutions are complex). If $\delta = 0$, the line and the quadric are tangent. If $\delta > 0$ and $\sum_{i=1}^{6}(\mathbf{u}\cdot\mathbf{e}_{oi})(\mathbf{q}^*\cdot\mathbf{e}_{\infty i}) = 0$, we have only one intersection point (linear equation). Otherwise, we have two different intersection points $\mathbf{x}_\pm$ computed by

$$\mathbf{x}_\pm = \mathbf{u}(-\beta\pm\sqrt{\delta})/\Big(2\sum_{i=1}^{6}(\mathbf{u}\cdot\mathbf{e}_{oi})(\mathbf{q}^*\cdot\mathbf{e}_{\infty i})\Big) + \mathbf{x}_{2\epsilon}. \tag{6.6}$$

7. Discussion

7.1. Limitations

The construction of quadric surfaces by the wedge of conformal points presented in Sections 3 and 4 is a distinguished property of QCGA that is missing

Table 2. Comparison of properties between DPGA, DCGA, and QCGA. The symboli $\bullet$ stands for "capable", $\circ$ for "incapable" and $\oslash$ for "unknown".

Framework	DPGA [6]	DCGA [8]	**QCGA**
vector space dimensions	8	10	15
construction from wedge of points	$\circ$	$\circ$	$\bullet$
quadrics intersection	$\circ$	$\circ$	$\bullet$
quadric plane intersection	$\bullet$	$\bullet$	$\bullet$
versors	$\bullet$	$\bullet$	$\oslash$
Darboux cyclides	$\circ$	$\bullet$	$\circ$

in DPGA and DCGA. However, QCGA also faces some limitations that do not affect DPGA and DCGA, as summarized in Table 2.

First, DPGA and DCGA are known to be capable of transforming all objects by versors [6, 8] whereas it is not yet clear whether objects in QCGA can be transformed using versors. An extended version of CGA versors can be used to transform lines in QCGA (and probably all round and flat objects of CGA), but more investigation is needed. Second, the number of basis elements spanned by QCGA is $2^{15}(\simeq 32,000)$ components for a full multivector. Although multivectors of QCGA are in reality almost always very sparse, this large number of elements may cause implementation issues (see Section 7.2). It also requires some numerical care in computation, especially during the wedge of 9 points. This is because some components are likely to be multiplied at the power of 9.

7.2. Implementations

There exist many different implementations of Geometric Algebra, however, very few can handle dimensions higher than 8 or 10. This is because higher dimensions bring a large number of elements of the multivector used, resulting in expensive computation. In many cases, the computation then becomes impossible in practice. QCGA has a 15 vector space dimension and hence requires some specific care during the computation.

We conducted our tests with an enhanced version of Breuils et al. [1, 2] which is based on a recursive framework. We remark that most of the products involved in our tests were the outer products between 14-vectors and 1-vectors, applying one of the less time consuming products of QCGA. Indeed, QCGA with vector space dimension of 15 has 2^{15} elements and this number is 1,000 times as large as that of elements for CGA with vector space dimension of 5 (CGA with vector space dimension of 5 is needed for the equivalent operations with QCGA with dimension of 15). The computational time required for QCGA, however, did not need 1,000 times but only 70 times of that for CGA. This means that the computation of QCGA runs in reasonable time on the enhanced version of Breuils et al. [1, 2]. More detailed analysis in this direction is left for future work. Figure 1 depicts a few examples generated with our OpenGL renderer based on the outer product null-space voxels and our ray-tracer.

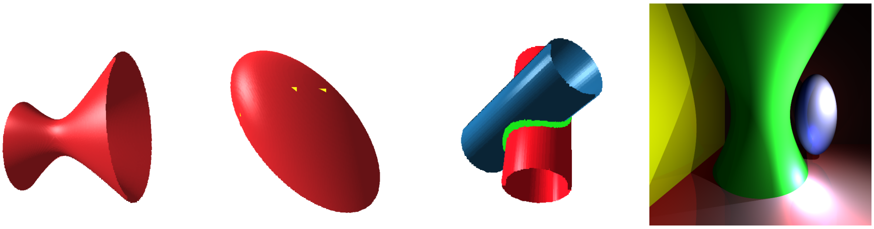

FIGURE 1. Example of our construction of QCGA objects. From left to right: a dual hyperboloid built from its equation, an ellipsoid built from its control points (in yellow), the intersection between two cylinders, and a hyperboloid with an ellipsoid and planes (the last one was computed with our ray-tracer).

8. Conclusion

This paper presented a new Geometric Algebra framework, Quadric Conformal Geometric Algebra (QCGA), that handles the construction of quadric surfaces using the implicit equation and/or control points. QCGA naturally includes CGA objects and generalizes some dedicated constructions. The intersection between objects in QCGA is computed using only outer products. This paper also detailed the computation of the tangent plane and the normal vector at a point on a quadric surface. Although QCGA is defined in a high dimensional space, most of the computations run in relatively low dimensional subspaces of this framework. Therefore, QCGA can be used for numerical computations in applications such as computer graphics.

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