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## Teaching and learning continuity with technologies

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*We developed a digital tool aiming at introducing the concept of – local - continuity together with its formal definition for Tunisian students at the end of secondary school. Our approach is a socioconstructivist one, mixing conceptualisation in the sense of Vergnaud together with Vygotski's concepts of mediation and ZPD. In the paper, we focus on the design of the tool and we give some flashes about students' productions with the tool and teachers' discourses in order to foster students' understanding of the continuity.*

*Keywords: teaching and learning of analysis and calculus, novel approaches to teaching, continuity, digital technologies*

The definition of continuity of functions at a given point, together with the concept of continuity, remains a major difficulty in the teaching and learning of analysis. There is a dialectic between the definition and the concept itself which make necessary the introduction of the two aspects together.

The definition of continuity brings FUG aspects in the sense of Robert (1982). This means first that it permits to formalize (F) the concept of continuity. But it also allows to unify (U) several different images (or situations) of continuity encountered by students: in Tall and Vinner (1981), several emblematic situations of continuity are established (see below) and the definition aims at unifying all these different kinds of continuity. Moreover, the definition of continuity allows generalisations (G) to all other numerical functions, not already encountered and not necessarily with graphical representations, or more general functions inside other spaces of functions. As Robert (1982) stresses for the definition of limit of sequences, notions which bring FUG aspects must be introduced with a specific attention to mediations and especially the role of the teacher.

Our ambition is then to design a technological tool which allows on one hand students activities concerning the two aspects of continuity and, on the other hand, allows the teacher to introduce the concept of continuity with its formal definition, referring to the activities developed on the technological tool. As it was noticed in the first INDRUM conference, papers about introduction of technologies in the teaching of analysis remain very few.

We first come back to well-known concept images and concept definitions of continuity. Then, we explain our theoretical frame about conceptualisation and mathematical activities. This theoretical frame leads us to the design of the technological tool which brings most of the aspects we consider important for the conceptualisation of continuity. Due to the text constraints, the results of the paper

are mostly in term of the design itself and the way the tool encompasses our theoretical frame and our hypotheses about conceptualisation (with tasks, activities and opportunities for mediations). Then, we can give some flashes about students' activities with the software and also teachers' discourses to introduce the definition of continuity, based on students' mathematical activities on the software.

## CONCEPT IMAGES AND CONCEPT DEFINITIONS OF CONTINUITY

No one can speak about continuity without referring to Tall and Vinner's paper about concept images and concept definitions in mathematics, whose particular reference is about limits and continuity (Tall and Vinner, 1981). Tall considers that the concept definition is one part of the total concept image that exists in our mind. Additionally, it is understood that learners enter their acquisition process of a newly introduced concept with preexisting concept images.

Sierpiska (1992) used the notion of epistemological obstacles regarding some properties of functions and especially the concept of limit. Epistemological obstacles for continuity are very close to those observed for the concept of limit and they can be directly related to students' concept images, as a specific origin of these conceptions (El Bouazzaoui, 1988). One of these obstacles can be associated to what we call a *primitive concept image*: it is a geometrical and very intuitive conception of continuity, related to the aspects of the curve. With this concept image, continuity and derivability are often mixed and continuity means mainly that the curve is smooth and have no angles. Historically, this primitive conception leads Euler to introduce a definition of continuity based on algebraic representations of functions. This leads to a second epistemological obstacle: a continuous function is given by only one algebraic expression, which can be called the *algebraic concept image* of continuity. This conception has led to a new obstacle with the beginning of Fourier's analysis. Then, a clear definition is necessary. This definition comes with Cauchy and Weierstrass and it is close to our actual formal definition.

We also refer to Bkouche (1996) who identifies three points of view about continuity of functions which are more or less connected to the epistemological obstacles we have highlighted. The first one is a *cinematic point of view*. Bkouche says that the variable pulls the function with this *dynamic* concept image. The other one is an *approximation point of view*: the desired degree of approximation of the function pulls the variable. This last point of view is more *static* and leads easily to the formal definition of continuity. These two points of view are also introduced by Robert (1982) when she studies the introduction of the formal definition of limit (for sequences). A third point of view is also identified by Bkouche that is the *algebraic point of view*, which is about algebraic rules, without any idea of the meaningful of these rules.

At last, we refer to more recent papers and specifically the one of Hanke and Schafer (2017) about continuity in the last CERME congress. Their review of central papers

on concept images about students' conceptions of continuity leads to a classification of the eight possible mental images that are reported in the literature: *I : Look of the graph of the function : "A graph of a continuous function must be connected"* - *II : Limits and approximation : "The left hand side and right hand side limit at each point must be equal"* - *III : Controlled wiggling : "If you wiggle a bit in  $x$ , the values will only wiggle a bit, too"* - *IV : Connection to differentiability : "Each continuous function is differentiable"* - *V : General properties of functions : "A continuous function is given by one term and not defined piecewise"* - *VI : Everyday language : "The function continues at each point and does not stop"* - *VII : Reference to a formal definition : "I have to check whether the definition of continuity applies at each point"* - *VIII : Miscellaneous*

We can recognize some of the previous categories, even if some refinements are brought. Mainly, concept images I, II, IV and VI can be close to the primitive concept image whereas VII refers to the formal definition and V seems to refer to the algebraic approach of continuity.

## CONCEPTUALISATION OF CONTINUITY

We base our research work on these possible concepts image and concepts definition of continuity. However, we are more interested in conceptualisation, as the process which describes the development of students' mathematical knowledge. Conceptualisation in our sense has been mainly introduced by Vergnaud (1990) and it has been extended within an activity theoretical frame developed in the French didactic of mathematics. These developments articulate two epistemological approaches: that of mathematics didactics and that of developmental cognitive psychology as it is discussed and developed in Vandebrouck (2018).

Broadly, conceptualisation means that the developmental process occurs within students' actions over a class of mathematical situations, characteristic of the concept involved. This class of situations brings technical tasks – direct application of the concept involved - as well as tasks with adaptations of this concept. A list of such adaptations can be found in Horoks and Robert (2007): for instance mix between the concept and other knowledge, *conversions between several registers of representations* (Duval 1995), *use of different points of view*, etc. Tasks that require these adaptations of knowledge or concepts are called complex tasks. These ones encourage conceptualisation, because students become able to develop high level activities allowing availability and flexibility around the relevant concept.

A level of conceptualisation refers to such a class of situations, in a more modest sense and with explicit references to scholar curricula. In this paper, the level of conceptualisation refers to the end of scientific secondary school in Tunisia or the beginning of scientific university in France. It supposes enough activities which can permit the teacher to introduce the formal definition of continuity together with the sense of the continuity concept. The aim is not to obtain from students a high

technicity about the definition itself – students are not supposed to establish or to manipulate the negation of the definition for instance. However, this level of conceptualisation supposes students to access the FUG aspects of the definition of continuity.

Of course, we also build on instrumental approach and instrumentation as a sub process of conceptualisation (Rabardel, 1995). Students' cognitive construction of knowledge (specific schemes) arise during the complex process of instrumental genesis in which they transform the artifact into an instrument that they integrate within their activities. Artigue (2002) says that it is necessary to identify the new potentials offered by instrumented work, but she also highlights the importance of identifying the constraints induced by the instrument and the instrumental distance between instrumented activities and traditional activities (in paper and pencil environment). Instrumentation theory also deals with the complexity of instrumental genesis.

We also refer to Duval's idea of visualisation as a contribution of the conceptualisation process (even if Duval and Vergnaud have not clearly discussed this point inside their frames). However, the technological tool brings new dynamic representations, which are different from static classical figures in paper and pencil environment. These new representations lead to enrich students' activities – mostly in term of recognition - bringing specific visualization processes. Duval argues that visualization is linked to visual perception, and can be produced in any register of representation. He introduces two types of visualization, namely the iconic and the non-iconic, saying that in mathematical activities, visualization does not work with iconic representations (Duval, 1999).

At last, we refer on Vygotsky (1986) who stresses the importance of mediations within a student's zone of proximal developmental (ZPD) for learning (scientific concepts). Here, we also draw on the double approach of teaching practices as a part of French activity theory coming from Robert and Rogalski (2005). The role of the teacher' mediations is specifically important in the conceptualisation process, especially because of the FUG aspects of the definition of continuity (as we have recalled above).

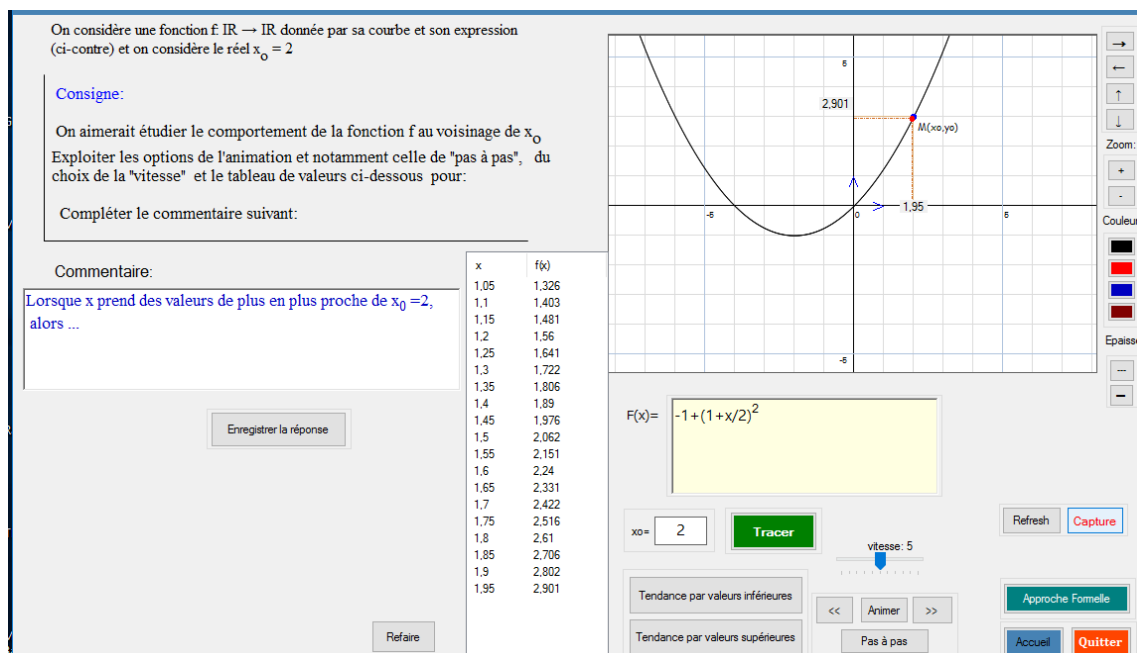
First of all, we refine the notion of mediation by adding a distinction between procedural and constructive mediations in the context of the dual regulation of activity. Procedural mediations are object oriented (oriented towards the resolution of the tasks), while constructive mediations are more subject oriented. We also distinguish individual (to pairs of students) and collective mediations (to the whole class).

Secondly, we use the notion of proximities (Bridoux, Grenier-Boley, Hache and Robert, 2016) which are discourses' elements that can foster students' understanding – and then conceptualisation - according to their ZPD and their own activities in

progress. In this sense, our approach is close to the one of Bartolini Bussi and Mariotti (2008) with their Theory of Semiotic Mediations. However, we do not refer explicitly at this moment to this theory which supposes a focus on signs and a more complex methodology than ours. According to us, the proximities characterize the attempts of alignment that the teacher operates between students' activities (what has been done in class) and the concept at stake. We therefore study the way the teacher organizes the movements between the general knowledge and its contextualized uses: we call ascending proximities those comments which explicit the transition from a particular case to a general theorem/property; descending proximities are the other way round; horizontal proximities consist in repeating in another way the same idea or in illustrating it.

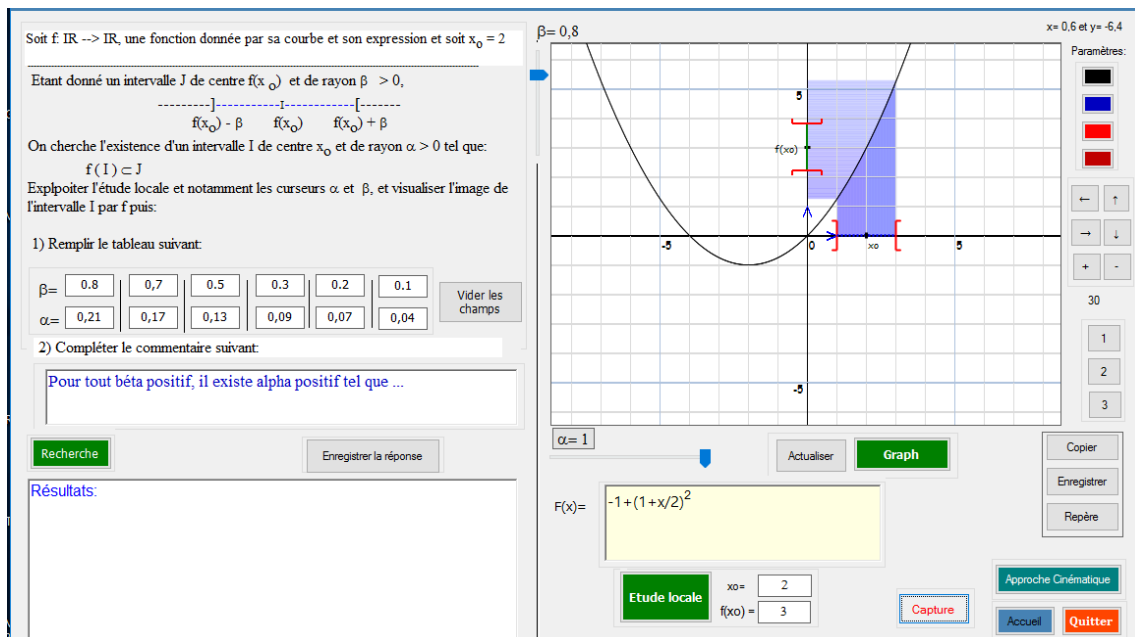
## DESIGN OF THE TECHNOLOGICAL TOOL

The technological tool called "TIC-Analyse" is designed to grasp most of the aspects which have been highlighted above. First of all, it is designed to foster students' activities about continuity aspects in the two first points of view identified by Bkouche: several functions are manipulated – continuous or not – and for each of them, two windows are in correspondence. In one of the window, the cinematic-dynamical point of view is highlighted (figure 1) whereas in the second window the approximation-static point of view is highlighted (figure 2).



**Figure 1: two windows for a function, the dynamic point of view about continuity**

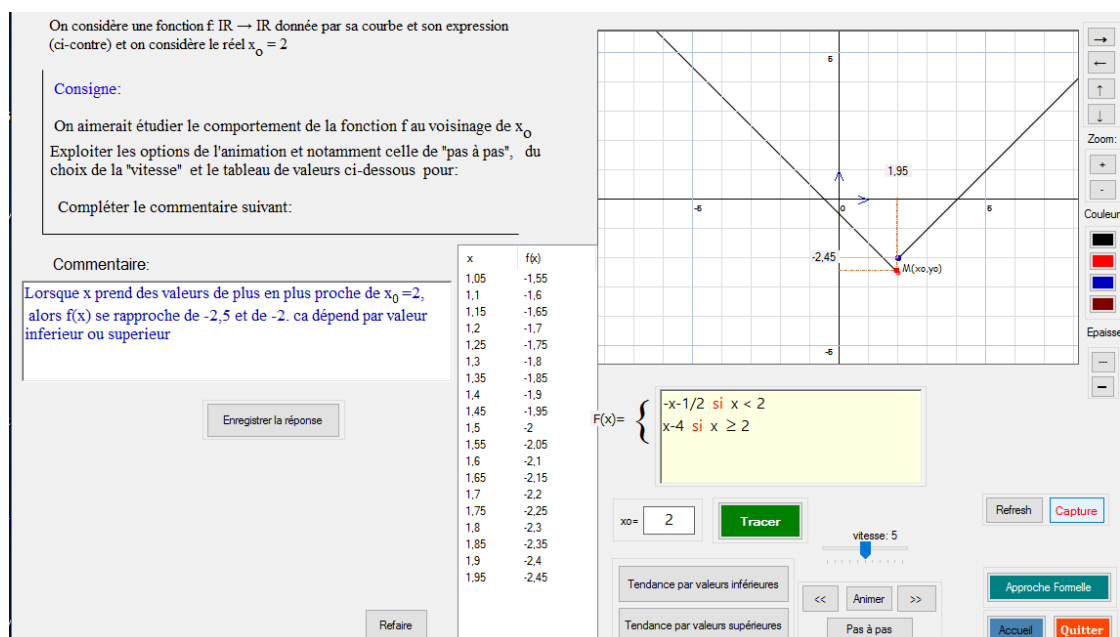
The correspondence between the two points of view is in coherence with Tall's idea of incorporation of the formal definition into the pre-existing students' concept images. It is also in coherence with the importance for students to deal with several points of view for the conceptualisation of continuity (adaptations).



**Figure 2: two windows for a function, the static points of view about continuity**

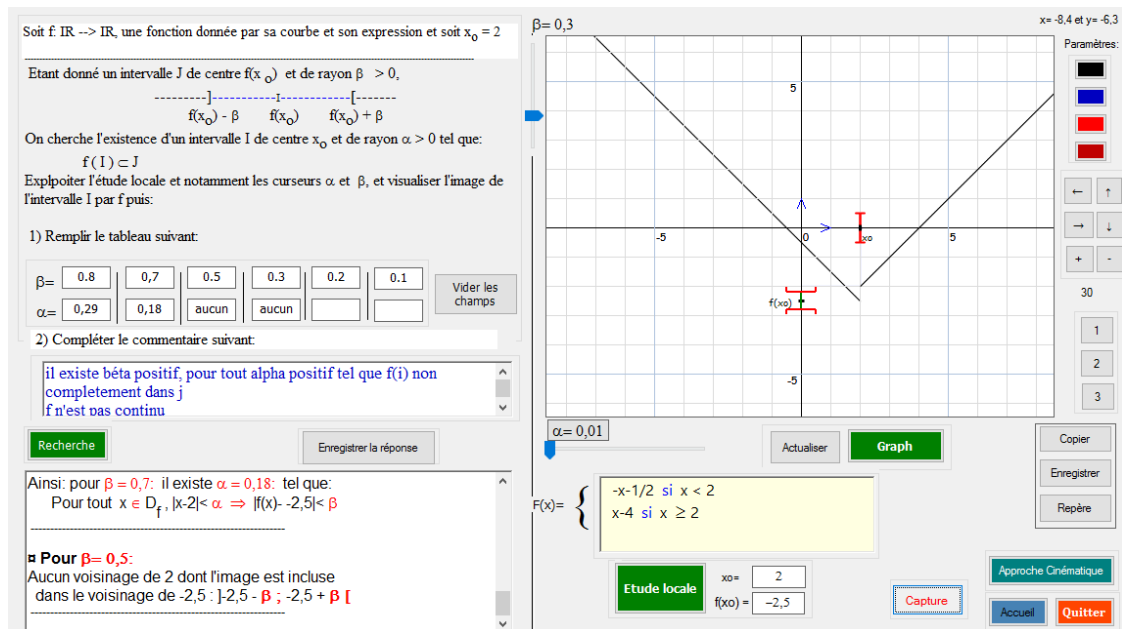
In second, the functions at stake in the software are extracted from the categories of Tall and Vinner (1981). For instance, we have chosen a continuous function which is defined by two different algebraic expressions, to avoid the algebraic concept image of continuity and to avoid the amalgam between continuity and derivability. We also have two kinds of discontinuity, smooth and with angle.

There is an emphasis not only on algebraic representations of functions in order to avoid algebraic conceptions of functions. Three registers of representations of functions (numerical, graphical and algebraic) are coordinated to promote students' activities about conversions between registers (adaptations).



**Figure 3: example of commentary given by a pair of students in the dynamic window**

The design of the software is coherent with the instrumental approach mostly in the sense that the instrumental distance between the technological environment, the given tasks, and the traditional paper and pencil environment is reduced. However the software produces dynamic new representations – a moving point on the curve associated to a numerical table of values within the dynamic window; two static intervals, one being included or not in the other, for the static window – occurring non iconic visualisations which intervene in the conceptualisation process.



**Figure 4: example of commentary given by a pair of students in the static window**

The software promotes students' actions and activities about given tasks: in the dynamic window, they are supposed to command the dynamic point on the given curve – corresponding to the given algebraic expression. They can observe the numerical values of coordinates corresponding to several discrete positions of the point and they must fill a commentary with free words about continuity aspects of the function at the given point (figures 1, 3). In the static window, they must fill the given array with values of  $\alpha$ , the  $\beta$  being given by the software (figures 2, 4). Then, they have to fill a commentary which begins differently according to the situation (continuity or not) and the  $\alpha$  they have found (figures 4, 5).

As we have mentioned in our theoretical frame, students are not supposed with these tasks and activities to get the formal definition by themselves. However, students are supposed to have developed enough knowledge in their ZPD so that the teacher can introduce the definition together with the sense and FUG aspects of continuity.

## STUDENTS ACTIVITIES AND TEACHER'S PROXIMITIES

The students work by pair on the tool. The session is a one hour session but four secondary schools with four teachers are involved. Students have some concept images of continuity but nothing has been thought about the formal definition. The



teacher is supposed to mediate students' activities on the given tasks. Students are not supposed to be in a total autonomy during the session according to our socio constructivist approach.

**Figure 5: example of commentary given by a pair of students in the static window**

We have collected video screen shots, videos of the session (for each schools) and recording of students' exchanges in some pairs. Students' activities on each tasks are identified, according to the tasks' complexity (mostly kinds of adaptations), their actions and interactions with computers and papers (written notes), the mediations they receive (procedural or constructive mediations, individual or collective, from the tool, the pairs or the teacher) and the discourses' elements seen as "potential" proximities proposed by the teacher.

It appears that the teacher mostly gives collective procedural mediations to introduce the given tasks, to assure an average progression of the students and to take care of the instrumental process. Some individuals mediations are only technical ones ("you can click on this button"). Some collective mediations are most constructive such as "now, we are going to see a formal approach. We are going to see again the four activities (ie tasks) but with a new approach which we are going to call formal approach...". The constructive mediations are not tasks oriented but they aim at helping students to organize their new knowledge and they contribute to the aimed conceptualisation according to our theoretical approach.

As examples of students' written notes (as traces of activities), we can draw on figure 3 and 4. A pair of students explains the dynamic non-continuity with their words "when x takes values more and more close to 2 then f(x) takes values close to -2,5 and -2. It depends whether it's lower or higher" (figure 3) which is in coherence with the primitive concept image of continuity. The same pair of students explains the non-continuity in relation to what they can observe on the screen: "there exists beta

positive, for all  $\alpha$  positive – already proposed by the tool in case of non-continuity – such that  $f(i)$  not completely in  $j$ ...  $f$  is not continuous”. We can note that the students are using “completely” to verbalize that the intersection of the two intervals is not empty. However, the inclusiveness of an interval into another one is not expected as a formalized knowledge at this level of conceptualisation. Their commentary is acceptable. Students are expressing what they have experimented several times : for several values of  $\beta$  ( $\beta = 0,3$  in figure 4), even with  $\alpha$  very small ( $\alpha = 0,01$  in figure 4), the image of the interval  $]2- \alpha, 2+ \alpha[$  is not included in  $] -2,5- \beta, -2,5+ \beta[$ . Concerning a case of continuity, the students are also able to write an acceptable commentary (figure 5) “for all  $\beta$  positive, their exists  $\alpha$  positive – already proposed by the tool in case of continuity – such that  $f(i)$  is included in  $j$ .”

Students’ activities on the given tasks are supposed to help the teacher to develop proximities with the formal definition. It is really observed that some students are able to interact spontaneously with the teacher when he wants to write the formal definition on the blackboard. This is interpreted as a sign that the teacher’s discourse encounters these students’ ZPD. Then the observed proximities seem to be horizontal ones: the teacher reformulates several times the students’ propositions in a way which lead gradually to the awaited formal definition, for instance “so, we are going to reformulate, for all  $\beta$  positive, their exists  $\alpha$  positive, such that if  $x$  belong to a neighbour of  $\alpha$  ... we can note it  $x_0 - \alpha, x_0 + \alpha$ ...”

Of course, it is insufficient to ensure proof and effectiveness of our experimentation. The conceptualisation of continuity is an ongoing long process with is only initiated by our teaching process. However, we want to highlight here the important role of the teacher and more generally the importance of mediations in the conceptualisation process of such a complex concepts. We only have presented the beginning of our experimentation. It is completed by new tasks on the tool which are designed to come back on similar activities and to continue the conceptualisation process.

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