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SPECIAL VALUES OF THE BELL POLYNOMIALS OF THE SECOND KIND FOR SOME SEQUENCES AND FUNCTIONS

FENG QI, DA-WEI NIU, DONGKYU LIM, AND YONG-HONG YAO

ABSTRACT. In the paper, the authors concisely review some closed formulas and applications of special values of the Bell polynomials of the second kind for some special sequences and elementary functions, explicitly present closed formulas for those sequences investigated in [F. T. Howard, *A special class of Bell polynomials*, Math. Comp. **35** (1980), no. 151, 977–989; Available online at <https://doi.org/10.2307/2006208>], and newly establish some closed formulas for some special values of the Bell polynomials of the second kind.

1. REVIEW OF CLOSED FORMULAS FOR THE BELL POLYNOMIALS

In [4, Definition 11.2] and [5, p. 134, Theorem A], the Bell polynomials of the second kind, denoted by $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $n \geq k \geq 0$, are defined by

$$B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) = \sum_{\substack{1 \leq i \leq n, \ell_i \in \{0\} \cup \mathbb{N} \\ \sum_{i=1}^n i \ell_i = n \\ \sum_{i=1}^n \ell_i = k}} \frac{n!}{\prod_{i=1}^{n-k+1} \ell_i!} \prod_{i=1}^{n-k+1} \left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula, see [4, Theorem 11.4] and [5, p. 139, Theorem C], can be described in terms of $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ by

$$\frac{d^n}{dx^n} f \circ h(x) = \sum_{k=0}^n f^{(k)}(h(x)) B_{n,k}(h'(x), h''(x), \dots, h^{(n-k+1)}(x)). \quad (1.1)$$

The identity

$$B_{n,k}(abx_1, ab^2x_2, \dots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \quad (1.2)$$

for $n \geq k \geq 0$ and $a, b \in \mathbb{C}$ can be found in [4, p. 412] and [5, p. 135].

There have been a number of literature, such as [1, 4, 5, 16, 17, 18, 19, 20, 83, 84, 85, 86], dedicated to the general theory of the Bell polynomials of the second kind $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$.

In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit.

In this paper, we pay our main attention on closed and explicit formulas, rather than recursive identities like (1.2), of the Bell polynomials of the second kind

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$B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $x_1, x_2, \dots, x_{n-k+1}$ to be replaced by some special sequences and elementary functions.

We now start out to concisely review some known closed formulas and applications of $B_{n,k}(x_1, x_2, \dots, x_{n-k+1})$ for $x_1, x_2, \dots, x_{n-k+1}$ to be replaced by some special sequences and elementary functions.

1.1. To find general formulas of the n th derivatives for functions of the type $f(e^x)$, such as $\frac{1}{e^{\pm x} \pm 1}$, by the Faà di Bruno formula (1.1), we need to compute

$$\begin{aligned} & B_{n,k}((e^{\pm x})', (e^{\pm x})'', \dots, (e^{\pm x})^{(n-k+1)}) \\ &= B_{n,k}(\pm e^{\pm x}, (\pm 1)^2 e^{\pm x}, \dots, (\pm 1)^{n-k+1} e^{\pm x}) \\ &= (\pm 1)^n e^{\pm kx} B_{n,k}(1, 1, \dots, 1), \end{aligned}$$

where we used in the last step the identity (1.2). In [5, p. 135], it is given that

$$B_{n,k}(1, 1, \dots, 1) = S(n, k), \quad (1.3)$$

where $S(n, k)$ denotes the Stirling numbers of the second kind. Consequently, by the Faà di Bruno formula (1.1) and the identities (1.2) and (1.3), for example, we can easily obtain

$$\left(\frac{1}{e^{\pm x} \pm 1} \right)^{(n)} = (\pm 1)^n \sum_{k=0}^n (-1)^k k! S(n, k) \frac{e^{\pm kx}}{(e^{\pm x} \pm 1)^{k+1}}.$$

For more information on applications of the identity (1.3), please read the papers [8, 9, 10, 12, 15, 23, 25, 35, 40, 41, 49, 52, 62, 65, 70, 81, 82, 87, 88, 90] and closely related references therein.

1.2. To acquire general formulas of the n th derivatives for functions of the type $f(\ln x)$, such as $\frac{1}{\ln(1+x)}$, by the Faà di Bruno formula (1.1) and the identity (1.2), we need to compute

$$\begin{aligned} & B_{n,k}([\ln(1+x)]', [\ln(1+x)]'', \dots, [\ln(1+x)]^{(n-k+1)}) \\ &= B_{n,k}\left(\frac{1}{1+x}, \frac{-1}{(1+x)^2}, \dots, \frac{(-1)^{n-k}(n-k)!}{(1+x)^{n-k+1}}\right) \\ &= \frac{(-1)^{n-k}}{(1+x)^n} B_{n,k}(0!, 1!, \dots, (n-k)!). \end{aligned}$$

In [5, p. 135], it is listed that

$$B_{n,k}(0!, 1!, 2!, \dots, (n-k)!) = (-1)^{n-k} s(n, k), \quad (1.4)$$

where $s(n, k)$ denotes the Stirling numbers of the first kind. Consequently, by virtue of the Faà di Bruno formula (1.1) and the identities (1.2) and (1.4), as an example, we have

$$\left[\frac{1}{\ln(1+x)} \right]^{(n)} = \frac{1}{(1+x)^n} \sum_{k=0}^n \frac{(-1)^k k! s(n, k)}{\ln^{k+1}(1+x)}.$$

For more information on applications of the identity (1.4), please refer to the papers [22, 29, 30, 32, 33, 34, 36, 62, 66, 75, 76, 77] and closely related references therein.

1.3. To establish general formulas of the n th derivatives for functions of the type $f(x^\alpha)$ for $\alpha \in \mathbb{R}$, such as e^{x^2} and $\sin \sqrt{a+bx}$, by the Faà di Bruno formula (1.1) and the identity (1.2), we need to calculate

$$B_{n,k}((x^\alpha)', (x^\alpha)'', \dots, (x^\alpha)^{(n-k+1)}) = x^{k\alpha-n} B_{n,k}(\langle \alpha \rangle_1, \langle \alpha \rangle_2, \dots, \langle \alpha \rangle_{n-k+1}),$$

where

$$\langle \alpha \rangle_n = \prod_{k=0}^{n-1} (\alpha - k) = \begin{cases} \alpha(\alpha-1) \cdots (\alpha-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

is called [73] the falling factorial. In [69], it was discovered that

$$B_{n,k}(\langle \alpha \rangle_1, \langle \alpha \rangle_2, \dots, \langle \alpha \rangle_{n-k+1}) = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \langle \alpha \ell \rangle_n. \quad (1.5)$$

The formula (1.5) has an equivalent form

$$\begin{aligned} B_{n,k} \left(1, 1-\lambda, (1-\lambda)(1-2\lambda), \dots, \prod_{\ell=0}^{n-k} (1-\ell\lambda) \right) \\ = \frac{(-1)^k}{k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \prod_{q=0}^{n-1} (\ell - q\lambda) \end{aligned} \quad (1.6)$$

which was earlier obtained in [61]. For more information on applications of the formula (1.6), please see [39, 44, 45, 56, 61, 63, 67, 69] and closely related references.

Taking $\lambda = 0$ in (1.6) recovers 1.3. Letting $\lambda = -1$ in (1.6) and simplifying give

$$\begin{aligned} B_{n,k}(1!, 2!, 3!, \dots, (n-k+1)!) \\ = (-1)^k \frac{n!}{k!} \sum_{\ell=1}^k (-1)^\ell \binom{k}{\ell} \binom{n+\ell-1}{\ell-1} = \binom{n-1}{k-1} \frac{n!}{k!} \end{aligned} \quad (1.7)$$

which is the Lah numbers collected in [4, p. 450] and [5, p. 135]. For more information on applications of the formula (1.7), please refer to the papers [14, 28, 30, 37, 38, 48, 51, 55, 58, 60, 61, 62, 63, 69, 71, 73, 74] and closely related references therein.

By the way, the n th derivatives for functions of the type $f(x^n)$ were recursively, not explicitly, discussed in [20, Section 4].

1.4. In [54, Theorem 4.1] and [78, Section 3], it was found little by little that

$$B_{n,k}(x, 1, 0, \dots, 0) = \frac{1}{2^{n-k}} \frac{n!}{k!} \binom{k}{n-k} x^{2k-n}, \quad (1.8)$$

where $\binom{0}{0} = 1$ and $\binom{p}{q} = 0$ for $q > p \geq 0$. Consequently, as an example, we have

$$(e^{x^2})^{(n)} = e^{x^2} \frac{n!}{(2x)^n} \sum_{k=0}^n \binom{k}{n-k} \frac{(2x)^{2k}}{k!}.$$

Taking $\alpha = 2$ in (1.5) or taking $\lambda = \frac{1}{2}$ in (1.6), making use of (1.2), letting $x = 1$ in (1.8), and comparing yield the sum

$$\sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \langle 2\ell \rangle_n = \frac{n!}{2^{n-2k}} \binom{k}{n-k}.$$

For detailed information on applications of the formula (1.8), please refer to the papers [24, 31, 43, 45, 51, 54, 56, 59, 61, 68, 69, 72, 74, 78, 79, 82, 92] and closely related references therein.

1.5. In [74, Theorem 1.4], it was presented that

$$\begin{aligned} & B_{n,k}(g'(x), g''(x), \dots, g^{(n-k+1)}(x)) \\ &= (-1)^{n+k} [2(n-k) - 1]!! \left(\frac{b}{2}\right)^n \binom{2n-k-1}{2(n-k)} \frac{1}{(a+bx)^{n-k/2}}, \end{aligned} \quad (1.9)$$

where $g(x) = \sqrt{a+bx}$ for $a, b \in \mathbb{R}$ and $b \neq 0$. Setting $a = 0$ and $b = 1$ and letting $x \rightarrow 1$ in (1.9) result in

$$B_{n,k}((-1)!!, 1!!, 3!!, \dots, [2(n-k) - 1]!!) = \binom{2n-k-1}{2(n-k)} [2(n-k) - 1]!!. \quad (1.10)$$

Taking $\alpha = \frac{1}{2}$ in (1.5) or taking $\lambda = 2$ in (1.6), simplifying, and comparing with (1.10) arrive at the sum

$$\sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \left\langle \frac{\ell}{2} \right\rangle_n = (-1)^n \frac{k! [2(n-k) - 1]!!}{2^n} \binom{2n-k-1}{2(n-k)}.$$

By virtue of the formula (1.1), the identity (1.2), and the formula (1.10) in sequence, we derived in [69] five closed formulas for the (modified) spherical Bessel functions $j_n(z)$, $y_n(z)$, $i_n^{(1)}(z)$, $i_n^{(2)}(z)$, and $k_n(z)$, whose definitions can be found in [21, p. 262–266], from considering the n th derivatives of the functions

$$\begin{aligned} & \cos \sqrt{z^2 - 2zt}, \quad \sin \sqrt{z^2 - 2zt}, \quad \cosh \sqrt{z^2 + 2izt}, \\ & \sinh \sqrt{z^2 + 2izt}, \quad \exp(-\sqrt{z^2 + 2izt}) \end{aligned}$$

with respect to t . For more information on applications of the formulas (1.9) and (1.10), please refer to the papers [39, 57, 62, 69, 71, 74] and closely related references therein.

1.6. In [26, Theorem 1.2], it was derived that

$$\begin{aligned} & B_{n,k} \left(-\sin x, -\cos x, \sin x, \cos x, \dots, \cos \left[x + \frac{(n-k+1)\pi}{2} \right] \right) \\ &= \frac{(-1)^k \cos^k x}{k!} \sum_{\ell=0}^k \frac{(-1)^\ell}{(2 \cos x)^\ell} \binom{k}{\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q-\ell)^n \cos \left[(2q-\ell)x + \frac{n\pi}{2} \right] \end{aligned} \quad (1.11)$$

and

$$\begin{aligned} & B_{n,k} \left(\cos x, -\sin x, -\cos x, \sin x, \dots, \sin \left[x + \frac{(n-k+1)\pi}{2} \right] \right) \\ &= \frac{(-1)^k \sin^k x}{k!} \sum_{\ell=0}^k \binom{k}{\ell} \frac{1}{(2 \sin x)^\ell} \sum_{q=0}^{\ell} (-1)^q \binom{\ell}{q} (2q-\ell)^n \cos \left[(2q-\ell)x + \frac{(n-\ell)\pi}{2} \right]. \end{aligned} \quad (1.12)$$

Since $\sin(x \pm \frac{\pi}{2}) = \pm \cos x$ and $\cos(x \pm \frac{\pi}{2}) = \mp \sin x$, the formulas (1.11) and (1.12) are equivalent to each other. These two formulas can be applied to establish general formulas of the n th derivatives for functions of the types $f(\sin x)$ and $f(\cos x)$, such

as $(\sin x)^\alpha$, $(\cos x)^\alpha$, $(\sec x)^\alpha$, $(\csc x)^\alpha$, $e^{\pm \sin x}$, $e^{\pm \cos x}$, $\ln(\cos x)$, $\ln(\sin x)$, $\ln(\sec x)$, $\ln(\csc x)$, $\sin(\sin x)$, $\cos(\sin x)$, $\sin(\cos x)$, $\cos(\cos x)$, $\tan x$, and $\cot x$. For more information on related applications and approaches, please refer to the papers [26, 46, 50, 82, 87, 89] and closely related references therein.

Recall from [2, p. 243, Eq. (10) and p. 250, Theorem 16] and [3, p. 152, Eq. (3.9)] that the associate Stirling numbers of the second kind or weighted Stirling numbers, whose definitions can be found in [2, 3, 6], can be generated by

$$\frac{(e^t - 1)^k}{k!} e^{rt} = \sum_{n=0}^{\infty} S_r(n, k) \frac{t^n}{n!} = \sum_{n=k}^{\infty} S_r(n, k) \frac{t^n}{n!}. \quad (1.13)$$

Recall from [3, p. 152, Eq. (3.8)] that

$$S_r(n, k) = \frac{1}{k!} \sum_{j=0}^k (-1)^{k-j} \binom{k}{j} (r+j)^n, \quad n \geq k \geq 0. \quad (1.14)$$

It is clear that $S_0(n, k) = S(n, k)$ and $S_r(n, m) = 0$ for $m > n$. See also [7, 54] and closely related reference therein.

Taking $x \rightarrow 0$ in (1.11) and (1.12) leads to

$$\begin{aligned} B_{n,k} \left(0, -1, 0, 1, \dots, \cos \frac{(n-k+1)\pi}{2} \right) \\ = \frac{(-1)^k}{k!} \left(\cos \frac{n\pi}{2} \right) \sum_{\ell=0}^k \frac{(-1)^\ell}{2^\ell} \binom{k}{\ell} \sum_{q=0}^{\ell} \binom{\ell}{q} (2q - \ell)^n \end{aligned} \quad (1.15)$$

and

$$\begin{aligned} B_{n,k} \left(1, 0, -1, 0, \dots, \sin \frac{(n-k+1)\pi}{2} \right) \\ = \frac{(-1)^k}{k! 2^k} \left[\cos \frac{(n-k)\pi}{2} \right] \sum_{q=0}^k (-1)^q \binom{k}{q} (2q - k)^n \\ = \left[\cos \frac{(n-k)\pi}{2} \right] 2^{n-k} S_{-k/2}(n, k). \end{aligned} \quad (1.16)$$

Similar to the formulas (1.15) and (1.16), another two formulas

$$\begin{aligned} B_{n,k} \left(1, 0, 1, \dots, \frac{1 - (-1)^{n-k+1}}{2} \right) &= \frac{1}{2^k k!} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} (k - 2\ell)^n \\ &= (-1)^{n-k} 2^{n-k} S_{-k/2}(n, k) = 2^{n-k} S_{-k/2}(n, k) \end{aligned}$$

and

$$\begin{aligned} B_{n,k} \left(0, 1, 0, \dots, \frac{1 + (-1)^{n-k+1}}{2} \right) \\ = \frac{1}{2^k k!} \sum_{\ell=0}^{2k} (-1)^\ell \binom{2k}{\ell} (k - \ell)^n = (-1)^n \frac{(2k)!}{(2k)!!} S_{-k}(n, 2k), \end{aligned}$$

where 0^0 is regarded as 1, were established, discussed, and applied to derive a closed formula for the Euler numbers in [54, 82].

1.7. In [28, Theorem 1.1], the formulas

$$B_{n,k}\left(\frac{1!}{2}, \frac{2!}{3}, \dots, \frac{(n-k+1)!}{n-k+2}\right) = (-1)^{n-k} \frac{1}{k!} \sum_{m=1}^k (-1)^m \frac{\binom{k}{m}}{\binom{n+m}{n}} s(n+m, m)$$

and

$$B_{n,k}(0, 1!, 2!, \dots, (n-k)!) = (-1)^n \binom{n}{k} \sum_{\ell=0}^k (-1)^\ell \frac{\binom{k}{\ell}}{\binom{n-k+\ell}{\ell}} s(n-k+\ell, \ell)$$

were proved for $n \geq k \geq 1$. These two formulas were used in [28, 48] to discover diagonal recurrence relations for the Stirling numbers of the first kind.

1.8. In [11, 13, 27, 42, 47, 91], the formulas

$$B_{n,k}(0, 1, \dots, 1) = \sum_{\ell=0}^k (-1)^\ell \binom{n}{\ell} S(n-\ell, k-\ell)$$

and

$$B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2}\right) = \frac{n!}{(n+k)!} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{n+k}{k-\ell} S(n+\ell, \ell) \quad (1.17)$$

were established and applied to investigate functions of the type $f\left(\frac{e^t-1}{t}\right)$ such as generating functions for the Bernoulli numbers and polynomials and for the Stirling numbers and polynomials.

1.9. In [72, Lemma 6], among other things, the formula

$$B_{n,k}(2!, 3!, \dots, (n-k+2)!) = \frac{n!}{k!} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \binom{n+2\ell-1}{n},$$

where $\binom{q}{0} = 1$ for all $q \in \mathbb{C}$, was deduced for establishing a closed formula for the large Schröder numbers. See also [51] and closely related references therein.

1.10. In [53, Lemma 2], the formula

$$\begin{aligned} & B_{n,k}(x, 1+x, 2+x, \dots, n-k+x) \\ &= k^{n-k} \binom{n}{k} \sum_{\ell=0}^k \binom{k}{\ell} \left[\sum_{q=0}^{n-k} \frac{(-1)^q}{k^q} \binom{n-k}{q} \frac{S(\ell+q, \ell)}{\binom{\ell+q}{\ell}} \right] (x-1)^\ell \end{aligned} \quad (1.18)$$

was proved and applied to acquire closed formulas for derangement numbers and their generating function. From (1.18), the formula

$$B_{n,k}(1, 2, 3, \dots, n-k+1) = \binom{n}{k} k^{n-k} \quad (1.19)$$

listed in [4, p. 451] and [5, p. 135] can be recovered straightforwardly.

1.11. In [4, p. 451], the formulas

$$\begin{aligned} B_{2r,k}(0, 2!, \dots, 0, (2r)!) &= \frac{(2r)!}{k!} \binom{r-1}{k-1}, \\ B_{2r-1,k}(0, 2!, \dots, (2r-2)!, 0) &= 0, \\ B_{2r,2s}(1!, 0, \dots, (2r-1)!, 0) &= \frac{(2r)!}{(2s)!} \binom{r+s-1}{2s-1}, \\ B_{2r,2s-1}(1!, 0, \dots, (2r-1)!, 0) &= 0, \\ B_{2r-1,2s-1}(1!, 0, \dots, (2r-1)!, 0) &= \frac{(2r-1)!}{(2s-1)!} \binom{r+s-2}{2s-2}, \\ B_{2r-1,2s}(1!, 0, \dots, 0, (2r-1)!) &= 0 \end{aligned}$$

were stated as exercises. These six formulas can be uniformly written as

$$B_{n,k}\left(0, 2!, 0, 4!, \dots, (n-k+1)! \frac{1+(-1)^{n-k+1}}{2}\right) = \frac{1+(-1)^n}{2} \frac{n!}{k!} \binom{\frac{n}{2}-1}{k-1}$$

and

$$B_{n,k}\left(1!, 0, 3!, 0, \dots, (n-k+1)! \frac{1-(-1)^{n-k+1}}{2}\right) = \frac{1+(-1)^{n+k}}{2} \frac{n!}{k!} \binom{\frac{n+k}{2}-1}{k-1}.$$

For details on discussion of these quantities, please refer to [64] and closely related references therein.

1.12. A class of the Bell polynomials of the second kind

$$B_{n,k}(a_1, a_2, 3-\alpha, 4-\alpha, \dots, (n-k+1)-\alpha) \quad (1.20)$$

were investigated in [16, Section 5]. In [16], the following closed formulas were presented:

$$B_{n,k}(0, 2, 3, \dots, n-k+1) = \frac{n!}{(n-k)!} S(n-k, k), \quad (1.21)$$

$$B_{n,k}(0, 0, 3, 4, \dots, n-k+1) = \frac{n!}{(n-k)!} S_1(n-k, k), \quad (1.22)$$

$$B_{n,k}(x, 2, 3, 4, \dots, n-k+1) = \binom{n}{k} \sum_{r=0}^k \binom{k}{r} (k-r)^{n-k} (x-1)^r, \quad (1.23)$$

$$B_{n,k}(0, 1, 2, 3, \dots, n-k) = \sum_{r=0}^n (-1)^{n-r} \binom{n}{r} k^r S_1(n-r, k), \quad (1.24)$$

where $S_r(n, k)$ can be generated by (1.13) and can be computed by (1.14). The formula (1.19) gives a closed formula for the special case $a_1 = 1$, $a_2 = 2$, and $\alpha = 0$ in (1.20). Substituting $1 - \alpha$ for x in (1.18) leads to

$$\begin{aligned} &B_{n,k}(1-\alpha, 2-\alpha, 3-\alpha, \dots, n-k+1-\alpha) \\ &= k^{n-k} \binom{n}{k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \left[\sum_{q=0}^{n-k} \frac{(-1)^q}{k^q} \binom{n-k}{q} \frac{S(q+\ell, \ell)}{\binom{q+\ell}{\ell}} \right] \alpha^\ell. \quad (1.25) \end{aligned}$$

Letting $x = 0$ in (1.18) or taking $\alpha = 1$ in (1.25) results in

$$B_{n,k}(0, 1, 2, 3, \dots, n-k) = k^{n-k} \binom{n}{k} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{q=0}^{n-k} \frac{(-1)^q}{k^q} \binom{n-k}{q} \frac{S(q+\ell, \ell)}{\binom{q+\ell}{\ell}}$$

which, due to useless of associate Stirling numbers of the second kind, is more elementary than (1.24).

1.13. In the paper [16] and closely related references therein, the following numbers and their generating functions were considered:

$$\begin{aligned} \frac{x^2}{2} \frac{1}{e^x - x - 1} &= \sum_{n=0}^{\infty} A_n \frac{x^n}{n!}, & \frac{x}{e^x - 1} &= \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}, \\ \frac{2x}{e^x + 1} &= \sum_{n=1}^{\infty} G_n \frac{x^n}{n!}, & -\frac{x^2}{(x-2)(e^x - x - 1)} &= \sum_{n=0}^{\infty} A'_n \frac{x^n}{n!}, \\ -\frac{2x}{(x-2)(e^x - 1)} &= \sum_{n=0}^{\infty} B'_n \frac{x^n}{n!}, & -\frac{4}{(x-2)(e^x + 1)} &= \sum_{n=0}^{\infty} G'_n \frac{x^n}{n!}, \\ \frac{x^3}{6} \frac{1}{x(e^x + 1) - 2(e^x - 1)} &= \sum_{n=0}^{\infty} V_n \frac{x^n}{n!}, & -\frac{4}{x(e^x - 1) - 2(e^x + 1)} &= \sum_{n=0}^{\infty} W_n \frac{x^n}{n!}, \end{aligned}$$

and

$$\begin{aligned} \left(1 - \frac{x}{2}\right) [x(e^x + 1) - 2(e^x - 1)]^k &= k! \sum_{n=3k}^{\infty} A(n, k) \frac{x^n}{n!}, \\ [x(e^x + 1) - 2(e^x - 1)]^k &= k! \sum_{n=3k}^{\infty} V(n, k) \frac{x^n}{n!}, \\ [(x-2)(e^x - 1)]^k &= k! \sum_{n=k}^{\infty} W(n, k) \frac{x^n}{n!}. \end{aligned}$$

Among them, the numbers B_n , G_n , and V_n are called the Bernoulli, Genocchi, and van der Pol numbers respectively. In [16], among other things, the closed formulas and identities

$$W_n = \sum_{k=1}^n \sum_{r=0}^n (-1)^{k+r} \frac{k!r!}{2^{k+r}} \binom{n}{r} \binom{k}{r} S(n-r, k), \quad (1.26)$$

$$V(n, k) = B_{n,k}(0, 0, 1, 2, \dots, n-k-1), \quad (1.27)$$

$$\begin{aligned} W(n, k) &= B_{n,k}(-2, 0, 1, 2, 3, \dots, n-k-1) \\ &= \sum_{r=0}^n r! \binom{n}{r} \binom{k}{r} (-2)^{k-r} S(n-r, k) \end{aligned} \quad (1.28)$$

were obtained.

2. CLOSED FORMULAS FOR SOME NUMBERS DISCUSSED BY HOWARD

In this section, we will discover or recover some relations and closed formulas for those numbers A_n , B_n , G_n , A'_n , B'_n , G'_n , V_n , W_n , $A(n, k)$, $V(n, k)$, and $W(n, k)$ which have been discussed in [16].

Theorem 2.1. For $n \geq k \geq 0$, we have

$$\begin{aligned} B_{n,k} & \left(\frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \dots, \frac{1}{(n-k+2)(n-k+3)} \right) \\ & = \frac{(-1)^k}{[2(n+k)]!!} \binom{n}{k} \sum_{m=0}^{n-k} m! \langle -2k \rangle_{n-k-m} \binom{n-k}{m} \binom{n+k}{m} \\ & \quad \times \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{q=0}^{n+k} 2^q \langle k-\ell \rangle_{n+k-q} \binom{n+k}{q} \frac{S(\ell+q, \ell)}{\binom{\ell+q}{\ell}} \end{aligned} \quad (2.1)$$

and

$$\begin{aligned} A_n & = \frac{1}{2^n n!} \sum_{k=0}^n \frac{\binom{n}{k}}{\binom{n+k}{k}} \sum_{m=0}^{n-k} m! \langle -2k \rangle_{n-k-m} \binom{n-k}{m} \binom{n+k}{m} \\ & \quad \times \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{q=0}^{n+k} 2^q \langle k-\ell \rangle_{n+k-q} \binom{n+k}{q} \frac{S(\ell+q, \ell)}{\binom{\ell+q}{\ell}}. \end{aligned} \quad (2.2)$$

Proof. Let

$$u = u(x) = \frac{2(e^x - x - 1)}{x^2}.$$

It is easy to see that

$$\lim_{x \rightarrow 0} \frac{1}{u(x)} = \lim_{x \rightarrow 0} \left(\frac{x^2}{2(e^x - x - 1)} \right) = 1$$

and

$$\begin{aligned} \lim_{x \rightarrow 0} u^{(n)}(x) & = \lim_{x \rightarrow 0} \left[\frac{2(e^x - x - 1)}{x^2} \right]^{(n)} = \lim_{x \rightarrow 0} \left(\frac{2}{x^2} \sum_{k=2}^{\infty} \frac{x^k}{k!} \right)^{(n)} \\ & = 2 \lim_{x \rightarrow 0} \left[\sum_{k=0}^{\infty} \frac{x^k}{(k+2)!} \right]^{(n)} = 2 \lim_{x \rightarrow 0} \sum_{k=n}^{\infty} \frac{\langle k \rangle_n}{(k+2)!} x^{k-n} \\ & = 2 \frac{\langle n \rangle_n}{(n+2)!} = 2 \frac{n!}{(n+2)!} = \frac{2}{(n+1)(n+2)}. \end{aligned}$$

Therefore, by virtue of the Faà di Bruno formula (1.1), we obtain

$$\begin{aligned} A_n & = \lim_{x \rightarrow 0} \left[\frac{1}{u(x)} \right]^{(n)} = \lim_{x \rightarrow 0} \sum_{k=0}^n \left(\frac{1}{u} \right)^{(k)} B_{n,k}(u'(x), u''(x), \dots, u^{(n-k+1)}(x)) \\ & = \lim_{x \rightarrow 0} \sum_{k=0}^n \frac{(-1)^k k!}{u^{k+1}} B_{n,k}(u'(x), u''(x), \dots, u^{(n-k+1)}(x)) \\ & = \sum_{k=0}^n (-1)^k k! 2^k B_{n,k} \left(\frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \dots, \frac{1}{(n-k+2)(n-k+3)} \right). \end{aligned}$$

In [5, p. 133], it was listed that

$$\frac{1}{k!} \left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!} \right)^k = \sum_{n=k}^{\infty} B_{n,k}(x_1, x_2, \dots, x_{n-k+1}) \frac{t^n}{n!} \quad (2.3)$$

for $k \geq 0$. From (2.3) it can be deduced that

$$\begin{aligned} & \sum_{n=k}^{\infty} B_{n,k} \left(\frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \dots, \frac{1}{(n-k+2)(n-k+3)} \right) \frac{t^n}{n!} \\ &= \frac{1}{k!} \left[\sum_{m=1}^{\infty} \frac{1}{(m+1)(m+2)} \frac{t^m}{m!} \right]^k = \frac{1}{k!} \left[\sum_{m=1}^{\infty} \frac{t^m}{(m+2)!} \right]^k \\ &= \frac{1}{k!} \left(\frac{e^t - 1 - t - \frac{t^2}{2}}{t^2} \right)^k = \frac{t^k}{k!} \left(\frac{e^t - 1 - t - \frac{t^2}{2}}{t^3} \right)^k \end{aligned}$$

which is equivalent to

$$\begin{aligned} & \sum_{n=0}^{\infty} B_{n+k,k} \left(\frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \dots, \frac{1}{(n+2)(n+3)} \right) \frac{k!}{(n+k)!} t^n = \left(\frac{e^t - 1 - t - \frac{t^2}{2}}{t^3} \right)^k \\ &= \frac{1}{t^{2k}} \left[\frac{e^t - 1}{t} - \left(1 + \frac{t}{2} \right) \right]^k = \frac{1}{t^{2k}} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \left(\frac{e^t - 1}{t} \right)^{\ell} \left(1 + \frac{t}{2} \right)^{k-\ell}. \end{aligned}$$

This implies that

$$\begin{aligned} & B_{n+k,k} \left(\frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \dots, \frac{1}{(n+2)(n+3)} \right) \frac{n!k!}{(n+k)!} \\ &= \lim_{t \rightarrow 0} \left[\frac{1}{t^{2k}} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \left(\frac{e^t - 1}{t} \right)^{\ell} \left(1 + \frac{t}{2} \right)^{k-\ell} \right]^{(n)} \\ &= \lim_{t \rightarrow 0} \sum_{m=0}^n \binom{n}{m} \left(\frac{1}{t^{2k}} \right)^{(m)} \left[\sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \left(\frac{e^t - 1}{t} \right)^{\ell} \left(1 + \frac{t}{2} \right)^{k-\ell} \right]^{(n-m)} \\ &= \lim_{t \rightarrow 0} \sum_{m=0}^n \binom{n}{m} \frac{\langle -2k \rangle_m}{t^{2k+m}} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \left[\left(\frac{e^t - 1}{t} \right)^{\ell} \left(1 + \frac{t}{2} \right)^{k-\ell} \right]^{(n-m)} \\ &= \lim_{t \rightarrow 0} \frac{1}{t^{2k+n}} \sum_{m=0}^n \binom{n}{m} \langle -2k \rangle_m \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} t^{n-m} \\ &\quad \times \left[\left(\frac{e^t - 1}{t} \right)^{\ell} \left(1 + \frac{t}{2} \right)^{k-\ell} \right]^{(n-m)} \\ &= \frac{1}{(2k+n)!} \lim_{t \rightarrow 0} \left(\sum_{m=0}^n \langle -2k \rangle_{n-m} \binom{n}{m} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} t^m \right. \\ &\quad \left. \times \left[\left(\frac{e^t - 1}{t} \right)^{\ell} \left(1 + \frac{t}{2} \right)^{k-\ell} \right]^{(m)} \right)^{(2k+n)} \\ &= \frac{(-1)^k}{(2k+n)!} \lim_{t \rightarrow 0} \sum_{m=0}^n \langle -2k \rangle_{n-m} \binom{n}{m} \sum_{\ell=0}^k (-1)^{\ell} \binom{k}{\ell} \\ &\quad \times \sum_{q=0}^{2k+n} \binom{2k+n}{q} (t^m)^{(q)} \left[\left(\frac{e^t - 1}{t} \right)^{\ell} \left(1 + \frac{t}{2} \right)^{k-\ell} \right]^{(m+2k+n-q)} \end{aligned}$$

$$\begin{aligned}
&= \frac{(-1)^k}{(2k+n)!} \sum_{m=0}^n \langle -2k \rangle_{n-m} \binom{n}{m} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \binom{2k+n}{m} \\
&\quad \times m! \lim_{t \rightarrow 0} \left[\left(\frac{e^t - 1}{t} \right)^\ell \left(1 + \frac{t}{2} \right)^{k-\ell} \right]^{(2k+n)} \\
&= \frac{(-1)^k}{(2k+n)!} \sum_{m=0}^n m! \langle -2k \rangle_{n-m} \binom{n}{m} \binom{2k+n}{m} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \\
&\quad \times \lim_{t \rightarrow 0} \sum_{q=0}^{2k+n} \binom{2k+n}{q} \left[\left(\frac{e^t - 1}{t} \right)^\ell \right]^{(q)} \left[\left(1 + \frac{t}{2} \right)^{k-\ell} \right]^{(2k+n-q)} \\
&= \frac{(-1)^k}{(2k+n)!} \sum_{m=0}^n m! \langle -2k \rangle_{n-m} \binom{n}{m} \binom{2k+n}{m} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \\
&\quad \times \lim_{t \rightarrow 0} \sum_{q=0}^{2k+n} \binom{2k+n}{q} \left[\ell! \sum_{p=q}^{\infty} \frac{S(p+\ell, \ell)}{(p+\ell)!} \langle p \rangle_q t^{p-q} \right] \\
&\quad \times \frac{\langle k-\ell \rangle_{2k+n-q}}{2^{2k+n-q}} \left(1 + \frac{t}{2} \right)^{k-\ell-(2k+n-q)} \\
&= \frac{(-1)^k}{(2k+n)!} \sum_{m=0}^n m! \langle -2k \rangle_{n-m} \binom{n}{m} \binom{2k+n}{m} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \\
&\quad \times \sum_{q=0}^{2k+n} \binom{2k+n}{q} \left[\ell! \frac{S(q+\ell, \ell)}{(q+\ell)!} \langle q \rangle_q \right] \frac{\langle k-\ell \rangle_{2k+n-q}}{2^{2k+n-q}},
\end{aligned}$$

where we used the L'Hôpital rule and the generating function

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!} \quad (2.4)$$

for the Stirling numbers of the second kind $S(n, k)$. As a result, we procure

$$\begin{aligned}
&B_{n+k, k} \left(\frac{1}{2 \cdot 3}, \frac{1}{3 \cdot 4}, \dots, \frac{1}{(n+2)(n+3)} \right) \\
&= \frac{(-1)^k}{[2(n+2k)]!!} \binom{n+k}{k} \sum_{m=0}^n m! \langle -2k \rangle_{n-m} \binom{n}{m} \binom{n+2k}{m} \\
&\quad \times \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{q=0}^{n+2k} 2^q \langle k-\ell \rangle_{n+2k-q} \binom{n+2k}{q} \frac{S(q+\ell, \ell)}{\binom{q+\ell}{\ell}}
\end{aligned}$$

which is equivalent to (2.1). From (2.1), we derive (2.2). The closed formula (2.2) was numerically verified to be true by the software MATHEMATICA. The proof of Theorem 2.1 is thus complete. \square

Theorem 2.2 ([11, Theorem 1]). *For $n \geq 0$, we have*

$$B_n = \sum_{i=0}^n (-1)^i \frac{\binom{n+1}{i+1}}{\binom{n+i}{i}} S(n+i, i). \quad (2.5)$$

Proof. Let

$$v = v(x) = \frac{e^x - 1}{x}.$$

Then

$$\begin{aligned} \lim_{x \rightarrow 0} [v(x)]^{(n)} &= \lim_{x \rightarrow 0} \left(\frac{e^x - 1}{x} \right)^{(n)} = \lim_{x \rightarrow 0} \left(\sum_{k=1}^{\infty} \frac{x^{k-1}}{k!} \right)^{(n)} \\ &= \lim_{x \rightarrow 0} \left[\sum_{k=0}^{\infty} \frac{x^k}{(k+1)!} \right]^{(n)} = \lim_{x \rightarrow 0} \sum_{k=n}^{\infty} \frac{\langle k \rangle_n}{(k+1)!} x^{k-n} = \frac{\langle n \rangle_n}{(n+1)!} = \frac{1}{n+1} \end{aligned}$$

and, in the light of (1.1) and (1.17) in sequence,

$$\begin{aligned} B_n &= \lim_{x \rightarrow 0} \left(\frac{x}{e^x - 1} \right)^{(n)} = \lim_{x \rightarrow 0} \left[\frac{1}{v(x)} \right]^{(n)} \\ &= \lim_{x \rightarrow 0} \sum_{k=0}^n \left(\frac{1}{v} \right)^{(k)} B_{n,k}(v'(x), v''(x), \dots, v^{(n-k+1)}(x)) \\ &= \sum_{k=0}^n \lim_{x \rightarrow 0} \frac{(-1)^k k!}{v^{k+1}} \lim_{x \rightarrow 0} B_{n,k}(v'(x), v''(x), \dots, v^{(n-k+1)}(x)) \\ &= \sum_{k=0}^n (-1)^k k! B_{n,k} \left(\frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n-k+2} \right) \\ &= \sum_{k=0}^n \frac{1}{\binom{n+k}{n}} \sum_{\ell=0}^k (-1)^\ell \binom{n+k}{k-\ell} S(n+\ell, \ell) = \sum_{i=0}^n (-1)^i \frac{\binom{n+1}{i+1}}{\binom{n+i}{i}} S(n+i, i). \end{aligned}$$

The proof of Theorem 2.2 is complete. \square

Theorem 2.3 ([8, Theorem 1.1]). *For $n \in \mathbb{N}$, we have*

$$G_n = n \sum_{k=0}^{n-1} (-1)^k \frac{k!}{2^k} S(n-1, k).$$

Proof. The generating function of the Genocchi numbers G_n can be rewritten as

$$\frac{2}{e^x + 1} = \sum_{n=1}^{\infty} G_n \frac{x^{n-1}}{n!} = \sum_{n=0}^{\infty} \frac{G_{n+1}}{n+1} \frac{x^n}{n!}$$

which means that, when denoting $w = w(x) = e^x + 1$,

$$\begin{aligned} \frac{G_{n+1}}{n+1} &= \lim_{x \rightarrow 0} \left(\frac{2}{e^x + 1} \right)^{(n)} = 2 \lim_{x \rightarrow 0} \sum_{k=0}^n \left(\frac{1}{w} \right)^{(k)} B_{n,k}(e^x, e^x, \dots, e^x) \\ &= 2 \lim_{x \rightarrow 0} \sum_{k=0}^n \frac{(-1)^k k!}{w^{k+1}} e^{kx} B_{n,k}(1, 1, \dots, 1) \\ &= \sum_{k=0}^n (-1)^k \frac{k!}{2^k} B_{n,k}(1, 1, \dots, 1) = \sum_{k=0}^n (-1)^k \frac{k!}{2^k} S(n, k) \end{aligned}$$

which gives

$$G_{n+1} = (n+1) \sum_{k=0}^n (-1)^k \frac{k!}{2^k} S(n, k), \quad n \geq 0$$

and then recovers [8, Theorem 1.1]. The proof of Theorem 2.3 is complete. \square

Theorem 2.4. For $n \geq 0$, we have

$$A'_{n+1} - \frac{n+1}{2}A'_n = A_{n+1}$$

and

$$\begin{aligned} A'_n &= \sum_{k=0}^n \frac{\langle n \rangle_k}{2^k} A_{n-k} = \sum_{p=0}^n \frac{\langle n \rangle_{n-p}}{2^{n-p}} A_p \\ &= \frac{1}{2^n} \sum_{p=0}^n \frac{\langle n \rangle_{n-p}}{p!} \sum_{k=0}^p \frac{\binom{p}{k}}{\binom{p+k}{k}} \sum_{m=0}^{p-k} m! \langle -2k \rangle_{p-k-m} \binom{p-k}{m} \binom{p+k}{m} \\ &\quad \times \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{q=0}^{p+k} 2^q \langle k-\ell \rangle_{p+k-q} \binom{p+k}{q} \frac{S(q+\ell, \ell)}{\binom{q+\ell}{\ell}}. \end{aligned} \quad (2.6)$$

Proof. The generating function of the numbers A'_n can be rearranged as

$$\begin{aligned} \frac{x^2}{2(e^x - x - 1)} &= \left(1 - \frac{x}{2}\right) \sum_{n=0}^{\infty} A'_n \frac{x^n}{n!} = \sum_{n=0}^{\infty} A'_n \frac{x^n}{n!} - \frac{1}{2} \sum_{n=0}^{\infty} A'_n \frac{x^{n+1}}{n!} \\ &= \sum_{n=0}^{\infty} A'_n \frac{x^n}{n!} - \frac{1}{2} \sum_{n=1}^{\infty} A'_{n-1} \frac{x^n}{(n-1)!} = A'_0 + \sum_{n=1}^{\infty} \left(A'_n - \frac{n}{2}A'_{n-1}\right) \frac{x^n}{n!} \end{aligned}$$

which implies that $A'_0 = 1$ and

$$A'_n - \frac{n}{2}A'_{n-1} = \lim_{x \rightarrow 0} \left[\frac{x^2}{2(e^x - x - 1)} \right]^{(n)} = A_n$$

for $n \in \mathbb{N}$. Recursing consecutively and making use of (2.1), the formula (2.6) follows. The proof of Theorem 2.4 is complete. \square

Theorem 2.5. For $n \geq 0$, we have

$$B'_0 = 1, \quad B'_{n+1} - \frac{n+1}{2}B'_n = B_{n+1}, \quad (2.7)$$

and

$$B'_n = \sum_{p=0}^n \frac{\langle n \rangle_{n-p}}{2^{n-p}} \sum_{i=0}^p (-1)^i \binom{p+1}{p+i} S(p+i, i).$$

Proof. The generating function of the numbers B'_n can be formulated as

$$\frac{x}{e^x - 1} = \left(1 - \frac{x}{2}\right) \sum_{n=0}^{\infty} B'_n \frac{x^n}{n!} = B'_0 + \sum_{n=1}^{\infty} \left(B'_n - \frac{n}{2}B'_{n-1}\right) \frac{x^n}{n!}$$

which means that the relation (2.7) is true. Recursing and utilizing (2.5), we obtain

$$B'_n = \sum_{k=0}^n \frac{\langle n \rangle_k}{2^k} B_{n-k} = \sum_{p=0}^n \frac{\langle n \rangle_{n-p}}{2^{n-p}} B_p = \sum_{p=0}^n \frac{\langle n \rangle_{n-p}}{2^{n-p}} \sum_{i=0}^p (-1)^i \binom{p+1}{p+i} S(p+i, i).$$

The proof of Theorem 2.5 is complete. \square

Theorem 2.6. For $n \geq 0$, we have

$$G'_0 = 1, \quad G'_{n+1} - \frac{n+1}{2}G'_n = \frac{G_{n+2}}{n+2},$$

and

$$G'_n = \sum_{\ell=0}^n \frac{\langle n \rangle_{n-\ell}}{2^{n-\ell}} \sum_{k=0}^{\ell} (-1)^k \frac{k!}{2^k} S(\ell, k).$$

Proof. It is easy to see that

$$\begin{aligned} \frac{2x}{e^x + 1} &= x \left(1 - \frac{x}{2}\right) \left[-\frac{4}{(x-2)(e^x+1)} \right] = x \left(1 - \frac{x}{2}\right) \sum_{n=0}^{\infty} G'_n \frac{x^n}{n!} \\ &= G'_0 x + \sum_{n=2}^{\infty} n \left(G'_{n-1} - \frac{n-1}{2} G'_{n-2} \right) \frac{x^n}{n!} \end{aligned}$$

which leads to $G'_0 = G_1 = 1$ and

$$G'_{n-1} - \frac{n-1}{2} G'_{n-2} = \frac{G_n}{n}, \quad n \geq 2.$$

Recursively, it follows that

$$G'_n = \sum_{k=0}^n \frac{\langle n \rangle_k}{2^k} \frac{G_{n-k+1}}{n-k+1} = \sum_{\ell=0}^n \frac{\langle n \rangle_{n-\ell}}{2^{n-\ell}} \frac{G_{\ell+1}}{\ell+1} = \sum_{\ell=0}^n \frac{\langle n \rangle_{n-\ell}}{2^{n-\ell}} \sum_{k=0}^{\ell} (-1)^k \frac{k!}{2^k} S(\ell, k).$$

The proof of Theorem 2.6 is complete. \square

Theorem 2.7. For $n \geq k \geq 0$, we have

$$\begin{aligned} B_{n,k} \left(\frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \dots, \frac{1}{(n-k+3)(n-k+4)} \right) &= \frac{(-1)^k}{(n+2k)!} 6^k \binom{n}{k} \\ &\times \sum_{q=0}^{n-k} \langle -3k \rangle_q (n-k-q)! \binom{n-k}{q} \binom{n+2k}{n-k-q} \sum_{\ell=0}^k 12^\ell \binom{k}{\ell} \sum_{p=0}^{n+2k} (n+2k-p)! \\ &\times \binom{n+2k}{p} \frac{S(p+\ell, \ell)}{\binom{p+\ell}{\ell}} \sum_{s=0}^{k-\ell} \binom{k-\ell}{s} \frac{(-6)^s}{(n-p+2\ell+2s)!} \sum_{\beta=1}^s \binom{s}{\beta} \beta^{n-p+2\ell+2s} \end{aligned}$$

and

$$\begin{aligned} V_n &= \sum_{k=0}^n \frac{k!}{(n+2k)!} \binom{n}{k} \sum_{q=0}^{n-k} \langle -3k \rangle_q (n-k-q)! \binom{n-k}{q} \binom{n+2k}{n-k-q} \\ &\times \sum_{\ell=0}^k 12^\ell \binom{k}{\ell} \sum_{p=0}^{n+2k} (n+2k-p)! \binom{n+2k}{p} \frac{S(p+\ell, \ell)}{\binom{p+\ell}{\ell}} \\ &\times \sum_{s=0}^{k-\ell} \binom{k-\ell}{s} \frac{(-6)^s}{(n-p+2\ell+2s)!} \sum_{\beta=1}^s \binom{s}{\beta} \beta^{n-p+2\ell+2s}. \end{aligned}$$

Proof. The generating function of the numbers V_n can be rewritten as

$$\left[\frac{x(e^x + 1) - 2(e^x - 1)}{x^3/6} \right]^{-1} = \sum_{n=0}^{\infty} V_n \frac{x^n}{n!}.$$

It is easy to see that

$$\begin{aligned}
\frac{x(e^x + 1) - 2(e^x - 1)}{x^3/6} &= \frac{6}{x^3} \left[x + \sum_{k=0}^{\infty} \frac{x^{k+1}}{k!} - 2 \sum_{k=1}^{\infty} \frac{x^k}{k!} \right] \\
&= \frac{6}{x^3} \left[x + \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} - 2 \sum_{k=1}^{\infty} \frac{x^k}{k!} \right] = \frac{6}{x^3} \sum_{k=2}^{\infty} \left[\frac{1}{(k-1)!} - \frac{2}{k!} \right] x^k \\
&= \frac{6}{x^3} \sum_{k=3}^{\infty} (k-2) \frac{x^k}{k!} = 6 \sum_{k=3}^{\infty} (k-2) \frac{x^{k-3}}{k!} = 6 \sum_{k=0}^{\infty} (k+1) \frac{x^k}{(k+3)!}
\end{aligned}$$

and that

$$\begin{aligned}
\lim_{x \rightarrow 0} \left[\frac{x(e^x + 1) - 2(e^x - 1)}{x^3/6} \right]^{(n)} &= 6 \lim_{x \rightarrow 0} \left[\sum_{k=0}^{\infty} (k+1) \frac{x^k}{(k+3)!} \right]^{(n)} \\
&= 6 \lim_{x \rightarrow 0} \sum_{k=n}^{\infty} (k+1) \frac{\langle k \rangle_n x^{k-n}}{(k+3)!} = 6(n+1) \frac{\langle n \rangle_n}{(n+3)!} = \frac{6}{(n+2)(n+3)}.
\end{aligned}$$

Accordingly, by similar arguments as above, it follows that

$$\begin{aligned}
V_n &= \lim_{x \rightarrow 0} \frac{d^n}{d x^n} \left[\frac{x(e^x + 1) - 2(e^x - 1)}{x^3/6} \right]^{-1} \\
&= \sum_{k=0}^n (-1)^k k! 6^k B_{n,k} \left(\frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \dots, \frac{1}{(n-k+3)(n-k+4)} \right).
\end{aligned}$$

By virtue of the formula (2.3), we have

$$\begin{aligned}
&\sum_{n=k}^{\infty} B_{n,k} \left(\frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \dots, \frac{1}{(n-k+3)(n-k+4)} \right) \frac{t^n}{n!} \\
&= \frac{1}{k!} \left[\sum_{m=1}^{\infty} \frac{1}{(m+2)(m+3)} \frac{t^m}{m!} \right]^k \\
&= \frac{1}{k!} \left[\frac{6t(e^t + 1) - t^3 - 12(e^t - 1)}{6t^3} \right]^k \\
&= \frac{1}{k!} \frac{2^k}{t^{2k}} \left(\frac{e^t + 1}{2} - \frac{t^2}{12} - \frac{e^t - 1}{t} \right)^k \\
&= \frac{1}{k!} \frac{2^k}{t^{2k}} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \left(\frac{e^t - 1}{t} \right)^\ell
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\sum_{\ell=0}^{\infty} B_{\ell+k,k} \left(\frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \dots, \frac{1}{(\ell+3)(\ell+4)} \right) \frac{1}{\binom{\ell+k}{k}} \frac{t^\ell}{\ell!} \\
&= \frac{2^k}{t^{3k}} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \left(\frac{e^t - 1}{t} \right)^\ell.
\end{aligned}$$

This implies that, by virtue of the L'Hôpital rule and the generating function (2.4) for the Stirling numbers of the second kind $S(n, k)$,

$$\begin{aligned}
& \frac{1}{\binom{m+k}{k}} B_{m+k,k} \left(\frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \dots, \frac{1}{(m+3)(m+4)} \right) \\
&= \lim_{t \rightarrow 0} \left[\frac{2^k}{t^{3k}} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \left(\frac{e^t - 1}{t} \right)^\ell \right]^{(m)} \\
&= 2^k \lim_{t \rightarrow 0} \sum_{q=0}^m \binom{m}{q} \left(\frac{1}{t^{3k}} \right)^{(q)} \left[\sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \left(\frac{e^t - 1}{t} \right)^\ell \right]^{(m-q)} \\
&= 2^k \lim_{t \rightarrow 0} \sum_{q=0}^m \binom{m}{q} \frac{\langle -3k \rangle_q}{t^{3k+q}} \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \left[\left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \left(\frac{e^t - 1}{t} \right)^\ell \right]^{(m-q)} \\
&= 2^k \lim_{t \rightarrow 0} \frac{1}{t^{3k+m}} \sum_{q=0}^m \binom{m}{q} \langle -3k \rangle_q \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} t^{m-q} \\
&\quad \times \left[\left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \left(\frac{e^t - 1}{t} \right)^\ell \right]^{(m-q)} \\
&= \frac{2^k}{(3k+m)!} \lim_{t \rightarrow 0} \sum_{q=0}^m \binom{m}{q} \langle -3k \rangle_q \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \\
&\quad \times \left(t^{m-q} \left[\left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \left(\frac{e^t - 1}{t} \right)^\ell \right]^{(m-q)} \right)^{(3k+m)} \\
&= \frac{2^k}{(3k+m)!} \lim_{t \rightarrow 0} \sum_{q=0}^m \binom{m}{q} \langle -3k \rangle_q \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{p=0}^{3k+m} \binom{3k+m}{p} \\
&\quad \times (t^{m-q})^{(p)} \left(\left[\left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \left(\frac{e^t - 1}{t} \right)^\ell \right]^{(m-q)} \right)^{(3k+m-p)} \\
&= \frac{2^k}{(3k+m)!} \sum_{q=0}^m \binom{m}{q} \langle -3k \rangle_q \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \binom{3k+m}{m-q} (m-q)! \\
&\quad \times \lim_{t \rightarrow 0} \left[\left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \left(\frac{e^t - 1}{t} \right)^\ell \right]^{(3k+m)} \\
&= \frac{2^k}{(3k+m)!} \sum_{q=0}^m \binom{m}{q} \langle -3k \rangle_q \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \binom{3k+m}{m-q} (m-q)! \\
&\quad \times \lim_{t \rightarrow 0} \sum_{p=0}^{3k+m} \binom{3k+m}{p} \left[\left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \right]^{(3k+m-p)} \left[\left(\frac{e^t - 1}{t} \right)^\ell \right]^{(p)} \\
&= \frac{2^k}{(3k+m)!} \sum_{q=0}^m \binom{m}{q} \langle -3k \rangle_q \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \binom{3k+m}{m-q} (m-q)!
\end{aligned}$$

$$\begin{aligned}
& \times \lim_{t \rightarrow 0} \sum_{p=0}^{3k+m} \binom{3k+m}{p} \left[\left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \right]^{(3k+m-p)} \ell! \sum_{r=p}^{\infty} \frac{S(r+\ell, \ell)}{(r+\ell)!} \langle r \rangle_p t^{r-p} \\
& = \frac{2^k}{(3k+m)!} \sum_{q=0}^m \binom{m}{q} \langle -3k \rangle_q \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \binom{3k+m}{m-q} (m-q)! \\
& \times \sum_{p=0}^{3k+m} \binom{3k+m}{p} \lim_{t \rightarrow 0} \left[\left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \right]^{(3k+m-p)} \frac{S(p+\ell, \ell)}{\binom{p+\ell}{\ell}},
\end{aligned}$$

where

$$\begin{aligned}
& \lim_{t \rightarrow 0} \left[\left(\frac{e^t + 1}{2} - \frac{t^2}{12} \right)^{k-\ell} \right]^{(\tau)} \\
& = \lim_{t \rightarrow 0} \left[\sum_{s=0}^{k-\ell} (-1)^{k-\ell-s} \binom{k-\ell}{s} \left(\frac{e^t + 1}{2} \right)^s \left(\frac{t^2}{12} \right)^{k-\ell-s} \right]^{(\tau)} \\
& = \lim_{t \rightarrow 0} \sum_{s=0}^{k-\ell} \frac{(-1)^{k-\ell-s}}{12^{k-\ell-s}} \binom{k-\ell}{s} \left[\left(\frac{e^t + 1}{2} \right)^s t^{2(k-\ell-s)} \right]^{(\tau)} \\
& = \lim_{t \rightarrow 0} \sum_{s=0}^{k-\ell} \frac{(-1)^{k-\ell-s}}{12^{k-\ell-s} 2^s} \binom{k-\ell}{s} \left[t^{2(k-\ell-s)} \sum_{\beta=0}^s \binom{s}{\beta} e^{\beta t} \right]^{(\tau)} \\
& = \lim_{t \rightarrow 0} \sum_{s=0}^{k-\ell} \frac{(-1)^{k-\ell-s}}{12^{k-\ell-s} 2^s} \binom{k-\ell}{s} \left[t^{2(k-\ell-s)} \sum_{\beta=0}^s \binom{s}{\beta} \sum_{\gamma=0}^{\infty} \frac{\beta^\gamma t^\gamma}{\gamma!} \right]^{(\tau)} \\
& = \lim_{t \rightarrow 0} \sum_{s=0}^{k-\ell} \frac{(-1)^{k-\ell-s}}{12^{k-\ell-s} 2^s} \binom{k-\ell}{s} \left[\sum_{\beta=0}^s \binom{s}{\beta} \sum_{\gamma=0}^{\infty} \frac{\beta^\gamma}{\gamma!} t^{2(k-\ell-s)+\gamma} \right]^{(\tau)} \\
& = \lim_{t \rightarrow 0} \sum_{s=0}^{k-\ell} \frac{(-1)^{k-\ell-s}}{12^{k-\ell-s} 2^s} \binom{k-\ell}{s} \sum_{\beta=0}^s \binom{s}{\beta} \sum_{\gamma=0}^{\infty} \frac{\beta^\gamma}{\gamma!} \langle 2(k-\ell-s) + \gamma \rangle_\tau t^{2(k-\ell-s)+\gamma-\tau} \\
& = \sum_{s=0}^{k-\ell} \frac{(-1)^{k-\ell-s}}{12^{k-\ell-s} 2^s} \binom{k-\ell}{s} \sum_{\beta=0}^s \binom{s}{\beta} \frac{\beta^{\tau-2(k-\ell-s)}}{[\tau-2(k-\ell-s)]!} \langle \tau \rangle_\tau.
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
& B_{m+k, k} \left(\frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \dots, \frac{1}{(m+3)(m+4)} \right) \\
& = \frac{(-1)^k}{(3k+m)! 6^k} \binom{m+k}{k} \sum_{q=0}^m \langle -3k \rangle_q \binom{m}{q} \binom{3k+m}{m-q} (m-q)! \\
& \times \sum_{\ell=0}^k 12^\ell \binom{k}{\ell} \sum_{p=0}^{3k+m} (3k+m-p)! \binom{3k+m}{p} \frac{S(p+\ell, \ell)}{\binom{p+\ell}{\ell}} \\
& \times \sum_{s=0}^{k-\ell} \binom{k-\ell}{s} \frac{(-6)^s}{(k+m-p+2\ell+2s)!} \sum_{\beta=1}^s \binom{s}{\beta} \beta^{k+m-p+2\ell+2s}.
\end{aligned}$$

The proof of Theorem 2.7 is complete. \square

Theorem 2.8. For $n \geq k \geq 0$, we have

$$\begin{aligned} W(n, k) &= B_{n,k}(-2, 0, 1, 2, \dots, n-k-1) \\ &= \binom{n}{k} \sum_{\ell=0}^{n-k} (-2)^\ell \langle k \rangle_{n-k-\ell} \binom{n-k}{\ell} \frac{S(\ell+k, k)}{\binom{\ell+k}{k}} \end{aligned} \quad (2.8)$$

and

$$W_n = \sum_{k=0}^n \frac{k!}{4^k} W(n, k) = \sum_{k=0}^n \frac{k!}{4^k} \binom{n}{k} \sum_{\ell=0}^{n-k} (-2)^\ell \langle k \rangle_{n-k-\ell} \binom{n-k}{\ell} \frac{S(\ell+k, k)}{\binom{\ell+k}{k}}. \quad (2.9)$$

Proof.

$$-\frac{4}{x(e^x-1) - 2(e^x+1)} = \sum_{n=0}^{\infty} W_n \frac{x^n}{n!},$$

It is not difficult to find that

$$\begin{aligned} x(e^x-1) - 2(e^x+1) &= \sum_{k=1}^{\infty} \frac{x^{k+1}}{k!} - 2 \sum_{k=0}^{\infty} \frac{x^k}{k!} - 2 \\ &= \sum_{k=2}^{\infty} \frac{x^k}{(k-1)!} - 2 \sum_{k=2}^{\infty} \frac{x^k}{k!} - 2x - 4 = \sum_{k=3}^{\infty} (k-2) \frac{x^k}{k!} - 2x - 4. \end{aligned}$$

Consequently, it follows that

$$[x(e^x-1) - 2(e^x+1)]'|_{x=0} = -2, \quad [x(e^x-1) - 2(e^x+1)]''|_{x=0} = 0,$$

and

$$[x(e^x-1) - 2(e^x+1)]^{(n)}|_{x=0} = n-2, \quad n \geq 3.$$

Therefore, we can derive that

$$\begin{aligned} W_n &= \lim_{x \rightarrow 0} \frac{d^n}{dx^n} \left[-\frac{4}{x(e^x-1) - 2(e^x+1)} \right] \\ &= -4 \sum_{k=0}^n (-1)^k \frac{k!}{(-4)^{k+1}} B_{n,k}(-2, 0, 1, 2, \dots, n-k-1) \\ &= \sum_{k=0}^n \frac{k!}{4^k} B_{n,k}(-2, 0, 1, 2, \dots, n-k-1) \end{aligned}$$

for $n \geq 0$. Since

$$\begin{aligned} \sum_{n=k}^{\infty} W(n, k) \frac{x^n}{n!} &= \frac{1}{k!} [(x-2)(e^x-1)]^k = \frac{1}{k!} \left[(x-2) \sum_{n=1}^{\infty} \frac{x^n}{n!} \right]^k \\ &= \frac{1}{k!} \left[\sum_{n=2}^{\infty} \frac{x^n}{(n-1)!} - 2 \sum_{n=1}^{\infty} \frac{x^n}{n!} \right]^k = \frac{1}{k!} \left[-2x + \sum_{n=2}^{\infty} (n-2) \frac{x^n}{n!} \right]^k, \end{aligned}$$

comparing with (2.3) recovers the first equality in (1.28). Consequently, we have

$$\begin{aligned} \frac{W(n+k, k)}{\binom{n+k}{k}} &= \frac{B_{n+k,k}(-2, 0, 1, 2, \dots, n-1)}{\binom{n+k}{k}} \\ &= \lim_{t \rightarrow 0} \frac{d^n}{dt^n} \left[(t-2) \left(\frac{e^t-1}{t} \right) \right]^k = \lim_{t \rightarrow 0} \sum_{\ell=0}^n \binom{n}{\ell} [(t-2)^k]^{(\ell)} \left[\left(\frac{e^t-1}{t} \right)^k \right]^{(n-\ell)} \end{aligned}$$

$$\begin{aligned}
&= \lim_{t \rightarrow 0} \sum_{\ell=0}^n \binom{n}{\ell} \langle k \rangle_{\ell} (t-2)^{k-\ell} \left[k! \sum_{p=n-\ell}^{\infty} \frac{S(p+k, k)}{(p+k)!} \langle p \rangle_{n-\ell} t^{p-(n-\ell)} \right] \\
&= \sum_{\ell=0}^n \binom{n}{\ell} \langle k \rangle_{\ell} (-2)^{k-\ell} k! \frac{S(n-\ell+k, k)}{(n-\ell+k)!} \langle n-\ell \rangle_{n-\ell} \\
&= k! \sum_{\ell=0}^n \binom{n}{\ell} \langle k \rangle_{n-\ell} (-2)^{\ell} \frac{S(\ell+k, k)}{(\ell+k)!} \langle \ell \rangle_{\ell}.
\end{aligned}$$

As a result, it follows that

$$B_{n+k, k}(-2, 0, 1, 2, \dots, n-1) = \binom{n+k}{k} \sum_{\ell=0}^n (-2)^{\ell} \langle k \rangle_{n-\ell} \binom{n}{\ell} \frac{S(\ell+k, k)}{\binom{\ell+k}{k}}.$$

The proof of Theorem 2.8 is complete. \square

Theorem 2.9. For $n \geq 3k$, we have

$$A(3k, k) = V(3k, k), \quad A(n+1, k) = V(n+1, k) - \frac{n+1}{2} V(n, k) \quad (2.10)$$

and

$$\begin{aligned}
V(n, k) &= B_{n, k}(0, 0, 1, 2, \dots, n-k-1) = 2^k \binom{n}{k} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \\
&\quad \times \sum_{q=0}^{n-k} \binom{n-k}{q} \frac{S(n-q-\ell, k-\ell)}{\binom{n-q-\ell}{n-q}} \sum_{m=0}^q \frac{\langle \ell \rangle_m}{2^m} S(q, m). \quad (2.11)
\end{aligned}$$

Proof. Since

$$\begin{aligned}
&k! \sum_{n=3k}^{\infty} A(n, k) \frac{x^n}{n!} = \left(1 - \frac{x}{2}\right) [x(e^x + 1) - 2(e^x - 1)]^k \\
&= \left(1 - \frac{x}{2}\right) k! \sum_{n=3k}^{\infty} V(n, k) \frac{x^n}{n!} = k! \left[\sum_{n=3k}^{\infty} V(n, k) \frac{x^n}{n!} - \frac{1}{2} \sum_{n=3k}^{\infty} V(n, k) \frac{x^{n+1}}{n!} \right] \\
&= k! \left[\sum_{n=3k}^{\infty} V(n, k) \frac{x^n}{n!} - \frac{1}{2} \sum_{n=3k+1}^{\infty} V(n-1, k) \frac{x^n}{(n-1)!} \right] \\
&= k! V(3k, k) + k! \sum_{n=3k+1}^{\infty} \left[V(n, k) - \frac{n}{2} V(n-1, k) \right] \frac{x^n}{n!},
\end{aligned}$$

then the relations in (2.10) follow immediately.

Combining the formula (2.3) with

$$\sum_{n=3k}^{\infty} V(n, k) \frac{x^n}{n!} = \frac{1}{k!} [x(e^x + 1) - 2(e^x - 1)]^k = \frac{1}{k!} \left[\sum_{\ell=3}^{\infty} (\ell-2) \frac{x^{\ell}}{\ell!} \right]^k$$

reveals that the formula (1.27) for $n \geq 3k$. By virtue of (2.3), we have

$$\sum_{n=k}^{\infty} B_{n, k}(0, 0, 1, 2, \dots, n-k-1) \frac{t^n}{n!} = \frac{1}{k!} \left[\sum_{m=3}^{\infty} (m-2) \frac{t^m}{m!} \right]^k,$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{B_{n+k,k}(0, 0, 1, 2, \dots, n-1) t^n}{\binom{n+k}{k} n!} = 2^k \left(\frac{e^t + 1}{2} - \frac{e^t - 1}{t} \right)^k, \\
& \frac{B_{n+k,k}(0, 0, 1, 2, \dots, n-1)}{\binom{n+k}{k}} = 2^k \lim_{t \rightarrow 0} \frac{d^n}{d t^n} \left(\frac{e^t + 1}{2} - \frac{e^t - 1}{t} \right)^k \\
& = 2^k \lim_{t \rightarrow 0} \frac{d^n}{d t^n} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \left(\frac{e^t + 1}{2} \right)^\ell \left(\frac{e^t - 1}{t} \right)^{k-\ell} \\
& = 2^k \lim_{t \rightarrow 0} \sum_{\ell=0}^k (-1)^{k-\ell} \binom{k}{\ell} \sum_{q=0}^n \binom{n}{q} \left[\left(\frac{e^t + 1}{2} \right)^\ell \right]^{(q)} \left[\left(\frac{e^t - 1}{t} \right)^{k-\ell} \right]^{(n-q)} \\
& = \lim_{t \rightarrow 0} \sum_{\ell=0}^k (-2)^{k-\ell} \binom{k}{\ell} \sum_{q=0}^n \binom{n}{q} \sum_{m=0}^q \langle \ell \rangle_m (e^t + 1)^{\ell-m} B_{q,m}(e^t, e^t, \dots, e^t) \\
& \quad \times \left[(k-\ell)! \sum_{p=n-q}^{\infty} \frac{S(p+k-\ell, k-\ell)}{(p+k-\ell)!} \langle p \rangle_{n-q} t^{p-(n-q)} \right] \\
& = \sum_{\ell=0}^k (-2)^{k-\ell} \binom{k}{\ell} \sum_{q=0}^n \binom{n}{q} \sum_{m=0}^q \langle \ell \rangle_m 2^{\ell-m} B_{q,m}(1, 1, \dots, 1) \\
& \quad \times (k-\ell)! \frac{S(n-q+k-\ell, k-\ell)}{(n-q+k-\ell)!} \langle n-q \rangle_{n-q} \\
& = (-2)^k \sum_{\ell=0}^k (-1)^\ell \binom{k}{\ell} \sum_{q=0}^n \binom{n}{q} \frac{S(n-q+k-\ell, k-\ell)}{\binom{n-q+k-\ell}{n-q}} \sum_{m=0}^q \frac{\langle \ell \rangle_m}{2^m} S(q, m).
\end{aligned}$$

The formula (2.11) follows readily. The proof of Theorem 2.9 is complete. \square

3. A CLOSED FORMULA FOR A SPECIAL BELL POLYNOMIAL

The quantity $B_{n,k}(a_1, a_2, 3-\alpha, 4-\alpha, \dots, (n-k+1)-\alpha)$ was specially investigated in [16], but there was no closed formula developed for it there. We now start out to derive a closed and explicit formula for $B_{n,k}(a_1, a_2, 3-\alpha, 4-\alpha, \dots, (n-k+1)-\alpha)$.

Theorem 3.1. *For $n \geq k \geq 0$, we have*

$$\begin{aligned}
& B_{n,k}(a_1, a_2, 3-\alpha, 4-\alpha, \dots, (n-k+1)-\alpha) \\
& = \frac{n!}{k!} \sum_{m=0}^k \binom{k}{m} \sum_{p+q+r=n-k} (-\alpha)^{m-q} (a_1 + \alpha)^{k-m-p} \\
& \quad \times \left(\frac{a_2 + \alpha}{2} - 1 \right)^p \frac{\langle k-m \rangle_p \langle m \rangle_q S(r+m, m)}{p!q!r!} \frac{S(r+m, m)}{\binom{r+m}{r}}. \quad (3.1)
\end{aligned}$$

Proof. By virtue of (2.3), we have

$$\begin{aligned}
& \sum_{n=k}^{\infty} B_{n,k}(a_1, a_2, 3-\alpha, 4-\alpha, \dots, (n-k+1)-\alpha) \frac{t^n}{n!} \\
& = \frac{1}{k!} \left(a_1 t + \frac{a_2}{2} t^2 + \sum_{m=3}^{\infty} (m-\alpha) \frac{t^m}{m!} \right)^k
\end{aligned}$$

$$= \frac{t^k}{k!} \left[a_1 + \alpha + \left(\frac{a_2 + \alpha}{2} - 1 \right) t + (t - \alpha) \left(\frac{e^t - 1}{t} \right) \right]^k$$

which is equivalent to

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{B_{n+k,k}(a_1, a_2, 3 - \alpha, 4 - \alpha, \dots, (n+1) - \alpha) t^n}{\binom{n+k}{k} n!} \\ = \left[a_1 + \alpha + \left(\frac{a_2}{2} + \frac{\alpha}{2} - 1 \right) t + (t - \alpha) \left(\frac{e^t - 1}{t} \right) \right]^k. \end{aligned}$$

Therefore, it follows that, when $A = a_1 + \alpha$ and $B = \frac{a_2 + \alpha}{2} - 1$,

$$\begin{aligned} & B_{n+k,k}(a_1, a_2, 3 - \alpha, 4 - \alpha, \dots, (n+1) - \alpha) \\ &= \binom{n+k}{k} \lim_{t \rightarrow 0} \frac{d^n}{d t^n} \left[a_1 + \alpha + \left(\frac{a_2}{2} + \frac{\alpha}{2} - 1 \right) t + (t - \alpha) \left(\frac{e^t - 1}{t} \right) \right]^k \\ &= \binom{n+k}{k} \lim_{t \rightarrow 0} \frac{d^n}{d t^n} \sum_{\ell=0}^k \binom{k}{\ell} (A + Bt)^\ell (t - \alpha)^{k-\ell} \left(\frac{e^t - 1}{t} \right)^{k-\ell} \\ &= \binom{n+k}{k} \lim_{t \rightarrow 0} \sum_{\ell=0}^k \binom{k}{\ell} \sum_{p+q+r=n} \frac{n!}{p!q!r!} [(A + Bt)^\ell]^{(p)} \\ &\quad \times [(t - \alpha)^{k-\ell}]^{(q)} \left[\left(\frac{e^t - 1}{t} \right)^{k-\ell} \right]^{(r)} \\ &= \binom{n+k}{k} \lim_{t \rightarrow 0} \sum_{\ell=0}^k \binom{k}{\ell} \sum_{p+q+r=n} \frac{n!}{p!q!r!} \langle \ell \rangle_p B^p (A + Bt)^{\ell-p} \langle k - \ell \rangle_q (t - \alpha)^{k-\ell-q} \\ &\quad \times (k - \ell)! \sum_{s=r}^{\infty} \frac{S(s + k - \ell, k - \ell)}{(s + k - \ell)!} \langle s \rangle_r t^{s-r} \\ &= \binom{n+k}{k} \sum_{\ell=0}^k \binom{k}{\ell} \sum_{p+q+r=n} \frac{n!}{p!q!r!} \langle \ell \rangle_p B^p A^{\ell-p} \langle k - \ell \rangle_q \\ &\quad \times (-\alpha)^{k-\ell-q} (k - \ell)! \frac{S(r + k - \ell, k - \ell)}{(r + k - \ell)!} \langle r \rangle_r \\ &= n! \binom{n+k}{k} \sum_{\ell=0}^k \binom{k}{\ell} \\ &\quad \times \sum_{p+q+r=n} \frac{(-\alpha)^{k-\ell-q} B^p A^{\ell-p} \langle \ell \rangle_p \langle k - \ell \rangle_q S(r + k - \ell, k - \ell)}{p!q!r! \binom{r+k-\ell}{r}}. \end{aligned}$$

The equality (3.1) is thus proved. The proof of Theorem 3.1 is thus complete. \square

4. REMARKS AND COMPARISONS

We now list several remarks and comparisons about our main results and some known results.

Remark 4.1. Although those formulas in Theorem 2.7 are a little complex, but they are surely closed formulas for

$$B_{n,k} \left(\frac{1}{3 \cdot 4}, \frac{1}{4 \cdot 5}, \dots, \frac{1}{(n-k+3)(n-k+4)} \right) \quad (4.1)$$

and V_n . If computing along alternative approaches the n th derivative of the function

$$\frac{1}{t^{2k}} \left(\frac{e^t + 1}{2} - \frac{t^2}{12} - \frac{e^t - 1}{t} \right)^k$$

with respect to t , then we can find alternative closed formulas for the quantity (4.1) and the sequence V_n .

Remark 4.2. The formulas (2.8) and (2.9) are slightly different from (1.28) and (1.26) respectively.

Remark 4.3. Taking $a_1 = x$, $a_2 = 1 + x$, and $\alpha = 1 - x$ in Theorem 3.1 gives

$$\begin{aligned} B_{n,k}(x, 1+x, 2+x, 3+x, \dots, n-k+x) \\ = \frac{n!}{k!} \sum_{\ell=0}^k \binom{k}{\ell} \sum_{q+r=n-k} \frac{\langle \ell \rangle_q}{q!r!} \frac{S(r+\ell, \ell)}{\binom{r+\ell}{r}} (x-1)^{\ell-q}. \end{aligned}$$

Comparing this with (1.18) demonstrates that

$$\begin{aligned} & \frac{k^{n-k}}{(n-k)!} \sum_{\ell=0}^k \binom{k}{\ell} \left[\sum_{q=0}^{n-k} \frac{(-1)^q}{k^q} \binom{n-k}{q} \frac{S(q+\ell, \ell)}{\binom{q+\ell}{\ell}} \right] (x-1)^\ell \\ &= \sum_{\ell=0}^k \binom{k}{\ell} \sum_{m=0}^{n-k} \frac{\langle \ell \rangle_{n-k-m}}{(n-k-m)!m!} \frac{S(m+\ell, \ell)}{\binom{m+\ell}{m}} (x-1)^{\ell+m-(n-k)} \\ &= \sum_{\ell=0}^k \binom{k}{\ell} \sum_{m=n-k-\ell}^{n-k} \frac{\langle \ell \rangle_{n-k-m}}{(n-k-m)!m!} \frac{S(m+\ell, \ell)}{\binom{m+\ell}{m}} (x-1)^{\ell+m-(n-k)} \\ &= \sum_{\ell=0}^k \binom{k}{\ell} \sum_{q=0}^{\ell} \frac{\langle \ell \rangle_{\ell-q}}{(\ell-q)![(n-k)-(\ell-q)]!} \frac{S(n-k+q, \ell)}{\binom{n-k+q}{(n-k)-(\ell-q)}} (x-1)^q \\ &= \sum_{\ell=0}^k \sum_{q=0}^{\ell} \binom{k}{\ell} \binom{\ell}{q} \frac{\ell! S(n-k+q, \ell)}{(n-k+q)!} (x-1)^q \\ &= \sum_{q=0}^k \sum_{\ell=q}^k \binom{k}{\ell} \binom{\ell}{q} \frac{\ell! S(n-k+q, \ell)}{(n-k+q)!} (x-1)^q \\ &= \sum_{\ell=0}^k \sum_{m=\ell}^k \binom{k}{m} \binom{m}{\ell} \frac{m! S(n-k+\ell, m)}{(n-k+\ell)!} (x-1)^\ell \\ &= \sum_{\ell=0}^k \sum_{q=0}^{k-\ell} \binom{k}{\ell+q} \binom{\ell+q}{\ell} \frac{(\ell+q)! S(n-k+\ell, \ell+q)}{(n-k+\ell)!} (x-1)^\ell \end{aligned}$$

which implies

$$k^n \frac{(n+\ell)!}{n!} \binom{k}{\ell} \sum_{q=0}^n \frac{(-1)^q}{k^q} \binom{n}{q} \frac{S(q+\ell, \ell)}{\binom{q+\ell}{\ell}}$$

$$= \sum_{q=0}^{k-\ell} \binom{k}{\ell+q} \binom{\ell+q}{\ell} (\ell+q)! S(n+\ell, \ell+q). \quad (4.2)$$

In particular, letting $\ell = k$ in (4.2) results in an interesting identity

$$\sum_{q=0}^n (-1)^q \binom{n}{q} \frac{S(q+k, k)}{k^q \binom{q+k}{k}} = \frac{S(n+k, k)}{k^n \binom{n+k}{n}}.$$

Remark 4.4. Letting $a_1 = 1$, $a_2 = 2$, and $\alpha = 0$ in Theorem 3.1 yields

$$B_{n,k}(1, 2, 3, 4, \dots, n-k+1) = \frac{n!}{(n-k)!} \sum_{\ell=0}^k \frac{S(n-k, \ell)}{(k-\ell)!}.$$

Comparing this with (1.19) leads to an identity

$$\sum_{\ell=0}^k \frac{S(n, \ell)}{(k-\ell)!} = \sum_{m=0}^k \frac{S(n, k-m)}{m!} = \frac{k^n}{k!} \quad (4.3)$$

which is a special case of the formula

$$x^n = \sum_{j=0}^n \binom{x}{j} j! S(n, j)$$

in [80, p. 116, (9.11)].

Remark 4.5. Setting $a_1 = x$, $a_2 = 2$, and $\alpha \rightarrow 0$ in Theorem 3.1 reduces to

$$\begin{aligned} & B_{n,k}(x, 2, 3, 4, \dots, n-k+1) \\ &= \frac{n!}{k!} \sum_{\ell=0}^k \binom{k}{\ell} \sum_{k-\ell+r=n-k} \frac{\langle k-\ell \rangle_{k-\ell}}{(k-\ell)! r!} x^\ell \frac{S(r+k-\ell, k-\ell)}{\binom{r+k-\ell}{r}} \\ &= \frac{n!}{k!} \sum_{\ell=0}^k \binom{k}{\ell} \frac{1}{(n-2k+\ell)!} \frac{S(n-k, k-\ell)}{\binom{n-k}{n-2k+\ell}} x^\ell \\ &= \binom{n}{k} \sum_{\ell=0}^k \binom{k}{\ell} (k-\ell)! S(n-k, k-\ell) x^\ell. \end{aligned}$$

Letting $x \rightarrow 0$ in the above equation recovers (1.21). Comparing (1.23) with the above equation reveals

$$\sum_{r=0}^k \binom{k}{r} (k-r)^{n-k} (x-1)^r = \sum_{\ell=0}^k \binom{k}{\ell} (k-\ell)! S(n-k, k-\ell) x^\ell \quad (4.4)$$

In particular, when $x = 1$ in (4.4), we obtain the equation (4.3) once again.

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