PI controllers for 1-D nonlinear transport equation

Jean-Michel Coron, Amaury Hayat

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Jean-Michel Coron[*], Amaury Hayat[†]

April 13, 2018

Abstract

In this paper, we introduce a method to get necessary and sufficient stability conditions for systems governed by 1-D nonlinear hyperbolic partial-differential equations with closed-loop integral controllers, when the linear frequency analysis cannot be used anymore. We study the stability of a general nonlinear transport equation where the control input and the measured output are both located on the boundaries. The principle of the method is to extract the limiting part of the stability from the solution using a projector on a finite-dimensional space and then use a Lyapunov approach. This paper improves a result of Trinh, Andrieu and Xu, and gives an optimal condition for the design of the controller. The results are illustrated with numerical simulations where the predicted stable and unstable regions can be clearly identified.

1 Introduction

Stabilization of systems with Proportional-Integral (PI) controllers has been well-studied in the last decades as it is the most famous boundary control in engineering applications. The use of PI controllers in practical applications goes back to the end of the 18th century with the Perier brothers' pump regulator [11, Pages 50-51 and figure 231, Plate 26], [7, Chapter 2] and later on with Fleeming Jenkin's regulator studied by Maxwell in [18]. Of course these regulators were not yet referred as PI control but in practice they worked similarly. Mathematically the PI control was studied first by Minorsky at the beginning of the 20th century for finite-dimensional systems [19]. In the last decades, the stability of 1-D linear systems with PI control has been well-investigated both for finite-dimensional systems [2, 1] and infinite-dimensional systems (see for instance [5, 12, 26, 24, 16, 21, 25] for hyperbolic systems) and is now very well-known. For infinite-dimensional nonlinear systems, however, only few results are known comparatively, most of them conservative [3, Theorem 2.10], [24]. From a mathematical point of view, dealing nonlinear systems is a challenging and very interesting issue. From a practical point of view, it can be seen as a necessity as numerous physical systems are based on infinite dimensional nonlinear models that are sometimes linearized afterward. The intuitive belief that the stability condition for a nonlinear system should be the same as the stability condition for its linearized counterpart when close to the equilibrium is wrong in general, as shown for example in [10].

The reason for this gap in knowledge between linear and nonlinear systems in infinite dimension is that the main method to obtain the stability of 1-D linear systems with PI control is the frequency (or spectrum) analysis (e.g. [26]), a powerful tool based on the Spectral Mapping Property which gives, among other things, the limit of stability from the differential operator's eigenvalues (e.g. [17, 22, 20]). This powerful tool is not anymore available when dealing with nonlinear systems. Thus, most studies use instead a Lyapunov approach that has the advantage of enabling robust results [9, 15] but as a counterpart is often conservative, meaning that the stability conditions raised are only sufficient and not necessary. Among the necessary and sufficient condition one can refer for instance to [3, Theorem 2.9]. Another point to mention is that, for nonlinear systems, the exponential stability in the different topologies are not equivalent [10].

[*]Sorbonne Université, Université Paris-Diderot SPC, CNRS, INRIA, Laboratoire Jacques-Louis Lions, LJLL, équipe CAGE, F-75005 Paris, France e-mail: coron@ann.jussieu.fr.

[†]Sorbonne Université, Université Paris-Diderot SPC, CNRS, INRIA, Laboratoire Jacques-Louis Lions, LJLL, équipe CAGE, F-75005 Paris, France e-mail: hayat@ljll.math.upmc.fr.

In this article we introduce a method to get a necessary and sufficient condition on the stability. We study the general scalar transport equation with a PI boundary controller which was studied in [24], and in which the authors obtained a sufficient, although conservative, stability condition.

Not only is this equation interesting in itself [8] but it is also interesting as, even if it is the most simple nonlinear evolution equation, it already has some of the key features of nonlinear hyperbolic models whose stabilization has been quite studied in the recent years using various methods [14, 3, 13] This problem has an associated linearized problem where the first eigenvalues making the system unstable are discrete and in finite number. We first extract from the solution of the nonlinear problem the part that would be associated to these eigenvalues in the linear case, using a projector on a finite-dimensional space. In the linearized problem this projected part of the solution is the limiting factor on the stability and it is therefore natural to think that it can also be the limiting factor in the non-linear case. Besides, we know precisely the dynamic of this projection and we can control precisely its decay. Then, a key point is to find a good Lyapunov function for the remaining part of the solution. As the remaining part of the solution is not the limiting factor, the Lyapunov function can be conservative with no harm provided that it gives a sufficient condition that goes beyond the limiting condition corresponding to the projected part.

2 Stability of non-linear transport equation with PI boundary condition

We are interested with the following problem

$$\partial_t z + \lambda(z)\partial_x z = 0, \tag{1}$$

$$z(0,t) = -k_I X(t), \tag{2}$$

$$\dot{X} = z(L,t), \tag{3}$$

where λ is a C^2 function with $\lambda(0) = \lambda_0 > 0$ and k_I is a constant. Let $T > 0$, one can show that the system is well-posed in $C^0([0,T], H^2(0,L)) \times C^2([0,T])$ for initial conditions small enough and sufficiently regular. More precisely one has [24]

Theorem 2.1. *Let $T > 0$. There exists $\delta(T) > 0$ such that for any $\phi_0 \in H^2(0,L)$ satisfying $|\phi_0|_{H^2} \leq \delta$, the system* (1)–(3) *with initial condition (ϕ_0, X_0) such that*

$$X_0 = -k_I^{-1}\phi_0(0),\ \phi_0(L) = k_I^{-1}\lambda(\phi_0(0))\phi_0'(0), \tag{4}$$

has a unique solution $(\phi, X) \in C^0([0,T], H^2(0,L)) \times C^2([0,T])$. Moreover there exists $C(T) > 0$ such that

$$|\phi(t,\cdot)|_{H^2} \leq C(T)\left(|\phi_0(\cdot)|_{H^2}\right). \tag{5}$$

The interest of this system comes from the fact that it is the most simple nonlinear system with a proportional integral control. However it already constitutes a challenge and, to our knowledge, the most advanced result so far is the following result developed in the recent years [24]:

Theorem 2.2. *If $0 < k_I < \lambda(0)\Pi(2-\sqrt{2})/2L$, then the nonlinear system* (1)–(3) *is exponentially stable for the H^2 norm, where*

$$\Pi(x) = \sqrt{x(2-x)}e^{-x/2}. \tag{6}$$

Note that $\Pi(2-\sqrt{2})/2 \approxeq 0.34$. In [24] it is also shown that this result is conservative. In order to study this system, it is interesting to compare it with the corresponding linear case namely the case where λ does not depend on z and (1) is replaced by

$$\partial_t z + \lambda_0 \partial_x z = 0. \tag{7}$$

In this case, a necessary and sufficient condition for the stability can be simply obtained from the frequency analysis, by looking at the eigenvalues of the system (7), (2), (3). It is easy to see that these eigenvalues satisfy the following equation [24].

$$k_I + \varrho e^{\frac{\varrho L}{\lambda_0}} = 0. \tag{8}$$

This implies from [6] that the linear system (7), (2), (3) is exponentially stable if and only if

$$k_I \in \left(0, \frac{\pi\lambda_0}{2L}\right). \tag{9}$$

In the nonlinear case, it is not possible anymore to use a frequency analysis method. One has to use other methods, as for instance the Lyapunov method, which is one of the most famous as it guarantees some robustness of the result. This method was for instance used in [24] to prove Theorem 2.2. However, this method is often conservative as, except in simple cases, it is often difficult to find the right Lyapunov function leading to an optimal condition. As stated in the introduction, we tackle this problem by extracting from the solution the part that limits the stability with a projector and apply our Lyapunov function to the remaining part. Our main result is the following

Theorem 2.3. *The nonlinear system* (1)–(3) *is exponentially stable for the* H^2 *norm if*

$$k_I \in \left(0, \frac{\pi\lambda(0)}{2L}\right). \tag{10}$$

The sharpness of this nonlinear result is suggested from the linear condition (9). This sharpness can also be illustrated by the following proposition

Proposition 2.4. *There exists* $k_1 > \pi\lambda(0)/2L$, *such that for any* $k_I \in (\pi\lambda(0)/2L, k_1)$ *the nonlinear system* (1)–(3) *is unstable for the* H^2 *norm.*

In Section 3 we introduce a new Lyapunov function that can be seen as a good Lyapunov function for this system but we show why it still leads to a conservative result. In Section 4 we introduce a projector to extract from the solution the limiting part for the stability. In Section 5 we prove Theorem 2.3 and Proposition 2.4 using the Lyapunov function and the projector respectively introduced in Section 3 and Section 4. In Section 6 we illustrate these results with a numerical simulation.

3 A quadratic Lyapunov function

In this section we first introduce a new Lyapunov function for the system (1)–(3). This Lyapunov function can be seen as a good candidate to study the stability for the H^2 norm, but, although it already gives a sufficient condition relatively close to the linear condition (9), we will show that it is not enough to achieve the optimal condition (10), which will be the motivation for the next section. As this part is only here to motivate the method of this paper, we will give a sketch of proof for a Lyapunov function equivalent to the L^2 norm, but the same would apply for a similar Lyapunov function equivalent to the H^2 norm (see Section 5).

Let us define $V_0 : L^2(0, L) \times \mathbb{R} \to \mathbb{R}$ by

$$V_0(Z, X) := \int_0^L f(x) e^{-\frac{\mu}{\lambda_0}x} Z^2(x)dx + \left(\int_0^L \alpha Z dx + \beta X\right)^2, \tag{11}$$

where f is a positive C^1 function to be determined later on and α and β are non-zero constants to be determined later on as well. For any $(Z, X) \in L^2(0, L) \times \mathbb{R}$ one has from Cauchy-Schwarz inequality:

$$\begin{aligned} &\frac{\min\{f(x)e^{-\frac{\mu}{\lambda_0}x} : x \in [0, L]\}}{L} (\int_0^L Zdx)^2 + \alpha^2 \left(\int_0^L Zdx\right)^2 + 2\beta\alpha X \left(\int_0^L Zdx\right) + (\beta X)^2 \\ &\leq V_0(Z, X) \leq C_1 \left(|Z|^2_{L^2(0,L)} + \beta X^2\right). \end{aligned} \tag{12}$$

Using that for any $p > 0$, there exists $n_1 \in \mathbb{N}^*$ such that

$$(p + 1)a^2 + b^2 - 2ab \geq \frac{p}{n_1}\left(a^2 + b^2\right), \ \forall\ (a, b) \in \mathbb{R}^2, \tag{13}$$

there exists $C_2 > 0$ such that

$$\frac{1}{C_2}\left(|Z|^2_{L^2(0,L)} + \beta X^2\right) \leq V_0(Z,X) \leq C_2\left(|Z|^2_{L^2(0,L)} + \beta X^2\right). \tag{14}$$

Thus, our function V_0 is equivalent to the norm on $L^2(0,L)\times\mathbb{R}$ defined by $|(Z,X)| = \left(|Z|^2_{L^2(0,L)} + \beta X^2\right)$. It is therefore enough to find $f \in C^1([0,L],(0,+\infty))$, α and β such that V_0 is exponentially decreasing along all $C^0([0,T],H^2\times\mathbb{R})$ solutions of system (1)–(3) to prove that the null steady-state of the system (1)–(3) is exponentially stable for the L^2 norm. Let $T > 0$, and let (z,X) be a $C^3([0,T]\times[0,L])\times C^3([0,T])$ solution of the system (1)–(3) (we could get the result for $C^0([0,T],H^2\times\mathbb{R})$ later on by density as in [4, Section 4], this will not be done in this section as it is only a sketch proof). Let us denote $V_0(z(x,\cdot),X(t))$ by $V_0(t)$. Differentiating V_0 with respect to t, using (1), (3) and integrating by parts one has

$$\begin{aligned}
\frac{dV_0}{dt} = &-\left[\lambda(z(t,x))f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)\right]_0^L + \int_0^L \lambda(0)f'(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx - \mu\int_0^L f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx \\
&+ \mu\int_0^L \frac{\lambda_0-\lambda(z(t,x))}{\lambda_0}f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx \\
&+ \int_0^L f(x)e^{-\frac{\mu}{\lambda_0}x}\frac{\partial\lambda}{\partial z}z_x z^2 + (\lambda(z(t,x))-\lambda(0))f'(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx \\
&+ 2\left(\int_0^L \alpha z dx + \beta X(t)\right)\left(-[\alpha\lambda z]_0^L + \int_0^L \alpha\frac{\partial\lambda}{\partial z}z_x z dx + \beta z(t,L)\right).
\end{aligned} \tag{15}$$

Thus using (2), one has

$$\begin{aligned}
\frac{dV_0}{dt} &= -\lambda(z(t,L))f(L)e^{-\frac{\mu}{\lambda_0}L}z^2(t,L) - \mu\int_0^L f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx + \lambda(z(t,0))f(0)X^2(t)k_I^2 \\
&- \int_0^L (-\lambda(0)f'(x))e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx + \mu\int_0^L \frac{\lambda_0-\lambda(z(t,x))}{\lambda_0}f(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx \\
&+ 2\left(\int_0^L \alpha z dx + \beta X(t)\right)(-\alpha\lambda_0 z(t,L) + \beta z(t,L) \; -\alpha k_I X(t)\lambda(z(t,0)) - \alpha(\lambda(z(t,L))-\lambda_0)z(t,L)) \\
&+ \int_0^L f(x)\frac{\partial\lambda}{\partial z}z_x e^{-\frac{\mu}{\lambda_0}x}z^2 + (\lambda(z(t,x))-\lambda(0))f'(x)e^{-\frac{\mu}{\lambda_0}x}z^2(t,x)dx \\
&+ 2\left(\int_0^L \alpha z dx + \beta X(t)\right)\int_0^L \alpha\frac{\partial\lambda}{\partial z}z_x z dx.
\end{aligned} \tag{16}$$

We can now choose $\beta = \lambda_0 \alpha$. Equation (16) becomes

$$\begin{aligned}
\frac{dV_0}{dt} &= -\lambda(z(t,L))e^{-\frac{\mu}{\lambda_0}L} f(L) z^2(t,L) - \mu\left(\int_0^L f(x) e^{-\frac{\mu}{\lambda_0}x} z^2(t,x)dx + X(t)^2\right) \\
&+ (\lambda(0)f(0)k_I^2 + \mu)X^2(t) - \int_0^L (-\lambda(0)f'(x))e^{-\frac{\mu}{\lambda_0}x} z^2(t,x)dx \\
&- 2\int_0^L \alpha^2 k_I \lambda(0) z X(t) dx - 2\alpha^2\lambda(0)^2 k_I X^2(t) \\
&- 2\int_0^L \alpha^2 k_I (\lambda(z(t,0)) - \lambda(0)) z X(t) dx \\
&- 2\alpha^2(\lambda(z(t,0)) - \lambda(0))(\lambda(z(t,0)) + \lambda(0)) k_I X^2(t) \\
&+ (\lambda(z(t,0)) - \lambda(0)) f(0) k_I^2 X^2(t) \\
&+ 2\left(\int_0^L \alpha z dx + \beta X(t)\right)(-\alpha(\lambda(z(t,L)) - \lambda_0) z(t,L)) \\
&+ \int_0^L f(x)\frac{\partial \lambda}{\partial z} z_x e^{-\frac{\mu}{\lambda_0}x} z^2 + (\lambda(z) - \lambda(0)) f'(x) e^{-\frac{\mu}{\lambda_0}x} z^2(t,x) dx \\
&+ 2\left(\int_0^L \alpha z dx + \beta X(t)\right)\int_0^L \alpha \frac{\partial \lambda}{\partial z} z_x z dx.
\end{aligned} \tag{17}$$

Using the equivalence between V_0 and $|z(t,\cdot)|_{L^2} + |X|$, there exists a constant $C_3 > 0$, maybe depending continuously on μ but positive for $\mu \in [0,\infty)$ such that

$$\mu\left(\int_0^L f(x)e^{-\frac{\mu}{\lambda_0}x} z^2(t,x)dx + X(t)^2\right) \geq \mu C_3 V_0, \tag{18}$$

and as λ is C^1, (17) can be simplified in

$$\begin{aligned}
\frac{dV_0}{dt} \leq &- \mu C_3 V_0 - \lambda(z(t,L)) f(L) e^{-\frac{\mu}{\lambda_0}L} z^2(t,L) \\
&- I + O\left((|z(t,\cdot)|_{H^2} + |X(t)|)^3\right),
\end{aligned} \tag{19}$$

where $O(r)$ means that there exist $\eta > 0$ and $C > 0$, both independent of ϕ, X, T and $t \in [0,T]$, such that

$$(|r| \leq \eta) \implies (|O(r)| \leq C_1 |r|),$$

and where I is the quadratic form defined by

$$\begin{aligned}
I := &X^2(t)\left(2\alpha^2\lambda(0)^2 k_I - \lambda(0) f(0) k_I^2 - \mu\right) \\
&+ \int_0^L (-\lambda(0)f'(x))e^{-\frac{\mu}{\lambda_0}x} z^2(t,x) + 2\alpha^2 z k_I \lambda(0) X(t) dx.
\end{aligned} \tag{20}$$

To ensure the decay of V_0, we would like to make this quadratic form in ϕ and X positive definite with $f > 0$. This implies that f is decreasing and $k_I > 0$. If we place ourselves in the limiting favourable case where I is only semi-definite positive, and $f(L) = \mu = 0$, one has

$$f'(x)\left(2\alpha^2\lambda(0)k_I - f(0)k_I^2\right) = -L\alpha^4 k_I^2. \tag{21}$$

Thus f' is constant and, as $f(L) = 0$,

$$-2\lambda(0)\alpha^2 f(0) k_I + f^2(0) k_I^2 + L^2\alpha^4 k_I^2 = 0. \tag{22}$$

With $\lambda(0) = \lambda_0$, this equation has a positive solution if and only if

$$4\alpha^4 k_I^2\left(\lambda_0^2 - k_I^2 L^2\right) \geq 0. \tag{23}$$

This is equivalent to $|k_I| \leq \lambda_0/L$. This is the limiting case, to get I definite positive and V_0 exponentially decreasing we would need to add $V_{0,1}(t) = V_0(z_t, \dot{X})$ and $V_{0,2}(t) = V(z_{tt}, \ddot{X})$ to make the Lyapunov function equivalent to the H^2 norm to deal with $O(|z(t,\cdot)|_{H^2} + |X(t)|)$ as in Section 5, and we would get the following sufficient condition: $k_I \in (0, \lambda_0/L)$ which is better that the condition given by Theorem 2.2, but conservative compared to the necessary condition (9). This motivates the next section.

4 Extracting the limiting part of the solution

In this section we introduce the projector that will enable us to extract from the solution the limiting part for the stability. We start by introducing the operator A,

$$A\begin{pmatrix}\phi\\X\end{pmatrix} := \begin{pmatrix}-\lambda_0\phi_x\\ \phi(L)\end{pmatrix} \tag{24}$$

defined on the domain $\mathcal{D}(A) = \{(\phi, X)^\intercal | \phi \in H^2(0,L), X \in \mathbb{R}, \phi(0) = -k_I X\}$. And we note that looking for solutions to the linearized problem (7), (2), (3) can be seen as looking for solutions $(\phi, X)^\intercal \in C^0([0,T], \mathcal{D}(A))$ to the differential problem

$$\begin{pmatrix}\dot{\phi}\\ \dot{X}\end{pmatrix} = A\begin{pmatrix}\phi\\X\end{pmatrix}. \tag{25}$$

As mentioned in Section 2, we know that any eigenvalue ϱ of this projector satisfies (8) which, denoting $\varrho\lambda_0^{-1} = \sigma_\varrho + i\omega_\varrho$ with $(\sigma_\varrho, \omega_\varrho) \in \mathbb{R}^2$, is equivalent to

$$\begin{aligned}\lambda_0 e^{\sigma_\varrho L}\left(\omega_\varrho \sin(\omega_\varrho L) - \sigma_\varrho\cos(\omega_\varrho L)\right) =& k_I,\\ \omega_\varrho\cos(\omega_\varrho L) + \sigma_\varrho \sin(\omega_\varrho L) =& 0.\end{aligned} \tag{26}$$

Assuming (9), there is a unique solution to (26) that also satisfies $\omega \in (-\pi/2L, \pi/2L)$ [23, Page 22]. We denote by ϱ_1 the corresponding eigenvalue. In [23] it was shown that this eigenvalue and its conjugate are the eigenvalues with the largest real part and are the limiting factor to the stability in the linear case. Although we do not need this claim in what follows, it explains why we consider this eigenvalue. We suppose that $\omega := \omega_{\varrho_1} \neq 0$. The special case $\omega_{\varrho_1} = 0$ is simpler can be treated similarly (see Remark 1).

We introduce the following projector:

$$p := \begin{pmatrix}p_1\\p_2\end{pmatrix} \in \mathcal{L}(\mathcal{D}(A), \mathrm{Span}\{e^{-\frac{\varrho_1}{\lambda_0}x}, e^{-\frac{\bar{\varrho}_1}{\lambda_0}x}\}) \tag{27}$$

defined by

$$\begin{aligned}p_1\begin{pmatrix}\phi\\X\end{pmatrix} :=& \alpha_1\left(\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx + \lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}X\right)e^{-\frac{\varrho_1}{\lambda_0}x}\\ &+ \bar{\alpha}_1\left(\int_0^L \phi(x)e^{\frac{\bar{\varrho}_1}{\lambda_0}x}dx + \lambda_0 e^{\frac{\bar{\varrho}_1}{\lambda_0}L}X\right)e^{-\frac{\bar{\varrho}_1}{\lambda_0}x},\end{aligned} \tag{28}$$

$$\begin{aligned}p_2\begin{pmatrix}\phi\\X\end{pmatrix} :=& \frac{\alpha_1}{\varrho_1}\left(\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx + \lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}X\right)e^{-\frac{\varrho_1}{\lambda_0}L}\\ &+ \frac{\bar{\alpha}_1}{\bar{\varrho}_1}\left(\int_0^L \phi(x)e^{\frac{\bar{\varrho}_1}{\lambda_0}x}dx + \lambda_0 e^{\frac{\bar{\varrho}_1}{\lambda_0}L}X\right)e^{-\frac{\bar{\varrho}_1}{\lambda_0}L},\end{aligned} \tag{29}$$

where $\bar{z}$ stands for the conjugate of z and $\alpha_1 := \varrho_1/(\varrho_1 L + \lambda_0)$. Here we used a slight abuse of notation and the notation $e^{-\frac{\varrho_1}{\lambda_0}x}$ outside the brackets refers actually to the function $x \to e^{-\frac{\varrho_1}{\lambda_0}x}$ defined on

$[0,L]$. One can see that p is real even though ϱ_1 is complex, as p is the sum of a function and its conjugate. Denoting $\varrho_1\lambda_0^{-1}=\sigma+i\omega$, the formulation (28)–(29) is equivalent to

$$\begin{aligned}
p_1\begin{pmatrix}\phi\\X\end{pmatrix} &= \left(\int_0^L \phi(x)e^{\sigma x}\cos(\omega x)dx+\lambda_0 X(t)\cos(\omega L)e^{\sigma L}\right)\left(\mathrm{Re}(\alpha_1)\sin(\omega x)e^{-\sigma x}+\mathrm{Im}(\alpha_1)\cos(\omega x)e^{-\sigma x}\right)\\
&\quad+\left(\int_0^L \phi(x)e^{\sigma x}\sin(\omega x)dx+\lambda_0 X(t)\sin(\omega L)e^{\sigma L}\right)\left(\mathrm{Re}(\alpha_1)\cos(\omega x)e^{-\sigma x}-\mathrm{Im}(\alpha_1)\sin(\omega x)e^{-\sigma x}\right).\\
p_2\begin{pmatrix}\phi\\X\end{pmatrix} &= \left(\int_0^L \phi(x)e^{\sigma x}\cos(\omega x)dx+\lambda_0 X(t)\cos(\omega L)e^{\sigma L}\right)\left(\mathrm{Re}(\frac{\alpha_1}{\varrho_1})\sin(\omega L)e^{-\sigma L}+\mathrm{Im}(\frac{\alpha_1}{\varrho_1})\cos(\omega x)e^{-\sigma x}\right)\\
&\quad+\left(\int_0^L \phi(x)e^{\sigma x}\sin(\omega x)dx+\lambda_0 X(t)\sin(\omega L)e^{\sigma L}\right)\left(\mathrm{Re}(\frac{\alpha_1}{\varrho_1})\cos(\omega L)e^{-\sigma L}-\mathrm{Im}(\frac{\alpha_1}{\varrho_1})\sin(\omega L)e^{-\sigma L}\right).
\end{aligned}\tag{30}$$

However in the following, for simplicity, we will keep the complex formulation. We first show that p commutes with the differential operator A given by (24). Indeed one can check that, with

$$p_{1,\varrho_1}:=\alpha_1\left(\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}X\right)e^{-\frac{\varrho_1}{\lambda_0}x},\tag{31}$$

one has

$$\begin{aligned}
p_{1,\varrho_1}\left(A\begin{pmatrix}\phi\\X\end{pmatrix}\right) &= p_{1,\varrho_1}\left(\begin{pmatrix}-\lambda_0\phi_x\\ \phi(L)\end{pmatrix}\right)=\alpha_1\left(-\lambda_0\int_0^L \phi_x(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}\phi(L)\right)e^{-\frac{\varrho_1}{\lambda_0}x}\\
&=\alpha_1\left(-\lambda_0\phi(L)e^{\frac{\varrho_1}{\lambda_0}L}+\lambda_0\phi(0)+\varrho_1\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}\phi(L)\right)e^{-\frac{\varrho_1}{\lambda_0}x}.
\end{aligned}\tag{32}$$

Using that $(\phi,X)^\intercal$ belongs to the space $\{(\phi,X)\in L^2(0,L)\times\mathbb{R}|\phi(0)=-k_I X\}$, together with (8), one gets that

$$\begin{aligned}
p_{1,\varrho_1}\left(A\begin{pmatrix}\phi\\X\end{pmatrix}\right) &=\alpha_1\varrho_1\left(\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}X\right)e^{-\frac{\varrho_1}{\lambda_0}x}\\
&=-\lambda_0\left(p_{1,\varrho_1}\begin{pmatrix}\phi\\X\end{pmatrix}\right)_x.
\end{aligned}\tag{33}$$

As $\bar{\varrho}_1$ also verifies (8) we get the same for $p_{1,\bar{\varrho}_1}$, which is defined as p_{1,ϱ_1} in (31) with ϱ_1 instead of ϱ_1. Thus from (28) and (31)

$$p_1\left(A\begin{pmatrix}\phi\\X\end{pmatrix}\right)=\left(A\left(p\begin{pmatrix}\phi\\X\end{pmatrix}\right)\right)_1.\tag{34}$$

Then from (8) and (29), one easily gets that, for any $(\phi,X)\in L^2(0,L)\times\mathbb{R}$, $p_2((\phi,X)^\intercal)=-k_I^{-1}p_1((\phi,X)^\intercal)(0)$, thus $p\begin{pmatrix}\phi\\X\end{pmatrix}\in\mathcal{D}(A)$ and

$$p\left(A\begin{pmatrix}\phi\\X\end{pmatrix}\right)=A\left(p\begin{pmatrix}\phi\\X\end{pmatrix}\right).\tag{35}$$

Now, we show that p is a projector, meaning that $p\circ p=p$. To avoid overloading the computations, we denote

$$d_1=\alpha_1\left(\int_0^L \phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}X\right),\tag{36}$$

and $\bar{d}_1$ is defined similarly with $\bar{\varrho}_1$ instead of ϱ_1. Therefore one has

$$\begin{aligned}
p_1\left(p\begin{pmatrix}\phi\\X\end{pmatrix}\right) &=\alpha_1\left(\int_0^L d_1+\bar{d}_1e^{(\frac{\varrho_1}{\lambda_0}-\frac{\bar{\varrho}_1}{\lambda_0})x}dx+\lambda_0 e^{\frac{\varrho_1}{\lambda_0}L}\left(d_1\frac{e^{-\frac{\varrho_1}{\lambda_0}L}}{\varrho_1}+\bar{d}_1\frac{e^{-\frac{\bar{\varrho}_1}{\lambda_0}L}}{\bar{\varrho}_1}\right)\right)e^{-\frac{\varrho_1}{\lambda_0}x}\\
&+\bar{\alpha}_1\left(\int_0^L \bar{d}_1+d_1e^{(\frac{\bar{\varrho}_1}{\lambda_0}-\frac{\varrho_1}{\lambda_0})x}dx+\lambda_0 e^{\frac{\bar{\varrho}_1}{\lambda_0}L}\left(d_1\frac{e^{-\frac{\varrho_1}{\lambda_0}L}}{\varrho_1}+\bar{d}_1\frac{e^{-\frac{\bar{\varrho}_1}{\lambda_0}L}}{\bar{\varrho}_1}\right)\right)e^{-\frac{\bar{\varrho}_1}{\lambda_0}x}.
\end{aligned}\tag{37}$$

Integrating and using (8), one has

$$
\begin{aligned}
p_1\left(p\begin{pmatrix}\phi\\X\end{pmatrix}\right) =& \alpha_1\left(d_1 L+\lambda_0\bar{d}_1\frac{e^{(\frac{\varrho_1}{\lambda_0}-\frac{\bar{\varrho}_1}{\lambda_0})L}-1}{\varrho_1-\bar{\varrho}_1}-\lambda_0\frac{k_I}{\varrho_1}\left(-\frac{d_1}{k_I}-\frac{\bar{d}_1}{k_I}\right)\right)e^{-\frac{\varrho_1}{\lambda_0}x}\\
&+\bar{\alpha}_1\left(\bar{d}_1 L+\lambda_0 d_1\frac{e^{(\frac{\bar{\varrho}_1}{\lambda_0}-\frac{\varrho_1}{\lambda_0})L}-1}{\bar{\varrho}_1-\varrho_1}-\lambda_0\frac{k_I}{\bar{\varrho}_1}\left(-\frac{d_1}{k_I}-\frac{\bar{d}_1}{k_I}\right)\right)e^{-\frac{\bar{\varrho}_1}{\lambda_0}x}\\
=&\alpha_1\left(d_1 L+\lambda_0\frac{d_1}{\varrho_1}+\lambda_0\bar{d}_1\frac{e^{(\frac{\varrho_1}{\lambda_0}-\frac{\bar{\varrho}_1}{\lambda_0})L}-1}{\varrho_1-\bar{\varrho}_1}+\lambda_0\frac{\bar{d}_1}{\varrho_1}\right)e^{-\frac{\varrho_1}{\lambda_0}x}\\
&+\bar{\alpha}_1\left(\bar{d}_1 L+\lambda_0\frac{\bar{d}_1}{\bar{\varrho}_1}+\lambda_0 d_1\frac{e^{(\frac{\bar{\varrho}_1}{\lambda_0}-\frac{\varrho_1}{\lambda_0})L}-1}{\bar{\varrho}_1-\varrho_1}+\lambda_0\frac{d_1}{\bar{\varrho}_1}\right)e^{-\frac{\bar{\varrho}_1}{\lambda_0}x}.
\end{aligned}
\tag{38}
$$

But, still from (8), observe that

$$
\frac{e^{(\frac{\varrho_1}{\lambda_0}-\frac{\bar{\varrho}_1}{\lambda_0})L}-1}{\varrho_1-\bar{\varrho}_1}=\frac{\left(-\frac{k_I}{\varrho_1}\right)\left(-\frac{\bar{\varrho}_1}{k_I}\right)-1}{\varrho_1-\bar{\varrho}_1}=-\frac{1}{\varrho_1},
\tag{39}
$$

and recall that $\alpha_1=\varrho_1/(\varrho_1 L+\lambda_0)$, thus

$$
p_1\left(p\begin{pmatrix}\phi\\X\end{pmatrix}\right)=d_1e^{-\frac{\varrho_1}{\lambda_0}x}+\bar{d}_1e^{-\frac{\bar{\varrho}_1}{\lambda_0}x}=p_1\begin{pmatrix}\phi\\X\end{pmatrix}.
\tag{40}
$$

Besides we have from (8) and (29)

$$
\begin{aligned}
p_2\left(p\begin{pmatrix}\phi\\X\end{pmatrix}\right)&=-k_I p_1\left(p\begin{pmatrix}\phi\\X\end{pmatrix}\right)(0)\\
&=-k_I p_1\begin{pmatrix}\phi\\X\end{pmatrix}(0)=p_2\begin{pmatrix}\phi\\X\end{pmatrix}.
\end{aligned}
\tag{41}
$$

Therefore $p\circ p=p$. As p is a linear application, this implies in particular that

$$
p\left(\begin{pmatrix}\phi\\X\end{pmatrix}-p\begin{pmatrix}\phi\\X\end{pmatrix}\right)=0.
\tag{42}
$$

Thus, let $(\phi,X)^\intercal\in\mathcal{D}(A)$, if we define $\phi_1=p_1(\phi,X)^\intercal$, $X_1:=p_2(\phi,X)^\intercal$ and $\phi_2:=\phi-\phi_1$ and $X_2:=X-X_1$, one has from (42) and (28), as $\alpha_1\neq 0$

$$
\int_0^L\phi_2(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0e^{\frac{\varrho_1}{\lambda_0}L}X_2=\int_0^L\phi_2(x)e^{\frac{\bar{\varrho}_1}{\lambda_0}x}dx+\lambda_0e^{\frac{\bar{\varrho}_1}{\lambda_0}L}X_2=0.
\tag{43}
$$

Thus

$$
\int_0^L\phi_2(x)\left(e^{\frac{\varrho_1}{\lambda_0}(x-L)}-e^{\frac{\bar{\varrho}_1}{\lambda_0}(x-L)}\right)dx=0.
\tag{44}
$$

Or equivalently, denoting as previously $\varrho_1\lambda_0^{-1}=\sigma+i\omega$,

$$
\int_0^L\phi_2(x)e^{\sigma(x-L)}\sin(\omega(x-L))dx=0.
\tag{45}
$$

Remark 1.
- *In the special case $\omega=0$, we can define p similarly as previously but with $\alpha_1=\bar{\alpha}_1=1/2$ instead. Then (35) still holds, but, as $\varrho_1=\bar{\varrho}_1$, p is now a projector on the one-dimensional space* $\mathrm{Span}\{e^{-\frac{\varrho_1}{\lambda_0}x}\}$ *and is defined by*

$$
p_1((\phi,X)^\intercal)=\left(\int_0^L\phi(x)e^{\frac{\varrho_1}{\lambda_0}x}dx+\lambda_0e^{\frac{\varrho_1}{\lambda_0}L}X\right)e^{-\frac{\varrho_1}{\lambda_0}x},
\tag{46}
$$

and $p_2((\phi,X)^\intercal)=\varrho_1^{-1}p_1((\phi,X)^\intercal)(L)$. Nevertheless (45) still holds and is straightforward. Indeed, we can still define $(\phi_1,X_1)^\intercal=p((\phi,X)^\intercal)$ and $(\phi_2,X_2)=(\phi-\phi_1,X-X_1)$, and, as $\omega=0$, (45) holds directly.

- *Note that, when $\omega \neq 0$,* (43) *contains two equations, as p is a projector on a space of dimension 2. Therefore another relation can be inferred from* (43) *in addition to* (45)*, namely*

$$\int_0^L \phi_2(x) e^{\sigma(x-L)} \cos(\omega(x-L)) dx = -\lambda_0 X_2. \tag{47}$$

However this relation will not be used in the following.

5 Exponential stability analysis

In this section we use the results of the above sections to prove Theorem 2.3. We first separate the solution of the system in a projected part and a remaining part using the projector defined in Section 4. Then we use the Lyapunov function defined in Section 3 to deal with the remaining part.

Proof of Theorem 2.3. Let $T > 0$ and let ϕ be a solution to the nonlinear system (1)–(3). We suppose in the following that

$$|\phi(t,\cdot)|_{H^2} \leq \varepsilon, \ \forall\, t \in [0,T], \tag{48}$$

with $\varepsilon \in (0,1)$ to be chosen later on. This assumption can be done as we are looking for a local result with respect to the perturbations (i.e. the initial conditions), and, from (5), for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|\phi_0|_{H^2} \leq \delta$ then (48) holds. Let us assume in addition that $\phi \in C^3([0,1] \times [0,T])$ (we will relax this assumption later on using a density argument). Using the last section, we define the following functions

$$\begin{pmatrix} \phi_1(t,x) \\ X_1(t,x) \end{pmatrix} = p \begin{pmatrix} \phi(t,x) \\ X(t) \end{pmatrix}, \tag{49}$$

$$\begin{pmatrix} \phi_2(t,x) \\ X_2(t,x) \end{pmatrix} = \begin{pmatrix} \phi(t,x) \\ X(t) \end{pmatrix} - \begin{pmatrix} \phi_1(t,x) \\ X_1(t) \end{pmatrix}. \tag{50}$$

We expect to have extracted from (ϕ, X) the limiting factor for the stability that is now contained in (ϕ_1, X_1). The function (ϕ_1, X_1) is a simple projection on a space of finite dimension, it has therefore a simple dynamic and is easy to control, while we will use our Lyapunov function introduced earlier in Section 3 to deal with (ϕ_2, X_2). In other words we will consider the following total Lyapunov function

$$V(t) = V_1(t) + V_2(t), \tag{51}$$

where V_1 is a Lyapunov function for (ϕ_1, X_1) to be defined and $V_2(t) = V_{2,1}(t) + V_{2,2}(t) + V_{2,3}(t)$, with $V_{2,k}(t) = V_0(\partial_t^{k-1}\phi_2(t,\cdot), \partial_t^{k-1}X(t))$. Recall that the definition of V_0 is given in (11).

Remark 2. *Note that, strictly speaking, this Lyapunov function can be expressed as a functional on time-independent functions belonging to $H^2(0,L) \times \mathbb{R}$, using for instance the following notations for $(\phi, X) \in H^2(0,L) \times \mathbb{R}$:*

$$\begin{aligned} &\dot{X} := \phi(t,L), \ \ddot{X} := -\lambda(\phi(L))\partial_x\phi(L), \\ &\partial_t\phi := -\lambda(\phi)\partial_x\phi, \ \partial_t^2\phi := -\lambda'(\phi)\left(\partial_x\phi\right)^2 - \lambda(\phi)\partial_x^2\phi. \end{aligned} \tag{52}$$

Of course these notations correspond to the time-derivatives of the functions when (X,ϕ) is time-dependent and a solution of (1)–(3)*. The same remark will apply later on for the definition of V_1 given by* (56)*.*

Let us look at ϕ_1. From the definition of p_1 given by (28), $p_1 = p_{1,\varrho_1} + p_{1,\bar{\varrho}_1}$ where p_{1,ϱ_1} is given by (31) and $p_{1,\bar{\varrho}_1}$ is given by the same definition with $\bar{\varrho}_1$ instead of ϱ_1. Similarly $p_2 = p_{2,\varrho_1} + p_{2,\bar{\varrho}_1}$ with

$$p_{2,\varrho_1}((\phi,X)^{\mathsf{T}}) = \frac{p_{1,\varrho_1}((\phi,X)^{\mathsf{T}})(L)}{\varrho_1}, \tag{53}$$

and $p_{2,\bar{\varrho}_1}$ defined similarly but with $\bar{\varrho}_1$ instead of ϱ_1. Therefore we can define

$$\begin{pmatrix} \phi_{\varrho_1}(t,x) \\ X_{\varrho_1}(t) \end{pmatrix} := p_{\varrho_1} \begin{pmatrix} \phi(t,x) \\ X(t) \end{pmatrix} := \begin{pmatrix} p_{1,\varrho_1}(\phi,X)^{\mathsf{T}}(t,x) \\ p_{2,\varrho_1}(\phi,X)^{\mathsf{T}}(t,L) \end{pmatrix}, \tag{54}$$

and we can define its conjugate $(\phi_{\bar{\lambda}1}, X_{\bar{\lambda}1})^\intercal$ similarly. Thus we can decompose $(\phi_1, X_1)^\intercal$ in

$$\begin{pmatrix}\phi_1(t,x)\\ X_1(t)\end{pmatrix} = \begin{pmatrix}\phi_{\varrho_1}(t,x)\\ X_{\varrho_1}(t)\end{pmatrix} + \begin{pmatrix}\phi_{\bar{\varrho}_1}(t,x)\\ X_{\bar{\varrho}_1}(t)\end{pmatrix}. \tag{55}$$

Let us now define $V_1(t)$ by

$$\begin{aligned} V_1(t) :=& \int_0^L |\phi_{\varrho_1}(t,x)|^2 + |\partial_t\phi_{\varrho_1}(t,x)|^2 + |\partial_t^2\phi_{\varrho_1}(t,x)|^2 dx \\ &+ |X_{\varrho_1}(t)|^2 + |\dot{X}_{\varrho_1}(t)|^2 + |\ddot{X}_{\varrho_1}(t)|^2. \end{aligned} \tag{56}$$

There exists $\varepsilon_1 \in (0,1)$ such that for $\varepsilon < \varepsilon_1$, one has

$$\begin{aligned} &\frac{1}{2}\min(1,\lambda_0^4)\left(|\phi_{\varrho_1}|_{H^2}^2 + |X_{\varrho_1}|^2 + |\dot{X}_{\varrho_1}|^2 + |\ddot{X}_{\varrho_1}|^2\right) \\ &\leq V_1 \leq 2\max(1,\lambda_0^4)\left(|\phi_{\varrho_1}|_{H^2}^2 + |X_{\varrho_1}|^2 + |\dot{X}_{\varrho_1}|^2 + |\ddot{X}_{\varrho_1}|^2\right), \end{aligned} \tag{57}$$

and therefore

$$\begin{aligned} &|\phi_1|_{H^2}^2 + |X_1|^2 + |\dot{X}_1|^2 + |\ddot{X}_1|^2 \\ &\leq 4|\phi_{\varrho_1}|_{H^2}^2 + 4|X_{\varrho_1}|^2 + 4|\dot{X}_{\varrho_1}|^2 + 4|\ddot{X}_{\varrho_1}|^2 \\ &\leq 8\max(1,\lambda_0^4)V_1. \end{aligned} \tag{58}$$

Differentiating V_1 one has

$$\begin{aligned} \frac{dV_1}{dt} =& \int_0^L 2\mathrm{Re}\left(\partial_t\phi_{\varrho_1}\phi_{\bar{\varrho}_1}\right) + 2\mathrm{Re}\left(\partial_t^2\phi_{\varrho_1}\partial_t\phi_{\bar{\varrho}_1}\right) + 2\mathrm{Re}\left(\partial_t^3\phi_{\varrho_1}\partial_t^2\phi_{\bar{\varrho}_1}\right)dx \\ &+ 2\mathrm{Re}\left(\dot{X}_{\varrho_1}X_{\bar{\varrho}_1}\right) + 2\mathrm{Re}\left(\ddot{X}_{\varrho_1}\dot{X}_{\bar{\varrho}_1}\right) + 2\mathrm{Re}\left(\dddot{X}_{\varrho_1}\ddot{X}_{\bar{\varrho}_1}\right). \end{aligned}. \tag{59}$$

From (28), (29), and (49)

$$\begin{pmatrix}\partial_t\phi_{\varrho_1}(t,x)\\ \dot{X}_{\varrho_1}(t)\end{pmatrix} = p_{\varrho_1}\begin{pmatrix}\partial_t\phi(t,x)\\ \dot{X}(t)\end{pmatrix} = p_{\varrho_1}\left(A_1\begin{pmatrix}\phi(t,x)\\ X(t)\end{pmatrix}\right), \tag{60}$$

where A_1 is now defined for any $(\phi, X)^\intercal \in \mathcal{D}(A)$ by

$$A_1\begin{pmatrix}\phi\\ X\end{pmatrix} := \begin{pmatrix}-\lambda(\phi)\phi_x\\ \phi(L)\end{pmatrix} = A\begin{pmatrix}\phi\\ X\end{pmatrix} + \begin{pmatrix}(\lambda_0 - \lambda(\phi))\,\phi_x\\ 0\end{pmatrix}. \tag{61}$$

Observe that the commutation property (34) still holds with p_{ϱ_1} instead of p, and that p_{ϱ_1} is still a linear operator, thus

$$\begin{aligned} \begin{pmatrix}\partial_t\phi_{\varrho_1}(t,x)\\ \dot{X}_{\varrho_1}(t)\end{pmatrix} &= A\left(p_{\varrho_1}\begin{pmatrix}\phi(t,x)\\ X(t)\end{pmatrix}\right) + p_{\varrho_1}\begin{pmatrix}(\lambda_0 - \lambda(\phi))\,\partial_x\phi(t,x)\\ 0\end{pmatrix} \\ &= \begin{pmatrix}-\lambda_0(\phi_{\varrho_1})_x(t,x)\\ \phi_{\varrho_1}(t,L)\end{pmatrix} + \begin{pmatrix}\alpha_1 e^{-\frac{\varrho_1}{\lambda_0}x}\int_0^L(\lambda_0 - \lambda(\phi))\partial_x(\phi(t,x))e^{\frac{\varrho_1}{\lambda_0}x}dx\\ \frac{\alpha_1}{\varrho_1}e^{-\frac{\varrho_1}{\lambda_0}L}\int_0^L(\lambda_0 - \lambda(\phi))\partial_x(\phi(t,x))e^{\frac{\varrho_1}{\lambda_0}x}dx\end{pmatrix} \\ &= \varrho_1\begin{pmatrix}\phi_{\varrho_1}(t,x)\\ X_{\varrho_1}(t)\end{pmatrix} + \begin{pmatrix}\alpha_1 e^{-\frac{\varrho_1}{\lambda_0}x}\int_0^L(\lambda_0 - \lambda(\phi))\partial_x(\phi(t,x))e^{\frac{\varrho_1}{\lambda_0}x}dx\\ \frac{\alpha_1}{\varrho_1}e^{-\frac{\varrho_1}{\lambda_0}L}\int_0^L(\lambda_0 - \lambda(\phi))\partial_x(\phi(t,x))e^{\frac{\varrho_1}{\lambda_0}x}dx\end{pmatrix}. \end{aligned} \tag{62}$$

Besides, let $k \in \{0,1,2\}$, as λ is C^1, integrating by parts and using (2),

$$
\begin{aligned}
&\left|\int_0^L (\lambda_0-\lambda(\phi))\partial_t^k \partial_x(\phi(t,x)) e^{\frac{\varrho_1}{\lambda_0}x}dx\right| \\
&= \Big|(\lambda_0-\lambda(\phi(t,L)))\partial_t^k\phi(t,L)e^{\frac{\varrho_1}{\lambda_0}L} - (\lambda_0-\lambda(\phi(t,0)))\partial_t^k\phi(t,0) \\
&- \int_0^L \frac{\varrho_1}{\lambda_0}(\lambda_0-\lambda(\phi(t,x)))\partial_t^k\phi(t,x)e^{\frac{\varrho_1}{\lambda_0}x} \\
&- \lambda'(\phi(t,x))\phi_t(t,x)\partial_t^k\phi(t,x)e^{\frac{\varrho_1}{\lambda_0}x}dx\Big| \\
\leq& \left|e^{\frac{\varrho_1}{\lambda_0}x}\right|_0 \frac{\varrho_1}{\lambda_0}\left(\int_0^L |\lambda_0-\lambda(\phi(t,x))-\lambda'(\phi(t,x))\phi_t(t,x)|^2\,dx\right)^{1/2} \\
&\times\left(\int_0^L |\partial_t^k\phi(t,x)|^2 dx\right)^{1/2} + O\left(\left|\partial_t^k\phi(t,L)\right||\phi(t,L)| + |\partial_t^k\phi(t,0)||\phi(t,0)|\right) \\
\leq& C_0\left(|\partial_t^k\phi(t,L)||\phi(t,L)| + |\partial_t^k X||X| + (|\phi|_0+|\phi_x|_0)\left|\partial_t^k\phi\right|_{L^2}\right),
\end{aligned}
\tag{63}
$$

where $|\cdot|_0$ denotes the C^0 norm or equivalently the L^∞ norm and C_0 is a constant independent of ϕ that depends only on λ, ϱ_1, L and k_I. Thus, using (63) with $k=0$, and noting that $|\phi|_0+|\phi_x|_0$ can be bounded by $|\phi|_{H^2}$ from Sobolev inequality, the last term of (62) is a quadratic perturbation that can be bounded by $\left(|\phi|_{H^2}^2+|X|^2+\phi(t,L)^2\right)$. One can do similarly with the second and third time-derivative noticing that

$$
\begin{aligned}
\begin{pmatrix}\partial_t^2\phi\\ \ddot{X}\end{pmatrix} &= \begin{pmatrix}-\lambda_0\partial_x(\partial_t\phi)\\ \partial_t\phi(t,L)\end{pmatrix} \\
&+\begin{pmatrix}-\lambda'(\phi)\phi_t\phi_x+(\lambda_0-\lambda(\phi))\partial_x(\partial_t\phi)\\ 0\end{pmatrix} \\
\begin{pmatrix}\partial_t^3\phi\\ \dddot{X}\end{pmatrix} &= \begin{pmatrix}-\lambda_0\partial_x(\partial_t^2\phi)\\ \partial_t^2\phi(t,L)\end{pmatrix} \\
&+\begin{pmatrix}-\lambda''(\phi)(\phi_t)^2\phi_x-2\lambda'(\phi)\phi_t\phi_{tx}-\lambda'(\phi)\phi_{tt}\phi_x\\ +(\lambda_0-\lambda(\phi))\partial_x(\partial_t^2\phi)\\ 0\end{pmatrix},
\end{aligned}
\tag{64}
$$

and noticing that all the quadratic terms in ϕ involve at most a second derivative in ϕ. Thus as λ is C^2, all the quadratic terms belong to L^1 and their L^1 norm can be bounded by $|\phi|_{H^2}^2$. The L^1 norm of the third order derivative can be bounded by $(|\phi|_{H^2}+|\ddot{X}|^2+(\partial_{tt}\phi(t,L))^2)$ using (63) and $k=2$. Therefore, noting from (28) that $|\partial_t^k\phi_1(t,L)| \leq 2|\bar{\varrho}_1||\partial_t^k X_{\varrho_1}|$,

$$
\begin{aligned}
\frac{dV_1}{dt} =& 2\mathrm{Re}(\varrho_1)\lambda_0\int_0^L |\phi_{\varrho_1}|^2+|\partial_t\phi_{\varrho_1}|^2+|\partial_t^2\phi_{\varrho_1}|^2dx + 2\mathrm{Re}(\varrho_1)\left(|X_{\varrho_1}(t)|^2+|\dot{X}_{\varrho_1}(t)|^2+|\ddot{X}_{\varrho_1}(t)|^2\right) \\
&+O\left(|\phi_{\varrho_1}|_{H^2}\left(|\phi|_{H^2}^2+|X_{\varrho_1}|^2+|\dot{X}_{\varrho_1}|^2+|\ddot{X}_{\varrho_1}|^2+|X|^2+|\dot{X}|^2+|\ddot{X}|^2\right)\right) \\
&+C(|\phi_{\varrho_1}|_{H^2}+|X_{\varrho_1}|)(\phi_2^2(t,L)+(\partial_t\phi_2)^2(t,L)+(\partial_t^2\phi_2)^2(t,L)),
\end{aligned}
\tag{65}
$$

where C is a positive constant that only depends on λ, ϱ_1, k_I, L. As $\mathrm{Re}(\varrho_1)<0$ from (10) and (8),

$$
\begin{aligned}
\frac{dV_1}{dt} \leq& -2|\mathrm{Re}(\varrho_1)|\min(\lambda_0,1)V_1 \\
&+O\left((|\phi_{\varrho_1}|_{H^2}+|\phi_2|_{H^2}+|X_{\varrho_1}|+|\dot{X}_{\varrho_1}|+|\ddot{X}_{\varrho_1}|+|X_2|+|\dot{X}_2|+|\ddot{X}_2|)^3\right) \\
&+C(|\phi_{\varrho_1}|_{H^2}+|X_{\varrho_1}|)(\phi_2^2(t,L)+\partial_t\phi_2^2(t,L)+\partial_t^2\phi_2^2(t,L)).
\end{aligned}
\tag{66}
$$

The first term will imply the exponential decay, while the two other terms will be compensated using V_2.

Let us now look at V_2. From (1)–(3), (8), (29), (50), and (62), (ϕ_2, X_2) is a solution to the following system

$$\begin{aligned}
\partial_t \phi_2 + \lambda(\phi)\partial_x \phi_2 &= (\lambda_0 - \lambda(\phi))\partial_x \phi_1 + p_1\left(\begin{pmatrix}(\lambda_0-\lambda(\phi))\partial_x\phi \\ 0\end{pmatrix}\right)\\
\phi_2(t,0) &= -k_I X_2(t)\\
\dot{X}_2 &= \phi_2(t,L).
\end{aligned} \tag{67}$$

Thus acting similarly as in Section 3, (15)–(19), and using (63), we have

$$\begin{aligned}
\frac{dV_{2,1}}{dt} \leq &-\mu C_3 V_{2,1} - \lambda(\phi(t,L))f(L)e^{-\frac{\mu}{\lambda_0}L}\phi_2^2(t,L) - X_2^2(t)\left(2\alpha^2\lambda(0)^2 k_I - \lambda(0)f(0)k_I^2 - \mu\right)\\
&- \int_0^L (-\lambda(0)f'(x))e^{-\frac{\mu}{\lambda_0}x}\phi_2^2(t,x)dx + 2\alpha^2 k_I \lambda(0)\phi_2 X(t)dx\\
&+ O\left(\left(|\phi_{\varrho_1}|_{H^2} + |X_{\varrho_1}| + |\phi_2|_{H^2} + |X_2| + |\dot{X}_2| + |\ddot{X}_2|\right)^3\right) + C_{2,1}|\phi|_0\left|\phi_2^2(t,L)\right|,
\end{aligned} \tag{68}$$

where $C_{2,1}$ is a positive constant independent of ϕ and X. If we look now at the quadratic form in X_2 and ϕ_2 that appears, we can see that it is exactly the same as previously in (20). However, since ϕ_2 is the complementary of ϕ_1 in ϕ, we now have an additional information on ϕ_2 given by (45). Thus, denoting again this quadratic form by I, recalling that $\lambda(0) = \lambda_0$, and using (45) we have

$$\begin{aligned}
I = &\int_0^L (-\lambda_0 f'(x))e^{-\frac{\mu}{\lambda_0}x}\phi_2^2(t,x)dx + 2\alpha^2 k_I\lambda_0 X(t)\int_0^L \phi_2(1-\kappa\theta(x))dx\\
&+ X^2(t)\left(2\alpha^2\lambda_0^2 k_I - \lambda_0 f(0)k_I^2 - \mu\right)\\
\geq &\inf_{x\in[0,L]}(-\lambda_0 f'(x))e^{-\frac{\mu}{\lambda_0}L}\left(\int_0^L \phi_2^2(t,x)dx\right)\\
&- 2\alpha^2 k_I\lambda_0|X(t)|\left(\int_0^L \phi_2^2 dx\right)^{1/2}\left(\int_0^L (1-\kappa\theta(x))^2 dx\right)^{1/2}\\
&+ X^2(t)\left(2\alpha^2\lambda_0^2 k_I - \lambda_0 f(0)k_I^2 - \mu\right)
\end{aligned} \tag{69}$$

where

$$\theta(x) := e^{\sigma(x-L)}\sin(\omega(x-L)) \tag{70}$$

and κ is a constant that can be chosen arbitrarily. As the right-hand side is now a quadratic form in $|\phi_2|_{L^2}$ and X, a sufficient condition for I to be positive is

$$\begin{aligned}
&\inf_{x\in[0,L]}(-\lambda_0 f'(x))e^{-\frac{\mu}{\lambda_0}L}\left(2\alpha^2\lambda_0^2 k_I - \lambda_0 f(0)k_I^2 - \mu\right)\\
&> \left(\alpha^2 k_I\lambda_0\right)^2\left(\int_0^L (1-\kappa\theta(x))^2 dx\right).
\end{aligned} \tag{71}$$

Of course we have all interest in choosing κ such that it minimizes the integral of $(1-\kappa\theta(x))^2$. We have

$$\int_0^L (1-\kappa\theta(x))^2 dx = \kappa^2\left(\int_0^L \theta^2(x)dx\right) - 2\kappa\left(\int_0^L \theta(x)dx\right) + L. \tag{72}$$

This is a second order polynomial in κ thus, assuming $\omega \neq 0$, its minimum is

$$\begin{aligned}
&L + \frac{\left(\int_0^L \theta(x)dx\right)^2}{\left(\int_0^L \theta^2(x)dx\right)^2}\left(\int_0^L \theta^2(x)dx\right) - 2\frac{\left(\int_0^L \theta(x)dx\right)}{\left(\int_0^L \theta^2(x)dx\right)}\left(\int_0^L \theta(x)dx\right)\\
&= L - \frac{\left(\int_0^L \theta(x)dx\right)^2}{\left(\int_0^L \theta^2(x)dx\right)}.
\end{aligned} \tag{73}$$

Choosing such κ, and f' constant, condition (71) becomes

$$\begin{aligned}
&e^{-\frac{\mu}{\lambda_0}L}\lambda_0\frac{f(0)-f(L)}{L}\left(2\alpha^2\lambda_0^2k_I-\mu-\lambda_0 f(0)k_I^2\right)\\
&-\left(\alpha^2k_I\lambda_0\right)^2L\left(1-\frac{\left(\int_0^L\theta(x)dx\right)^2}{\left(L\int_0^L\theta^2(x)dx\right)}\right)>0.
\end{aligned}\tag{74}$$

which is equivalent to

$$\begin{aligned}
&-\lambda^2(0)k_I^2f^2(0)+\left(2\alpha^2\lambda_0^2k_I-\mu+f(L)\lambda_0k_I^2\right)\lambda_0f(0)-(2\alpha^2\lambda_0^2k_I-\mu)\lambda_0f(L)\\
&-e^{\frac{\mu}{\lambda_0}L}\left(\alpha^2k_I\lambda_0\right)^2L^2\left(1-\frac{\left(\int_0^L\theta(x)dx\right)^2}{L\left(\int_0^L\theta^2(x)dx\right)}\right)>0.
\end{aligned}\tag{75}$$

We place ourselves in the limiting case, when $\mu=0$ and $f(L)=0$. As the left-hand side is a second order polynomial in $f(0)$, there exists a positive solution $f(0)$ to the inequality if and only if

$$\left(1-\frac{\left(\int_0^L\theta(x)dx\right)^2}{L\left(\int_0^L\theta^2(x)dx\right)}\right)k_I^2L^2<\lambda_0^2.\tag{76}$$

Under assumption (10) we can show that this is always verified, this is done in the Appendix. When $\omega=0$, taking again f' constant and the limiting case where $f(L)=0$ and $\mu=0$, I is definite positive provided that

$$\begin{aligned}
&-\lambda^2(0)k_I^2f^2(0)+\left(2\alpha^2\lambda_0^2k_I-\mu+f(L)\lambda_0k_I^2\right)\lambda_0f(0)\\
&-(2\alpha^2\lambda_0^2k_I-\mu)\lambda_0f(L)-e^{\frac{\mu}{\lambda_0}L}\left(\alpha^2k_I\lambda_0\right)^2L^2>0.
\end{aligned}\tag{77}$$

There exists a positive solution $f(0)$ to this inequality if and only if

$$k_I^2<\left(\frac{\lambda_0}{L}\right)^2,\tag{78}$$

but, as ϱ_1 is real and k_I is positive, $k_I=-(\lambda_0/L)(\varrho_1L/\lambda_0)e^{-\varrho_1L/\lambda_0}<\lambda_0/L$, thus (78) is satisfied. Thus, by continuity, there always exists $\mu_1>0$ and f positive such that $I>0$ and therefore

$$\begin{aligned}
\frac{dV_{2,1}}{dt}\leq&-\mu_1C_3V_{2,1}-(\lambda(\phi(t,L))f(L)e^{-\frac{\mu}{\lambda_0}L}-C_{2,1}|\phi|_0)\phi_2^2(t,L)\\
&+O\left(\left(|\phi_{\varrho_1}|_{H^2}+|X_{\varrho_1}|+|\phi_2|_{H^2}+|X_2(t)|+|\dot X_2(t)|+|\ddot X_2(t)|\right)^3\right).
\end{aligned}\tag{79}$$

Let us now deal with $V_{2,2}$ and $V_{2,3}$. Observe that from (67), one has for $\phi_2\in C^3$,

$$\begin{aligned}
\partial_t^2\phi_2+\lambda(\phi)\partial_x(\partial_t\phi_2)=&(\lambda_0-\lambda(\phi))\partial_{tx}^2\phi_1-p_1\left(\begin{pmatrix}(\lambda_0-\lambda(\phi))\partial_{tx}^2\phi-\lambda'(\phi)\partial_t\phi\partial_x\phi\\0\end{pmatrix}\right)\\
&-\lambda'(\phi)\partial_t\phi\partial_x\phi_1-\lambda'(\phi)\partial_t\phi\partial_x\phi_2,\\
\partial_t^3\phi_2+\lambda(\phi)\partial_x(\partial_t^2\phi_2)=&(\lambda_0-\lambda(\phi))\partial_{ttx}^3\phi_1-\lambda'(\phi)\partial_t^2\phi\partial_x\phi_1-2\lambda'(\phi)\partial_t\phi\partial_{tx}^2\phi_1\\
&-\lambda''(\phi)(\partial_t\phi)^2\partial_x\phi_1\\
&-p_1\left(\begin{pmatrix}(\lambda_0-\lambda(\phi))\partial_{ttx}^3\phi-\lambda''(\phi)(\partial_t\phi)^2\partial_x\phi-2\lambda'(\phi)\partial_t\phi\partial_{tx}^2\phi-\lambda'(\phi)\partial_{tt}^2\phi\partial_x\phi\\0\end{pmatrix}\right)\\
&-2\lambda'(\phi)\partial_t\phi\partial_{tx}\phi_2-\lambda'(\phi)\partial_t^2\phi\partial_x\phi_2-\lambda''(\phi)(\partial_t\phi)^2\partial_x\phi_2.
\end{aligned}\tag{80}$$

and

$$\begin{aligned}
\partial_t\phi_2(t,0)=-k_I\dot X_2(t),\qquad&\partial_t^2\phi_2(t,0)=-k_I\ddot X_2(t),\\
\ddot X_2=\partial_t\phi_2(t,L),\qquad&\dddot X_2=\partial_t^2\phi_2(t,L).
\end{aligned}\tag{81}$$

From (63), $p_1((\lambda_0-\lambda(\phi))\partial^3_{ttx}\phi)$ can be bounded by $(|\phi|^2_{H^2}+|\ddot{X}|^2+(\partial^2_{tt}\phi(t,L))^2)$ and, from (28), $\partial_{ttx}\phi_1$ is proportional to $\partial_{tt}\phi_1$. Thus, as all the other terms in the right hand sides are quadratic perturbations and include at most a second order derivative, the L^1 norm of the right-hand sides can be bounded by $(|\phi_{\varrho_1}|^2_{H^2}+|\phi_2|^2_{H^2}+|X_{\varrho_1}|^2+|X_2|^2+|\dot{X}_2|^2+|\ddot{X}_2|^2+\phi^2(t,L)+(\partial_t\phi(t,L))^2+(\partial^2_{tt}\phi(t,L))^2)$, which is small compared to the first-order term in the left-hand sides. Therefore we have, as previously

$$
\begin{aligned}
&\frac{dV_{2,k}}{dt}\leq -\mu C_3 V_{2,k}-\lambda(\phi(t,L))f(L)e^{-\frac{\mu}{\lambda_0}L}(\partial_t^k\phi_2)^2(t,L)\\
&-(\partial_t^k X_2)^2(t)\left(2\alpha^2\lambda(\phi(t,0))^2k_I-\lambda(\phi(t,0))f(0)k_I^2-\mu\right)\\
&-\int_0^L(-\lambda_0 f'(x))e^{-\frac{\mu}{\lambda_0}x}(\partial_t^k\phi_2)^2(t,x)+2\alpha^2k_I\lambda(\phi(t,0))\partial_t^k\phi_2X(t)dx\\
&+O\left(\left(|\phi_{\varrho_1}|_{H^2}+|X_{\varrho_1}|+|\phi_2|_{H^2}+|X_2(t)|+|\dot{X}_2(t)|+|\ddot{X}_2(t)|\right)^3\right)\\
&+C_{2,k}|\phi|_0\left|(\partial_t^{k-1}\phi_2)(t,L)\right|^2,\ \text{for } k=2,\ 3,
\end{aligned}
\tag{82}
$$

where $C_{2,k}$ are positive constants independent of ϕ and X. Besides, from (45),

$$
\int_0^L\partial_t^{k-1}\phi_2(t,x)\left(e^{\sigma(x-L)}\sin(\omega(x-L))\right)dx=0,\ \text{for } k=2,\ 3.
\tag{83}
$$

Thus we can perform exactly as for $V_{2,1}$ and consequently

$$
\begin{aligned}
&\frac{dV_{2,k}}{dt}\leq -\mu C_3V_{2,k}-\left(\lambda(\phi(t,L))f(L)e^{-\frac{\mu}{\lambda_0}L}-C_{2,k}|\phi|_0\right)\left|\partial_t^{k-1}\phi_2(t,L)\right|^2\\
&+O\left(\left(|\phi_{\varrho_1}|_{H^2}+|X_{\varrho_1}|+|\phi_2|_{H^2}+|X_2|+|\dot{X}_2|+|\ddot{X}_2|\right)^3\right),\ \text{for } k=2,3,
\end{aligned}
\tag{84}
$$

thus, from (79) and (84),

$$
\begin{aligned}
&\frac{dV_2}{dt}\leq -\mu C_3V_2-\sum_{k=1}^{3}\left(\lambda(\phi(t,L))f(L)e^{-\frac{\mu}{\lambda_0}L}-C_{2,k}|\phi|_0|\partial_t^{k-1}\phi|_{L^2}\right)\left|\partial_t^{k-1}\phi_2(t,L)\right|^2\\
&+O\left(\left(|\phi_{\varrho_1}|_{H^2}+|X_{\varrho_1}|+|\phi_2|_{H^2}+|X_2|+|\dot{X}_2|+|\ddot{X}_2|\right)^3\right).
\end{aligned}
\tag{85}
$$

This implies from (51) and (66) that

$$
\begin{aligned}
&\frac{dV}{dt}\leq -\min\left(2\mathrm{Re}(\varrho_1),2\mathrm{Re}(\varrho_1)\lambda_0,\mu\right)V\\
&-\left(\lambda(\phi(t,L))f(L)e^{-\frac{\mu}{\lambda_0}L}-C_4|\phi|_{H^2}\right)\left(|\phi_2(t,L)|^2+|\partial_t\phi_2(t,L)|^2+\left|\partial_t^2\phi_2(t,L)\right|^2\right)\\
&+O\left(\left(|\phi_{\varrho_1}|_{H^2}+|X_{\varrho_1}|+|X_{\dot{\varrho}_1}|+|\ddot{X}_{\varrho_1}|+|\phi_2|_{H^2}+|X_2(t)|+|\dot{X}_2(t)|+|\ddot{X}_2(t)|\right)^3\right).
\end{aligned}
\tag{86}
$$

But from (14) and (57), V is equivalent to the norm $\Big(|\phi_{\varrho_1}|_{H^2}+|X_{\varrho_1}|+|X_{\dot{\varrho}_1}|+|\ddot{X}_{\varrho_1}|+|\phi_2|_{H^2}+|X_2(t)|$ $+|\dot{X}_2(t)|+|\ddot{X}_2(t)|\Big)^2$. Besides, we have from (31), (56), and using Cauchy-Schwarz inequality

$$
\begin{aligned}
V_1(t)&\leq\left(\int_0^L|\alpha_1e^{-\frac{\varrho_1}{\lambda_0}x}|^2dx+\left|\frac{\alpha_1e^{\frac{\varrho_1}{\lambda_0}L}}{\varrho_1}\right|^2\right)\sum_{k=1}^{3}\left|\left(\int_0^L e^{2\varrho_1x}dx\right)^{1/2}\left(\int_0^L\partial_t^k\phi^2dx\right)^{1/2}+\partial_t^kXe^{\frac{\varrho_1}{\lambda_0}L}\right|^2\\
&\leq C_5\left(|\phi(t,\cdot)|^2_{H_2}+|X(t)|^2+|\dot{X}(t)|^2+|\ddot{X}(t)|^2\right),
\end{aligned}
\tag{87}
$$

where C_5 is a constant that does not depend on X or ϕ. Also, from (14), (87) and noting that $\phi_2=\phi-\phi_1$ and $X_2=X-X_1$,

$$
\begin{aligned}
V_2(t)&\leq C_2\left(|\phi_2(t,\cdot)|^2_{H_2}+|X_2(t)|^2+|\dot{X}_2(t)|^2+|\ddot{X}_2(t)|^2\right)\\
&\leq C_6\left(|\phi(t,\cdot)|^2_{H_2}+|X(t)|^2+|\dot{X}(t)|^2+|\ddot{X}(t)|^2\right).
\end{aligned}
\tag{88}
$$

This implies that

$$\begin{aligned}&\left(|\phi_{\varrho_1}|_{H^2}+|X_{\varrho_1}|+|X_{\dot\varrho_1}|+|\ddot X_{\varrho_1}|+|\phi_2|_{H^2}+|X_2(t)|+|\dot X_2(t)|+|\ddot X_2(t)|\right)\\&=O\left(|\phi|_{H^2}+|X|+|\dot X|+|\ddot X|\right).\end{aligned}\tag{89}$$

But from (2)–(3) and Sobolev inequality,

$$\left(|\phi|_{H^2}+|X|+|\dot X|+|\ddot X|\right)=O\left(|\phi|_{H^2}\right).\tag{90}$$

Therefore, from (86), (89)–(90), and (48), there exists $\gamma>0$ and $\varepsilon_2\in(0,\varepsilon_1]$ such that for any $\varepsilon\in(0,\varepsilon_2)$, one has

$$\frac{dV}{dt}\leq-\gamma V.\tag{91}$$

This shows the exponential decay for V. It remains now only to show that it also implies the exponential decay for (ϕ,X) in the H^2 norm. Observe first that from (87)–(88) and (90) there exists $C_7>0$ independent of ϕ and X such that

$$V(t)\leq C_7|\phi(t,\cdot)|_{H^2},\forall\ t\in[0,T].\tag{92}$$

And from (14), (58), and (91),

$$\begin{aligned}&|\phi(t,\cdot)|_{H^2}+|X(t)|+|\dot X(t)|+|\ddot X(t)|\\&\leq 4\max(1,\lambda_0^2)V_1(t)+C_2V_2(t)\\&\leq\max(4,4\lambda_0^2,C_2)e^{-\gamma t}V(0).\end{aligned}\tag{93}$$

Thus, there exists $C_8>0$ independent of ϕ and X such that

$$|\phi(t,\cdot)|_{H^2}\leq C_8e^{-\gamma t}\left(|\phi(0,\cdot)|_{H^2}\right).\tag{94}$$

So far ϕ is assumed to be of class C^3, however since this inequality only involves the H^2 norm of ϕ, this can be extended to any solution $(\phi,X)\in C^0([0,T],H^2(0,L))\times C^1([0,T])$ of the system (1)–(3) (see for instance [4] for more details). This concludes the proof of Theorem 2.3. □

We now prove Proposition 2.4, which follows rapidly from the proof of Theorem 2.3.

Proof of Proposition 2.4. From (113) in the Appendix, one can see that (76) still holds with $k_I=\pi\lambda_0/2L$. Thus by continuity there exists $k_1>\pi\lambda_0/2L$ such that for any $k_I\in(\pi\lambda_0/2L,k_1)$ (76) still holds and consequently the quadratic form I given by (69) is still definite positive. Suppose now by contradiction that the system is stable for the H^2 norm. Then for any $\varepsilon>0$, there exists $\delta_1>0$ such that for any initial condition $(\phi_0,X_0)\in H^2(0,L)\times\mathbb{R}$ such that $(|\phi_0|_{H^2}+|X_0|)\leq\delta_1$ and satisfying the compatibility condition $X_0=-k_I^{-1}\phi_0(0)$ and $\phi(L)=k_I^{-1}\lambda(\phi_0(0))\phi_0'(0)$, the associated solution (ϕ,X) is defined on $[0,+\infty)$ and

$$(|\phi|_{H^2}+|X|)\leq\varepsilon,\ \forall\ t\in[0,+\infty).\tag{95}$$

Let $\Theta>0$, from (65) and (85), using that $I>0$,

$$\begin{aligned}&\frac{dV_1-\Theta V_2}{dt}\geq 2\mathrm{Re}(\varrho_1)\min(\lambda_0,1)V_1+\mu\Theta C_3V_2\\&+\left(\Theta f(L)\lambda_0e^{-\mu\frac{L}{\lambda_0}}-C_9(1+\Theta)\left(|\phi|_{H^2}+|X|+|\dot X|+|\ddot X|\right)\right)\left(\sum_{k=1}^{k=3}\left|(\partial_t^{k-1}\phi_2)(t,L)\right|^2\right)\\&+O\left(\left(|\phi_{\varrho_1}|_{H^2}+|X_{\varrho_1}|+|\phi_2|_{H^2}+|X_2(t)|+|\dot X_2(t)|+|\ddot X_2(t)|\right)^3\right),\end{aligned}\tag{96}$$

where C_9 is a constant independent of ϕ and X. We can choose (ϕ_0,X_0) satisfying the compatibility conditions and $\Theta>0$ such that $c:=(V_1-\Theta V_2)(0)>0$, and $(|\phi_0|_{H^2}+|X_0|)\leq\delta$ with δ to be chosen. Actually Θ only depends on the ratio between V_1 and V_2 thus it can be made independent of δ by simply rescaling $|\phi_0|_{H^2}$ and $|X_0|$. Using (96) and (90) there exists $\gamma_2 > 0$ and $\varepsilon > 0$ such that, if $(|\phi|_{H^2} + |X|) \leq \varepsilon$, then

$$\frac{dV_1 - \Theta V_2}{dt} \geq \gamma_2(V_1 - \Theta V_2). \tag{97}$$

Thus, from (95) and the stability hypothesis, we can choose $\delta > 0$ such that (97) holds. This implies that

$$(V_1 - \Theta V_2)(t) \geq ce^{\gamma_2 t}, \ \forall\, t \in [0, +\infty), \tag{98}$$

which contradicts (95). This ends the proof of Proposition 2.4. $\square$

Remark 3. *This last proof is limited by the limit value of k_I for which I is not positive definite anymore. This is due to the fact that we have only extracted the first limiting eigenvalues from the solution. It is natural to think that we could apply the same method to extract a finite number of eigenvalues instead and separate (ϕ, X) in (ϕ_1, X_1), its projection on a n-dimensional space, and (ϕ_2, X_2). Then we would deduce more constraints like* (45) *on (ϕ_2, X_2), which would increase the upper bound of k_I for which I defined in* (69) *is definite positive, and thus the bound k_1 for which Proposition* (2.4) *holds, and maybe, by increasing this number of eigenvalues, prove that this proposition holds for arbitrary large k_1.*

6 Numerical simulations

In this section we give a numerical simulation that illustrates Theorem 2.3 and Proposition2.4.

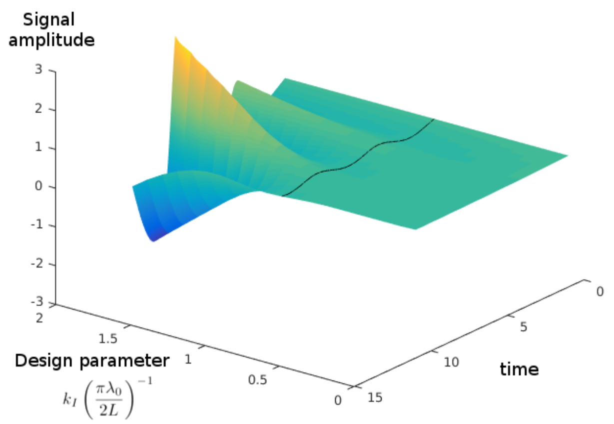

Figure 1: Example of numerical simulations of $\phi(t, 0)$ with respect to t varying between 0 and 10 for various values of k_I between $0.1k_{I,c}$ to $2k_{I,c}$, where $k_{I,c} = \pi\lambda_0/2L$ is the critical value of Theorem 2.3 and Proposition 2.4. The black line represents the trajectory for $k_I = k_{I,c}$. On the left k_I is larger and the system is unstable, and on the right k_I is smaller and the system is stable. The system parameters are chosen such that $\lambda(x) = 1 + x$, $\lambda_0 = L = 1$, and $\phi_0(x) = 0.1$ on $[0, L/2]$ and $\phi_0(L) = 0$ so that ϕ_0 satisfies the compatibility conditions (4) for any $k_I \in [0.1k_{I,c}, 2k_{I,c}]$. The simulations are obtained by a finite-difference method.

7 Conclusion

In this article we studied the exponential stability of a general nonlinear transport equation with integral boundary controllers and we introduced a method to get an optimal stability condition through a Lyapunov approach, by extracting first the limiting part of the stability from the solution using a projector on a finite-dimension space. We believe that this method could be used for many other systems and could be useful in the futur as, for many nonlinear systems governed by partial differential equations, the stability conditions that are known today are only sufficient and may still be improved.

In this section we prove (76) under assumption (10). Note that this is equivalent to

$$\frac{\left(\int_0^L \theta(x)dx\right)^2}{L\left(\int_0^L \theta^2(x)dx\right)} > 1 - \frac{\lambda_0^2}{k_I^2 L^2}. \tag{99}$$

By definition of ϱ_1 (see Section 4) and (26), we have

$$\sigma = -\frac{\omega}{\tan(\omega L)}, \tag{100}$$

and using (26) and (100)

$$\frac{\lambda_0}{k_I L} = -\frac{\sin(\omega L)}{\omega L} e^{\frac{\omega L}{\tan(\omega L)}}. \tag{101}$$

Condition (99) thus becomes

$$\frac{\left(\int_0^L \theta(x)dx\right)^2}{L\left(\int_0^L \theta^2(x)dx\right)} + \frac{\sin^2(\omega L)}{(\omega L)^2} e^{2\frac{\omega L}{\tan(\omega L)}} - 1 > 0. \tag{102}$$

From (8) and the definition of θ given by (70), we have

$$\int_0^L \theta(x)dx = \frac{\omega}{\sigma^2+\omega^2}. \tag{103}$$

Using (100),

$$\left(\int_0^L \theta(x)dx\right)^2 = \frac{\sin^4(\omega L)}{\omega^2}. \tag{104}$$

Similarly we have

$$\int_0^L \theta^2(x)dx = \frac{\sigma e^{-2\sigma L}(\sigma\cos(2\omega L) - \omega\sin(2\omega L)) + (\omega^2+\sigma^2) - \sigma^2 - e^{-2\sigma L}(\sigma^2+\omega^2)}{4\sigma(\sigma^2+\omega^2)}. \tag{105}$$

Therefore, using again (100) and the fact that $(1+\tan^{-2}(\omega L)) = \sin^{-2}(\omega L)$,

$$\begin{aligned}\int_0^L \theta^2(x)dx &= \left(\frac{\frac{\cos^2(\omega L)}{\sin^2(\omega L)} e^{2\frac{\omega L}{\tan(\omega L)}}(\cos^2(\omega L)+\sin^2(\omega L)) - e^{2\frac{\omega L}{\tan(\omega L)}}\frac{1}{\sin^2(\omega L)} + 1}{-4\omega\sin^{-2}(\omega L)}\right)\tan(\omega L) \\ &= \frac{(e^{2\frac{\omega L}{\tan(\omega L)}} - 1)}{4\omega}\sin^2(\omega L)\tan(\omega L).\end{aligned} \tag{106}$$

Therefore using (104) and (106), condition (102) becomes

$$\frac{4\sin^2(\omega L)}{(\omega L)\tan(\omega L)(e^{2\frac{\omega L}{\tan(\omega L)}} - 1)} + \frac{\sin^2(\omega L)}{(\omega L)^2} e^{2\frac{\omega L}{\tan(\omega L)}} - 1 > 0, \tag{107}$$

which is equivalent to

$$\left(2\frac{\frac{2\omega L}{\tan(\omega L)}}{(e^{\frac{2\omega L}{\tan(\omega L)}} - 1)} + e^{\frac{2\omega L}{\tan(\omega L)}}\right)\frac{\sin^2(\omega L)}{(\omega L)^2} - 1 > 0. \tag{108}$$

Note that, under assumption (10) and from the definition of ϱ_1, $\omega L \in (-\pi/2, \pi/2)$, which implies that $2(\omega L)/\tan(\omega L) \in (0,2)$. Hence, let us study the function $g : X \to (2X/(e^X - 1) + e^X)$ on $(0,2)$. Taking its derivative one has

$$g'(X) = \frac{(e^X-1)(2+e^X(e^X-1)) - 2Xe^X}{(e^X-1)^2}. \tag{109}$$

Taking again the derivative of the numerator of the right-hand side of (109), one has

$$\begin{aligned}((e^X-1)(2+e^X(e^X-1))-2Xe^X))' =& (e^X-1)(e^X(e^X-1)+e^{2X}) \\ &+e^X(2+e^X(e^X-1))-2e^X-2Xe^X.\end{aligned} \tag{110}$$

Thus using that $X < e^X - 1$ on $(0,+\infty)$ and in particular on $(0,2)$, we get

$$((e^X-1)(2+e^X(e^X-1))-2Xe^X))' > (e^X-1)(e^X(e^X-1)+2e^{2X}-2e^X) > 0. \tag{111}$$

Hence g' is non-decreasing on $(0,2)$. But, from (109), $g'(0)=0$, therefore g is non-decreasing on $(0,2)$. As $\lim_{X\to 0} g(X) = 3$, we have

$$\left(2\frac{\frac{2\omega L}{\tan(\omega L)}}{(e^{2\frac{\omega L}{\tan(\omega L)}}-1)}+e^{2\frac{\omega L}{\tan(\omega L)}}\right)\frac{\sin^2(\omega L)}{(\omega L)^2}-1 \geq 3\frac{\sin^2(\omega L)}{(\omega L)^2}-1, \tag{112}$$

and, as $x \to \sin(x)/x$ is positive and decreasing on $[0,\pi/2]$, we have

$$\left(2\frac{\frac{2\omega L}{\tan(\omega L)}}{(e^{2\frac{\omega L}{\tan(\omega L)}}-1)}+e^{2\frac{\omega L}{\tan(\omega L)}}\right)\frac{\sin^2(\omega L)}{(\omega L)^2}-1 \geq \frac{12}{\pi^2}-1 > 0. \tag{113}$$

Hence (108) holds and therefore condition (99) holds as well. This ends the proof of (76) under assumption (10).

Acknowledgment

The authors would like to thank Peipei Shang and Sébastien Boyaval for their valuable remarks. The authors would also like to thank the ETH-FIM and the ETH-ITS for their support and their warm welcome. Finally, the authors would also like to thank the ANR project Finite 4SoS ANR 15-CE23-0007 and the French Corps des IPEF.

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