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SHARP TIME NULL CONTROLLABILITY OF A POPULATION DYNAMICS MODEL WITH AGE STRUCTURING AND DIFFUSION

DEBAYAN MAITY, MARIUS TUCSNAK, AND ENRIQUE ZUAZUA

ABSTRACT. In this article we study the null controllability of a linear system coming from a population dynamics model with age structuring and spatial diffusion (of Lotka-McKendrick type). The control is localized in the space variable as well as with respect to the ages. The first novelty we bring in is that the age interval in which the control needs to be active can be arbitrarily small and does not need to contain a neighborhood of 0. The second one is that we prove that the whole population can be steered into zero in an uniform time, without, as in the existing literature, excluding some interval of low ages. Finally, we improve the existing estimates of the controllability time and we show that our estimates are sharp, at least when the control is active for very low ages. The method of proof, combining final-state observability estimates with the use of characteristics, avoids the explicit use of parabolic Carleman estimates.

Key words. Population dynamics, Null controllability.

AMS subject classifications. 93B03, 93B05, 92D25

1. INTRODUCTION AND MAIN RESULTS

In this article, we study the null-controllability of an infinite dimensional linear system describing the dynamics of a single species age-structured population with spatial diffusion. In these models, going back to Gurtin [6] and generalizing the classical Lotka-McKendrick system, the state space of the system is $H = L^2([0, a_+] \times \Omega)$, where a_+ denotes the maximal age an individual can attain and $\Omega \subset \mathbb{R}^n$ (with $n \in \mathbb{N}$ in general but with $n = 3$ for real life applications) is an open bounded set which represents the spatial environment occupied by the individuals. Let $p(t, a, x)$ be the distribution density of individuals with respect to age $a \geq 0$ and spatial position $x \in \Omega$ at some time $t \geq 0$. Then, according to the above reference, the function p satisfies the degenerate parabolic partial differential equation

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial a} - Lp + \mu(a)p = mv \quad (t, a, x) \in (0, \infty) \times (0, a_+) \times \Omega, \quad (1.1)$$

where the operator L is defined by

$$Lp = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\sigma_{ij} \frac{\partial p}{\partial x_j} \right), \quad (1.2)$$

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with $\sigma_{ij} = \sigma_{ji} \in C^2(\overline{\Omega})$, for $1 \leq i, j \leq n$, and we assume there exists a constant $c > 0$ such that

$$\sum_{i,j=1}^n \sigma_{ij}(x) \xi_i \xi_j \geq c |\xi|^2 \quad (x \in \overline{\Omega}, \xi \in \mathbb{R}^n).$$

Moreover, the positive function μ denotes the natural mortality rate of individuals of age a , supposed to be independent of the spatial position x and of time. The control function is v , depending on t , a and x , whereas m is the characteristic function of $(a_1, a_2) \times \omega$, with $0 \leq a_1 < a_2 \leq a_\dagger$ and $\omega \subset \Omega$ an open set. Thus the control is localized both in age and with respect to the spatial variable. This control process corresponds to harvesting of adding individuals of age between a_1 and a_2 from the spatial domain ω . Note that equation (1.1) is a slight generalization of the one proposed in [6], where the operator L is just the standard Laplacian. We denote by β the positive function describing the fertility rate at age a , supposed to be independent of the spatial position x and of time, so that the density of new born individuals at the point x at time t is given by

$$p(t, 0, x) = \int_0^{a_\dagger} \beta(a) p(t, a, x) da \quad (t, x) \in (0, \infty) \times \Omega. \quad (1.3)$$

We assume that the individuals never leave the set Ω , so that p satisfies the Neumann boundary condition

$$\frac{\partial p}{\partial \nu_L} = \sum_{i,j=1}^n \sigma_{ij} \frac{\partial p}{\partial x_j} n_i = 0 \quad (t, a, x) \in (0, \infty) \times (0, a_\dagger) \times \partial\Omega, \quad (1.4)$$

where n denotes the unit outer normal to $\partial\Omega$. To complete the model, we introduce the initial condition

$$p(0, a, x) = p_0(a, x) \quad (a, x) \in (0, a_\dagger) \times \Omega. \quad (1.5)$$

We assume that the fertility rate β and the mortality rate μ satisfy the conditions

(H1) $\beta \in L^\infty(0, a_\dagger)$, $\beta \geq 0$ for almost every $a \in (0, a_\dagger)$.

(H2) $\mu \in L^1[0, a^*]$ for every $a^* \in (0, a_\dagger)$, $\mu \geq 0$ for almost every $a \in (0, a_\dagger)$.

(H3) $\int_0^{a_\dagger} \mu(a) da = +\infty$.

For more details about the modelling of such system and the biological significance of the hypotheses we refer to Webb [15].

Our first main result is

Theorem 1.1. *Assume that β and μ satisfy the conditions (H1)-(H3) above. Moreover, suppose that the fertility rate β is such that*

$$\beta(a) = 0 \text{ for all } a \in (0, a_b), \quad (1.6)$$

for some $a_b \in (0, a_\dagger)$ and that $a_1 < a_b$. Recall that m is the characteristic function of $(a_1, a_2) \times \omega$, with $0 \leq a_1 < a_2 \leq a_\dagger$ and that $\omega \subset \Omega$ is an open set. Then for every $\tau > a_1 + \max\{a_1, a_\dagger - a_2\}$ and for every $p_0 \in L^2((0, a_\dagger) \times \Omega)$ there exists a control $v \in L^2((0, \tau) \times (a_1, a_2) \times \omega)$ such that the solution p of (1.1)-(1.5) satisfies

$$p(\tau, a, x) = 0 \text{ for all } a \in (0, a_\dagger), x \in \Omega. \quad (1.7)$$

Let us now mention some related works from the literature. The null controllability results of the diffusion free age-dependent population dynamics model was first obtained by Barbu, Iannelli and Martcheva [4]. They proved the population can be steered into any quasi steady state, except for a small interval of ages near zero. Recently, Hegoburu, Magal and Tucsnak [7] proved that this restriction is not necessary provided individuals do not reproduce at the age close to zero. They also proved there exists controls which preserves the positivity of the state trajectory. However, in the both works, the control is supported in the interval $(0, a_0)$, for some $a_0 < a_+$. Recently, Maity [11] proved that null controllability can be achieved by controls supported in any subinterval of $(0, a_+)$, provided we control before the individuals start to reproduce.

Concerning the models with spatial diffusion, namely for system (1.1) - (1.5), as far as we know, the first result was obtained by Ainseba and Anița [2]. They proved null controllability of the system (1.1) - (1.5) in any arbitrary time $\tau > 0$, provided the initial data is small and the control acts in a spatial subdomain $\omega \subset \Omega$ but for all ages. Later, Ainseba [1] removed the smallness condition of initial data, but he proved null controllability except for a small interval of ages near zero. In [3], Ainseba and Anița proved null controllability when the control acts only in a spatial subdomain and only for small ages, provided controllability time is large enough and initial data is small. Recently, Hegoburu and Tucsnak [8] proved that the system (1.1) - (1.5) is null controllable for all ages and in any time by controls localized with respect to the spatial variable but active for all ages. Their method is based on Lebeau-Robbiano type strategy, originally developed for the null-controllability of the heat equation. Traore [13] considered similar model with nonlinear distributions of the newborns. He proved null controllability except for small ages with controls localized in space variable and active for all ages. Martinez et. al [12] considered linearized Croco-type equation, which is similar to the system (1.1) -(1.5), with $\beta = \mu = 0$. They proved regional null controllability of such system.

The main novelties brought in by our paper are:

- We improve the existing estimates on the time necessary to control the system to zero and we show that our global controllability result applies to individuals of all ages, without needing to exclude ages in a neighborhood of zero.
- We are able to tackle the case of a control which is active for ages $a \in [a_1, a_2]$, with arbitrary $a_1 \in [0, a_+)$ and $a_2 \in (a_1, a_+]$. Thus, unlike in the existing literature, we do not need to control arbitrarily low ages or all the ages.
- Unlike most of the approaches in the literature, our methodology does not require adaptations of the existing parabolic Carleman estimates to the adjoint system of (1.1) - (1.5). We just combine characteristics method with existing observability estimates for parabolic equations. Thus our approach applies independently of the method used to derive final-state observability for the associated parabolic system (moment methods, local or global Carleman estimates, Lebeau-Robbiano strategy,...).

The remaining part of this work is organized as follows:

- In Section 2 we first recall some basic facts about the Lotka-McKendrick semigroup with diffusion. We next formulate our control problem in a semigroup setting and we define the associated adjoint semigroup.

- In Section 3 we prove the final state observability for the adjoint system and, as a consequence, we obtain the proof of the main result in Theorem 3.2.
- Section 4 is devoted to the description of possible extensions and open questions.

2. LOTKA-MCKENDRICK SEMIGROUP WITH DIFFUSION

In this section we provide some basic results on the population semigroup for the linear age structured model with diffusion and its adjoint operator. Most of them were existing in the literature, so we just give the statements and the appropriate references. In some cases, namely when the adjoint operator is involved, we did not find detailed justifications in the existing literature, so, with no claim of originality, we felt necessary to give a more detailed presentation.

We write below equations (1.1)-(1.5) as an abstract control system with input space

$$H = L^2(0, a_+; L^2(\Omega))$$

Before introducing the semigroup generator, we consider the diffusion free population operator $A_1 : \mathcal{D}(A_1) \rightarrow H$ defined by

$$\begin{aligned} \mathcal{D}(A_1) = \left\{ \varphi \in H \mid \varphi(\cdot, x) \text{ is locally absolutely continuous on } [0, a_+), \right. \\ \left. \varphi(0, x) = \int_0^{a_+} \beta(a)\varphi(a, x) da \text{ for a.e. } x \in \Omega, \frac{\partial \varphi}{\partial a} + \mu\varphi \in H \right\}, \\ A_1\varphi = -\frac{\partial \varphi}{\partial a} - \mu\varphi, \end{aligned} \quad (2.1)$$

and the diffusion operator $A_2 : \mathcal{D}(A_2) \rightarrow H$ defined by

$$\mathcal{D}(A_2) = \left\{ \varphi \in L^2(0, a_+; H^2(\Omega)) \mid \frac{\partial \varphi}{\partial \nu_L} = 0 \text{ on } \partial\Omega \right\}, \quad A_2\varphi = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(\sigma_{ij} \frac{\partial \varphi}{\partial x_j} \right). \quad (2.2)$$

We also introduce the input space $U = H$ and the control operator $B \in \mathcal{L}(U, H)$ defined by

$$Bu = mu \quad (u \in U). \quad (2.3)$$

With the above notation, we rewrite the system (1.1)-(1.5) as:

$$\dot{z}(t) = \mathcal{A}z(t) + Bu(t), \quad (2.4)$$

$$z(0) = p_0, \quad (2.5)$$

where we have set $p(t, \cdot) = z(t)$, $v(t, \cdot) = u(t)$ and the population operator with diffusion $\mathcal{A} : \mathcal{D}(\mathcal{A}) \rightarrow H$ is defined by

$$\mathcal{D}(\mathcal{A}) = \mathcal{D}(A_1) \cap \mathcal{D}(A_2), \quad \mathcal{A}\varphi = A_1\varphi + A_2\varphi. \quad (2.6)$$

The fact that the system we consider is well-posed follows from the following result:

Lemma 2.1. *The operator $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is the infinitesimal generator of a strongly continuous semigroup \mathbb{T} on H .*

Proof. A proof of this lemma can be found in [9, Theorem 2.8]. □

With the above notation, our main result in Theorem 1.1 can be rephrased to : the pair (\mathcal{A}, B) , with \mathcal{A} defined in (2.6) and B defined in (2.3) is null controllable in ant time $\tau > a_1 + \max\{a_1, a_\dagger - a_2\}$. It is well-known that, the null controllability in time τ of (\mathcal{A}, B) is equivalent to the final-state observability in time τ of the pair (\mathcal{A}^*, B^*) , wher \mathcal{A}^* and B^* are the adjoint operators of \mathcal{A} and B , respectively (see, for instance, [14, Section 11.2]). It is thus important to determine the adjoint of the operator \mathcal{A} . To this aim, we introduce an auxiliary unbounded operator $(\mathcal{A}_0, \mathcal{D}(\mathcal{A}_0))$ defined by

$$\begin{aligned} \mathcal{D}(\mathcal{A}_0) = & \left\{ \varphi \in H \mid \varphi(\cdot, x) \text{ is locally absolutely continuous on } [0, a_\dagger), \varphi \in L^2(0, a_\dagger; H^2(\Omega)), \right. \\ & \left. \lim_{a \rightarrow a_\dagger^-} \varphi(a, x) = 0 \text{ for a.e. } x \in \Omega, \frac{\partial \varphi}{\partial \nu_L} = 0 \text{ on } \partial\Omega, \frac{\partial \varphi}{\partial a} - \mu\varphi + L\varphi \in H \right\}, \\ \mathcal{A}_0\varphi = & \frac{\partial \varphi}{\partial a} - \mu\varphi + L\varphi. \end{aligned} \quad (2.7)$$

We prove the following lemma

Lemma 2.2. *The operator $(\mathcal{A}_0, \mathcal{D}(\mathcal{A}_0))$ is the infinitesimal generator of a strongly continuous semigroup \mathbb{T}^0 on H .*

Proof. For $\varphi \in \mathcal{D}(\mathcal{A}_0)$, we have

$$\begin{aligned} (\mathcal{A}_0\varphi, \varphi)_H = & \lim_{a \rightarrow a_\dagger^-} \int_0^a \left(\frac{\partial \varphi}{\partial a} - \mu\varphi + B\varphi \right) \varphi \\ = & \lim_{a \rightarrow a_\dagger^-} \frac{\varphi^2(a)}{2} - \frac{\varphi^2(0)}{2} - \lim_{a \rightarrow a_\dagger^-} \int_0^a \left(\mu\varphi^2 + \sum a_{ij} \frac{\partial \varphi}{\partial x_i} \frac{\partial \varphi}{\partial x_j} \right) \leq 0. \end{aligned}$$

Let $f \in H$ and we consider the following problem

$$\varphi - \frac{\partial \varphi}{\partial a} + \mu\varphi - L\varphi = f \text{ in } (0, a_\dagger) \times \Omega, \quad \lim_{a \rightarrow a_\dagger^-} \varphi(a, x) = 0, \quad \frac{\partial \varphi}{\partial \nu_L} = 0 \text{ on } \partial\Omega.$$

Let us set $\tilde{\varphi}(a, x) = \exp\left(-a - \int_0^a \mu(r) dr\right) \varphi(a, x)$. Then $\tilde{\varphi}$ solves

$$-\frac{\partial \tilde{\varphi}}{\partial a} - L\tilde{\varphi} = \tilde{f} \text{ in } (0, a_\dagger) \times \Omega, \quad \tilde{\varphi}(a_\dagger, x) = 0 \text{ in } \Omega, \quad \frac{\partial \tilde{\varphi}}{\partial \nu_L} = 0 \text{ on } \partial\Omega,$$

where $\tilde{f}(a, x) = \exp\left(-a - \int_0^a \mu(r) dr\right) f(a, x)$. It is easy to see that $\tilde{f} \in H$. Thus $\tilde{\varphi} \in L^2(0, a_\dagger; H^2(\Omega)) \cap H^1(0, a_\dagger; L^2(\Omega))$ and

$$\|\tilde{\varphi}(a, \cdot)\|_{L^2(\Omega)} \leq C \left(-a - \int_0^a \mu(r) dr \right) \|f\|_H.$$

With the above estimate it is easy to see that $\varphi \in \mathcal{D}(\mathcal{A}_0)$ and hence \mathcal{A}_0 generates a C^0 semigroup on H . \square

We are now in a position to define the adjoint of the unbounded operator \mathcal{A} .

Proposition 2.3. *The adjoint of $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ in H is defined by*

$$\mathcal{D}(\mathcal{A}^*) = \mathcal{D}(\mathcal{A}_0), \quad \mathcal{A}^*\psi = \frac{\partial\psi}{\partial a} - \mu\psi + \beta\psi(0, x) + L\psi. \quad (2.8)$$

Proof. We can easily verify that $\mathcal{D}(\mathcal{A}_0) \subset \mathcal{D}(\mathcal{A}^*)$. We want to show the reverse inclusion.

To this aim, we note that for $\lambda > 0$ large enough, the operator, $I - \mathcal{F}(\lambda)$, where $\mathcal{F}(\lambda) \in \mathcal{L}(L^2(\Omega))$ is defined by

$$\mathcal{F}(\lambda)g(x) = \left[(\lambda I - \mathcal{A}_0)^{-1}(\beta(a)g(x)) \right](0, x), \quad (2.9)$$

is invertible. Indeed, this follows from the fact that $\lim_{\lambda \rightarrow \infty} \|\mathcal{F}(\lambda)\|_{\mathcal{L}(L^2(\Omega), H)} = 0$.

For λ as above, we define $\mathcal{G}_\lambda : H \mapsto H$ defined by $\mathcal{G}_\lambda f = \varphi_\lambda$ where φ_λ solves

$$\lambda\varphi_\lambda - \frac{\partial\varphi_\lambda}{\partial a} + \mu\varphi_\lambda - L\varphi_\lambda - \beta(a)\varphi_\lambda(0, x) = f \text{ in } (0, a_+) \times \Omega, \quad \lim_{a \rightarrow a_+^-} \varphi_\lambda(a, x) = 0, \quad \frac{\partial\varphi_\lambda}{\partial\nu_L} = 0 \text{ on } \partial\Omega. \quad (2.10)$$

The fact that the operator \mathcal{G}_λ is well defined follows from the fact that the unique solution of (2.10) is clearly given by

$$\varphi_\lambda(a, x) = (\lambda I - \mathcal{A}_0)^{-1} (f(a, x) + V_{\lambda, f}(a, x)), \quad (2.11)$$

where

$$V_{\lambda, f}(a, x) = \beta(a) (I - \mathcal{F}(\lambda))^{-1} \left(\left[(\lambda I - \mathcal{A}_0)^{-1} f \right](0, x) \right).$$

This means, in particular, that $\varphi_\lambda \in \mathcal{D}(\mathcal{A}_0)$.

We are now in the position to prove the inclusion $\mathcal{D}(\mathcal{A}^*) \subset \mathcal{D}(\mathcal{A}_0)$. To this aim, take λ as above and let $\psi \in \mathcal{D}((\lambda I - \mathcal{A})^*)$. Then there exists $f \in H$ such that

$$\int_0^{a_+} \int_\Omega \psi(\lambda I - \mathcal{A})\varphi = \int_0^{a_+} \int_\Omega f\varphi \text{ for all } \varphi \in \mathcal{D}(\mathcal{A}).$$

Let $\eta_\lambda = \mathcal{G}_\lambda f$, with \mathcal{G}_λ defined several lines above. Then, using (2.10) and integrating by parts we obtain

$$\begin{aligned} \int_0^{a_+} \int_\Omega f\varphi &= \lim_{a \rightarrow a_+^-} \int_0^a \int_\Omega \left(\lambda\eta_\lambda - \frac{\partial\eta_\lambda}{\partial a} + \mu\eta_\lambda - L\eta_\lambda - \beta(a)\eta_\lambda(0, x) \right) \varphi \\ &= \lim_{a \rightarrow a_+^-} \int_0^a \int_\Omega \eta_\lambda \left(\lambda\varphi + \frac{\partial\varphi}{\partial a} + \mu\varphi - L\varphi \right) = \int_0^{a_+} \int_\Omega \eta_\lambda(\lambda I - \mathcal{A})\varphi. \end{aligned}$$

Therefore,

$$\int_0^{a_+} \int_\Omega (\psi - \eta_\lambda)(\lambda I - \mathcal{A})\varphi = 0, \text{ for all } \varphi \in \mathcal{D}(\mathcal{A}). \quad (2.12)$$

By choosing $\varphi = (\lambda I - \mathcal{A})^{-1}(\psi - \eta_\lambda)$ we get

$$\int_0^{a_+} \int_\Omega |\psi - \eta_\lambda|^2 = 0.$$

Thus $\psi \in \mathcal{D}(\mathcal{A}_0)$ and ψ solves (2.10). This completes the proof of the proposition. \square

3. OBSERVABILITY INEQUALITY

As mentioned above, the null-controllability of a pair (\mathcal{A}, B) is equivalent to the final state observability of the pair (\mathcal{A}^*, B^*) , see [14, Theorem 11.2.1]. Recall that that final-state observability of (\mathcal{A}^*, B^*) is defined as

Definition 3.1. [14, Definition 6.1.1] *The pair (\mathcal{A}^*, B^*) is final state observable in time τ if there exists a $k_\tau > 0$ such that*

$$\int_0^\tau \|B^* \mathbb{T}_t^* q_0\|_H \geq k_\tau^2 \|\mathbb{T}_\tau^* q_0\|^2, \quad \text{for all } q_0 \in \mathcal{D}(\mathcal{A}^*).$$

For \mathcal{A} defined in (2.6) and $q_0 \in H$ we set

$$q(t) = \mathbb{T}_t^* q_0 \quad (t \geq 0),$$

where \mathbb{T} is the semigroup generated by \mathcal{A} . According to Proposition 2.3 we have:

$$\begin{cases} \frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - Lq - \beta(a)q(t, 0, x) + \mu(a)q = 0, & t \geq 0, (a, x) \in (0, a_\dagger) \times \Omega, \\ q(t, a_\dagger, x) = 0, & t \geq 0, x \in \Omega, \\ \frac{\partial q}{\partial \nu_L} = 0, & t \geq 0, (a, x) \in (0, a_\dagger) \times \partial\Omega, \\ q(0, a, x) = q_0(a, x), & (a, x) \in (0, a_\dagger) \times \Omega. \end{cases} \quad (3.1)$$

In view of [14, Theorem 11.2.1], the statement in Theorem 1.1 is equivalent to the following theorem:

Theorem 3.2. *Under the assumption of Theorem 1.1, the pair (\mathcal{A}^*, B^*) is final-state observable for every $\tau > a_1 + \max\{a_1, a_\dagger - a_2\}$. In other words, for every $\tau > a_1 + \max\{a_1, a_\dagger - a_2\}$, there exists $k_\tau > 0$ such that the solution q of (3.1) satisfies*

$$\int_0^{a_\dagger} \int_\Omega q^2(\tau, a, x) \, dx da \leq k_\tau^2 \int_0^\tau \int_{a_1}^{a_2} \int_\omega q^2(t, a, x) \, dx da dt \quad (q_0 \in \mathcal{D}(\mathcal{A}^*)). \quad (3.2)$$

Before we begin the proof, let us briefly describe its main steps. The first one is rewriting the system (3.1) as

$$\dot{q}(t) = \mathcal{A}_0 q(t) + V(t), \quad t \geq 0, \quad q(0) = q_0,$$

where \mathcal{A}_0 is defined in (2.7) and

$$V(t, a, x) = \beta(a)q(t, 0, x) \quad (t \geq 0, x \in \Omega).$$

From Duhamel's formula and recalling that \mathbb{T}^0 stands for the semigroup generated by \mathcal{A}_0 , it easily follows that

$$\|q(\tau)\|_H^2 \leq C \left(\|\mathbb{T}_\tau^0 q_0\|_H^2 + \int_0^\tau \|V(t)\|_H^2 \, dt \right) \leq C \left(\|\mathbb{T}_\tau^0 q_0\|_H^2 + \int_0^\tau \int_\Omega q^2(t, 0, x) \, dx dt \right), \quad (3.3)$$

for some constant C depending only on τ .

The second step consists in deriving upper appropriate bounds for each one of the terms in the right hand side of (3.3). This is accomplished by combining some change of variables using the characteristics of the free diffusion problem with some known observability inequalities for parabolic equations.

To accomplish this program, we first recall the following observability inequality for parabolic equations (see, for instance, Imanuvilov and Fursikov [5]) :

Proposition 3.3. *Let $T > 0$, $0 \leq t_0 < \tau$ and $t_1 \in (t_0, T]$. Then for every $w_0 \in L^2(\Omega)$, the solution w of the initial and boundary problem*

$$\begin{cases} \frac{\partial w}{\partial s}(s, x) - Lw(s, x) = 0 & ((s, x) \in (t_0, T) \times \Omega), \\ \frac{\partial w}{\partial \nu_L} = 0 & ((s, x) \in (t_0, T) \times \partial\Omega), \\ w(t_0, x) = w_0(x), & (x \in \Omega), \end{cases} \quad (3.4)$$

satisfies the estimate

$$\int_{\Omega} w^2(T, x) dx \leq \int_{\Omega} w^2(t_1, x) dx \leq c_1 e^{\frac{c_2}{t_1 - t_0}} \int_{t_0}^{t_1} \int_{\omega} w^2(s, x) dx ds, \quad (3.5)$$

where the constants c_1 and c_2 depend on L , on Ω and on τ .

3.1. Estimate of $\|\mathbb{T}_{\tau}^0 q_0\|_H^2$. In this subsection we give estimate of the term $\|\mathbb{T}_{\tau}^0 q_0\|_H^2$ appearing in (3.3), where \mathbb{T}^0 is the semigroup generated by the operator \mathcal{A}_0 defined in (2.7). Taking $q_0 \in H$ and denoting

$$z(t) = \mathbb{T}_t^0 q_0 \quad (t \geq 0),$$

it follows from (2.7)

$$\begin{cases} \frac{\partial z}{\partial t} - \frac{\partial z}{\partial a} - Lz + \mu(a)z = 0, & t \geq 0, (a, x) \in (0, a_{\dagger}) \times \Omega, \\ z(t, a_{\dagger}, x) = 0 & (t \geq 0, x \in \Omega), \\ \frac{\partial z}{\partial \nu_L} = 0 & (t \geq 0, (a, x) \in (0, a_{\dagger}) \times \partial\Omega), \\ z(0, a, x) = q_0(a, x) & ((a, x) \in (0, a_{\dagger}) \times \Omega). \end{cases} \quad (3.6)$$

This means that z is a solution of the adjoint of a system called of ‘‘Crocco type’’ in [12], where the authors proved a regional controllability result. This result cannot be used directly for our purposes, so we derive the following observability inequality.

Proposition 3.4. *Let $0 \leq a_1 < a_2 \leq a_{\dagger}$. Then for every $\tau > \max\{a_1, a_{\dagger} - a_2\}$, there exists $C_{\tau} > 0$ such that, for every $q_0 \in \mathcal{D}(\mathcal{A}_0)$, the solution z of (3.6), satisfies*

$$\int_0^{a_{\dagger}} \int_{\Omega} z^2(\tau, a, x) dx da \leq C_{\tau} \left(\int_0^{\tau} \int_{a_1}^{a_2} \int_{\omega} |z(t, a, x)| dx da dt \right)^2. \quad (3.7)$$

Proof. We set

$$\tilde{z}(t, a, x) = e^{-\int_0^a \mu(r) dr} z(t, a, x) \quad (t \geq 0, (a, x) \in (0, a_{\dagger}) \times \Omega).$$

Then \tilde{z} satisfies

$$\begin{cases} \frac{\partial \tilde{z}}{\partial t} - \frac{\partial \tilde{z}}{\partial a} - L\tilde{z} = 0 & (t \geq 0, (a, x) \in (0, a_\dagger) \times \Omega), \\ \tilde{z}(t, a_\dagger, x) = 0, & t \geq 0, x \in \Omega, \\ \frac{\partial \tilde{z}}{\partial \nu_L} = 0, & t \geq 0, (a, x) \in (0, a_\dagger) \times \partial\Omega, \\ \tilde{z}(0, a, x) := \tilde{q}_0(a, x), & (a, x) \in (0, a_\dagger) \times \Omega, \end{cases} \quad (3.8)$$

where $\tilde{q}_0 = e^{-\int_0^a \mu(r) dr} q_0 \in H^1(0, a_\dagger; H^2(\Omega))$. It is easily shown that the desired conclusion of this Proposition follows as soon as we show that there exists a constant $C_\tau > 0$ such that

$$\int_0^{a_\dagger} \int_\Omega \tilde{z}^2(\tau, a, x) dx da \leq C_\tau \int_\omega \tilde{z}^2(t, a, x) dx da dt, \quad (3.9)$$

for every $q_0 \in H$, and $\tau > \max\{a_1, a_\dagger - a_2\}$. Indeed, using characteristics, it is easily seen that $z(t, a, x) = 0$ for $t > a_\dagger - a$ and for every $x \in \Omega$. Therefore, there exists $a_* \in (0, a_\dagger)$ with

$$z(\tau, a, x) = 0 \quad (a \in (a_*, a_\dagger), x \in \Omega \text{ and for } \tau > \max\{a_1, a_\dagger - a_2\}).$$

Therefore, using (3.9) we get

$$\begin{aligned} \int_0^{a_\dagger} \int_\Omega z^2(\tau, a, x) dx da &= \int_0^{a_*} \int_\Omega z^2(\tau, a, x) dx da \leq \left(e^{2 \int_0^{a_*} \mu(r) dr} \right) \int_0^{a_*} \int_\Omega \tilde{z}^2(\tau, a, x) dx da \\ &\leq \left(e^{2 \|\mu\|_{L^1[0, a_*]}} \right) \int_0^{a_\dagger} \int_\Omega \tilde{z}^2(\tau, a, x) dx da \leq C_\tau \int_0^\tau \int_{a_1}^{a_2} \int_\omega \tilde{z}^2(t, a, x) dx da dt \\ &\leq C_\tau \int_0^\tau \int_{a_1}^{a_2} \int_\omega z^2(t, a, x) dx da dt, \end{aligned}$$

where C_τ is a generic constant depending only on τ . We have thus shown that (3.9) implies (3.7).

We can thus concentrate in the remaining part of the proof in checking (3.9). We give below specific arguments for each one of several cases.

Case 1: $\tau > a_\dagger$. In this case $\tilde{z}(\tau, a, x) = 0$ for all $(a, x) \in (0, a_\dagger) \times \Omega$ and the estimate (3.9) holds trivially.

Case 2: $a_1 \leq a_\dagger - a_2 < \tau \leq a_\dagger$.

In this case $\tilde{z}(\tau, a, x) = 0$ for all $a > a_\dagger - \tau$ and $x \in \Omega$. We set $b_0 = \max\{a_2 - \tau, 0\}$. If $b_0 > 0$, then either $b_0 < a_1$ or $b_0 \geq a_1$. Without loss of generality, let us assume that $0 < b_0 < a_1$. Next choose $a_0 \in (a_\dagger - \tau, a_2)$ such that $a_1 < a_0$ and we split the interval $(0, a_\dagger)$ as

$$(0, a_\dagger) = (0, b_0) \cup (b_0, a_1) \cup (a_1, a_0) \cup (a_0, a_\dagger). \quad (3.10)$$

As explained in the introduction, we are going to use Proposition 3.3 along the characteristics. Before we do that, let us explain why we have divided the interval $(0, a_\dagger)$ in the above way. Basically, the division depends on when the trajectory $\gamma(s) = (\tau - s, a + s)$, $s \in [0, \tau]$ (or equivalently the backward characteristics starting from (τ, a)) enters the observation region $(a_1, a_2) \times (0, \tau)$ and exits from the same region (see Fig. 1). More precisely,

- For $a \in (0, b_0)$, the trajectory $\gamma(s)$ enters the observation region at $s = a_1 - a$. As $b_0 + \tau < a_2$, at $s = \tau$, $\gamma(s)$ hits the line $t = 0$ without leaving the observation region (blue region in Fig. 1).
- For $a \in (b_0, a_1)$, the trajectory $\gamma(s)$ enters the observation domain at $s = a_1 - a$ and exits the observation region at $s = a_2 - a < \tau$ (red region in Fig. 1).
- For $a \in (a_1, a_0)$ the trajectory $\gamma(s)$ starts inside the observation region but it exits the region in time $s = a_2 - a < \tau$ (green region in Fig. 1).

In the remaining part of the proof we give upper bounds for $\int_I \int_{\Omega} \tilde{z}^2(\tau, a, x) dx da$ where I is successively each one of the intervals appearing in the decomposition (3.10).

Upper bound on $(0, b_0)$:

For a.e $a \in (0, b_0)$, we first set

$$w(s, x) = \tilde{z}(s, a + \tau - s, x), \quad s \in (0, \tau), x \in \Omega.$$

Then w solves the following problem

$$\begin{cases} \frac{\partial w}{\partial s} - Lw = 0, & (s, x) \in (0, \tau) \times \Omega, \\ \frac{\partial w}{\partial \nu_L} = 0, & (s, x) \in (0, \tau) \times \partial\Omega, \\ w(0, x) = \tilde{z}(0, \tau + a, x), & x \in \Omega. \end{cases} \quad (3.11)$$

Applying Proposition 3.3, with the choice $t_0 = 0$, $t_1 = \tau + a - a_1$ and $T = \tau$, w satisfy the following estimate

$$\int_{\Omega} w^2(\tau, x) dx \leq \int_{\Omega} w^2(\tau + a - a_1, x) dx \leq c_1 e^{\frac{c_2}{\tau + a - a_1}} \int_0^{\tau + a - a_1} \int_{\omega} w^2(s, x) dx ds.$$

In terms of \tilde{z} , the above inequality writes

$$\begin{aligned} \int_{\Omega} \tilde{z}^2(\tau, a, x) dx &\leq c_1 e^{\frac{c_2}{\tau + a - a_1}} \int_0^{\tau + a - a_1} \int_{\omega} \tilde{z}^2(s, a + \tau - s, x) dx ds \\ &= c_1 e^{\frac{c_2}{\tau + a - a_1}} \int_{a_1}^{\tau + a} \int_{\omega} \tilde{z}^2(\tau + a - s, s, x) dx ds. \end{aligned}$$

Integrating with respect to a over $(0, b_0)$ we obtain

$$\begin{aligned} \int_0^{b_0} \int_{\Omega} \tilde{z}^2(\tau, a, x) dx da &\leq c_1 e^{\frac{c_2}{\tau - a_1}} \int_0^{b_0} \int_{a_1}^{\tau + a} \int_{\omega} \tilde{z}^2(\tau + a - s, s, x) dx ds da \\ &= C_{\tau} \int_{a_1}^{a_2} \int_{s-\tau}^{b_0} \int_{\omega} \tilde{z}^2(\tau + a - s, s, x) dx dads = C_{\tau} \int_{a_1}^{a_2} \int_0^{a_2-s} \int_{\omega} \tilde{z}^2(r, s, x) dx dr ds \\ &\leq C_{\tau} \int_0^{\tau} \int_{a_1}^{a_2} \int_{\omega} \tilde{z}^2(t, a, x) dx dadt. \end{aligned} \quad (3.12)$$

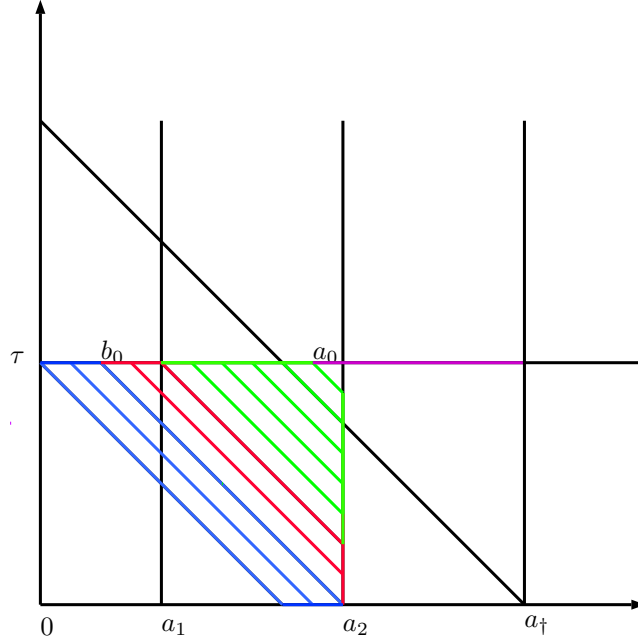


FIGURE 1. An illustration of Case 2: Blue region corresponds to the interval $(0, b_0)$, Red corresponds to the interval (b_0, a_1) , Green corresponds to the interval (a_1, a_0) and Purple corresponds to the interval (a_0, a_{\dagger}) .

Upper bound on (b_0, a_1) :

For a.e. $a \in (b_0, a_1)$, we define

$$w(s, x) = \tilde{z}(s, a + \tau - s, x), \quad s \in (\tau + a - a_2, \tau), x \in \Omega.$$

Then w solves the following problem

$$\begin{cases} \frac{\partial w}{\partial s} - Lw = 0, & (s, x) \in (\tau + a - a_2, \tau) \times \Omega, \\ \frac{\partial w}{\partial \nu_L} = 0, & (s, x) \in (\tau + a - a_2, \tau) \times \partial\Omega, \\ w(\tau + a - a_2, x) = \tilde{z}(\tau + a - a_2, a_2, x), & x \in \Omega. \end{cases} \quad (3.13)$$

Applying Proposition 3.3 with the choice $t_0 = \tau + a - a_2$, $t_1 = \tau + a - a_1$ and $T = \tau$, it is easy to see that, w satisfy the following estimate

$$\int_{\Omega} w^2(\tau, x) dx \leq \int_{\Omega} w^2(\tau + a - a_1, x) dx \leq c_1 e^{a_2 - a_1} \frac{c_2}{\int_{\tau+a-a_2}^{\tau+a-a_1}} \int_{\omega} w^2(s, x) dx ds.$$

In terms of \tilde{z} , the above inequality becomes

$$\int_{\Omega} \tilde{z}^2(\tau, a, x) dx \leq c_1 e^{a_2 - a_1} \frac{c_2}{\int_{\tau+a-a_2}^{\tau+a-a_1}} \int_{\omega} \tilde{z}^2(s, a + \tau - s, x) dx ds$$

$$= C \int_{\omega} \tilde{z}^2(\tau + a - s, s, x) \, dx ds.$$

Integrating with respect to a over (b_0, a_1) we get

$$\begin{aligned} \int_{b_0}^{a_1} \int_{\Omega} \tilde{z}^2(\tau, a, x) \, dx da &\leq C \int_{b_0}^{a_1} \int_{a_1}^{a_2} \int_{\omega} \tilde{z}^2(\tau + a - s, s, x) \, dx ds da \\ &= C \int_{a_1}^{a_2} \int_{b_0}^{a_1} \int_{\omega} \tilde{z}^2(\tau + a - s, s, x) \, dx dad s = C \int_{\tau+b_0-s}^{\tau+a_1-s} \int_{\omega} \tilde{z}^2(r, s, x) \, dx dr ds \\ &\leq C \int_{a_1}^{a_2} \int_0^{\tau} \int_{\omega} \tilde{z}^2(r, s, x) \, dx dr ds = C \int_0^{\tau} \int_{a_1}^{a_2} \int_{\omega} \tilde{z}^2(t, a, x) \, dx dad t. \end{aligned} \quad (3.14)$$

Upper bound on (a_1, a_0) :

For a.e. $a \in (a_1, a_0)$, we define

$$w(s, x) = \tilde{z}(s, a + \tau - s, x), \quad s \in (\tau + a - a_2, \tau), x \in \Omega.$$

Then w satisfies the system (3.13). Applying Proposition 3.3 with $t_0 = \tau + a - a_2$ and $t_1 = T = \tau$, we have that, w satisfy the following estimate

$$\int_{\Omega} w^2(\tau, x) \, dx \leq c_1 e^{\frac{c_2}{a_2 - a}} \int_{\tau+a-a_2}^{\tau} \int_{\omega} w^2(s, x) \, dx ds.$$

In terms of \tilde{z} , the above inequality reads as follows

$$\begin{aligned} \int_{\Omega} \tilde{z}^2(\tau, a, x) \, dx &\leq c_1 e^{\frac{c_2}{a_2 - a}} \int_{\tau+a-a_2}^{\tau} \int_{\omega} \tilde{z}^2(s, a + \tau - s, x) \, dx ds \\ &= c_1 e^{\frac{c_2}{a_2 - a}} \int_a^{a_2} \int_{\omega} \tilde{z}^2(\tau + a - s, s, x) \, dx ds. \end{aligned}$$

Integrating with respect to a over (a_1, a_0) we get

$$\begin{aligned} \int_{a_1}^{a_0} \int_{\Omega} \tilde{z}^2(\tau, a, x) \, dx da &\leq c_1 e^{\frac{c_2}{a_2 - a_0}} \int_{a_1}^{a_0} \int_a^{a_2} \int_{\omega} \tilde{z}^2(\tau + a - s, s, x) \, dx ds da \\ &= C_{\tau} \int_{a_1}^{a_2} \int_{a_1}^s \int_{\omega} \tilde{z}^2(\tau + a - s, s, x) \, dx dad s = C_{\tau} \int_{\tau+a_1-s}^{\tau} \int_{\omega} \tilde{z}^2(r, s, x) \, dx dr ds \\ &\leq C_{\tau} \int_0^{\tau} \int_{a_1}^{a_2} \int_{\omega} \tilde{z}^2(t, a, x) \, dx dad t. \end{aligned} \quad (3.15)$$

Upper bound on (a_0, a_{\dagger}) :

In this case $\tilde{z}(T, a, x) = 0$ for all $(a, x) \in (a_0, a_{\dagger}) \times \Omega$.

Therefore, combining (3.12), (3.14) and (3.15) we get

$$\int_0^{a_{\dagger}} \int_{\Omega} \tilde{z}^2(\tau, a, x) \, dx da \leq C_{\tau} \int_{\omega} \tilde{z}^2(t, a, x) \, dx dad t, \quad (3.16)$$

Case 3: $a_{\dagger} - a_2 < a_1 < \tau \leq a_{\dagger}$. Proceeding as case 2, one can similarly obtain the following estimate

$$\int_0^{a_{\dagger}} \int_{\Omega} \tilde{z}^2(\tau, a, x) \, dx da \leq C_{\tau} \int_0^{\tau} \int_{a_1}^{a_2} \int_{\omega} \tilde{z}^2(t, a, x) \, dx da dt. \quad (3.17)$$

Finally, combining (3.16) and (3.17), we obtain (3.9). \square

3.2. Estimate of $q(t, 0, x)$. In this subsection, we estimate $q(t, 0, x)$ from the right hand side of (3.3). More precisely, we prove the following proposition

Proposition 3.5. *Let us assume the hypothesis of Theorem 1.1 and let $\tau > a_1$ and $\eta \in (a_1, \tau)$. Then there exists a constant $C > 0$ such that, for every $q_0 \in \mathcal{D}(\mathcal{A}^*)$, the solution q of the system (3.1), satisfies*

$$\int_{\eta}^{\tau} \int_{\Omega} q^2(t, 0, x) \, dx dt \leq C \int_0^{\tau} \int_{a_1}^{a_2} \int_{\omega} q^2(t, a, x) \, dx da dt. \quad (3.18)$$

Proof. We set

$$\tilde{q}(t, a, x) = q(t, a, x) e^{-\int_0^a \mu(r) dr}. \quad (3.19)$$

Since $\beta(a) = 0$ for all $a \in (0, a_b)$, \tilde{q} satisfies

$$\frac{\partial \tilde{q}}{\partial t} - \frac{\partial \tilde{q}}{\partial a} - L\tilde{q} = 0 \quad (t \geq 0, (a, x) \in (0, a_b) \times \Omega). \quad (3.20)$$

Here also we are going to use Proposition 3.3 along the characteristics. Since we want to estimate $q(t, 0, x)$ we need to consider the trajectory $\gamma(s) = (t-s, s)$, $s \leq t \leq \tau$ (or equivalently the backward characteristic starting from $(t, 0)$). If $\tau < a_1$, the trajectory $\gamma(s)$ never reaches the observation region $(0, \tau) \times (a_1, a_2)$ (see Fig. 2). This is why we choose $\tau > a_1$. Without loss of generality, let us assume that $\tau > a_b$, $\eta < a_b$ and $a_2 \leq a_b$.

Case 1: For a.e. $t \in (a_b, \tau)$, we define

$$w(s, x) = \tilde{q}(s, t-s, x), \quad s \in (t-a_b, t), x \in \Omega. \quad (3.21)$$

Then w satisfies

$$\begin{cases} \frac{\partial w}{\partial s} - Lw = 0 & ((s, x) \in (t-a_b, t) \times \Omega), \\ \frac{\partial w}{\partial \nu_L} = 0 & ((s, x) \in (t-a_b, t) \times \partial\Omega), \\ w(t-a_b, x) = q(t-a_b, a_b, x) & (x \in \Omega). \end{cases} \quad (3.22)$$

Using Proposition 3.3, with $t_0 = t-a_b$, $t_1 = t-a_1$ and $T = t$, we obtain

$$\int_{\Omega} w^2(t, x) \, dx \leq \int_{\Omega} w^2(t-a_1, x) \, dx \leq c_1 e^{a_b - a_1} \int_{t-a_b}^{t-a_1} \int_{\omega} w^2(s, x) \, dx ds.$$

In terms of \tilde{q} the above inequality reads as

$$\int_{\Omega} \tilde{q}^2(t, 0, x) \, dx \leq c_1 e^{a_b - a_1} \int_{t-a_b}^{t-a_1} \int_{\omega} \tilde{q}^2(s, t-s, x) \, dx ds$$

$$= c_1 e^{a_b - a_1} \frac{c_2}{c_1} \int_{a_1}^{a_b} \int_{\omega} \tilde{q}^2(t-s, s, x) \, dx ds.$$

Integrating with respect to t over $[a_b, \tau]$ we obtain

$$\begin{aligned} \int_{a_b}^{\tau} \int_{\Omega} \tilde{q}^2(t, 0, x) \, dx dt &\leq c_1 e^{a_b - a_1} \frac{c_2}{c_1} \int_{a_b}^{\tau} \int_{a_1}^{a_b} \int_{\omega} \tilde{q}^2(t-s, s, x) \, dx ds dt \\ &= C \int_{a_1}^{a_b} \int_{a_b}^{\tau} \int_{\omega} \tilde{q}^2(t-s, s, x) \, dx dt ds = C \int_{a_1}^{a_b} \int_{a_b-s}^{\tau-s} \int_{\omega} \tilde{q}^2(r, s, x) \, dx dr ds \\ &\leq C \int_0^{\tau} \int_{a_1}^{a_b} \int_{\omega} \tilde{q}^2(t, a, x) \, dx da dt. \quad (3.23) \end{aligned}$$

Case 2: For a.e $t \in (\eta, a_b)$, we define

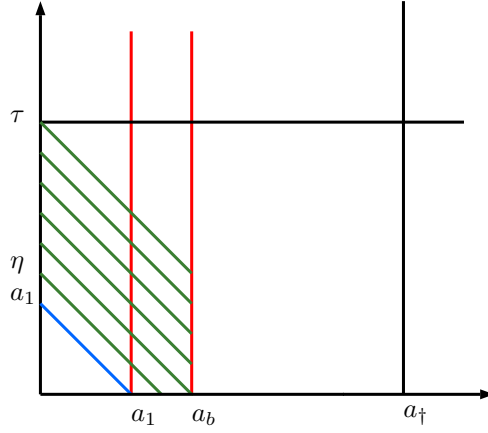


FIGURE 2. An illustration of the estimate of $\tilde{q}(t, 0, x)$. Here we have chosen $a_2 = a_b$. Since $\tau > a_1$ all the backward characteristics starting from $(t, 0)$ enters the observation domain (the green region).

$$w(s, x) = \tilde{q}(s, t-s, x), \quad s \in (0, t), x \in \Omega. \quad (3.24)$$

Then w solves the following problem

$$\begin{cases} \frac{\partial w}{\partial s} - Lw = 0, & (s, x) \in (0, t) \times \Omega, \\ \frac{\partial w}{\partial \nu_L} = 0, & (s, x) \in (0, t) \times \partial\Omega, \\ w(0, x) = \tilde{q}(0, t, x), & x \in \Omega. \end{cases}$$

Proposition 3.3, with $t_0 = 0$, $t_1 = t - a_1$ and $T = t$, we obtain

$$\int_{\Omega} w^2(t, x) \, dx \leq \int_{\Omega} w^2(t - a_1, x) \, dx \leq c_1 e^{t - a_1} \int_0^{t - a_1} \int_{\omega} w^2(s, x) \, dx ds.$$

This yields

$$\int_{\Omega} \tilde{q}^2(t, 0, x) \, dx \leq c_1 e^{t-a_1} \int_0^{t-a_1} \int_{\omega} \tilde{q}^2(s, t-s, x) \, dx ds = c_1 e^{t-a_1} \int_{a_1}^t \int_{\omega} \tilde{q}^2(t-s, s, x) \, dx ds.$$

Integrating with respect to t over $[\eta, a_b]$ we get

$$\begin{aligned} \int_{\eta}^{a_b} \int_{\Omega} \tilde{q}^2(t, 0, x) \, dx dt &\leq c_1 e^{\eta-a_1} \int_{\eta}^{a_b} \int_{a_1}^t \int_{\omega} \tilde{q}^2(t-s, s, x) \, dx ds dt \\ &\leq C_{\eta} \int_0^{a_b} \int_{a_1}^t \int_{\omega} \tilde{q}^2(t-s, s, x) \, dx ds dt = C \int_{a_1}^{a_b} \int_s^{a_b} \int_{\omega} q^2(t-s, s, x) \, dx dt ds \\ &= C \int_{a_1}^{a_b} \int_0^{a_b-s} \int_{\omega} q^2(r, s, x) \, dx dr ds \leq C \int_0^{\tau} \int_{a_1}^{a_b} \int_{\omega} \tilde{q}^2(t, a, x) \, dx da dt. \end{aligned} \quad (3.25)$$

Combining, (3.23) and (3.25) we obtain

$$\int_{\eta}^T \int_{\Omega} \tilde{q}^2(t, 0, x) \, dx dt \leq C \int_0^T \int_{a_1}^{a_2} \int_{\omega} \tilde{q}^2(t, a, x) \, dx da dt.$$

Note that, $\tilde{q}(t, 0, x) = q(t, 0, x)$. Thus from the above estimate we easily obtain (3.18). \square

3.3. Proof of Main results. Now, we are in a position to prove our main result

Proof of Theorem 3.2. Since $\tau > a_1 + \max\{a_1, a_{\dagger} - a_2\}$, we can choose $\eta \in (a_1, \tau)$ such that $\tau - \eta > \max\{a_1, a_{\dagger} - a_2\}$. For $t \in (\eta, \tau)$ and $a \in (0, a_{\dagger})$ we define

$$V(t, a, x) = \beta(a)q(t, 0, x). \quad (3.26)$$

Since $q(\cdot, 0, \cdot) \in L^2(0, \tau; L^2(\Omega))$, we have that $V \in L^2(\eta, \tau; H)$. It is easy to see that q solves the following problem

$$\begin{cases} \frac{\partial q}{\partial t} - \frac{\partial q}{\partial a} - Lq + \mu(a)q = V, & (t, a, x) \in (\eta, \tau) \times (0, a_{\dagger}) \times \Omega, \\ q(t, a_{\dagger}, x) = 0, & (t, x) \in (\eta, \tau) \times \Omega, \\ \frac{\partial q}{\partial \nu_L} = 0, & (t, a, x) \in (\eta, \tau) \times (0, a_{\dagger}) \times \partial\Omega. \end{cases} \quad (3.27)$$

Therefore,

$$q(t, a, x) = \mathbb{T}_{t-\eta}^0 q_{\eta} + \int_{\eta}^t \mathbb{T}_{t-s}^0 V(s, \cdot, \cdot) \, ds, \quad (3.28)$$

where $q_{\eta} = q(\eta, \cdot, \cdot)$ and \mathbb{T}_t^0 is the semigroup generated by the unbounded operator \mathcal{A}_0 defined in (2.7). Since \mathcal{A}_0 generates a C^0 semigroup on H , the above representation formula for q yields

$$\int_0^{a_{\dagger}} \int_{\Omega} q^2(\tau, a, x) \, dx da \leq C \left(\|\mathbb{T}_{\tau-\eta}^0 q_{\eta}\|_H^2 + \int_{\eta}^{\tau} \int_{\Omega} q^2(t, 0, x) \, dx dt \right). \quad (3.29)$$

Since $\tau - \eta > \max\{a_1, a_{\dagger} - a_2\}$, using Proposition 3.4 we have

$$\|\mathbb{T}_{\tau-\eta}^0 q_{\eta}\|_H^2 \leq C_{\tau} \int_{\eta}^{\tau} \int_{a_1}^{a_2} \int_{\omega} q^2(t, a, x) \, dx da dt \leq C_{\tau} \int_0^{\tau} \int_{a_1}^{a_2} \int_{\omega} q^2(t, a, x) \, dx da dt.$$

On the other hand, since $\tau > a_1$, applying Proposition 3.5 we get

$$\int_{\eta}^{\tau} \int_{\Omega} q^2(t, 0, x) \, dx dt \leq C \int_0^{\tau} \int_{a_1}^{a_2} \int_{\omega} q^2(t, a, x) \, dx da dt.$$

Finally combining the above two estimates together with (3.29), we obtain

$$\int_0^{a_{\dagger}} \int_{\Omega} q^2(\tau, a, x) \, dx da \leq C_{\tau} \int_0^{\tau} \int_{a_1}^{a_2} \int_{\omega} q^2(t, a, x) \, dx da dt. \quad (3.30)$$

This completes the proof of the theorem. \square

4. COMMENTS AND EXTENSIONS

The main result in this section gives lower bounds for the controllability time in Theorem 1.1. We show, in particular, that the controllability time in Theorem 1.1 is sharp in the case $a_1 = 0$. More precisely, we have:

Proposition 4.1. *Under the assumptions of Theorem 1.1, let $\tau < \max\{a_1, a_{\dagger} - a_2\}$. Then there exists $q_0 \in \mathcal{D}(\mathcal{A}^*)$ such that, the solution q of (3.1), satisfies*

- $q(t, a, x) = 0$ for all $(t, a, x) \in (0, \tau) \times (a_1, a_2) \times \Omega$.
- $\int_0^{a_{\dagger}} \int_{\Omega} q^2(\tau, a, x) \, dx da > 0$.

Proof. Let $(\varphi_j)_{j \geq 1}$ be an orthonormal basis of $L^2(\Omega)$ comprising of eigenvectors of the operator $-A_2$ and let $(\lambda_j)_{j \geq 1}$ be the corresponding eigenvalues. The solution q of (3.1) writes

$$q(t, a, \cdot) = \sum_{j=1}^{\infty} q^j(t, a) \varphi_j,$$

where

$$\begin{cases} \frac{\partial q^j}{\partial t} - \frac{\partial q^j}{\partial a} - \beta(a)q^j(t, 0) + (\mu(a) + \lambda_j)q^j = 0, & t \geq 0, a \in (0, a_{\dagger}), \\ q^j(t, a_{\dagger}) = 0, & t \geq 0, \\ q^j(0, a) = q_0^j(a), & a \in (0, a_{\dagger}) \end{cases} \quad (4.1)$$

and

$$q_0(a, \cdot) = \sum_{j=1}^{\infty} q_0^j(a) \varphi_j.$$

Integrating along the characteristic lines, we obtain the following expression of $q^j(t, a)$

$$q^j(t, a) = \begin{cases} \frac{\pi^j(a)}{\pi^j(a+t)} q_0^j(a+t) + \int_0^t \frac{\pi^j(a)}{\pi^j(a+t-s)} \beta(a+t-s) V^j(s) \, ds, & t \leq a_{\dagger} - a, \\ \int_{t+a-a_{\dagger}}^t \frac{\pi^j(a)}{\pi^j(a+t-s)} \beta(a+t-s) V^j(s) \, ds, & t > a_{\dagger} - a, \end{cases} \quad (4.2)$$

where

$$V^j(t) = q^j(t, 0) \quad \text{and} \quad \pi^j(a) = \exp\left(\lambda_j a + \int_0^a \mu(r) \, dr\right).$$

Without loss of generality, let us assume that $a_1 > 0$.

Case 1: $\tau < a_1$

Let us choose $a_* \in (\tau, a_1)$ and $\varepsilon > 0$ such that $(a_* - \varepsilon, a_* + \varepsilon) \subset (\tau, a_1)$. Let $q_0^1 \in C_c^\infty(0, a_1)$ such that

$$q_0^1 = 0 \text{ for all } a \in [0, \tau] \cup [a_1, a_1] \quad \text{and} \quad q_0^1 = 1 \text{ for all } a \in (a_* - \varepsilon, a_* + \varepsilon).$$

Since $\beta(a) = 0$ for $a \in [0, a_b)$, using the expression (4.2), we first have $V^1(t) = q^1(t, 0) = 0$ for all $t \in [0, \tau]$. Using this it is easy to see that

$$q^1(t, a) = \begin{cases} \frac{\pi^1(a)}{\pi^1(a+t)} & \text{if } a \in (a_* - t - \varepsilon, a_* - t + \varepsilon), t \in [0, \tau] \\ 0 & \text{if } a \geq a_1, t \in [0, \tau]. \end{cases}$$

Now set $q_0(a, x) = q_0^1(a)\varphi_1(x)$. Then $q(t, a, x) = 0$ for all $(t, a, x) \in (0, \tau) \times (a_1, a_1) \times \Omega$. Moreover,

$$\int_0^{a_1} \int_\Omega q^2(\tau, a, x) dx da \geq \int_{a_* - \tau - \varepsilon}^{a_* - \tau + \varepsilon} (q^1)^2(\tau, a) da > 0.$$

Case 2: $a_1 \leq \tau < a_1 - a_2$

Let us choose $a_* \in (a_2 + \tau, a_1)$ and $\varepsilon > 0$ such that $(a_* - \varepsilon, a_* + \varepsilon) \subset (a_2 + \tau, a_1)$. Let $q_0^1 \in C_c^\infty(0, a_1)$ such that

$$q_0^1 = 0 \text{ for all } a \in [0, a_2 + \tau] \quad \text{and} \quad q_0^1 = 1 \text{ for all } a \in (a_* - \varepsilon, a_* + \varepsilon).$$

As $\beta(a) = 0$, for $a \in [0, a_b)$, using the expression (4.2) we obtain

$$V^1(t, 0) = 0 \text{ for all } t \in [0, a_b).$$

Using the above identity and (4.2), we can easily obtain that

$$q^1(t, a) = 0 \text{ for all } t \in (0, a_b), a \in (0, a_2 + \tau - t),$$

and

$$q^1(a_b, a) = 0 \text{ for all } a \in [0, a_2 + \tau - a_b] \cup [a_1 - a_b, a_1]$$

$$q^1(a_b, a) = \frac{\pi^1(a)}{\pi^1(a+t)} \text{ for all } a \in (a_* - a_b - \varepsilon, a_* - a_b + \varepsilon).$$

Next we can calculate q^1 for $(t, a) \in (a_b, 2a_b) \times (0, a_1)$ with $q^1(a_b, \cdot)$ as initial data. Continuing this process, we obtain

$$q^1(t, a) = 0 \text{ for all } t \in (0, \tau), a \in (0, a_2 + \tau - t),$$

and

$$q^1(\tau, a) = \frac{\pi^1(a)}{\pi^1(a+t)} \text{ for all } a \in (a_* - \tau - \varepsilon, a_* - \tau + \varepsilon).$$

Then we can proceed as Case 1 to conclude the proof of the proposition. \square

As a consequence of the above proposition, the following theorem follows easily:

Theorem 4.2. *Under the assumption of Theorem 1.1, the system (1.1)-(1.5) is not null-controllable in time $\tau < \max\{a_1, a_1 - a_2\}$.*

Remark 4.3. *The above theorem shows that the controllability time in Theorem 1.1 is sharp in the case $a_1 = 0$.*

Before ending the paper, we describe some possible extensions of the results in this paper.

First, still in the case $a_1 = 0$, it would be interesting to make precise the dependence of the control cost on a_2 . This could allow the extension to the diffusive case of the singular perturbation result obtained in [7], which describes the behavior of the control problem when $a_2 \rightarrow 0$ (direct birth control) in the diffusion free case.

An important issue in view of applications is to design controls such that the corresponding state trajectories join two different non negative stationary states in some time τ , while preserving the positivity of the controlled trajectory for $t \in [0, \tau]$. This type of result has been proved in [7] for the diffusion free Lotka-McKendrick system (in a uniform time) and in Lohéac, Trélat and Zuazua [10] for purely parabolic problems (in a time depending on an appropriate norm of the difference of the two stationary states). We conjecture that the situation encountered in the latter case also applies to the problem considered in the present work. An essential ingredient in proving this conjecture should be proving the null controllability of the system in this paper by means of L^∞ controls and then “slowly” driving (s.t. positivity is preserved), the initial state towards the desired target.

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