



**HAL**  
open science

# CONVERGENCE OF A FINITE VOLUME SCHEME FOR A SYSTEM OF INTERACTING SPECIES WITH CROSS-DIFFUSION

José Carrillo, Francis Filbet, Markus Schmidtchen

► **To cite this version:**

José Carrillo, Francis Filbet, Markus Schmidtchen. CONVERGENCE OF A FINITE VOLUME SCHEME FOR A SYSTEM OF INTERACTING SPECIES WITH CROSS-DIFFUSION. *Numerische Mathematik*, 2020, 145, pp.473-511. hal-01764444

**HAL Id: hal-01764444**

**<https://hal.science/hal-01764444>**

Submitted on 12 Apr 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# CONVERGENCE OF A FINITE VOLUME SCHEME FOR A SYSTEM OF INTERACTING SPECIES WITH CROSS-DIFFUSION

JOSÉ A. CARRILLO

Department of Mathematics, Imperial College London  
London SW7 2AZ, United Kingdom

FRANCIS FILBET

Institut de Mathématiques de Toulouse , Université Paul Sabatier  
Toulouse, France

MARKUS SCHMIDTCHEN

Department of Mathematics, Imperial College London  
London SW7 2AZ, United Kingdom

ABSTRACT. In this work we present the convergence of a positivity preserving semi-discrete finite volume scheme for a coupled system of two non-local partial differential equations with cross-diffusion. The key to proving the convergence result is to establish positivity in order to obtain a discrete energy estimate to obtain compactness. We numerically observe the convergence to reference solutions with a first order accuracy in space. Moreover we recover segregated stationary states in spite of the regularising effect of the self-diffusion. However, if the self-diffusion or the cross-diffusion is strong enough, mixing occurs while both densities remain continuous.

## 1. INTRODUCTION

In this paper we develop and analyse a numerical scheme for the following non-local interaction system with cross-diffusion and self-diffusion

$$(1) \quad \begin{cases} \frac{\partial \rho}{\partial t} = \frac{\partial}{\partial x} \left( \rho \frac{\partial}{\partial x} (W_{11} \star \rho + W_{12} \star \eta + \nu(\rho + \eta)) + \frac{\epsilon}{2} \frac{\partial \rho^2}{\partial x} \right), \\ \frac{\partial \eta}{\partial t} = \frac{\partial}{\partial x} \left( \eta \frac{\partial}{\partial x} (W_{22} \star \eta + W_{21} \star \rho + \nu(\rho + \eta)) + \frac{\epsilon}{2} \frac{\partial \eta^2}{\partial x} \right), \end{cases}$$

governing the evolution of two species  $\rho$  and  $\eta$  on an interval  $(a, b) \subset \mathbb{R}$  for  $t \in [0, T)$ . The system is equipped with non-negative initial data  $\rho^0, \eta^0 \in L^1_+(a, b) \cap L^\infty_+(a, b)$ . We denote by  $m_1$  the mass of  $\rho_0$  and by  $m_2$  the mass of  $\eta_0$ , respectively,

$$m_1 = \int_a^b \rho_0(x) dx, \quad \text{and} \quad m_2 = \int_a^b \eta_0(x) dx.$$

---

2010 *Mathematics Subject Classification.* Primary: 74S10; 65M12; 92C15; Secondary: 45K05; 92D25; 47N60.

*Key words and phrases.* Finite volume methods; Integro-partial differential equations; Population dynamics (general); Developmental biology, pattern formation;

On the boundary  $x = a$  and  $b$ , we prescribe no-flux boundary conditions

$$\begin{cases} \rho \frac{\partial}{\partial x} (W_{11} \star \rho + W_{12} \star \eta + \nu(\rho + \eta) + \epsilon\rho) = 0, \\ \eta \frac{\partial}{\partial x} (W_{22} \star \eta + W_{21} \star \rho + \nu(\rho + \eta) + \epsilon\eta) = 0, \end{cases}$$

such that the total mass of each species is conserved with respect to time  $t \geq 0$ . While the self-interaction potentials  $W_{11}, W_{22} \in C_b^2(a, b)$  model the interactions among individuals of the same species (also referred to as *intraspecific* interactions), the cross-interaction potentials  $W_{12}, W_{21} \in C_b^2(a, b)$  encode the interactions between individuals belonging to different species, *i.e.* *interspecific* interactions. Here  $C_b^2(a, b)$  denotes the set of twice continuously differentiable functions on the interval  $[a, b]$  with bounded derivatives. The two positive parameters  $\epsilon, \nu > 0$  determine the strengths of the self-diffusion and the cross-diffusion of both species, respectively. It is the interplay between the non-local interactions of both species and their individual and joint size-exclusion, modelled by the non-linear diffusion [5, 4, 6, 33, 12, 10], that leads to a large variety of behaviours including complete phase separation or mixing of both densities in both stationary configurations and travelling pulses [11, 18].

While their single species counterparts have been studied quite intensively [30, 20, 34, 14] and references therein, related two-species models like the system of our interest, Eq. (1), have only recently gained considerable attention [24, 11, 18, 23, 16]. One of the most striking phenomena of these interaction models with cross-diffusion is the possibility of phase separation. Since the seminal papers [29, 5] established segregation effects for the first time for the purely diffusive system corresponding to (1) for  $W_{ij} \equiv 0$ ,  $i, j \in \{1, 2\}$  and  $\epsilon = 0$ , many generalisations were presented. They include reaction-(cross-)diffusion systems [3, 7, 16] and references therein, and by adding non-local interactions [11, 18, 23, 2] and references therein. Ref. [23] have established the existence of weak solutions to a class of non-local systems under a strong coercivity assumption on the cross-diffusion also satisfied by system (1).

Typical applications of these non-local models comprise many biological contexts such cell-cell adhesion [32, 31, 15], for instance, as well as tumour models [28, 25], but also the formation of the characteristic stripe patterns of zebrafish can be modelled by these non-local models [35]. Systems of this kind are truly ubiquitous in nature and we remark that ‘species’ may not only refer to biological species but also to a much wider class of (possibly inanimate) agents such as planets, physical or chemical particles, just to name a few.

Since system (1) is in conservative form a finite volume scheme is a natural choice as a numerical method. This is owing to the fact that, by construction, finite volume schemes are locally conservative: due the divergence theorem the change in density on a test cell has to equal the sum of the in-flux and the out-flux of the same cell. There is a huge literature on finite volume schemes, first and foremost [27]. They give a detailed description of the construction of such methods and address convergence issues. Schemes similar to the one proposed in Section 2 have been studied in [9] in the case of nonlinear degenerate diffusion equations in any dimension. A similar scheme for a system of two coupled PDEs was proposed in [21]. Later, the authors in [13] generalised the scheme proposed in [9] including both local and non-local drifts. The scheme was then extended to two species in [18]. All the aforementioned schemes have in common that they preserve non-negativity – a property that is also crucial for our analysis.

Before we define the finite volume scheme we shall present a formal energy estimate for the continuous system. The main difficulty in this paper is to establish positivity and reproducing the continuous energy estimate at the discrete level. The remainder of the introduction is dedicated to presenting the aforementioned energy estimate. Let us consider

$$\begin{aligned} \frac{d}{dt} \int_a^b \rho \log \rho \, dx &= \int_a^b \log \rho \frac{\partial \rho}{\partial t} \, dx \\ &= \int_a^b \log \rho \frac{\partial}{\partial x} \left( \rho \frac{\partial}{\partial x} (W_{11} \star \rho + W_{12} \star \eta + \nu(\rho + \eta) + \epsilon \rho) \right) \, dx \\ &= - \int_a^b \rho \frac{\partial}{\partial x} (W_{11} \star \rho + W_{12} \star \eta + \nu(\rho + \eta) + \epsilon \rho) \frac{\partial}{\partial x} (\log \rho) \, dx. \end{aligned}$$

Upon rearranging we get

$$\frac{d}{dt} \int_a^b \rho \log \rho \, dx + \nu \int_a^b \frac{\partial}{\partial x} (\rho + \eta) \frac{\partial \rho}{\partial x} \, dx + \epsilon \int_a^b \left| \frac{\partial \rho}{\partial x} \right|^2 \, dx \leq \int_a^b (W'_{11} \star \rho + W'_{12} \star \eta) \frac{\partial \rho}{\partial x} \, dx.$$

A similar computation for  $\eta$  yields

$$\frac{d}{dt} \int_a^b \eta \log \eta \, dx + \nu \int_a^b \frac{\partial}{\partial x} (\rho + \eta) \frac{\partial \eta}{\partial x} \, dx + \epsilon \int_a^b \left| \frac{\partial \eta}{\partial x} \right|^2 \, dx \leq \int_a^b (W'_{22} \star \eta + W'_{21} \star \rho) \frac{\partial \eta}{\partial x} \, dx,$$

whence, upon adding both, we obtain

$$\frac{d}{dt} \int_a^b [\rho \log \rho + \eta \log \eta] \, dx + \nu \int_a^b \left| \frac{\partial \sigma}{\partial x} \right|^2 \, dx + \epsilon \int_a^b \left( \left| \frac{\partial \rho}{\partial x} \right|^2 + \left| \frac{\partial \eta}{\partial x} \right|^2 \right) \, dx \leq \mathcal{D}_\rho + \mathcal{D}_\eta,$$

where  $\sigma = \rho + \eta$  and

$$\begin{cases} \mathcal{D}_\rho := \int_a^b (W'_{11} \star \rho + W'_{12} \star \eta) \frac{\partial \rho}{\partial x} \, dx, \\ \mathcal{D}_\eta := \int_a^b (W'_{22} \star \eta + W'_{21} \star \rho) \frac{\partial \eta}{\partial x} \, dx, \end{cases}$$

denote the advective parts associated to  $\rho$  and  $\eta$ , respectively. The advective parts can be controlled by using the weighted Young's inequality to get

$$\begin{aligned} |\mathcal{D}_\rho| &= \left| \int_a^b (W'_{11} \star \rho + W'_{12} \star \eta) \frac{\partial \rho}{\partial x} \, dx \right| \\ &\leq \frac{1}{2\alpha} \int_a^b |W'_{11} \star \rho + W'_{12} \star \eta|^2 \, dx + \frac{\alpha}{2} \int_a^b \left| \frac{\partial \rho}{\partial x} \right|^2 \, dx, \end{aligned}$$

for some  $\alpha > 0$ . In choosing  $0 < \alpha < \epsilon$  we obtain

$$(2) \quad \frac{d}{dt} \int_a^b [\rho \log \rho + \eta \log \eta] \, dx + \int_a^b \left| \frac{\partial \sigma}{\partial x} \right|^2 \, dx + \left( \epsilon - \frac{\alpha}{2} \right) \int_a^b \left| \frac{\partial \rho}{\partial x} \right|^2 + \left| \frac{\partial \eta}{\partial x} \right|^2 \, dx \leq \frac{C_\rho + C_\eta}{2\alpha},$$

where  $C_\rho = \|W'_{11} \star \rho + W'_{12} \star \eta\|_{L^2}$  and  $C_\eta = \|W'_{22} \star \eta + W'_{21} \star \rho\|_{L^2}$ . From the last line, Eq. (2), we may deduce bounds on the gradient of each species as well as on their sum. As mentioned above the crucial ingredient for this estimate is the positivity of solutions.

The rest of this paper is organised as follows. In the subsequent section we present a semi-discrete finite volume approximation of system (1) and we present the main result, Theorem 2.4. Section 3 is dedicated to establishing positivity and to the derivation of a priori estimates. In Section 4 we obtain compactness, pass to the limit, and identify the limiting functions as weak solutions to system (1). We conclude the paper with a numerical exploration in Section 5. We study the numerical order of accuracy and discuss stationary states and phase segregation phenomena.

## 2. NUMERICAL SCHEME AND MAIN RESULT

In this section we introduce the semi-discrete finite volume scheme for system (1). To begin with, let us introduce our notion of weak solutions.

**Definition 2.1** (Weak solutions.). *A couple of functions  $(\rho, \eta) \in L^2(0, T; H^1(a, b))^2$  is a weak solution to system (1) if it satisfies*

$$(3a) \quad - \int_a^b \rho_0 \varphi(0, \cdot) dx = \int_0^T \int_a^b \left[ \rho \left( \frac{\partial \varphi}{\partial t} + \left( -\nu \frac{\partial \sigma}{\partial x} + \frac{\partial V_1}{\partial x} \right) \frac{\partial \varphi}{\partial x} \right) + \frac{\epsilon}{2} \rho^2 \frac{\partial^2 \varphi}{\partial x^2} \right] dx dt,$$

and

$$(3b) \quad - \int_a^b \eta_0 \varphi(0, \cdot) dx = \int_0^T \int_a^b \left[ \eta \left( \frac{\partial \varphi}{\partial t} + \left( -\nu \frac{\partial \sigma}{\partial x} + \frac{\partial V_2}{\partial x} \right) \frac{\partial \varphi}{\partial x} \right) + \frac{\epsilon}{2} \eta^2 \frac{\partial^2 \varphi}{\partial x^2} \right] dx dt,$$

respectively, for any  $\varphi \in C_c^\infty([0, T] \times (a, b); \mathbb{R})$ . Here we have set  $V_k = -W_{k1} \star \rho - W_{k2} \star \eta$ , for  $k \in \{1, 2\}$ , and  $\sigma = \rho + \eta$ , as above.

Notice that the existence of weak solutions to system (1) will follow directly from the convergence of the numerical solution. Indeed, our analysis relies on a compactness argument which does not suppose a priori existence of solution to system (1).

To this end we first define the following space discretisation of the domain.

**Definition 2.2** (Space discretisation). *To discretise space, we introduce the mesh*

$$\mathcal{T} := \bigcup_{i \in I} C_i,$$

where the control volumes are given by  $C_i = [x_{i-1/2}, x_{i+1/2})$  for all  $i \in I := \{1, \dots, N\}$ . We assume that the measure of the control volumes are given by  $|C_i| = \Delta x_i = x_{i+1/2} - x_{i-1/2} > 0$ , for all  $i \in I$ . Note that  $x_{1/2} = a$ , and  $x_{N+1/2} = b$ .

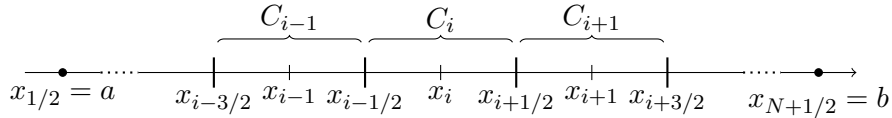


FIGURE 1. Space discretisation according to Definition 2.2.

We also define  $x_i = (x_{i+1/2} + x_{i-1/2})/2$  the centre of cell  $C_i$  and set  $\Delta x_{i+1/2} = x_{i+1} - x_i$  for  $i = 1, \dots, N-1$ . We assume that the mesh is admissible in the sense that there exists  $\xi \in (0, 1)$  such that for  $h := \max_{1 \leq i \leq N} \{\Delta x_i\}$

$$(4) \quad \xi h \leq \Delta x_i \leq h,$$

and, as a consequence,  $\xi h \leq \Delta x_{i+1/2} \leq h$ , as well.

On this mesh we shall now define the semi-discrete finite volume approximation of system (1). The discretised initial data are given by the cell averages of the continuous initial data, *i.e.*

$$(5) \quad \rho_i^0 := \frac{1}{\Delta x_i} \int_{C_i} \rho_0(x) dx, \quad \text{and} \quad \eta_i^0 := \frac{1}{\Delta x_i} \int_{C_i} \eta_0(x) dx,$$

for all  $i \in I$ . Next, we introduce the discrete versions of the cross-diffusion and the interaction terms. We set

$$(6) \quad \begin{cases} (V_1)_i := - \sum_{j=1}^N \Delta x_j (W_{11}^{i-j} \rho_j + W_{12}^{i-j} \eta_j), \\ (V_2)_i := - \sum_{j=1}^N \Delta x_j (W_{22}^{i-j} \eta_j + W_{21}^{i-j} \rho_j), \end{cases}$$

where

$$(7) \quad W_{kl}^{i-j} = \frac{1}{\Delta x_j} \int_{C_j} W_{kl}(|x_i - s|) ds,$$

for  $k, l = 1, 2$ , and

$$U_i := -(\rho_i + \eta_i),$$

for the cross-diffusion term, respectively. Then the scheme reads

$$(8a) \quad \begin{cases} \frac{d\rho_i}{dt}(t) = - \frac{\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}}{\Delta x_i}, \\ \frac{d\eta_i}{dt}(t) = - \frac{\mathcal{G}_{i+1/2} - \mathcal{G}_{i-1/2}}{\Delta x_i}, \end{cases}$$

for  $i \in I$ . Here the numerical fluxes are given by

$$(8b) \quad \begin{cases} \mathcal{F}_{i+1/2} = \left[ \nu (dU)_{i+1/2}^+ + (dV_1)_{i+1/2}^+ \right] \rho_i + \left[ \nu (dU)_{i+1/2}^- + (dV_1)_{i+1/2}^- \right] \rho_{i+1} \\ \quad - \frac{\epsilon}{2} \frac{\rho_{i+1}^2 - \rho_i^2}{\Delta x_{i+1/2}}, \\ \mathcal{G}_{i+1/2} = \left[ \nu (dU)_{i+1/2}^+ + (dV_2)_{i+1/2}^+ \right] \eta_i + \left[ \nu (dU)_{i+1/2}^- + (dV_2)_{i+1/2}^- \right] \eta_{i+1} \\ \quad - \frac{\epsilon}{2} \frac{\eta_{i+1}^2 - \eta_i^2}{\Delta x_{i+1/2}}, \end{cases}$$

for  $i = 1, \dots, N-1$ , with the numerical no-flux boundary condition

$$(8c) \quad \mathcal{F}_{1/2} = \mathcal{F}_{N+1/2} = 0, \quad \text{and} \quad \mathcal{G}_{1/2} = \mathcal{G}_{N+1/2} = 0,$$

where we introduced the discrete gradient  $du_{i+1/2}$  as

$$du_{i+1/2} := \frac{u_{i+1} - u_i}{\Delta x_{i+1/2}}.$$

As usual, we use  $(z)^\pm$  to denote the positive (resp. negative) part of  $z$ , *i.e.*

$$(z)^+ := \max(z, 0), \quad \text{and} \quad (z)^- := \min(z, 0).$$

At this stage, the numerical flux (8b) may look strange since

- the cross-diffusion term is approximated as a convective term using that

$$\frac{\partial}{\partial x} \left( \rho \frac{\partial}{\partial x} (\rho + \eta) \right) = \frac{\partial}{\partial x} \left( \rho \frac{\partial \sigma}{\partial x} \right)$$

with  $\sigma = \rho + \eta$  and  $\frac{\partial \sigma}{\partial x}$  is considered as a velocity field. This treatment has already been used in [9] and allows to preserve the positivity of both discrete densities  $(\rho, \eta)$  (see Lemma 3.1), which is crucial for the convergence analysis.

- In this new formulation, the velocity field is split in two parts both treated by an upwind scheme. One part comes from the cross-diffusion part, and the second one comes from the non-local interaction fields. This splitting is crucial to recovering a consistent dissipative term for the discrete energy estimate corresponding to Eq. (2).

**Definition 2.3** (Piecewise constant approximation). *For a given mesh  $\mathcal{T}_h$  we define the approximate solution to system (1) by*

$$\rho_h(t, x) := \rho_i(t), \quad \text{and} \quad \eta_h(t, x) := \eta_i(t),$$

for all  $(t, x) \in [0, T] \times C_i$ , with  $i = 1, \dots, N$ . Moreover, we define the following approximations of the gradients

$$d\rho_h(t, x) = \frac{\rho_{i+1} - \rho_i}{\Delta x_{i+1/2}}, \quad \text{and} \quad d\eta_h(t, x) = \frac{\eta_{i+1} - \eta_i}{\Delta x_{i+1/2}}$$

for  $(t, x) \in [0, T] \times [x_i, x_{i+1}]$ , for  $i = 1, \dots, N-1$ . Furthermore, in order to define  $d\rho_h$  and  $d\eta_h$  on the whole interval  $(a, b)$  we set them to zero on  $(a, x_1)$  and  $(x_N, b)$ .

Notice that the discrete gradients  $(d\rho_h, d\eta_h)$  are piecewise constant just like  $(\rho_h, \eta_h)$  however not on the same partition of the interval  $(a, b)$ . We have set out all definitions necessary to formulate the convergence of the numerical scheme (8).

**Theorem 2.4** (Convergence to a weak solution.). *Let  $\rho_0, \eta_0 \in L^1_+(a, b) \cap L^\infty_+(a, b)$  be some initial data and  $Q_T := (0, T) \times (a, b)$ . Then,*

- (i) *there exists a non-negative approximate solution  $(\rho_h, \eta_h)$  in the sense of Definition 2.3;*
- (ii) *up to a subsequence, this approximate solution converges strongly in  $L^2(Q_T)$  to  $(\rho, \eta) \in L^2(Q_T)$ , where  $(\rho, \eta)$  is a weak solution as in Definition 2.1. Furthermore we have  $\rho, \eta \in L^2(0, T; H^1(a, b))$ ;*
- (iii) *as a consequence system (1) has a weak solution.*

### 3. A PRIORI ESTIMATES

This section is dedicated to deriving a priori estimates for our system. In order to do so we require the positivity of approximate solutions and their conservation of mass, respectively. The following lemma guarantees these properties.

**Lemma 3.1** (Existence of non-negative solutions and conservation of mass). *Assume that the initial data  $(\rho_0, \eta_0)$  in non-negative. Then there exists a unique non-negative approximate solution  $(\rho_h, \eta_h)_{h>0}$  to the scheme (8a)-(8b). Furthermore, the finite volume scheme conserves the initial mass of both densities.*

*Proof.* On the one hand we notice that the right-hand side of (8a)-(8b) is locally Lipschitz with respect to  $(\rho_i, \eta_i)_{1 \leq i \leq N}$ . Hence, we may apply the Cauchy-Lipschitz theorem to obtain a unique local-in-time solution.

On the other hand to prove that this solution is global in time, we show the non-negativity of the solution together with the conservation of mass and argue by contradiction.

Let  $h > 0$  be fixed and some initial data,  $\rho_i(0), \eta_i(0) \geq 0$ , be given for  $i = 1, \dots, N$ . We rewrite the scheme in the following way.

$$(9) \quad \frac{d\rho_i}{dt}(t) = -\frac{\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}}{\Delta x_i} = \frac{1}{\Delta x_i}(A\rho_i + B\rho_{i+1} + C\rho_{i-1}),$$

where

$$\begin{cases} A = \nu(dU)_{i-1/2}^- + (dV_1)_{i-1/2}^- - \nu(dU)_{i+1/2}^+ - (dV_1)_{i+1/2}^+ - \epsilon \frac{\rho_i}{\Delta x_{i+1/2}}, \\ B = -\nu(dU)_{i+1/2}^- - (dV_1)_{i+1/2}^- + \epsilon \frac{\rho_{i+1}}{2\Delta x_{i+1/2}}, \\ C = \nu(dU)_{i-1/2}^+ + (dV_1)_{i-1/2}^+ + \epsilon \frac{\rho_{i-1}}{2\Delta x_{i+1/2}}. \end{cases}$$

Then let  $t^* \geq 0$  be the maximal time for all densities to remain non-negative, *i.e.*

$$t^* = \sup_{t \geq 0} \{\rho_i(t) \geq 0 \mid \text{for all } i = 1, \dots, N\}.$$

If  $t^* < \infty$ , then, by the continuity of the solution, there exists at least one index  $i \in \{1, \dots, N\}$  and some positive constant,  $\tau > 0$ , such that  $\rho_i(t^*) = 0$  and  $\rho_i(t) < 0$  for all  $t \in (t^*, t^* + \tau)$ .

Also note that the neighbours of  $\rho_i$  have to satisfy  $\rho_{i+1}(t^*) \geq 0$  and  $\rho_{i-1}(t^*) \geq 0$ , since otherwise the negativity of one of them would contradict the maximality of  $t^*$ .

By the above computation, Eq. (9), we see that, if  $\rho_{i+1}(t^*) > 0$  or  $\rho_{i-1}(t^*) > 0$ ,

$$\begin{aligned} \frac{d\rho_i}{dt}(t^*) &= \frac{1}{\Delta x_i}(A\rho_i(t^*) + B\rho_{i+1}(t^*) + C\rho_{i-1}(t^*)) \\ &= \frac{1}{\Delta x_i}(B\rho_{i+1}(t^*) + C\rho_{i-1}(t^*)) > 0, \end{aligned}$$

which cannot occur since  $\rho_i(t) < 0$  for all  $t \in (t^*, t^* + \tau)$ . If  $\rho_{i-1}(t^*) = \rho_i(t^*) = \rho_{i+1}(t^*) = 0$  then repeat the argument for  $\rho_{i+1}$  or  $\rho_{i-1}$  and apply the same argument. Noting that there is at least one index,  $i$ , such that  $\rho_i(t^*) > 0$ , for we would otherwise contradict the uniqueness of solutions, we eventually reach a contradiction.

Finally we get the conservation of mass,

$$\begin{aligned} \frac{d}{dt} \int_a^b \rho_h(t, x) dx &= \sum_{i=1}^N \Delta x_i \frac{d}{dt} \rho_i \\ &= \sum_{i=1}^N \Delta x_i \frac{\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}}{\Delta x_i} = \mathcal{F}_{N+1/2} - \mathcal{F}_{1/2} = 0, \end{aligned}$$

by the no-flux condition. Analogously, the second species remains nonnegative and its mass is conserved as well. As a consequence of the control of the  $L^1$ -norm of  $(\rho_h, \eta_h)$  we can extend the local solution to a global, nonnegative solution.  $\square$



Now, we are ready to study the evolution of the energy of the system on the semi-discrete level. The remaining part of this section is dedicated to proving the following lemma – an estimate similar to (2) for the semi-discrete scheme (8).

**Lemma 3.2** (Energy control). *Consider a solution of the semi-discrete scheme (8a)-(8b). Then we have*

$$\frac{d}{dt} \sum_{i=1}^N \Delta x_i [\rho_i \log \rho_i + \eta_i \log \eta_i] + \sum_{i=1}^{N-1} \Delta x_{i+1/2} \left[ \nu |dU_{i+1/2}|^2 + \frac{\epsilon}{4} (|d\rho_{i+1/2}|^2 + |d\eta_{i+1/2}|^2) \right] \leq C_\epsilon,$$

where the constant  $C_\epsilon > 0$  is given by

$$(10) \quad C_\epsilon = \frac{(b-a)}{\epsilon} \left( (\|W'_{11}\|_{L^\infty} + \|W'_{21}\|_{L^\infty}) m_1 + (\|W'_{12}\|_{L^\infty} + \|W'_{22}\|_{L^\infty}) m_2 \right).$$

*Proof.* Upon using the scheme, Eq. (8a), we get

$$\frac{d}{dt} \sum_{i=1}^N \Delta x_i \rho_i \log \rho_i = - \sum_{i=1}^N (\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}) \log \rho_i.$$

By discrete integration by parts and the no-flux condition, Eq. (8c), we obtain

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^N \Delta x_i \rho_i \log \rho_i &= \sum_{i=1}^{N-1} \Delta x_{i+1/2} \mathcal{F}_{i+1/2} \operatorname{dlog} \rho_{i+1/2} \\ &= \nu \sum_{i=1}^{N-1} \Delta x_{i+1/2} \left( (dU)_{i+1/2}^+ \rho_i + (dU)_{i+1/2}^- \rho_{i+1} \right) \operatorname{dlog} \rho_{i+1/2} \\ &\quad + \sum_{i=1}^{N-1} \Delta x_{i+1/2} \left( (dV_1)_{i+1/2}^+ \rho_i + (dV_1)_{i+1/2}^- \rho_{i+1} \right) \operatorname{dlog} \rho_{i+1/2}, \\ &\quad - \frac{\epsilon}{2} \sum_{i=1}^{N-1} (\rho_{i+1}^2 - \rho_i^2) \operatorname{dlog} \rho_{i+1/2}, \end{aligned}$$

where, in the last equality, we substituted the definition of the numerical flux, Eq. (8b). Let us define

$$(11) \quad \tilde{\rho}_{i+1/2} := \begin{cases} \frac{\rho_{i+1} - \rho_i}{\log \rho_{i+1} - \log \rho_i}, & \text{if } \rho_i \neq \rho_{i+1}, \\ \frac{\rho_i + \rho_{i+1}}{2}, & \text{else,} \end{cases}$$

for  $i \in \{1, \dots, N-1\}$ , and note that then  $\tilde{\rho}_{i+1/2} \in [\rho_i, \rho_{i+1}]$  by concavity of the log. Reordering the terms, we obtain

$$\begin{aligned}
 & \frac{d}{dt} \sum_{i=1}^N \Delta x_i \rho_i \log \rho_i - \sum_{i=1}^{N-1} \Delta x_{i+1/2} \left[ \nu dU_{i+1/2} \tilde{\rho}_{i+1/2} - \frac{\epsilon}{2} d\rho_{i+1/2}^2 \right] d\log \rho_{i+1/2} \\
 &= \nu \sum_{i=1}^{N-1} \Delta x_{i+1/2} \left( (dU)_{i+1/2}^+(\rho_i - \tilde{\rho}_{i+1/2}) + (dU)_{i+1/2}^-(\rho_{i+1} - \tilde{\rho}_{i+1/2}) \right) d\log \rho_{i+1/2} \\
 (12) \quad &+ \sum_{i=1}^{N-1} \Delta x_{i+1/2} \left( (dV_1)_{i+1/2}^+(\rho_i - \tilde{\rho}_{i+1/2}) + (dV_1)_{i+1/2}^-(\rho_{i+1} - \tilde{\rho}_{i+1/2}) \right) d\log \rho_{i+1/2} \\
 &+ \sum_{i=1}^{N-1} \Delta x_{i+1/2} \tilde{\rho}_{i+1/2} dV_{1,i+1/2} d\log \rho_{i+1/2}.
 \end{aligned}$$

Thus, using  $\tilde{\rho}_{i+1/2} \in [\rho_i, \rho_{i+1}]$  and the monotonicity of log, we note that

$$(13) \quad \begin{cases} (\rho_i - \tilde{\rho}_{i+1/2}) d\log \rho_{i+1/2} (\nu (dU)_{i+1/2}^+ + (dV_1)_{i+1/2}^+) \leq 0, \\ (\rho_{i+1} - \tilde{\rho}_{i+1/2}) d\log \rho_{i+1/2} (\nu (dU)_{i+1/2}^- + (dV_1)_{i+1/2}^-) \leq 0. \end{cases}$$

This is easy to see, for if  $\rho_i = \rho_{i+1}$  we observe  $d\log \rho_{i+1/2} = 0$  and Eqs. (13) hold with equality. In the case of  $\rho_i < \rho_{i+1}$  we observe

$$\underbrace{(\rho_i - \tilde{\rho}_{i+1/2})}_{\leq 0} \underbrace{d\log \rho_{i+1/2}}_{\geq 0} \underbrace{(\nu (dU)_{i+1/2}^+ + (dV_1)_{i+1/2}^+)}_{\geq 0} \leq 0,$$

while, for  $\rho_i > \rho_{i+1}$  there also holds

$$\underbrace{(\rho_i - \tilde{\rho}_{i+1/2})}_{\geq 0} \underbrace{d\log \rho_{i+1/2}}_{\leq 0} \underbrace{(\nu (dU)_{i+1/2}^+ + (dV_1)_{i+1/2}^+)}_{\geq 0} \leq 0,$$

whence we infer the inequality. The same argument can be applied in order to obtain the second line of Eq. (13). Thus we may infer from Eq. (12) that

$$\begin{aligned}
 & \frac{d}{dt} \sum_{i=1}^N \Delta x_i \rho_i \log \rho_i - \sum_{i=1}^{N-1} \Delta x_{i+1/2} \left( \nu \tilde{\rho}_{i+1/2} dU_{i+1/2} - \frac{\epsilon}{2} d\rho_{i+1/2}^2 \right) d\log \rho_{i+1/2} \\
 & \leq \sum_{i=1}^{N-1} \Delta x_{i+1/2} \tilde{\rho}_{i+1/2} d\log \rho_{i+1/2} dV_{1,i+1/2}.
 \end{aligned}$$

Note that the definition of  $\tilde{\rho}_{i+1/2}$  in Eq. (11), is consistent with the case  $\rho_i = \rho_{i+1}$  and there holds

$$\tilde{\rho}_{i+1/2} d\log \rho_{i+1/2} = d\rho_{i+1/2},$$

whence we get

$$\begin{aligned} & \frac{d}{dt} \sum_{i=1}^N \Delta x_i \rho_i \log \rho_i - \nu \sum_{i=1}^{N-1} \Delta x_{i+1/2} d\rho_{i+1/2} dU_{i+1/2} \\ & + \frac{\epsilon}{2} \sum_{i=1}^{N-1} \Delta x_{i+1/2} d\rho_{i+1/2}^2 d\log \rho_{i+1/2} \leq \sum_{i=1}^{N-1} \Delta x_{i+1/2} d\rho_{i+1/2} dV_{1,i+1/2}. \end{aligned}$$

Furthermore, we notice that

$$\frac{1}{2} d\rho_{i+1/2}^2 d\log \rho_{i+1/2} = \frac{\rho_{i+1} + \rho_i}{2 \tilde{\rho}_{i+1/2}} |d\rho_{i+1/2}|^2 \geq \frac{1}{2} |d\rho_{i+1/2}|^2,$$

where we employed Eq. (11). Hence we have

$$\begin{aligned} & \frac{d}{dt} \sum_{i=1}^N \Delta x_i \rho_i \log \rho_i - \nu \sum_{i=1}^{N-1} \Delta x_{i+1/2} d\rho_{i+1/2} dU_{i+1/2} \\ (14) \quad & + \frac{\epsilon}{2} \sum_{i=1}^{N-1} \Delta x_{i+1/2} |d\rho_{i+1/2}|^2 \leq \sum_{i=1}^{N-1} \Delta x_{i+1/2} d\rho_{i+1/2} dV_{1,i+1/2}. \end{aligned}$$

A similar computation can be applied to the second species, which yields

$$\begin{aligned} & \frac{d}{dt} \sum_{i=1}^N \Delta x_i \eta_i \log \eta_i - \nu \sum_{i=1}^{N-1} \Delta x_{i+1/2} d\eta_{i+1/2} dU_{i+1/2} \\ (15) \quad & + \frac{\epsilon}{2} \sum_{i=1}^{N-1} \Delta x_{i+1/2} |d\eta_{i+1/2}|^2 \leq \sum_{i=1}^{N-1} \Delta x_{i+1/2} d\eta_{i+1/2} dV_{2,i+1/2}. \end{aligned}$$

Upon adding up equations (14) and (15), we obtain

$$\begin{aligned} & \frac{d}{dt} \sum_{i=1}^N \Delta x_i [\rho_i \log \rho_i + \eta_i \log \eta_i] - \nu \sum_{i=1}^{N-1} \Delta x_{i+1/2} dU_{i+1/2} d(\rho + \eta)_{i+1/2} \\ (16) \quad & + \frac{\epsilon}{2} \sum_{i=1}^{N-1} \Delta x_{i+1/2} \left( |d\rho_{i+1/2}|^2 + |d\eta_{i+1/2}|^2 \right) \leq \mathcal{R}_h, \end{aligned}$$

where  $\mathcal{R}_h$  is given by

$$\mathcal{R}_h = \sum_{i=1}^{N-1} \Delta x_{i+1/2} [dV_{1,i+1/2} d\rho_{i+1/2} + dV_{2,i+1/2} d\eta_{i+1/2}].$$

Finally we notice that

$$(17) \quad \mathcal{R}_h \leq \frac{1}{2} \sum_{i=1}^{N-1} \Delta x_{i+1/2} \left[ \frac{|dV_{1,i+1/2}|^2}{\alpha} + \alpha |d\rho_{i+1/2}|^2 + \frac{|dV_{2,i+1/2}|^2}{\alpha} + \alpha |d\eta_{i+1/2}|^2 \right],$$

for any  $\alpha > 0$ , by Young's inequality. Observing, that for  $k = 1, 2$ ,

$$\begin{aligned} |dV_{k,i+1/2}| &\leq \left| \sum_{j=1}^{N-1} \rho_j \int_{C_j} \frac{W_{k1}(x_{i+1} - y) - W_{k1}(x_i - y)}{\Delta x_{i+1/2}} dy \right| \\ &\quad + \left| \sum_{j=1}^{N-1} \eta_j \int_{C_j} \frac{W_{k2}(x_{i+1} - y) - W_{k2}(x_i - y)}{\Delta x_{i+1/2}} dy \right| \\ &\leq \|W'_{k1}\|_{L^\infty} m_1 + \|W'_{k2}\|_{L^\infty} m_2 \end{aligned}$$

and by conservation of positivity and mass, it gives for  $k = 1, 2$ ,

$$\frac{1}{2\alpha} \sum_{i=1}^{N-1} \Delta x_{i+1/2} |dV_{k,i+1/2}|^2 \leq \frac{(b-a)}{2\alpha} (m_1 \|W'_{k1}\|_{L^\infty} + m_2 \|W'_{k2}\|_{L^\infty})^2.$$

Thus Eq. (17) becomes

$$\mathcal{R}_h \leq \frac{\alpha}{2} \sum_{i=1}^{N-1} \Delta x_{i+1/2} (|d\rho_{i+1/2}|^2 + |d\eta_{i+1/2}|^2) + C_{2\alpha},$$

where  $C_{2\alpha}$  is given in Eq. (10). Finally, substituting the latter estimate into Eq. (16), we obtain

$$\begin{aligned} \frac{d}{dt} \sum_{i=1}^N \Delta x_i [\rho_i \log \rho_i + \eta_i \log \eta_i] + \nu \sum_{i=1}^{N-1} \Delta x_{i+1/2} |dU_{i+1/2}|^2 \\ + \frac{\epsilon - \alpha}{2} \sum_{i=1}^{N-1} \Delta x_{i+1/2} (|d\rho_{i+1/2}|^2 + |d\eta_{i+1/2}|^2) \leq C_{2\alpha}, \end{aligned}$$

for any solution  $(\rho_i)_{i \in I}, (\eta_i)_{i \in I}$  of the semi-discrete scheme (8). Hence choosing  $\alpha = \epsilon/2$  concludes the proof.  $\square$

**Corollary 3.3** (A priori bounds). *Let  $(\rho_i)_{i \in I}, (\eta_i)_{i \in I}$  be solutions of the semi-discrete scheme (8). Then there exists a constant  $C > 0$ , independent of  $h > 0$ , such that*

$$\int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2} (|d\rho_{i+1/2}|^2 + |d\eta_{i+1/2}|^2 + |dU_{i+1/2}|^2) dt \leq C.$$

*Proof.* Using the fact that  $x \log x \geq -\log(e)/e$ , i.e.  $x \log x$  is bounded from below, yields

$$\sum_{i \in I} \Delta x_i [\rho_i \log \rho_i + \eta_i \log \eta_i](t) \geq -2 \frac{\log(e)}{e} (b-a) =: -C_1.$$

Hence the functional is bounded from below. Therefore, we integrate the inequality of Lemma 3.2 in time and get

$$\begin{aligned} \int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2} \left[ \nu |dU_{i+1/2}(t)|^2 + \frac{\epsilon}{4} (|d\rho_{i+1/2}(t)|^2 + |d\eta_{i+1/2}(t)|^2) \right] dt \\ \leq C_\epsilon T + C_1 + \int_a^b \rho_0 |\log \rho_0| + \eta_0 |\log \eta_0| dx, \end{aligned}$$

which proves the statement.  $\square$

Thanks to a classical discrete Poincaré inequality [8], we get uniform  $L^2$ -estimates on the discrete approximation  $(\rho_h, \eta_h)_{h>0}$ .

**Lemma 3.4.** *Let  $(\rho_i)_{i \in I}, (\eta_i)_{i \in I}$  be the numerical solutions obtained from scheme (8). Then there holds*

$$\|\rho_h\|_{L^2(Q_T)} + \|\eta_h\|_{L^2(Q_T)} \leq C,$$

for some constant  $C > 0$  independent of  $h > 0$ .

*Proof.* Let  $j, k \in I$  be arbitrary. We consider the difference and apply the Cauchy-Schwarz inequality

$$|\rho_k - \rho_j| = \left| \sum_{i=j}^{k-1} \Delta x_{i+1/2} d\rho_{i+1/2} \right| \leq \sqrt{(b-a)} \left( \sum_{i=1}^{N-1} \Delta x_{i+1/2} |d\rho_{i+1/2}|^2 \right)^{1/2}.$$

Upon squaring both sides of the equation, multiplying by  $\Delta x_k \Delta x_l$  and summing over  $k, j \in I$ , we get

$$\sum_{i=1}^N \Delta x_i |\rho_i|^2 \leq \frac{(b-a)^2}{2} \sum_{i=1}^{N-1} \Delta x_{i+1/2} |d\rho_{i+1/2}|^2 + \frac{1}{b-a} \left( \sum_{i=1}^N \Delta x_i \rho_i \right)^2.$$

Integrating over time, we get

$$\int_0^T \|\rho_h(t)\|_{L^2}^2 dt \leq (b-a)^2 \int_0^T \sum_{i=1}^{N-1} \Delta x_i |d\rho_{i+1/2}(t)|^2 dt + \frac{m_1^2 T}{b-a},$$

by Corollary 3.3, the conservation of mass, and by application of Lemma 3.1. The same argument applies for the second species,  $\eta_h$ , which concludes the proof.  $\square$

#### 4. PROOF OF THEOREM 2.4

This section is dedicated to proving compactness of both species, the fluxes, and the regularising porous-medium type diffusion. Upon establishing the compactness result we identify the limits as weak solutions in the sense of Definition 2.1.

First by application of Lemma 3.1, we get existence and uniqueness of a non-negative approximate solution  $(\rho_h, \eta_h)$  to (8a)-(8b). Hence the first item of Theorem 2.4 is proven. Now let us investigate the asymptotic  $h \rightarrow 0$ .

**4.1. Strong compactness of approximate solutions.** We shall now make use of the above estimates in order to obtain strong compactness of both species,  $(\rho_h, \eta_h)$  in  $L^2(Q_T)$ .

**Lemma 4.1** (Strong compactness in  $L^2(Q_T)$ ). *Let  $(\rho_h, \eta_h)_{h>0}$  be the approximation to system (1) obtained by the semi-discrete scheme (8). Then there exist functions  $\rho, \eta \in L^2(Q_T)$  such that*

$$\rho_h \rightarrow \rho, \quad \text{and} \quad \eta_h \rightarrow \eta,$$

strongly in  $L^2(Q_T)$ , up to a subsequence.

*Proof.* We invoke the compactness criterion by Aubin and Lions [22]. Accordingly, a set  $P \subset L^2(0, T; B)$  is relatively compact if  $P$  is bounded in  $L^2(0, T; X)$  and the set of derivatives  $\{\partial_t \rho | \rho \in P\}$  is bounded in a third space  $L^1(0, T; Y)$ , whenever the involved Banach spaces satisfy  $X \hookrightarrow Y \hookrightarrow B \hookrightarrow Y$ , i.e. the first embedding is compact and the second one continuous. For our purpose we

choose  $X := BV(a, b)$ ,  $B := L^2(a, b)$ , and  $Y := H^{-2}(a, b)$ . The first embedding is indeed compact, *e.g.* Ref. [1, Theorem 10.1.4] and the second one is continuous.

In the second step we show the time derivatives are bounded in  $L^1(0, T; H^{-2}(a, b))$ . To this end, let  $\varphi \in C_c^\infty((a, b))$ . Throughout, we write  $\langle \cdot, \cdot \rangle$  for  $\langle \cdot, \cdot \rangle_{H^{-2}, H^2}$  for the dual pairing. Making use of the scheme, there holds

$$\left\langle \frac{d\rho_h}{dt}, \varphi \right\rangle = \sum_{i=1}^N \int_{C_i} \frac{d\rho_i}{dt} \varphi \, dx = - \sum_{i=1}^N \frac{\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}}{\Delta x_i} \int_{C_i} \varphi \, dx,$$

having used the scheme, Eq. (8a). Next we set

$$\varphi_i := \frac{1}{\Delta x_i} \int_{C_i} \varphi \, dx,$$

perform a discrete integration by parts and use the no-flux boundary conditions, Eq. (8c), to obtain

$$\left\langle \frac{d\rho_h}{dt}, \varphi \right\rangle = \sum_{i=1}^{N-1} \mathcal{F}_{i+1/2} (\varphi_{i+1} - \varphi_i).$$

Using the definition of the numerical flux, Eq. (8b), we may simplify this expression further to get

$$\begin{aligned} \left\langle \frac{d\rho_h}{dt}, \varphi \right\rangle &= \sum_{i=1}^{N-1} \left[ \left( \nu(dU)_{i+1/2}^+ + (dV_1)_{i+1/2}^+ \right) \rho_i + \left( \nu(dU)_{i+1/2}^- + (dV_1)_{i+1/2}^- \right) \rho_{i+1} \right] (\varphi_{i+1} - \varphi_i) \\ &\quad - \frac{\epsilon}{2} \sum_{i=1}^{N-1} \frac{\rho_{i+1}^2 - \rho_i^2}{\Delta x_{i+1/2}} (\varphi_{i+1} - \varphi_i) \end{aligned}$$

Let us begin with the self-diffusion part. Using the Cauchy-Schwarz inequality, we estimate the discrete gradient and  $\rho$  itself by Corollary 3.3 and Lemma 3.4

(18)

$$\begin{aligned} &\frac{\epsilon}{2} \int_0^T \sum_{i=1}^{N-1} \frac{\rho_{i+1}^2 - \rho_i^2}{\Delta x_{i+1/2}} (\varphi_{i+1} - \varphi_i) \, dt \\ &\leq \frac{\epsilon}{2} \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^\infty} \int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2} |d\rho_{i+1/2}| (\rho_{i+1} + \rho_i) \, dt \\ &\leq \frac{\epsilon}{2} \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^\infty} \left( \int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2} |d\rho_{i+1/2}|^2 \, dt \right)^{1/2} \left( \int_0^T \sum_{i=1}^N 2\Delta x_{i+1/2} (|\rho_i|^2 + |\rho_{i+1}|^2) \, dt \right)^{1/2} \\ &\leq \frac{\epsilon}{2} \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^\infty} \left( \int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2} |d\rho_{i+1/2}|^2 \, dt \right)^{1/2} \left( \int_0^T \sum_{i=1}^N 4\xi^{-1} \Delta x_i |\rho_i|^2 \, dt \right)^{1/2} \\ &\leq \frac{\epsilon}{\sqrt{\xi}} \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^\infty} \left( \int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2} |d\rho_{i+1/2}|^2 \, dt \right)^{1/2} \left( \int_0^T \sum_{i=1}^N \Delta x_i |\rho_i|^2 \, dt \right)^{1/2} \\ &\leq C \|\varphi\|_{H^2(a,b)}, \end{aligned}$$

where we used that  $\varphi' \in H^1 \subset L^\infty$ .

Next, we address the cross-diffusion and non-local interactions terms using the same argument. For instance for the cross-diffusive part, we have

$$\begin{aligned} & \nu \int_0^T \left| \sum_{i=1}^{N-1} \left[ (dU)_{i+1/2}^+ \rho_i + (dU)_{i+1/2}^- \rho_{i+1} \right] \right| |\varphi_{i+1} - \varphi_i| dt \\ & \leq \frac{2\nu}{\sqrt{\xi}} \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^\infty} \left( \int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2} |dU_{i+1/2}|^2 dt \right)^{1/2} \left( \int_0^T \sum_{i=1}^N \Delta x_i \rho_i^2 dt \right)^{1/2} \\ & \leq C \|\varphi\|_{H^2(a,b)}, \end{aligned}$$

where we used Corollary 3.3 and Lemma 3.4 again. The non-local interaction term is estimated in the same way, thus there holds

$$\int_0^T \left| \left\langle \frac{d\rho_h}{dt}, \varphi \right\rangle \right| dt \leq C \|\varphi\|_{H^2(a,b)}.$$

By density of  $C_c^\infty((a, b))$  in  $H_0^2(a, b)$  we may infer the boundedness of  $(\frac{d\rho_h}{dt})_{h>0}$  in  $L^1(0, T; H^{-2}(a, b))$ , which concludes the proof.  $\square$

From the latter result we can prove the convergence of the discrete advection field  $dV_{1,h}$  and  $dV_{2,h}$  defined as in Definition 2.3.

**Lemma 4.2.** *For any  $1 \leq p \leq \infty$  and  $k \in \{1, 2\}$ , the piecewise constant approximation  $dV_{k,h}$  converges strongly in  $L^2(0, T; L^2(a, b))$  to  $-(W'_{k1} \star \rho + W'_{k2} \star \eta)$ , where  $(\rho, \eta)$  corresponds to the limit obtained in Lemma 4.1.*

*Proof.* Let  $k \in \{1, 2\}$ . For each  $i = 0, \dots, N-1$ , and  $x \in [x_i, x_{i+1})$  we have

$$\begin{aligned} dV_{k,h}(x) = (dV_k)_{i+1/2} &= - \sum_{j=1}^N \int_{C_j} \frac{W_{k1}(x_{i+1} - y) - W_{k1}(x_i - y)}{\Delta x_{i+1/2}} \rho_j dy, \\ &\quad - \sum_{j=1}^N \int_{C_j} \frac{W_{k2}(x_{i+1} - y) - W_{k2}(x_i - y)}{\Delta x_{i+1/2}} \eta_j dy. \end{aligned}$$

We define  $V'_{k,h}$  and  $V'_k$  as

$$\begin{cases} V'_{k,h}(x) & := -W'_{k1} \star \rho_h - W'_{k2} \star \eta_h, \\ V'_k(x) & := -W'_{k1} \star \rho - W'_{k2} \star \eta. \end{cases}$$

On the one hand from the strong convergence of  $(\rho_h, \eta_h)$  to  $(\rho, \eta)$  in  $L^2(0, T; L^2(a, b))$  and the convolution product's properties, we obtain

$$(19) \quad \|V'_{k,h} - V'_k\|_{L^2(0,T;L^2(a,b))} \rightarrow 0, \text{ when } h \rightarrow 0.$$

On the other hand, we have for any  $x \in [x_i, x_{i+1})$

$$\begin{aligned} |dV_{k,h}(x) - V'_{k,h}(x)| &\leq \sum_{j=1}^N \int_{C_j} \left| \frac{W_{k1}(x_{i+1}-y) - W_{k1}(x_i-y)}{\Delta x_{i+1/2}} - W'_{k1}(x-y) \right| \rho_j dy, \\ &\quad + \sum_{j=1}^N \int_{C_j} \left| \frac{W_{k2}(x_{i+1}-y) - W_{k2}(x_i-y)}{\Delta x_{i+1/2}} - W'_{k2}(x-y) \right| \eta_j dy, \\ &\leq (\|W''_{k1}\|_{L^\infty} m_1 + \|W''_{k2}\|_{L^\infty} m_2) h, \end{aligned}$$

hence there exists a constant  $C > 0$  such that

$$|dV_{k,h}(x) - V'_{k,h}(x)|^2 \leq C^p h^2.$$

Integrating over  $x \in [x_i, x_{i+1})$  and summing over  $i \in \{1, \dots, N-1\}$ , we get that

$$(20) \quad \|dV_{k,h} - V'_k\|_{L^2(0,T;L^2(a,b))} \rightarrow 0, \text{ when } h \rightarrow 0.$$

Notice that  $(x_1, x_N) \subset (a, b)$  where  $x_1 \rightarrow a$  and  $x_N \rightarrow b$  as  $h \rightarrow 0$ . From Eqs. (19) and (20) we get that  $\|dV_{k,h} - V'_k\|_{L^2(0,T;L^2(a,b))}$  goes to zero as  $h$  tends to zero.  $\square$

**4.2. Weak compactness for the discrete gradients.** In the previous section we have established the strong  $L^2$ -convergence of both species,  $(\rho_h)_{h>0}$  and  $(\eta_h)_{h>0}$ . However, in order to be able to pass to the limit in the cross-diffusion term  $\rho_h(d\rho_h + d\eta_h)$  we need to establish weak convergence in the discrete gradients in  $L^2$ . This is done in the following proposition.

**Proposition 1** (Weak convergence of the derivatives). *The discrete spatial derivatives, defined in Definition 2.3, satisfy  $d\beta_h$  converges weakly to  $\frac{\partial\beta}{\partial x}$  in  $L^2(Q_T)$  and  $\beta \in L^2(0, T; H^1(a, b))$ , where  $\beta \in \{\rho, \eta, U\}$*

*Proof.* Take  $\beta \in \{\rho, \eta, U\}$ , hence from Lemma 4.1, we know that  $\beta_h \rightarrow \beta$  strongly in  $L^2(Q_T)$ . Furthermore, from Corollary 3.3 we also deduce that  $d\beta_h$  weakly converges to some function  $r \in L^2(Q_T)$ .

Let us show that  $\beta \in L^2(0, T, H^1(a, b))$  and  $r = \frac{\partial\beta}{\partial x}$ . First, we have for any  $t \in [0, T]$  and any  $\varphi \in \mathcal{C}_c^\infty((0, T) \times (a, b))$ ,

$$\begin{aligned} \int_{Q_T} \beta_h(t) \frac{\partial\varphi}{\partial x} dx &= \int_0^T \sum_{i=1}^N \beta_i(t) [\varphi(t, x_{i+1/2}) - \varphi(t, x_{i-1/2})] dt \\ &= - \int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2} d\beta_{i+1/2}(t) \varphi(t, x_{i+1/2}) dt, \end{aligned}$$



having used discrete integration by parts and the fact that  $\varphi$  is compactly supported, *i.e.*  $\varphi(t, x_{N+1/2}) = \varphi(t, x_{1/2}) = 0$ . Then, by Definition 2.3 on the discrete gradient, we may consider

$$\begin{aligned}
& \left| \int_0^T \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} d\beta_{i+1/2} \varphi(t, x) dx dt + \int_0^T \int_a^b \beta_h \frac{\partial \varphi}{\partial x} dx dt \right| \\
& \leq \int_0^T \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} |d\beta_{i+1/2}| |\varphi(t, x) - \varphi(t, x_{i+1/2})| dx dt \\
& \leq \left\| \frac{\partial \varphi}{\partial x} \right\|_{\infty} \left( \int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2} |d\beta_{i+1/2}|^2 dt \right)^{1/2} \left( \int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2}^3 dt \right)^{1/2} \\
& \leq \left\| \frac{\partial \varphi}{\partial x} \right\|_{\infty} C^{1/2} T^{1/2} \sqrt{b-a} h,
\end{aligned}$$

This yields the statement, when  $h \rightarrow 0$ , for we have

$$(21) \quad \int_0^T \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} d\beta_{i+1/2} \varphi(t, x) dx dt + \int_0^T \int_a^b \beta_h \frac{\partial \varphi}{\partial x} dx dt \rightarrow 0,$$

which proves that  $d\beta_h$  converges weakly to  $\frac{\partial \beta}{\partial x}$ , as  $h \rightarrow 0$  and thus  $\beta \in L^2(0, T; H^1(a, b))$ .  $\square$

**4.3. Passing to the limit.** We have now garnered all information necessary to prove Theorem 2.4. For brevity we shall only show the convergence result for  $\rho$ , as it follows for  $\eta$  similarly, using the same arguments. Let  $\varphi \in C_c^\infty([0, T] \times (a, b))$  be a test function. We introduce the following notations:

$$\left\{ \begin{array}{l} \mathcal{E}_h := \int_0^T \int_a^b \rho_h \frac{\partial \varphi}{\partial t} dx dt + \int_a^b \rho_h(0) \varphi(0) dx, \\ \mathcal{A}_h := \int_0^T \int_a^b dV_{1,h} \rho_h \frac{\partial \varphi}{\partial x} dx dt, \\ \mathcal{C}_h := \nu \int_0^T \int_a^b dU_h \rho_h \frac{\partial \varphi}{\partial x} dx dt, \\ \mathcal{D}_h := \frac{\epsilon}{2} \int_0^T \int_a^b \rho_h^2 \frac{\partial^2 \varphi}{\partial x^2} dx dt. \end{array} \right.$$

and

$$\varepsilon(h) := \mathcal{E}_h + \mathcal{A}_h + \mathcal{C}_h + \mathcal{D}_h.$$

On the other hand, we set

$$\varphi_i(t) = \frac{1}{\Delta x_i} \int_{C_i} \varphi(t, x) dx,$$

and multiply the scheme, Eq. (8a), by the test function and integrate in time and space to get

$$(22) \quad \mathcal{E}_h + \mathcal{A}_{1,h} + \mathcal{C}_{1,h} + \mathcal{D}_{1,h} = 0,$$

where

$$\left\{ \begin{array}{l} \mathcal{A}_{1,h} := \sum_{i=1}^{N-1} \int_0^T \Delta x_{i+1/2} \left[ (dV_1)_{i+1/2}^+ \rho_i + (dV_1)_{i+1/2}^- \rho_{i+1} \right] d\varphi_{i+1/2}(t) dt, \\ \mathcal{C}_{1,h} := \nu \sum_{i=1}^{N-1} \int_0^T \Delta x_{i+1/2} \left[ (dU)_{i+1/2}^+ \rho_i + (dU)_{i+1/2}^- \rho_{i+1} \right] d\varphi_{i+1/2}(t) dt, \\ \mathcal{D}_{1,h} := -\frac{\epsilon}{2} \sum_{i=1}^{N-1} \int_0^T [\rho_{i+1}^2 - \rho_i^2] d\varphi_{i+1/2}(t) dt. \end{array} \right.$$

When  $h$  tends to zero and from the strong convergence of  $(\rho_h, \eta_h)_{h>0}$  to  $(\rho, \eta)$  in  $L^2(Q_T)$ , the strong convergence of  $(dV_{k,h})_{h>0}$  to  $V'_k$  in  $L^2(Q_T)$  and the weak convergence of the discrete gradient  $(dU_h)_{h>0}$  to  $-\frac{\partial \sigma}{\partial x}$  in  $L^2(Q_T)$ , it is easy to see that

$$\begin{aligned} \varepsilon(h) &\rightarrow \int_0^T \int_a^b \left\{ \rho \left[ \frac{\partial \varphi}{\partial t} + \left( \frac{\partial V_1}{\partial x} - \nu \frac{\partial}{\partial x} (\rho + \eta) \right) \frac{\partial \varphi}{\partial x} \right] + \frac{\epsilon}{2} \rho^2 \frac{\partial^2 \varphi}{\partial x^2} \right\} dx dt \\ &\quad + \int_a^b \rho(0) \varphi(0) dx, \end{aligned}$$

when  $h \rightarrow 0$ . Therefore it suffices to prove that  $\varepsilon(h) \rightarrow 0$ , as  $h$  goes to zero, which will be achieved by proving that  $\mathcal{A}_h - \mathcal{A}_{1,h}$ ,  $\mathcal{C}_h - \mathcal{C}_{1,h}$  and  $\mathcal{D}_h - \mathcal{D}_{1,h}$  vanish in the limit  $h \rightarrow 0$ .

*The self-diffusion part  $\mathcal{D}_h - \mathcal{D}_{1,h}$ .* On the one hand, after a simple integration we get

$$\begin{aligned} \mathcal{D}_h &= \frac{\epsilon}{2} \sum_{i=1}^N \int_0^T \rho_i^2(t) \left[ \frac{\partial \varphi}{\partial x}(t, x_{i+1/2}) - \frac{\partial \varphi}{\partial x}(t, x_{i-1/2}) \right] dt \\ &= -\frac{\epsilon}{2} \sum_{i=1}^{N-1} \int_0^T [\rho_{i+1}^2(t) - \rho_i^2(t)] \frac{\partial \varphi}{\partial x}(t, x_{i+1/2}) dt. \end{aligned}$$

Hence, we have

$$\mathcal{D}_h - \mathcal{D}_{1,h} = -\frac{\epsilon}{2} \sum_{i=1}^{N-1} \int_0^T [\rho_{i+1}^2(t) - \rho_i^2(t)] \left[ \frac{\partial \varphi}{\partial x}(t, x_{i+1/2}) - d\varphi_{i+1/2}(t) \right] dt$$

and observing that

$$\left| \frac{\partial \varphi}{\partial x}(t, x_{i+1/2}) - d\varphi_{i+1/2}(t) \right| \leq \left\| \frac{\partial^2 \varphi}{\partial x^2} \right\|_{L^\infty} h,$$

we obtain, in conjunction with the Cauchy-Schwarz inequality and the *a priori* bounds established in Corollary 3.3, that and Lemma 3.4,

$$\begin{aligned} (23) \quad |\mathcal{D}_h - \mathcal{D}_{1,h}| &\leq \frac{\epsilon}{2} \left\| \frac{\partial^2 \varphi}{\partial x^2} \right\|_{L^\infty} \left( \sum_{i=1}^{N-1} \int_0^T \Delta x_{i+1/2} |d\rho_{i+1/2}|^2 dt \right)^{1/2} \frac{2 \|\rho_h\|_{L^2(Q_T)}}{\xi^{1/2}} h \\ &\leq C h, \end{aligned}$$

in the virtue of the estimate Eq. (18).

*The cross-diffusion part.* Let us now treat the cross-diffusion part. This term is more complicated since it involves the piecewise constant functions  $\rho_h$  and  $dU_h$ , which are not defined on the same mesh. Thus, on the one hand we reformulate the discrete cross-diffusion term  $\mathcal{C}_{1,h}$  as  $\mathcal{C}_{1,h} = \mathcal{C}_{10,h} + \mathcal{C}_{11,h}$  with

$$\mathcal{C}_{10,h} = \nu \sum_{i=1}^{N-1} \int_0^T \Delta x_{i+1/2} (dU)_{i+1/2}^- [\rho_{i+1} - \rho_i] d\varphi_{i+1/2}(t) dt$$

and

$$\mathcal{C}_{11,h} = \nu \sum_{i=1}^{N-1} \int_0^T \Delta x_{i+1/2} \rho_i dU_{i+1/2} d\varphi_{i+1/2}(t) dt,$$

where a direct computation and the application of Corollary 3.3 and Lemma 3.4 yield

$$(24) \quad \begin{aligned} |\mathcal{C}_{10,h}| &\leq \nu \left\| \frac{\partial \varphi}{\partial x} \right\|_{L^\infty} \|dU_h\|_{L^2(Q_T)} \|d\rho_h\|_{L^2(Q_T)} h \\ &\leq C h. \end{aligned}$$

On the other hand, the term  $\mathcal{C}_h$  can be rewritten as

$$\mathcal{C}_h = \nu \int_0^T \sum_{i=1}^{N-1} dU_{i+1/2}(t) \int_{x_i}^{x_{i+1}} \rho_h \frac{\partial \varphi}{\partial x} dx dt.$$

Since

$$\begin{aligned} \int_{x_i}^{x_{i+1}} \rho_h \frac{\partial \varphi}{\partial x} dx &= \rho_i [\varphi(t, x_{i+1/2}) - \varphi(t, x_i)] + \rho_{i+1} [\varphi(t, x_{i+1}) - \varphi(t, x_{i+1/2})], \\ &= [\rho_i - \rho_{i+1}] [\varphi(t, x_{i+1/2}) - \varphi(t, x_i)] + \rho_{i+1} [\varphi(t, x_{i+1}) - \varphi(t, x_i)], \end{aligned}$$

the term  $\mathcal{C}_h$  can be decomposed as  $\mathcal{C}_h = \mathcal{C}_{00,h} + \mathcal{C}_{01,h}$  with

$$\mathcal{C}_{00,h} = -\nu \int_0^T \sum_{i=1}^{N-1} dU_{i+1/2} [\rho_{i+1} - \rho_i] [\varphi(t, x_{i+1/2}) - \varphi(t, x_i)] dt$$

and

$$\mathcal{C}_{01,h} = \nu \int_0^T \sum_{i=1}^{N-1} dU_{i+1/2} \rho_i [\varphi(t, x_{i+1}) - \varphi(t, x_i)] dt.$$

Similarly to (24), the first term  $\mathcal{C}_{00,h}$  can be estimated as

$$(25) \quad |\mathcal{C}_{00,h}| \leq C h,$$

whereas the second term  $\mathcal{C}_{01,h}$  is compared to  $\mathcal{C}_{11,h}$

$$|\mathcal{C}_{01,h} - \mathcal{C}_{11,h}| \leq \nu \int_0^T \sum_{i=1}^{N-1} \Delta x_{i+1/2} |dU_{i+1/2}| \rho_i \left| \frac{\varphi(t, x_{i+1}) - \varphi(t, x_i)}{\Delta x_{i+1/2}} - d\varphi_{i+1/2}(t) \right| dt.$$

Using a second order Taylor expansion of  $\varphi$  at  $x_i$  and  $x_{i+1}$ , it yields that

$$\left| \frac{\varphi(t, x_{i+1}) - \varphi(t, x_i)}{\Delta x_{i+1/2}} - d\varphi_{i+1/2}(t) \right| \leq C h,$$

hence we get from Corollary 3.3 and Lemma 3.4 that

$$(26) \quad |\mathcal{C}_{01,h} - \mathcal{C}_{11,h}| \leq C h.$$

from Corollary 3.3 and Lemma 3.4. Gathering Eqs. (24), (25), and (26), we finally obtain that

$$(27) \quad |\mathcal{C}_h - \mathcal{C}_{1,h}| \leq C h.$$

*The advective part.* The evaluation of  $\mathcal{A}_h - \mathcal{A}_{1,h}$  is along the same lines of the cross-diffusion terms  $\mathcal{C}_h - \mathcal{C}_{1,h}$  since the latter is treated as an advective term. Hence, thanks to Lemma 4.2, we get that

$$(28) \quad |\mathcal{A}_h - \mathcal{A}_{1,h}| \leq C h.$$

Finally by definition of  $\varepsilon(h)$  and using Eq. (22) together with Eqs. (23), (27), and (28), we obtain

$$\begin{aligned} |\varepsilon(h)| &= |-(\mathcal{A}_{1,h} + \mathcal{C}_{1,h} + \mathcal{D}_{1,h}) + \mathcal{A}_h + \mathcal{C}_h + \mathcal{D}_h| \\ &\leq |\mathcal{A}_h - \mathcal{A}_{1,h}| + |\mathcal{C}_h - \mathcal{C}_{1,h}| + |\mathcal{D}_h - \mathcal{D}_{1,h}| \\ &\leq C h, \end{aligned}$$

that is,  $\varepsilon(h) \rightarrow 0$ , when  $h \rightarrow 0$ , which proves that  $(\rho, \eta)$  is a weak solution to Eq. (1). This proves the second item of Theorem 2.4.

Finally the last item concerning the existence of solutions to (1) is a direct consequence of the convergence.

## 5. NUMERICAL EXAMPLES AND VALIDATION

In this section we perform some numerical simulations of system (1) using our scheme, Eqs. (8). In Section 5.1 we test our scheme by computing the error between the numerical simulation and a benchmark solution on a finer grid. Furthermore we determine the numerical convergence order. In Section 5.2 we compute the numerical stationary states of system (1) and discuss the implication of different cross-diffusivities and the self-diffusivities, respectively.

Let us note here, that in the case of no regularising porous-medium diffusion, *i.e.*  $\epsilon = 0$ , and certain singular potentials the stationary states of system (1) are even known explicitly [18]. This allows us to compare the numerical solution directly to the analytical stationary state in Section 5.2.1. Throughout the remainder of this section we apply the scheme (8) to system (1) using different self-diffusions,  $\epsilon$ , and cross-diffusions,  $\nu$ .

**5.1. Error and numerical order of convergence.** This section is dedicated to validating our main convergence result, Theorem 2.4. Due to the lack of explicit solutions we compute the numerical solution on a fine grid and consider it a benchmark solution. We then compute numerical approximations on coarser grids and study the error in order to obtain the numerical convergence order.

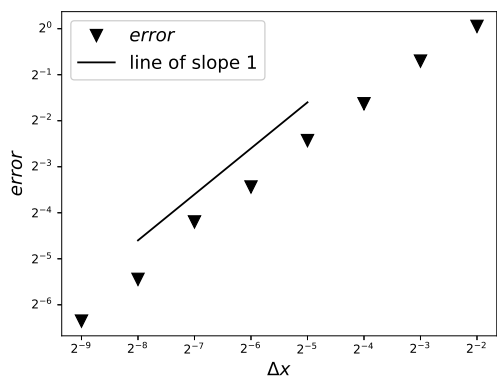
In all our simulations we use a fourth order Runge-Kutta scheme to solve the ordinary differential equations Eqs. (8) with initial data Eqs. (5). The discrepancy between the benchmark solution and numerical solutions on coarser grids is measured by the error

$$e := \left( \Delta t \sum_{k=1}^M \left( \Delta x \sum_{i=1}^N |\rho_{\text{ex}}(t_k, x_i) - \rho(t_k, x_i)|^2 + |\eta_{\text{ex}}(t_k, x_i) - \eta(t_k, x_i)|^2 \right) \right)^{1/2}.$$

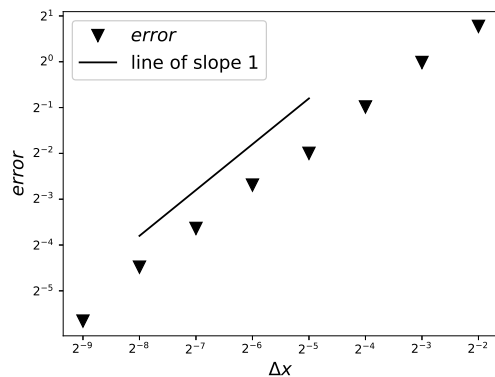
Here  $\rho_{\text{ex}}, \eta_{\text{ex}}$  denote the benchmark solutions. We use this quantity to study the convergence of our scheme as the grid size decreases.

5.1.1. *No non-local interactions.* Let us begin with the purely diffusive system. We consider system (1) without any interactions, *i.e.*  $W_{ij} \equiv 0$ , for  $i, j = 1, 2$ , and we choose  $\nu = 0.5$  and  $\epsilon = 0.1$ .

In Figure 2 we present the convergence result as we decrease the grid size. We computed a benchmark solution on a grid of  $\Delta x = 2^{-10}$  on the time interval  $[0, 10]$  with  $\Delta t = 0.05$ . Figure 2(a) shows the convergence for symmetric initial data whereas Figure 2(b) shows the convergence of the same system in case of asymmetric initial data. In both cases we overlay a line of slope one and we conclude that the numerical convergence is of order one.



(a) Symmetric initial data.  $\rho_0(x) = \eta_0(x) = \mathbb{1}_{[7,10]}$ .



(b) Asymmetric initial data.  $\rho_0(x) = \mathbb{1}_{[5,7]}$  and  $\eta_0(x) = \mathbb{1}_{[10,12]}$ .

FIGURE 2. In the purely diffusive system all interaction kernels are set to zero. Both graphs show the convergence to the benchmark solution. The triangular markers denote the discrepancy between the numerical solution and the benchmark solution. A line of slope one is superimposed for the ease of comparison. On the left we start the system with symmetric initial conditions, on the right we start with asymmetric initial data. In both cases the scheme has numerical convergence order 1.

5.1.2. *Gaussian cross-interactions.* Next, we add non-local self-interaction and cross-interactions. We choose smooth Gaussians with different variances. These potentials, like the related, more singular Morse potentials, are classical in mathematical biology since oftentimes the availability of sensory information such as sight, smell or hearing is spatially limited [26, 19, 17]. For the intraspecific interaction we use

$$W_{11}(x) = W_{22}(x) = 1 - \exp\left(-\frac{|x|^4}{4 \times 0.1}\right),$$

while we choose

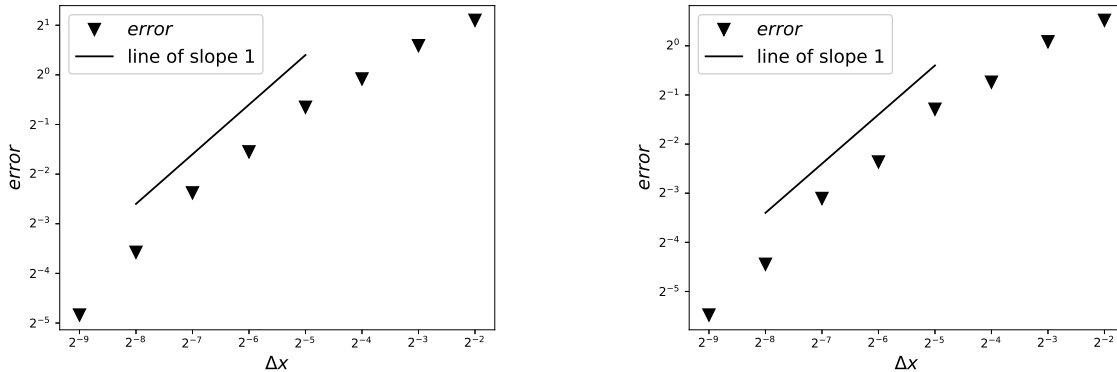
$$W_{12}(x) = -W_{21}(x) = 1 - \exp\left(-\frac{|x|^2}{2 \times 0.1}\right),$$

for the interspecific interactions.

We consider system (1) with the diffusive coefficients  $\nu = 0.4$  and  $\epsilon \in \{0.1, 0.5\}$ , and we initialise the system with

$$\rho(x) = \eta(x) = c((s - 6.5)(9.5 - s))^+,$$

on the domain  $[0, 9]$ . Here the constant  $c$  is such that  $\rho$  and  $\eta$  have unit mass. Figure 3 depicts the simulation with Gaussian kernels as interaction potentials. In Figures 3(a) & 3(b) we present the error plots corresponding to  $\epsilon \in \{0.1, 0.5\}$ . Again we observe convergence to the reference solution with a first order accuracy in space.



(a) Convergence to benchmark solution for individual diffusion constant  $\epsilon = 0.1$ .

(b) Convergence to benchmark solution for individual diffusion constant  $\epsilon = 0.5$ .

FIGURE 3. We choose Gaussian interaction kernels of different strengths and ranges for the self-interaction and the cross-interaction, respectively. The graphs show the numerical convergence order in the cases of  $\epsilon = 0.1$  (left), and  $\epsilon = 0.5$  (right), respectively.

**5.2. General behaviour of solutions and stationary states.** In this section we aim to study the asymptotic behaviour of system (1) numerically. Let us begin by going back to the set up of Section 5.1.2. We note that the potentials were chosen in such a way that there is an attractive intraspecific force, and the cross-interactions are chosen as attractive-repulsive explaining the segregation observed in Figure 4(a). For larger self-diffusivity,  $\epsilon = 0.5$ , we see that some mixing occurs. In the absence of any individual diffusion we would have expected adjacent species with a jump discontinuity at their shared boundary [18]. However, this phenomenon is no longer possible as we have a control on the gradients of each individual species, by Lemma 3.3, rendering jumps impossible thus explaining the continuous transition.

In the subsequent section we shall push our scheme even further by dropping the smoothness assumption on our potentials.

**5.2.1. Case of singular potentials.** In this section we go beyond the limit of what we could prove in this paper. On the one hand we consider more singular potentials and on the other hand we

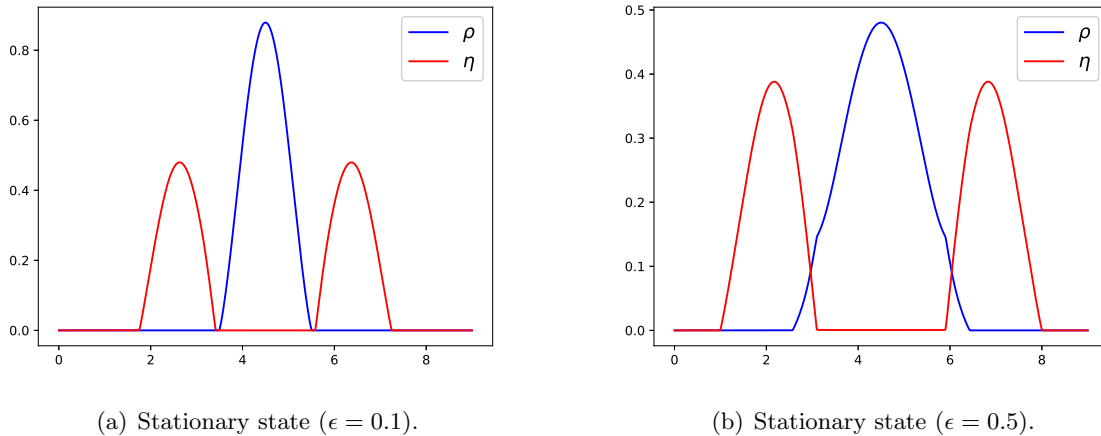


FIGURE 4. We choose Gaussian interaction kernels of different strengths and ranges for the self-interaction and the cross-interaction, respectively. The graphs correspond to the simulated stationary states for self-diffusivities  $\epsilon = 0.1$ , and  $\epsilon = 0.5$ , respectively.

consider vanishing individual diffusion. We study system (1) for  $\epsilon \in \{0, 0.02, 0.04, 0.06, 0.09\}$ , and  $\nu \in \{0.05, 0.5\}$ . Here the potentials are given by

$$W_{11}(x) = W_{22}(x) = x^2/2.$$

for the self-interaction terms and

$$W_{12}(x) = |x| = \pm W_{21}(x).$$

for the cross-interactions. The system is posed on the domain  $[0, 5]$  with a grid size of  $\Delta x = 2^{-8}$ . Note that the case of  $\epsilon = 0$  corresponds to the absence of individual diffusion, see Figures 5(a) & 5(c). By virtue of Corollary 3.3, it is the individual diffusion that regularises the stationary states, in the sense that we will not observe any discontinuities in either  $\rho$  or  $\eta$ . As we add individual diffusion we can see the immediate regularisation. While stationary states may still remain segregated, as it is shown in Figure 5(b), adjacent solutions are not possible anymore (see Figure 5(d)). In the case of attractive-repulsive interspecific interactions, *i.e.*

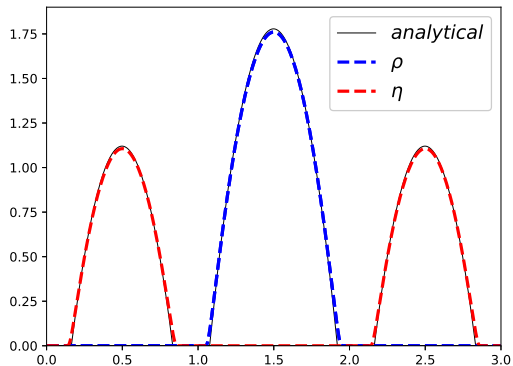
$$W_{12}(x) = |x| = -W_{21}(x).$$

we expect both species to segregate [18]. We initialise the system with the following symmetric initial data

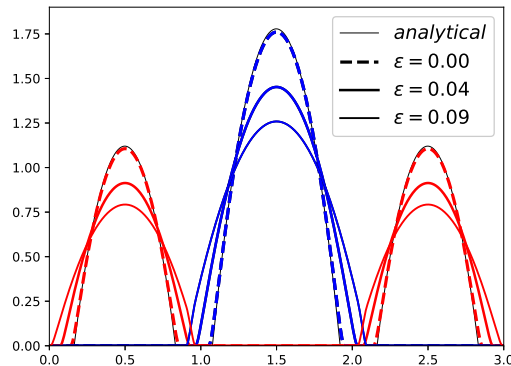
$$\rho(x) = \eta(x) = c((x-3)(5-x))^+,$$

as symmetric initial data are known to approach stationary states [18]. Here the constant  $c$  normalises the mass of  $\rho$  and  $\eta$  to one.

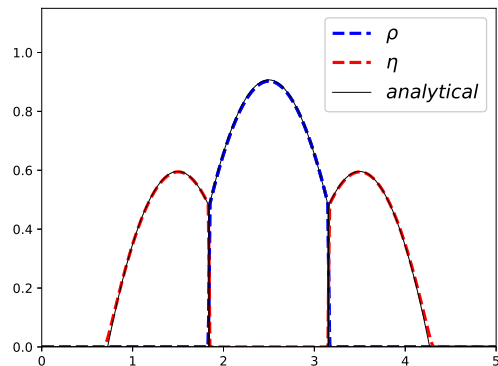
Figures 5(a) & 5(c) show our scheme performs well even in regimes we are unable to show convergence due to the lack of regularity in the potentials as well as the lack of regularity due to the absence of the porous medium type self-diffusion. We reproduce the steady states of [18] that exhibit phase separation phenomena.



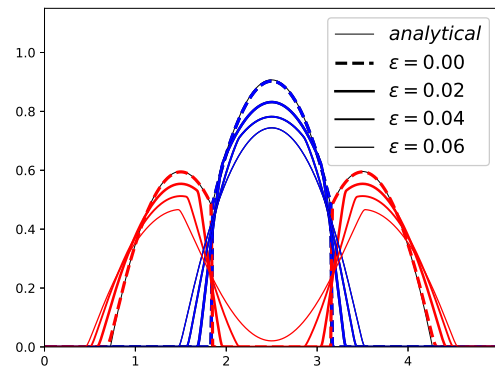
(a) In the case  $\nu = 0.05$ ,  $\epsilon = 0$ , we obtain a great agreement of the numerically computed stationary states and the analytical stationary states described in [18].



(b) Adding individual diffusion may still lead to segregated stationary states. However both species remain continuous as they mix.



(c) The case  $\nu = 0.5$ ,  $\epsilon = 0$  leads to adjacent stationary states. Again we see an excellent agreement of the numerical stationary states and the analytical ones [18].



(d) The regularising effect of the individual diffusion, by Corollary 3.3, becomes apparent immediately. Instantaneously both species become continuous as they start to intermingle in a small region. This region grows as we keep increasing the individual diffusion.

FIGURE 5. We pushed our numerical scheme to see how it performs in regimes in which we are unable to prove convergence. We chose Newtonian cross-interactions in the attractive-repulsive case. The red curves denote the symmetric stationary states of  $\eta$  while the blue curves are the stationary distributions of  $\rho$ . The different line widths and styles correspond to varying values of  $\epsilon$ .

In the case of attractive-attractive self-interactions, *i.e.*  $W_{12}(x) = |x| = W_{21}(x)$ , we observe an interesting phenomenon. Even in the absence of the individual diffusion, *i.e.*  $\epsilon = 0$ , some mixing occurs, see Figure 6, due to numerical diffusion. This is in contrast to the finite volume schemes proposed in [13, 18].



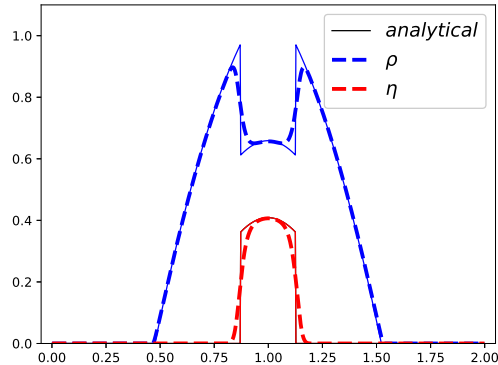


FIGURE 6. We choose  $m_1 = 0.6$  and  $m_2 = 0.1$  in order to be able to compare the stationary state with the explicit one given in [18]. We can see a strong resemblance between the numerical stationary state and the one obtained analytically. However there are some regimes of mixing due to numerical diffusion.

## 6. CONCLUSION

In this paper we presented a finite volume scheme for a system of non-local partial differential equations with cross-diffusion. We were able to reproduce a continuous energy estimate on the discrete level for our scheme. These discrete estimates for the approximate solution are enough to get compactness results and we are able to identify the limit of the approximate solutions as a weak solution of the equation. We complement the analytical part with numerical simulations. These back up our convergence result and we are also able to apply the scheme in cases in which we cannot show convergence. To this end we pushed the scheme to regimes of singular potentials also lacking the regularising porous medium type self-diffusion terms. Comparing them with the explicit stationary states from [18] we conclude the scheme performs well even in regimes it was not designed for.

## ACKNOWLEDGEMENTS

JAC was partially supported by EPSRC grant number EP/P031587/1.

## REFERENCES

- [1] H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces: applications to PDEs and optimization*. SIAM, 2014.
- [2] J. Berendsen, M. Burger, and J.-F. Pietschmann. On a cross-diffusion model for multiple species with nonlocal interaction and size exclusion. *Nonlinear Analysis*, 159:10–39, 2017.
- [3] M. Bertsch, R. Dal Passo, and M. Mimura. A free boundary problem arising in a simplified tumour growth model of contact inhibition. *Interfaces and Free Boundaries*, 12(2):235–250, 2010.
- [4] M. Bertsch, M. Gurtin, and H. D. On a degenerate diffusion equation of the form  $c(z)_t = \varphi(z_x)_x$  with application to population dynamics. *Journal of differential equations*, 67(1):56–89, 1987.
- [5] M. Bertsch, M. Gurtin, H. D., and L. Peletier. On interacting populations that disperse to avoid crowding: preservation of segregation. *Journal of mathematical biology*, 23(1):1–13, 1985.
- [6] M. Bertsch, M. Gurtin, and D. Hilhorst. On interacting populations that disperse to avoid crowding: the case of equal dispersal velocities. *Nonlinear Analysis: Theory, Methods & Applications*, 11(4):493–499, 1987.

- [7] M. Bertsch, D. Hilhorst, H. Izuhara, and M. Mimura. A nonlinear parabolic-hyperbolic system for contact inhibition of cell-growth. *Differ. Equ. Appl*, 4(1):137–157, 2012.
- [8] M. Bessemoulin-Chatard, C. Chainais-Hillairet, and F. Filbet. On discrete functional inequalities for some finite volume schemes. *IMA J. Num. Analysis*, 35(3):1125–1149, 2015.
- [9] M. Bessemoulin-Chatard and F. Filbet. A finite volume scheme for nonlinear degenerate parabolic equations. *SIAM Journal on Scientific Computing*, 34(5):B559–B583, 2012.
- [10] M. Bruna, M. Burger, H. Ranetbauer, and M.-T. Wolfram. Cross-diffusion systems with excluded-volume effects and asymptotic gradient flow structures. *Journal of Nonlinear Science*, 27(2):687–719, 2017.
- [11] M. Burger, M. Di Francesco, S. Fagioli, and A. Stevens. Sorting phenomena in a mathematical model for two mutually attracting/repelling species. *preprint arXiv:1704.04179*, 2017.
- [12] V. Calvez and J. A. Carrillo. Volume effects in the Keller-Segel model: energy estimates preventing blow-up. *J. Math. Pures Appl. (9)*, 86(2):155–175, 2006.
- [13] J. A. Carrillo, A. Chertock, and Y. Huang. A finite-volume method for nonlinear nonlocal equations with a gradient flow structure. *Communications in Computational Physics*, 17(01):233–258, 2015.
- [14] J. A. Carrillo, Y.-P. Choi, and M. Hauray. The derivation of swarming models: mean-field limit and wasserstein distances. In *Collective dynamics from bacteria to crowds*, pages 1–46. Springer, 2014.
- [15] J. A. Carrillo, A. Colombi, and M. Scianna. Adhesion and volume constraints via nonlocal interactions lead to cell sorting. *preprint arXiv:1706.08969*, 2017.
- [16] J. A. Carrillo, S. Fagioli, F. Santambrogio, and M. Schmidtchen. Splitting schemes & segregation in reaction-(cross)-diffusion systems. *arXiv preprint arXiv:1711.05434*, 2017.
- [17] J. A. Carrillo, Y. Huang, and S. Martin. Explicit flock solutions for Quasi-Morse potentials. *European J. Appl. Math.*, 25(5):553–578, 2014.
- [18] J. A. Carrillo, Y. Huang, and M. Schmidtchen. Zoology of a non-local cross-diffusion model for two species. *arXiv preprint arXiv:1705.03320*, 2017.
- [19] J. A. Carrillo, S. Martin, and V. Panferov. A new interaction potential for swarming models. *Phys. D*, 260:112–126, 2013.
- [20] J. A. Carrillo, R. J. McCann, C. Villani, et al. Kinetic equilibration rates for granular media and related equations: entropy dissipation and mass transportation estimates. *Revista Matemática Iberoamericana*, 19(3):971–1018, 2003.
- [21] C. Chainais-Hillairet and F. Filbet. Asymptotic behaviour of a finite-volume scheme for the transient drift-diffusion model. *IMA journal of numerical analysis*, 27(4):689–716, 2007.
- [22] X. Chen, A. Jüngel, and J.-G. Liu. A note on Aubin-Lions-Dubinskiĭ lemmas. *Acta Appl. Math.*, 133:33–43, 2014.
- [23] M. Di Francesco, A. Esposito, and S. Fagioli. Nonlinear degenerate cross-diffusion systems with nonlocal interaction. *Nonlinear Analysis*, 169:94–117, 2018.
- [24] M. Di Francesco and S. Fagioli. Measure solutions for non-local interaction pdes with two species. *Nonlinearity*, 26(10):2777, 2013.
- [25] P. Domschke, D. Trucu, A. Gerisch, and M. A. J. Chaplain. Mathematical modelling of cancer invasion: implications of cell adhesion variability for tumour infiltrative growth patterns. *J. Theoret. Biol.*, 361:41–60, 2014.
- [26] M. R. D’Orsogna, Y.-L. Chuang, A. L. Bertozzi, and L. S. Chayes. Self-propelled particles with soft-core interactions: patterns, stability, and collapse. *Physical review letters*, 96(10):104302, 2006.
- [27] R. Eymard, T. Gallouët, and R. Herbin. Finite volume methods. *Handbook of numerical analysis*, 7:713–1018, 2000.
- [28] A. Gerisch and M. A. J. Chaplain. Mathematical modelling of cancer cell invasion of tissue: Local and non-local models and the effect of adhesion. *Journal of Theoretical Biology*, 250(4):684–704, 2008.
- [29] M. E. Gurtin and A. Pipkin. A note on interacting populations that disperse to avoid crowding. *Quarterly of Applied Mathematics*, 42(1):87–94, 1984.
- [30] A. Mogilner and L. Edelstein-Keshet. A non-local model for a swarm. *Journal of Mathematical Biology*, 38(6):534–570, 1999.
- [31] H. Murakawa and H. Togashi. Continuous models for cell-cell adhesion. *Journal of theoretical biology*, 374:1–12, 2015.
- [32] K. J. Painter, J. M. Bloomfield, J. A. Sherratt, and A. Gerisch. A nonlocal model for contact attraction and repulsion in heterogeneous cell populations. *Bull. Math. Biol.*, 77(6):1132–1165, 2015.

- [33] K. J. Painter and T. Hillen. Volume-filling and quorum-sensing in models for chemosensitive movement. *Can. Appl. Math. Q.*, 10(4):501–543, 2002.
- [34] C. M. Topaz, A. L. Bertozzi, and M. A. Lewis. A nonlocal continuum model for biological aggregation. *Bulletin of mathematical biology*, 68(7):1601–1623, 2006.
- [35] A. Volkening and B. Sandstede. Modelling stripe formation in zebrafish: an agent-based approach. *Journal of The Royal Society Interface*, 12(112), 2015.

*E-mail address:* carrillo@imperial.ac.uk

*E-mail address:* francis.filbet@math.univ-toulouse.fr

*E-mail address:* m.schmidtchen15@imperial.ac.uk