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The orbit algebra of an oligomorphic permutation group with polynomial profile is Cohen-Macaulay

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Abstract. Let G be a group of permutations of a denumerable set E. The **profile** of G is the function φ_G which counts, for each n, the (possibly infinite) number $\varphi_G(n)$ of orbits of G acting on the n-subsets of E. Counting functions arising this way, and their associated generating series, form a rich yet apparently strongly constrained class. In particular, Cameron conjectured in the late seventies that, whenever $\varphi_G(n)$ is bounded by a polynomial, it is asymptotically equivalent to a polynomial. In 1985, Macpherson further asked if the **orbit algebra** of G – a graded commutative algebra invented by Cameron and whose Hilbert function is φ_G – is finitely generated.

In this paper we announce a proof of a stronger statement: the orbit algebra is Cohen-Macaulay. The generating series of the profile is a rational fraction whose numerator has positive coefficients and denominator admits a combinatorial description.

The proof uses classical techniques from group actions, commutative algebra, and invariant theory; it steps towards a classification of ages of permutation groups with profile bounded by a polynomial.

Keywords: Oligomorphic permutation groups, invariant theory, generating series

1 Introduction

Counting objects under a group action is a classical endeavor in algebraic combinatorics. If G is a permutation group acting on a finite set E, Burnside's lemma provides a formula for the number of orbits, while Pólya theory refines this formula to compute, for example, the **profile** of G, that is the function which counts, for each n, the number $\varphi_G(n)$ of orbits of G acting on subsets of size n of E.

In the seventies, Cameron initiated the study of the profile when G is instead a permutation group of an infinite set E. Of course the question makes sense mostly if $\varphi_G(n)$ is finite for all n; in that case, the group is called **oligomorphic**, and the infinite sequence $\varphi_G = (\varphi_G(n))_n$ an **orbital profile**. This setting includes, for example, counting integer partitions (with optional length and height restrictions) or graphs up to an isomorphism, and has become a whole research subject [4, 3]. One central topic is the description of general properties of orbital profiles. It was soon observed that the potential growth rates exhibited jumps. For example, the profile either grows at least as fast as the partition function, or is bounded by a polynomial [9, Theorem 1.2]. In the latter case, it was conjectured to be asymptotically polynomial:

Conjecture 1.1 (Cameron [4])**.** *Let G be an oligomorphic permutation group whose profile is bounded by a polynomial. Then $\varphi_G(n) \sim an^k$ for some $a > 0$ and $k \in \mathbb{N}$.*

As a tool in this study, Cameron introduced early on the **orbit algebra** $\mathbb{Q}\mathcal{A}(G)$ of G, a graded connected commutative algebra whose Hilbert function coincides with φ_G. Macpherson then asked the following

Question 1.2 (Macpherson [9] p. 286)**.** Let G be an oligomorphic permutation group whose profile is bounded by a polynomial. Is $\mathbb{Q}\mathcal{A}(G)$ finitely generated?

The point is that, by standard commutative algebra, whenever $\mathbb{Q}\mathcal{A}(G)$ is finitely generated, its Hilbert function is asymptotically polynomial, as conjectured by Cameron. It is in fact *eventually a quasi polynomial*. Equivalently, the generating series of the profile $\mathcal{H}_G = \sum_{n\in\mathbb{N}} \varphi_G(n) z^n$ is a rational fraction of the form

$$\mathcal{H}_G = \frac{P(z)}{\prod_{i\in I}(1 - z^{d_i})},$$

where $P(z)$ is a polynomial in $\mathbb{Z}[z]$ and the d_i's are the degrees of the generators.

In this paper, we report on a proof of Cameron's conjecture by answering positively to Macpherson's question, and even to a stronger question: is $\mathbb{Q}\mathcal{A}(G)$ Cohen-Macaulay?

Theorem 1.3 (Main Theorem)**.** *Let G be a permutation group whose profile is bounded by a polynomial. Then $\mathbb{Q}\mathcal{A}(G)$ is Cohen-Macaulay over a free subalgebra with generators of degrees $(d_i)_{i\in I}$ prescribed by the **block structure** of G.*

Corollary 1.4. *The generating series of the profile is of the (irreducible) form*

$$\mathcal{H}_G = \frac{P(z)}{\prod_{i\in I}(1 - z^{d_i})},$$

for some polynomial P in $\mathbb{N}[z]$; therefore $\varphi_G \sim an^{|I|-1}$ for some $a > 0$.

Investigating the Cohen-Macaulay property was inspired by the important special case of invariant rings of finite permutation groups. At this stage, we presume that this theorem can be obtained as a consequence of some classification result for ages of permutation groups with profile bounded by a polynomial. In addition, the orbit algebra would always be isomorphic to the invariant ring of some finite permutation group.

This research is part of a larger program initiated in the seventies: the study of the *profile of relational structures* [6, 10] and in general of the behavior of counting functions for hereditary classes of finite structures, like undirected graphs, posets, tournaments, ordered graphs, or permutations; see [8, 2] for surveys. For example, the analogue of Cameron's conjecture is proved in [1] for undirected graphs and in [7] for permutations.

In [11], the orbit algebra is proved to be integral (under a natural restriction). Cameron extends in [5] the definition of the orbit algebra to the general context of relational structures. The analogue of Theorem 1.3 holds when the profile is bounded (see [10, Theorem 26] and [12, Theorem 1.5]); it can fail as soon as the profile grows faster.

In a context which is roughly the generalization of transitive groups with a finite number of infinite blocks, the analogue of Cameron's conjecture holds [12, Theorem 1.7] and the finite generation admits a combinatorial characterization [13].

This paper is structured as follows. In Section 2, we review the basic definitions of orbit algebras and provide classical examples and operations. The central notion is that of block systems; we explain that each block system provides a lower bound on the growth of the profile and state the existence of a *canonical block system* $B(G)$ meant to maximize this lower bound. This block system splits into orbits of blocks; each such orbit consists either of infinitely many finite blocks or finitely many infinite blocks; there can be in addition some finite orbits of finite blocks; they form the *kernel* and are mostly harmless. The approach in the sequel is to treat each orbit of blocks separately, and then recombine the results.

In Section 3, we show that the finite generation property can be lifted from normal subgroups of finite index. This proves our Main Theorem when $B(G)$ consists of a single orbit of infinite blocks. This further enables to assume without loss of generality that the kernel is empty. In Section 4, we classify, up to taking subgroups of finite index, ages when $B(G)$ consists of a single infinite orbit of finite blocks. Finally, in Section 5, we combine the previous results to prove our Main Theorem in the general case.

Full proofs and additional examples and figures will be published in a long version of this extended abstract.

We would like to thank Maurice Pouzet for suggesting to work on this conjecture, and Peter Cameron and Maurice Pouzet for enlightening discussions.

2 Preliminaries

2.1 The age, profile and orbit algebra of a permutation group

Let G be a permutation group, that is a group of permutations of some set E. Unless stated otherwise, E is denumerable and G is infinite. The action of G on the elements of E induces an action on finite subsets. The **age** of G is the set $\mathcal{A}(G)$ of its orbits of finite subsets. Within an orbit, all subsets share the same cardinality, which is called the **degree** of the orbit. This gives a grading of the age according to the degree of the orbits:

$\mathcal{A}(G) = \sqcup_{n\in\mathbb{N}}\mathcal{A}(G)_n$. The **profile** of G is the function $\varphi_G : n \mapsto |\mathcal{A}(G)_n|$. In general, the profile may take infinite values; the group is called **oligomorphic** if it does not.

We call the growth rate of a profile bounded by a polynomial the smallest number r satisfying $\phi(n) = O(n^r)$ (for instance, the growth rate of $n^2 + n$ is 2). By extension, we speak of the **growth rate of a permutation group** (which is that of its profile).

Definition 2.1. We say that a permutation group is P**-oligomorphic** if its profile is bounded by a polynomial.

Examples 2.2. Let G be the infinite symmetric group $\mathfrak{S}_\infty$. For each n there is a single orbit containing all subsets of size n, hence $\varphi_G(n) = 1$. We say that G *has profile* 1.

Now take $E = E_1 \sqcup E_2$, where E_1 and E_2 are two copies of $\mathbb{N}$. Let G be the group acting on E by permuting the elements independently within E_1 and E_2 and by exchanging E_1 and E_2: G is the **wreath product** $\mathfrak{S}_\infty \wr \mathfrak{S}_2$. In that case, the orbits of subsets of cardinality n are in bijection with integer partitions of n with at most two parts.

The **orbit algebra** of G, as defined by Cameron, is the graded connected vector space $\mathbb{Q}\mathcal{A}(G)$ of formal finite linear combinations of elements of $\mathcal{A}(G)$, endowed with a commutative product as follows: embed $\mathbb{Q}\mathcal{A}(G)$ in the vector space of formal linear combinations of finite subsets of E by mapping each orbit to the sum of its elements; endow the latter with the disjoint product that maps two finite subsets to their union if they are disjoint and to 0 otherwise. Some care is to be taken to check that all is well defined.

2.2 Block systems and primitive groups

A key notion when studying permutation groups is that of **block systems**; they are the discrete analogues of quotient modules in representation theory. A **block system** is a partition of E into parts, called **blocks**, such that each $g \in G$ maps blocks onto blocks. The partitions $\{E\}$ and $\{\{e\} \mid e \in E\}$ are always block systems and are therefore called the trivial block systems. A permutation group is **primitive** if it admits no non trivial block system. By extension, an orbit of elements is **primitive** if the restriction of the group to this orbit is primitive. A block system is **transitive** if G acts transitively on its blocks; in this case, all the blocks are conjugated and thus share the same cardinality.

The collection of block systems of a permutation group forms a lattice with respect to the refinement order, with the two trivial block systems as top and bottom respectively.

The following two theorems will be central in our study.

Theorem 2.3 (Macpherson [9])**.** *The profile of an oligomorphic primitive permutation group is either* 1 *or grows at least as fast as the partition function.*

In our study of Macpherson's question, all groups have profile bounded by a polynomial, and therefore primitive groups always have profile 1.

Theorem 2.4 (Cameron [4])**.** *There are only five complete permutation groups with profile* 1*:*

1. *The automorphism group* $\mathrm{Aut}(\mathbb{Q})$ *of the rational chain (order-preserving bijections on* $\mathbb{Q}$*);*
2. $\mathrm{Rev}(\mathbb{Q})$ *(order-preserving or reversing bijections on* $\mathbb{Q}$*)*
3. $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$*, preserving the cyclic order (see* $\mathbb{Q}/\mathbb{Z}$ *as a circle);*
4. $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$*, generated by* $\mathrm{Cyc}(\mathbb{Q}/\mathbb{Z})$ *and a reflection;*
5. $\mathfrak{S}_\infty$*.*

The notion of completion refers here to the topology of simple convergence, described in Section 2.4 of [4]. Thanks to the following lemma, it plays only minor role for our purposes. Thus, for the sake of the simplicity of exposition in this extended abstract, we will most of the time blur the distinction between a permutation group and its completion.

Lemma 2.5. *A permutation group and its completion share the same profile and orbit algebra.*

In particular, we may now say that Theorem 2.4 essentially classifies primitive P-oligomorphic groups.

2.3 Operations and examples

Lemma 2.6 (Operations on groups, their ages and orbit algebras)**.**

1. *Let G be a permutation group acting on E, and F be a stable subset of E. Then,* $\mathbb{Q}\mathcal{A}(G_{|F})$ *is both a subalgebra and a quotient of* $\mathbb{Q}\mathcal{A}(G)$*.*
2. *Let G be a permutation group acting on E and H be a subgroup. Then,* $\mathbb{Q}\mathcal{A}(G)$ *is a subalgebra of* $\mathbb{Q}\mathcal{A}(H)$*.*
3. *Let G and H be permutation groups acting on E and F respectively. Take $G \times H$ endowed with its natural action on the disjoint union $E \sqcup F$. Then,* $\mathcal{A}(G \times H) \simeq \mathcal{A}(G) \times \mathcal{A}(H)$*, and* $\mathbb{Q}\mathcal{A}(G \times H) \simeq \mathbb{Q}\mathcal{A}(G) \otimes \mathbb{Q}\mathcal{A}(H)$*; it follows that* $\mathcal{H}_{G\times H} = \mathcal{H}_G\mathcal{H}_H$*.*
4. *Let G and H be permutation groups acting on E and F respectively. Intuitively, the* ***wreath product*** *$G \wr H$ acts on $|F|$ copies $(E_f)_{f\in F}$ of E, by permuting each copy of E independently according to G and permuting the copies according to H. By construction, the partition $(E_f)_{f\in F}$ forms a block system, and $G \wr H$ is not primitive (unless G or H is and F or E, respectively, is of size 1).*

Examples 2.7 (Wreath products)**.**

1. Let G be the wreath product $\mathfrak{S}_\infty \wr \mathfrak{S}_k$. The profile counts integer partitions with at most k parts. The orbit algebra is the algebra of symmetric polynomials over k variables, that is the free commutative algebra with generators of degrees $1, \ldots, k$. The generating series of the profile is given by $\mathcal{H}_G = \frac{1}{\prod_{d=1,\ldots,k}(1-z^d)}$.
2. Let G' be a finite permutation group. Then, the orbit algebra of $G = \mathfrak{S}_\infty \wr G'$ is the **invariant ring** $\mathbb{Q}[X]^{G'}$ which consists of the polynomials in $\mathbb{Q}[X] = \mathbb{Q}[X_1, \ldots, X_k]$ that are invariant under the action of G'.

3. Let G' be a finite permutation group. Then, the orbit algebra of $G = G' \wr \mathfrak{S}_\infty$ is the free commutative algebra generated by the G'-orbits of non trivial subsets. The generating series of the profile is given by $\mathcal{H}_G = \frac{1}{\prod_d (1-z^d)}$, where d runs through the degrees of the G'-orbits of non trivial subsets, taken with multiplicity.

2.4 The canonical block system

Let G be a P-oligomorphic permutation group. In this section we show how a block system provides a lower bound on the growth of the profile. Seeking to optimize this lower bound, we establish the existence of a canonical block system satisfying appropriate properties. The later sections will show that this block system minimizes synchronization and provides a tight lower bound.

Lemma 2.8. *Let G be a P-oligomorphic permutation group, endowed with a transitive block system B. Then,*

1. *Case 1: B has finitely many infinite blocks, as in Example 2.7 (1) and (2). Then G is a subgroup of $\mathfrak{S}_\infty \wr \mathfrak{S}_k$ (where k is the number of blocks), and $\mathbb{Q}\mathcal{A}(G)$ contains Sym_k which is a free algebra with generators of degrees $(1, \ldots, k)$.*
2. *Case 2: B has infinitely many finite blocks, as in Example 2.7 (3). Then, G is a subgroup of $\mathfrak{S}_k \wr \mathfrak{S}_\infty$, and $\mathbb{Q}\mathcal{A}(G)$ contains the free algebra with generators of degrees $(1, \ldots, k)$.*

Sketch of proof. Use Lemma 2.6 and Examples 2.7. □

Remark 2.9. Let G be an oligomorphic permutation group, and $E_1, \ldots, E_k$ be a partition of E such that each E_i is stable under G. In our use case, we have a block system B, and each E_i is the support of one of the orbits of blocks in B.

Then, G is a subgroup of $G_{|E_1} \times \cdots \times G_{|E_k}$. Therefore, by Lemma 2.6, $\mathbb{Q}\mathcal{A}(G)$ contains as subalgebra $\mathbb{Q}\mathcal{A}(G_{|E_1}) \otimes \cdots \otimes \mathbb{Q}\mathcal{A}(G_{|E_k})$. In particular, the algebraic dimension of the age algebra is bounded below by the sum of the algebraic dimensions for each orbit of blocks.

Combining Lemma 2.8 and Remark 2.9, each block system of G provides a lower bound on the algebraic dimension of $\mathbb{Q}\mathcal{A}(G)$, and therefore on the growth rate of the profile.

This bound is not necessarily tight. Consider indeed the group $\mathrm{Id}_2 \wr \mathfrak{S}_\infty$. There are three block systems: one with two infinite blocks, one with two orbits of blocks of size 1, one with one orbit of blocks of size 2. With the above considerations, all three block systems give a lower bound of 2. However, as we will see in Corollary 4.9, the lower bound for the latter block system can be refined to 3 which is tight.

This rightfully suggests that better lower bounds are obtained when maximizing the size of the finite blocks, and then maximizing the number of infinite blocks.

Theorem 2.10. *Let G be a P-oligomorphic permutation group. There exists a unique block system, denoted $B(G)$ and called the* ***canonical block system of*** *G, that maximizes the size of its finite blocks, and then maximizes the number of infinite blocks.*

Sketch of proof. In the lattice of block systems of G, consider the lower set I of the block systems whose blocks are all finite. The join of two such block systems turns out to be again in I. Furthermore, if B is coarser than B', then the aforementioned lower bound on the algebraic dimension increases strictly; since the growth of the profile is polynomial, there can be no infinite increasing chain. It follows that I is in fact a lattice with a maximal (coarsest) block system. The kernel of G, if non empty, is one of the blocks of this block system.

Consider now the collection of the remaining blocks of size 1 of B. Following similar arguments as above, they can be replaced by a canonical maximal collection of infinite blocks to produce $B(G)$. □

2.5 Subdirect products

The actions of a permutation group on two of its orbits need not be independent; intuitively, there may be partial or full synchronization, which influences the profile and the orbit algebra. A classical tool to handle this phenomenon is that of subdirect products.

Definition 2.11. Let G_1 and G_2 be groups. A **subdirect product** of G_1 and G_2 is a subgroup of $G_1 \times G_2$ which projects onto each factor under the canonical projections.

For instance, suppose G is a permutation group that has two orbits of elements E_1 and E_2. If G_i is the group induced on E_i by G, G is a subdirect product of G_1 and G_2.

Denote $N_1 = \mathrm{Fix}_G(E_2)$ and $N_2 = \mathrm{Fix}_G(E_2)$. Then N_1 and N_2 are normal subgroups of G; and $N_1 \cap N_2 = \{1\}$, so N_1 and N_2 generate their direct product. We have (after restriction of N_i when needed): $\frac{G_1}{N_1} \simeq \frac{G}{N_1 \times N_2} \simeq \frac{G_2}{N_2}$, this quotient representing the synchronized part of each group, and $G = \{(g_1, g_2) \mid g_1 N_1 = g_2 N_2\}$.

With the classification of the groups of profile 1 in mind (see Theorem 2.4), there are very few possible synchronizations between two primitive infinite orbits of G: the possibilities are indeed linked to the normal subgroups of the two restrictions G_1 and G_2. Namely, the synchronization may be total, limited to the reflection in the cases of $\mathrm{Rev}(\mathbb{Q})$ and $\mathrm{Rev}(\mathbb{Q}/\mathbb{Z})$ (synchronization *of order 2*) or absent.

Lemma 2.12 (Reduction 0)**.** *Synchronizations of order 2 between primitive orbits do not change the age of G (this would be false for orbits of tuples).*

This lemma implies that the synchronizations of order 2 can harmlessly be ignored in the study of the orbit algebra of a group; so we will assume from now on that only full synchronizations may exist.

This is also true regarding the infinite orbits of finite blocks (instead of just elements), with the last item of Lemma 2.8 in mind: taking two such orbits, either the permutations of their blocks fully synchronize (blockwise) or they do not at all. We derive the following remark.

Remark 2.13. By construction, the actions of G on the orbits of blocks of the canonical block system $B(G)$ are independant blockwise (potentially ignoring harmless synchronizations of order 2).

3 Lifting from subgroups of finite index

In this section, we study how the orbit algebra of an oligomorphic permutation group G relates to the orbit algebra of a normal subgroup K of G of finite index, and derive two important reductions for Macpherson's question.

Theorem 3.1. *Let G be an oligomorphic permutation group and K be a normal subgroup of finite index. If the orbit algebra $\mathbb{Q}\mathcal{A}(K)$ of K is finitely generated, then so is its subalgebra $\mathbb{Q}\mathcal{A}(G)$. If in addition $\mathbb{Q}\mathcal{A}(K)$ is Cohen-Macaulay, then $\mathbb{Q}\mathcal{A}(G)$ is Cohen-Macaulay too.*

This is a close variant of Hilbert's theorem stating that the ring of invariants of a finite group is finitely generated; the orbit algebra $\mathbb{Q}\mathcal{A}(K)$ plays the role of the polynomial ring $\mathbb{Q}[X]$, while the orbit algebra $\mathbb{Q}\mathcal{A}(G)$ plays the role of the invariant ring $\mathbb{Q}[X]^G$. The key ingredient in Hilbert's proof is the *Reynolds operator*, a finite averaging operator over the group. In the setting of orbit algebras, G is not finite; however, we will compensate by using the *relative Reynolds operator* with respect to K, which is a finite averaging operator over the coset representatives. Then we just proceed as in Hilbert's proof. The same approach can be used to prove that $\mathbb{Q}\mathcal{A}(G)$ is Cohen-Macaulay as soon as $\mathbb{Q}\mathcal{A}(K)$ is.

Corollary 3.2 (Reduction 1)**.** *Let G be an oligomorphic group that admits a non trivial finite transitive block system. Let K be the subgroup of the elements of G that stabilize each block. Assume that the orbit algebra of K is finitely generated. Then so is the orbit algebra of G.*

The second application is a reduction of Macpherson's question to groups that admit no finite orbits of elements.

Definition 3.3. The **kernel** of an oligomorphic permutation group G is the union $\ker(G)$ of its finite orbits of elements.

This terminology comes from the broader context of relational structures: it can be shown that $\ker(G)$ is indeed the kernel of the associated homogeneous relational structure. It is not to be confused with the notion of kernel from group theory.

Remark 3.4. The kernel $\ker(G)$ of an oligomorphic group is finite. Indeed, G has a finite number $\varphi_G(1)$ of orbits and thus of finite orbits; hence their union is finite as well.

Theorem 3.5 (Reduction 2)**.** *Let G be an oligomorphic permutation group with profile bounded by a polynomial. Assume that the orbit algebra of any group with the same profile growth and no finite orbit is finitely generated. Then, the orbit algebra of G is finitely generated as well.*

Sketch of proof. Let K be the subgroup of G fixing $\ker(G)$; it is normal and of finite index in G. Using the two upcoming simple lemmas together with Lemma 2.6 (3), the groups K and G share the same profile growth. Apply Theorem 3.1. □

Lemma 3.6. *Let G be a permutation group and K be a normal subgroup of finite index. Then,*

$$\varphi_G(n) \leq \varphi_K(n) \leq |G : K|\varphi_G(n).$$

Lemma 3.7. *Let G be an oligomorphic permutation group and K be a normal subgroup of finite index. Then* $\ker(K) = \ker(G)$.

Proof. Let O be a G-orbit of elements. Since K is a normal subgroup, O splits into K-orbits on which G – and actually G/K – acts transitively by permutation; there are thus finitely many such K-orbits, all of the same size. In particular, infinite G-orbits split into infinite K-orbits, and similarly for finite ones. □

In order to give an idea of the proof of Theorem 3.1, let us now turn to the *relative Reynolds operator* R_K^G. It is defined by choosing some representatives $(g_i)_i$ of the cosets of K in G:

$$R_K^G := \frac{1}{|G : K|} \sum_i g_i.$$

Lemma 3.8. *Let G be an oligomorphic permutation group, and K be a normal subgroup of finite index. Then, the relative Reynolds operator R_K^G defines a projection from $\mathbb{Q}\mathcal{A}(K)$ onto $\mathbb{Q}\mathcal{A}(G)$ which does not depend on the choice of the g_i's, and is a $\mathbb{Q}\mathcal{A}(G)$-module morphism.*

Sketch of proof of Theorem 3.1. Use the relative Reynolds operator to replay Hilbert's proof of finite generation for invariants of finite groups, as well as the classical proof of the Cohen-Macaulay property (see e.g. [**Stanley.1979**]). □

4 Case of a transitive block system with finite blocks

In this section, permutation groups are assumed to be P-oligomorphic and endowed with a non trivial transitive block system with infinitely many finite blocks, which we choose maximal. We bring a positive answer to Macpherson's question in this setting.

Lemma 4.1. *Let G be a P-oligomorphic permutation group having a non trivial transitive block system with infinitely many maximal finite blocks; then G acts on the blocks as $\mathfrak{S}_\infty$.*

Sketch of proof. Using their maximality, the action of G on the blocks is classified by Theorem 2.4; argue now that with any of the five actions but $\mathfrak{S}_\infty$ the profile would not be bounded by any polynomial. $\square$

Definition 4.2. Let $S_B = S_B^G = \mathrm{Stab}_G(B)$ and, for $i \geq 0$, $H_i = H_i^G = \mathrm{Fix}_{S_B}(B_1, \ldots, B_i)_{|B_{i+1}}$. The sequence $H_0\ H_1\ H_2 \cdots$ is called the **tower** of G with respect to the block system B. The groups H_i are considered up to a permutation group isomorphism.

Remark 4.3. By conjugation, using Lemma 4.1, the tower does not depend on the ordering of the blocks. Furthermore, H_{i+1} is a normal subgroup of H_i for all $i \in \mathbb{N}$. The above definition and this remark also apply to a permutation group of a finite set, if it acts on the blocks as the full symmetric group.

Example 4.4. Let H be a finite permutation group. The tower of $H \wr \mathfrak{S}_\infty$ (resp. $H \times \mathfrak{S}_\infty$) for its natural block system is $H\ H\ H \cdots$ (resp. $H\ Id\ Id \cdots$).

Lemma 4.5. *For all $k \in \mathbb{N}$, $\mathrm{Fix}_G(B_1, \ldots, B_k)$ acts on the remaining blocks as $\mathfrak{S}_\infty$.*

Sketch of proof. As $\mathrm{Fix}_G(B_1, \ldots, B_k)$ is a normal subgroup of finite index of $\mathrm{Stab}_G(B_1, \ldots, B_k)$, it acts on the remaining blocks as a subgroup of finite index of $\mathfrak{S}_\infty$. Use Theorem 2.4. $\square$

Proposition 4.6. *Two groups with the same tower have isomorphic ages.*

Sketch of proof. Start by restricting to the first k blocks, and use Lemma 4.5 to show that, up to conjugation within each block, the restrictions of the two groups have the same age. Use again Lemma 4.5 to derive that the age of each group coincides, up to degree k, with that of its restriction. Conclude by taking the limit at $k = \infty$. $\square$

Lemma 4.7. *Let G be a finite permutation group endowed with a block system composed of four blocks on which G acts by $\mathfrak{S}_4$; denote its tower by $H_0\ H_1\ H_2\ H_3$. Then, $H_1 = H_2$.*

Proof. An element s of S_B is determined by its action on each block, which we write as a quadruple. Let g be an element of H_1. Then S_B has an element x that may be written $(1, g, h, l)$, with h and l also in H_1. Let σ be an element of G that permutes the first two blocks and fixes the other two (it exists by hypothesis). By conjugating x with σ in G, we get an element y in S_B that we may write $(g', 1, h, l)$. Then we have $x^{-1}y = (g', g^{-1}, 1, 1)$; hence, using Remark 4.3, g^{-1} and therefore g are in H_2. $\square$

Corollary 4.8. *Let G be a P-oligomorphic permutation group having a non trivial transitive block system with infinitely many maximal finite blocks. Then, the tower of G has the form $H_0\ H\ H\ H \cdots$, where H_0 is a finite permutation group and H is a normal subgroup of H_0.*

Sketch of proof. Restrict to sequences of four consecutive blocks and use Lemma 4.7. $\square$

Corollary 4.9. *Let G be a P-oligomorphic permutation group endowed with a non trivial transitive block system with infinitely many (maximal) finite blocks. Then, G contains a finite index subgroup K whose age coincides with that of $H \wr \mathfrak{S}_\infty$ (where H is as in Corollary 4.8).*
It follows that its algebraic dimension is given by the number of H-orbits (of non trivial subsets).

5 Proof of the main theorem

Theorem 5.1. *Let G be a P-oligomorphic permutation group. Then, its orbit algebra $\mathbb{Q}\mathcal{A}(G)$ is finitely generated and Cohen-Macaulay.*

Sketch of proof. Consider the canonical block system $B(G)$ introduced in subsection 2.4. Recall that it consists of finitely many infinite blocks, a finite number of infinite orbits of finite blocks, and possibly one finite stable block.

We aim to prove the existence of a normal subgroup K of finite index of G with a simple form, ensuring that its orbit algebra is a finitely generated (almost free) algebra.

Start with $K = G$. If there exists a finite stable block B_k in $B(G)$, replace K by the kernel of its action on B_k (i.e. the kernel of the natural projection on this action). This ensures that K fixes B_k. Replace further K by the kernel of its action on the set of infinite blocks of $B(G)$. This ensures that K stabilizes each of the infinite blocks. Using that K is of finite index and the construction of $B(G)$, these blocks are now primitive orbits. Possibly replacing again K, we may further assume that (the completion of) K acts on each of them as one of the three primitive groups of Theorem 2.4 that admit no subgroup of finite index. Now take an orbit of finite blocks. Using Corollary 4.9, and replacing K if needed, we may assume that the restriction of K on the support of the orbit has the same age as some $H \wr \mathfrak{S}_\infty$. Repeat for the other orbits of finite blocks.

By construction of $B(G)$ and Subsection 2.5, argue that there is now no synchronization between K-orbits of blocks. Then, K has the same age as some direct product of groups of the form $\mathfrak{S}_\infty$, $\mathrm{Aut}(\mathbb{Q})$, $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$, and $G' \wr \mathfrak{S}_\infty$ (and possibly a finite identity group). From Remark 2.9, $\mathbb{Q}\mathcal{A}(K)$ is a finitely generated free algebra (possibly tensored with some finite dimensional diagonal algebra). Using Theorem 3.1, it follows that $\mathbb{Q}\mathcal{A}(G)$ is finitely generated and Cohen-Macaulay over some free subalgebra $\mathbb{Q}[\theta_i]$.

From the groups involved in the direct product, one may construct explicitly the generators θ_i, and therefore the degrees $(d_i)_i$ appearing in Corollary 1.4 in the denominator of the Hilbert series. $\square$

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