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ASYMPTOTIC AND COARSE LIPSCHITZ STRUCTURES OF QUASI-REFLEXIVE BANACH SPACES

G. LANCIEN AND M. RAJA

ABSTRACT. In this note, we extend to the setting of quasi-reflexive spaces a classical result of N. Kalton and L. Randrianarivony on the coarse Lipschitz structure of reflexive and asymptotically uniformly smooth Banach spaces. As an application, we show for instance, that for $1 \leq q < p$, a q -asymptotically uniformly convex Banach space does not coarse Lipschitz embed into a p -asymptotically uniformly smooth quasi-reflexive Banach space. This extends a recent result of B.M. Braga.

1. INTRODUCTION.

We start this note with some basic definitions on metric embeddings. Let (M, d) and (N, δ) be two metric spaces and f be a map from M into N . We define the *compression modulus* of f by

$$\rho_f(t) = \inf \{ \delta(f(x), f(y)), d(x, y) \geq t \},$$

and the *expansion modulus* of f by

$$\omega_f(t) = \sup \{ \delta(f(x), f(y)), d(x, y) \leq t \}.$$

We say that f is a *Lipschitz embedding* if there exist A, B in $(0, \infty)$ such that $\omega_f(t) \leq Bt$ and $\rho_f(t) \geq At$.

If the metric space M is unbounded, we say that f is a *coarse embedding* if $\lim_{t \rightarrow \infty} \rho_f(t) = \infty$ and $\omega_f(t) < \infty$ for all $t > 0$. Note that if M is a Banach space, we have automatically in that case that ω_f is dominated by an affine function. This last observation can be deduced from the proof of Proposition 1.11 in [3], which in turn can be found in the earlier paper [5].

We say that f is a *coarse Lipschitz embedding* if there exist A, B, C, D in $(0, +\infty)$ such that $\omega_f(t) \leq Bt + D$ and $\rho_f(t) \geq At - C$.

In order to refine the scale of coarse embeddings, E. Guentner and J. Kaminker introduced in [8] the following notion. Let X and Y be two Banach spaces. We define $\alpha_Y(X)$ as the supremum of all $\alpha \in [0, 1)$ for which there exists a

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coarse embedding $f : X \rightarrow Y$ and A, C in $(0, \infty)$ so that $\rho_f(t) \geq At^\alpha - C$ for all $t > 0$. Then, $\alpha_Y(X)$ is called the *compression exponent of X in Y* .

We now turn to the definitions of the uniform asymptotic properties of norms that will be considered in this paper. For a Banach space $(X, \|\cdot\|)$ we denote by B_X the closed unit ball of X and by S_X its unit sphere. The following definitions are due to V. Milman [14] and we follow the notation from [10]. For $t \in [0, \infty)$, $x \in S_X$ and Y a closed linear subspace of X , we define

$$\bar{\rho}_X(t, x, Y) = \sup_{y \in S_Y} (\|x + ty\| - 1) \quad \text{and} \quad \bar{\delta}_X(t, x, Y) = \inf_{y \in S_Y} (\|x + ty\| - 1).$$

Then

$$\bar{\rho}_X(t, x) = \inf_{\dim(X/Y) < \infty} \bar{\rho}_X(t, x, Y) \quad \text{and} \quad \bar{\delta}_X(t, x) = \sup_{\dim(X/Y) < \infty} \bar{\delta}_X(t, x, Y)$$

and

$$\bar{\rho}_X(t) = \sup_{x \in S_X} \bar{\rho}_X(t, x) \quad \text{and} \quad \bar{\delta}_X(t) = \inf_{x \in S_X} \bar{\delta}_X(t, x).$$

The norm $\|\cdot\|$ is said to be *asymptotically uniformly smooth* (in short AUS) if

$$\lim_{t \rightarrow 0} \frac{\bar{\rho}_X(t)}{t} = 0.$$

It is said to be *asymptotically uniformly convex* (in short AUC) if

$$\forall t > 0 \quad \bar{\delta}_X(t) > 0.$$

Let $p \in (1, \infty)$ and $q \in [1, \infty)$.

We say that the norm of X is *p -AUS* if there exists $c > 0$ such that for all $t \in [0, \infty)$, $\bar{\rho}_X(t) \leq ct^p$.

We say that the norm of X is *q -AUC* if there exists $c > 0$ such that for all $t \in [0, 1]$, $\bar{\delta}_X(t) \geq ct^q$.

Similarly, there is in X^* a modulus of weak* asymptotic uniform convexity defined by

$$\bar{\delta}_X^*(t) = \inf_{x^* \in S_{X^*}} \sup_E \inf_{y^* \in S_E} (\|x^* + ty^*\| - 1),$$

where E runs through all weak*-closed subspaces of X^* of finite codimension. The norm of X^* is said to be *weak* uniformly asymptotically convex* (in short weak*-AUC) if $\bar{\delta}_X^*(t) > 0$ for all t in $(0, \infty)$. If there exists $c > 0$ and $q \in [1, \infty)$ such that for all $t \in [0, 1]$ $\bar{\delta}_X^*(t) \geq ct^q$, we say that the norm of X^* is *q -weak*-AUC*.

Let us recall the following classical duality result concerning these moduli (see for instance [6] Corollary 2.3 for a precise statement).

Proposition 1.1. *Let X be a Banach space.*

Then $\|\cdot\|_X$ is AUS if and only if $\|\cdot\|_{X^}$ is weak*-AUC.*

If $p, q \in (1, \infty)$ are conjugate exponents, then $\|\cdot\|_X$ is p -AUS if and only if $\|\cdot\|_{X^}$ is q -weak*-AUC.*

The main purpose of this note is to extend to the quasi-reflexive case an important result obtained by N. Kalton and L. Randrianarivony in [12] about coarse Lipschitz embeddings into p -AUS reflexive spaces. In order to explain their result, we need to introduce special metric graphs that we shall call *Kalton-Randrianarivony's graphs*. For an infinite subset \mathbb{M} of \mathbb{N} and $k \in \mathbb{N}$, we denote

$$G_k(\mathbb{M}) = \{\bar{n} = (n_1, \dots, n_k), n_i \in \mathbb{M} \ n_1 < \dots < n_k\}.$$

Then we equip $G_k(\mathbb{M})$ with the distance $d(\bar{n}, \bar{m}) = |\{j, n_j \neq m_j\}|$. The fundamental result of their paper (Theorem 4.2 in [12]) can be rephrased as follows.

Theorem 1.2. (Kalton-Randrianarivony 2008) *Let $p \in (1, \infty)$ and assume that Y is a reflexive p -AUS Banach space. Then there exists a constant $C > 0$ such that for any $k \in \mathbb{N}$, any infinite subset \mathbb{M} of \mathbb{N} , any $f : (G_k(\mathbb{M}), d) \rightarrow Y$ Lipschitz map and any $\varepsilon > 0$, there exists an infinite subset \mathbb{M}' of \mathbb{M} , such that*

$$\text{diam } f(G_k(\mathbb{M}')) \leq C \text{Lip}(f) k^{1/p} + \varepsilon.$$

We refer the reader to [12] and [11] for the various applications derived by the authors. Very recently, B.M. Braga used the above theorem in [4] to develop many other applications. We will only mention one of them in this introduction (Corollary 4.5 in [4]).

Theorem 1.3. (Braga 2016) *Let $1 \leq q < p$, X be a q -AUC Banach space and Y be a p -AUS reflexive Banach space. Then $\alpha_Y(X) \leq q/p$.*

We recall that a Banach space is said to be *quasi-reflexive* if the image of its canonical embedding into its bidual is of finite codimension in its bidual. The aim of this note is to obtain a version of the above Theorem 1.2 for quasi-reflexive Banach spaces. This is done in section 2 and our main results are Theorem 2.2 and Theorem 2.4. In section 3, we apply them in order to extend some results from [4] to the quasi-reflexive setting, including the above Theorem 1.3. Our applications are stated in Corollary 3.2, Theorem 3.5, Corollary 3.6 and Corollary 3.8.

2. THE MAIN RESULT.

We first need the following simple property of the bidual of a Banach space X , in relation with the modulus of asymptotic uniform smoothness of X .

Proposition 2.1. *Let X be a Banach space. Then the bidual norm on X^{**} has the following property. For any $t \in (0, 1)$, any weak*-null sequence $(x_n^{**})_{n=1}^{\infty}$ in $B_{X^{**}}$ and any $x \in S_X$ we have:*

$$\limsup_{n \rightarrow \infty} \|x + tx_n^{**}\| \leq 1 + \bar{\rho}_X(t, x).$$

Proof. Let x in S_X and $\varepsilon > 0$. By definition, there exists a finite codimensional subspace Y of X such that

$$(1) \quad x + tB_Y \subset (1 + \bar{\rho}_X(t, x) + \varepsilon)B_X.$$

There exist $x_1^*, \dots, x_k^* \in X^*$ such that

$$Y = \bigcap_{i=1}^k \{x \in X, x_i^*(x) = 0\}.$$

Note that the weak*-closure of Y in X^{**} is

$$\bar{Y}^{w*} = Y^{\perp\perp} = \bigcap_{i=1}^k \{x^{**} \in X^{**}, x^{**}(x_i^*) = 0\}.$$

By Goldstine's Theorem we have that the weak*-closures in X^{**} of B_Y and B_X are respectively $B_{\bar{Y}^{w*}}$ and $B_{X^{**}}$. So, taking the weak*-closures in (1) we get that

$$x + tB_{\bar{Y}^{w*}} \subset (1 + \bar{\rho}_X(t, x) + \varepsilon)B_{X^{**}}.$$

Since the sequence $(x_n^{**})_{n=1}^\infty$ is weak*-null, we have that for all i in $\{1, \dots, k\}$, $\lim_{n \rightarrow \infty} x_n^{**}(x_i^*) = 0$. It follows easily that $\lim_{n \rightarrow \infty} d(x_n^{**}, B_{\bar{Y}^{w*}}) = 0$. We deduce, that for n large enough

$$x + tx_n^{**} \in (1 + \bar{\rho}_X(t) + 2\varepsilon)B_{X^{**}},$$

which concludes our proof. \square

We shall now give an analogue of Theorem 1.2 when Y is only assumed to be quasi-reflexive and p -AUS for some $p \in (1, \infty)$. The idea is to adapt techniques from a work by F. Nétilard [15] on the coarse Lipschitz embeddings between James spaces. To this end, for \mathbb{M} an infinite subset of \mathbb{N} , we denote $I_k(\mathbb{M})$ the set of strictly interlaced pairs in $G_k(\mathbb{M})$, namely :

$$I_k(\mathbb{M}) = \{(\bar{n}, \bar{m}) \in G_k(\mathbb{M}) \times G_k(\mathbb{M}), n_1 < m_1 < n_2 < m_2 < \dots < n_k < m_k\}.$$

Note that for $(\bar{n}, \bar{m}) \in I_k(\mathbb{M})$, $d(\bar{n}, \bar{m}) = k$. Our statement is then the following.

Theorem 2.2. *Let $p \in (1, \infty)$ and Y be a quasi-reflexive p -AUS Banach space. Then there exists a constant $C > 0$ such that for any $k \in \mathbb{N}$, any infinite subset \mathbb{M} of \mathbb{N} , any $f : (G_k(\mathbb{M}), d) \rightarrow Y^{**}$ Lipschitz and any $\varepsilon > 0$ there exists an infinite subset \mathbb{M}' of \mathbb{M} , such that*

$$\forall (\bar{n}, \bar{m}) \in I_k(\mathbb{M}') \quad \|f(\bar{n}) - f(\bar{m})\| \leq CLip(f)k^{\frac{1}{p}} + \varepsilon.$$

In fact, we will show a more general result, which follows the ideas of section 6 in [12]. Before to state it, we need some preparation. We briefly recall the setting of section 6 in [12].

So let Y be a Banach space and denote by $\bar{\rho}_Y$ its modulus of asymptotic

uniform smoothness. It is easily checked that $\bar{\rho}_Y$ is an Orlicz function. Then we define the Orlicz sequence space:

$$\ell_{\bar{\rho}_Y} = \left\{ x \in \mathbb{R}^{\mathbb{N}}, \exists r > 0 \sum_{n=1}^{\infty} \bar{\rho}_Y\left(\frac{|x_n|}{r}\right) < \infty \right\},$$

equipped with the norm

$$\|x\|_{\bar{\rho}_Y} = \inf \left\{ r > 0, \sum_{n=1}^{\infty} \bar{\rho}_Y\left(\frac{|x_n|}{r}\right) \leq 1 \right\}.$$

Next we construct a sequence of norms $(N_k)_{k=1}^{\infty}$, where N_k is a norm on \mathbb{R}^k , as follows.

For all $\xi \in \mathbb{R}$, $N_1(\xi) = |\xi|$.

$N_2(\xi, \eta) = |\eta|$ if $\xi = 0$ and

$$N_2(\xi, \eta) = |\xi| \left(1 + \bar{\rho}_Y\left(\frac{|\eta|}{|\xi|}\right) \right) \text{ if } \xi \neq 0.$$

Then, for $k \geq 3$, we define by induction the following norm on \mathbb{R}^k :

$$N_k(\xi_1, \dots, \xi_k) = N_2(N_{k-1}(\xi_1, \dots, \xi_{k-1}), \xi_k).$$

The following property is proved in [12].

Proposition 2.3. *For any $k \in \mathbb{N}$ and any $a \in \mathbb{R}^k$:*

$$N_k(a) \leq e \|a\|_{\bar{\rho}_Y}.$$

Fix now $a = (a_1, \dots, a_k)$ a sequence of non zero real numbers and define the following distance on $G_k(\mathbb{M})$, for \mathbb{M} infinite subset of \mathbb{N} :

$$\forall \bar{n}, \bar{m} \in G_k(\mathbb{M}), \quad d_a(\bar{n}, \bar{m}) = \sum_{j, n_j \neq m_j} |a_j|.$$

We can now state our general result, from which Theorem 2.2 is easily deduced.

Theorem 2.4. *Let Y be a quasi-reflexive Banach space, $a = (a_1, \dots, a_k)$ a sequence of non zero real numbers, \mathbb{M} an infinite subset of \mathbb{N} and let $f : (G_k(\mathbb{M}), d_a) \rightarrow Y^{**}$ be a Lipschitz map.*

Then for any $\varepsilon > 0$ there exists an infinite subset \mathbb{M}' of \mathbb{M} , such that

$$\forall (\bar{n}, \bar{m}) \in I_k(\mathbb{M}') \quad \|f(\bar{n}) - f(\bar{m})\| \leq 2e \text{Lip}(f) \|a\|_{\bar{\rho}_Y} + \varepsilon.$$

Proof. Under the assumptions of Theorem 2.4, we will show that there exists an infinite subset \mathbb{M}' of \mathbb{M} , such that

$$\forall (\bar{n}, \bar{m}) \in I_k(\mathbb{M}') \quad \|f(\bar{n}) - f(\bar{m})\| \leq 2\text{Lip}(f) N_k(a) + \varepsilon.$$

Then, the conclusion will follow from Proposition 2.3.

Since the graphs $G_k(\mathbb{N})$ are all countable, we may assume that Y is separable. We can write $Y^{**} = Y \oplus E$, where E is finite dimensional.

We will prove our statement by induction on $k \in \mathbb{N}$. It is clearly true for $k = 1$, so assume it is true for $k \in \mathbb{N}$ and let $a = (a_1, \dots, a_{k+1})$ be a sequence of

non zero real numbers and $f : (G_{k+1}(\mathbb{M}), d_a) \rightarrow Y^{**}$ be a Lipschitz map. Let $\varepsilon > 0$ and fix $\eta \in (0, \frac{\varepsilon}{2})$ (our initial choice of a small η will be made precise later).

Since Y is separable and quasi-reflexive, Y^* is also separable. So, using weak*-compactness in Y^{**} , we can find an infinite subset \mathbb{M}_0 of \mathbb{M} such that

$$\forall \bar{n} \in G_k(\mathbb{M}_0) \quad w^* - \lim_{n_{k+1} \in \mathbb{M}_0} f(\bar{n}, n_{k+1}) = g(\bar{n}) \in Y^{**}.$$

Using the weak* lower semi continuity of $\|\cdot\|_{Y^{**}}$ we get that the map $g : G_k(\mathbb{M}_0, d_{(a_1, \dots, a_k)}) \rightarrow Y^{**}$ satisfies $\text{Lip}(g) \leq \text{Lip}(f)$. For $\bar{n} \in G_k(\mathbb{M}_0)$, we can write $g(\bar{n}) = h(\bar{n}) + e(\bar{n})$, with $h(\bar{n}) \in Y$ and $e(\bar{n}) \in E$. It then follows from Ramsey's theorem and the norm compactness of bounded sets in E that there exists an infinite subset \mathbb{M}_1 of \mathbb{M}_0 such that

$$\forall \bar{n}, \bar{m} \in G_k(\mathbb{M}_1) \quad \|e(\bar{n}) - e(\bar{m})\| \leq \eta.$$

For $\bar{n}, \bar{m} \in G_k(\mathbb{M}_1)$ and $t, l \in \mathbb{M}_1$, set

$$u_{\bar{n}, \bar{m}, t, l} = (f(\bar{n}, t) - g(\bar{n})) - (f(\bar{m}, l) - g(\bar{m})).$$

Since $\|\cdot\|_{Y^{**}}$ is weak* lower semi continuous, we have that

$$\|u_{\bar{n}, \bar{m}, t, l}\| \leq 2\text{Lip}(f)|a_{k+1}|.$$

On the other hand, it follows from our induction hypothesis that there exists an infinite subset \mathbb{M}_2 of \mathbb{M}_1 such that

$$\forall (\bar{n}, \bar{m}) \in I_k(\mathbb{M}_2) \quad \|g(\bar{n}) - g(\bar{m})\| \leq 2\text{Lip}(f)N_k(a_1, \dots, a_k) + \eta.$$

Therefore

$$\forall (\bar{n}, \bar{m}) \in I_k(\mathbb{M}_2) \quad \|h(\bar{n}) - h(\bar{m})\| \leq 2\text{Lip}(f)N_k(a_1, \dots, a_k) + 2\eta.$$

Assume first that $h(\bar{n}) \neq h(\bar{m})$. Then it follows from Proposition 2.1 and the fact that $u_{\bar{n}, \bar{m}, t, l}$ is tending to 0 in the weak* topology, as t, l tend to ∞ , that there exists $N_{(\bar{n}, \bar{m})} \in \mathbb{M}_2$ such that for all $t, l \in \mathbb{M}_2$ satisfying $t, l \geq N_{(\bar{n}, \bar{m})}$:

$$\begin{aligned} \|h(\bar{n}) - h(\bar{m}) + u_{\bar{n}, \bar{m}, t, l}\| &\leq \|h(\bar{n}) - h(\bar{m})\| \left(1 + \bar{\rho}_Y \left(\frac{2\text{Lip}(f)|a_{k+1}|}{\|h(\bar{n}) - h(\bar{m})\|}\right)\right) + \eta \\ &\leq N_2(\|h(\bar{n}) - h(\bar{m})\|, 2\text{Lip}(f)|a_{k+1}|) + \eta. \end{aligned}$$

Note that if $h(\bar{n}) = h(\bar{m})$, the above inequality is clearly true for all $t, l \in \mathbb{M}_2$. Therefore, we have that for all $(\bar{n}, \bar{m}) \in I_k(\mathbb{M}_2)$ there exists $N_{(\bar{n}, \bar{m})} \in \mathbb{M}_2$ such that for all $t, l \geq N_{(\bar{n}, \bar{m})}$:

$$\begin{aligned} \|h(\bar{n}) - h(\bar{m}) + u_{\bar{n}, \bar{m}, t, l}\| &\leq N_2(2\text{Lip}(f)N_k(a_1, \dots, a_k) + 2\eta, 2\text{Lip}(f)|a_{k+1}|) + \eta \\ &\leq 2\text{Lip}(f)N_{k+1}(a_1, \dots, a_{k+1}) + \frac{\varepsilon}{2}, \end{aligned}$$

if η was initially chosen small enough.

So we have proved that for all $(\bar{n}, \bar{m}) \in I_k(\mathbb{M}_2)$ there exist $N_{(\bar{n}, \bar{m})} \in \mathbb{M}_2$ such that for all $t, l \geq N_{(\bar{n}, \bar{m})}$:

$$(2) \quad \|f(\bar{n}, t) - f(\bar{m}, l)\| \leq 2\text{Lip}(f)N_{k+1}(a_1, \dots, a_{k+1}) + \varepsilon.$$

We now wish to construct \mathbb{M}' infinite subset of \mathbb{M}_2 satisfying our conclusion. For that purpose, for a finite subset F of \mathbb{N} of cardinality at least $2k$, we denote $I_k(F)$ the set of all $(\bar{n}, \bar{m}) \in I_k(\mathbb{N})$ such that $n_1 < m_1 < \dots < n_k < m_k \in F$. Assume now $\mathbb{M}_2 = \{n_1 < \dots < n_j < \dots\}$. We shall define inductively the elements of our set $\mathbb{M}' = \{m_1 < \dots < m_j < \dots\}$. First, we set $m_1 = n_1, \dots, m_{2k} = n_{2k}$. Then, for $j > 2k$, we define $m_j = n_{\phi(j)} > m_{j-1}$ in such a way that for all $(\bar{n}, \bar{m}) \in I_k(\{m_1, \dots, m_{j-1}\})$ we have that $m_j \geq N_{(\bar{n}, \bar{m})}$. Then, it should be clear from equation (2) and the construction of \mathbb{M}' that

$$\forall (\bar{n}, \bar{m}) \in I_{k+1}(\mathbb{M}') \quad \|f(\bar{n}) - f(\bar{m})\| \leq 2\text{Lip}(f)N_{k+1}(a_1, \dots, a_{k+1}) + \varepsilon.$$

This finishes our induction. \square

Remark. Suppose now that Y is a non reflexive Banach space and fix $\theta \in (0, 1)$. Then, James' Theorem (see [9]) insures the existence of a sequence $(x_n)_{n=1}^\infty$ in S_X and a sequence $(x_n^*)_{n=1}^\infty$ in S_{X^*} such that

$$x_n^*(x_i) = \theta \text{ if } n \leq i \text{ and } x_n^*(x_i) = 0 \text{ if } n > i.$$

In particular, for all $n_1 < \dots < n_k < m_1 < \dots < m_k$:

$$(3) \quad \|x_{n_1} + \dots + x_{n_k} - (x_{m_1} + \dots + x_{m_k})\| \geq \theta k.$$

Define now, for $k \in \mathbb{N}$ and $\bar{n} \in G_k(\mathbb{N})$, $f(\bar{n}) = x_{n_1} + \dots + x_{n_k}$. Then f is clearly 2-Lipschitz. On the other hand, it follows from (3) that for any infinite subset \mathbb{M} of \mathbb{N} , $\text{diam}(f(G_k(\mathbb{M}))) \geq \theta k$. This shows that the conclusion of Theorem 1.2 cannot hold as soon as Y is non reflexive. This obstacle is overcome in our Theorems 2.2 and 2.4 by considering particular pairs of elements in $G_k(\mathbb{N})$ that are k -separated, namely the strictly interlaced pairs from $I_k(\mathbb{N})$. Note also that the pairs considered in our above application of James' Theorem are k -separated but at the "opposite" of being interlaced, since $n_1 < \dots < n_k < m_1 < \dots < m_k$.

3. APPLICATIONS.

Let us start with the following definitions.

Definition 3.1.

(i) A Banach space X has the *Banach-Saks property* if every bounded sequence in X admits a subsequence whose Cesàro means converge in norm.

(ii) A Banach space X has the *alternating Banach-Saks property* if for every bounded sequence $(x_n)_{n=1}^\infty$ in X , there exists a subsequence $(x_{n_k})_{k=1}^\infty$ of $(x_n)_{n=1}^\infty$ and a sequence $(\varepsilon_k)_{k=1}^\infty \in \{-1, 1\}^\mathbb{N}$ such that the Cesàro means of the sequence $(\varepsilon_k x_{n_k})_{k=1}^\infty$ converge in norm.

Our last remark of section 2 was used in [1] (Theorem 4.1), to show that if a Banach space coarse Lipschitz embeds into a reflexive AUS Banach space, then X is reflexive. This result, was recently improved by B.M Braga [4] (Theorem 1.3) who showed that actually X must have the Banach-Saks property (which clearly implies reflexivity). As a first application of our result we obtain.

Proposition 3.2. *Assume that X is a Banach space which coarse Lipschitz embeds into a quasi reflexive AUS Banach space Y . Then X has the alternating Banach-Saks property.*

Proof. Since Y is AUS, there exists $p \in (1, \infty)$ such that Y is p -AUS renormable. This a consequence of a result of Knaust, Odell and Schlumprecht [13] in the separable case and of the second named author for the general case [16]. So we may as well assume that Y is p -AUS for some $p > 1$.

Assume also that X does not have the alternating Banach-Saks property. Then it follows from the work of B. Beauzamy (Theorems II.2 and III.1 in [2]) that there exists a sequence $(x_n)_{n=1}^\infty$ in X such that for all $k \in \mathbb{N}$, all $\varepsilon_1, \dots, \varepsilon_k \in \{-1, 1\}$ and all $n_1 < \dots < n_k$:

$$(4) \quad \frac{1}{2} \leq \left\| \frac{1}{k} \sum_{i=1}^k \varepsilon_i x_{n_i} \right\| \leq \frac{3}{2}.$$

Assume now that X coarse Lipschitz embeds into Y . Then, after a linear change of variable if necessary, there exists $f : X \rightarrow Y$ and $A, B > 0$ such that

$$\forall x, x' \in X \quad \|x - x'\| \geq \frac{1}{2} \Rightarrow A\|x - x'\| \leq \|f(x) - f(x')\| \leq B\|x - x'\|.$$

We then define $\varphi_k : (G_k(\mathbb{N}), d) \rightarrow X$ as follows:

$$\varphi_k(\bar{n}) = x_{n_1} + \dots + x_{n_k}, \quad \text{for } \bar{n} = (n_1, \dots, n_k) \in G_k(\mathbb{N}).$$

We clearly have that $\text{Lip}(\varphi_k) \leq 3$. Moreover, for $\bar{n} \neq \bar{m} \in G_k(\mathbb{N})$ we have $\|\varphi_k(\bar{n}) - \varphi_k(\bar{m})\| \geq 1$. We deduce that for all $k \in \mathbb{N}$, $\text{Lip}(f \circ \varphi_k) \leq 3B$. It now follows from Theorem 2.2 that there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$, there is an infinite subset \mathbb{M}_k of \mathbb{N} so that

$$\forall (\bar{n}, \bar{m}) \in I_k(\mathbb{M}_k) \quad \|(f \circ \varphi_k)(\bar{n}) - (f \circ \varphi_k)(\bar{m})\| \leq Ck^{1/p}.$$

On the other hand, it follows from (4) that for all $(\bar{n}, \bar{m}) \in I_k(\mathbb{M}_k)$, we have: $\|\varphi_k(\bar{n}) - \varphi_k(\bar{m})\| \geq k$. Therefore

$$\forall (\bar{n}, \bar{m}) \in I_k(\mathbb{M}_k) \quad \|(f \circ \varphi_k)(\bar{n}) - (f \circ \varphi_k)(\bar{m})\| \geq Ak.$$

This yields a contradiction for k large enough. □

In order to state some more quantitative results, we will need the following definition.

Definition 3.3. Let $q \in (1, \infty)$ and X be a Banach space. We say that X has the q -co-Banach-Saks property if for every semi-normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X there exists a subsequence $(x'_n)_{n=1}^\infty$ of $(x_n)_{n=1}^\infty$ and $c > 0$ such that for all $k \in \mathbb{N}$ and all $k \leq n_1 < \dots < n_k$:

$$\|x'_{n_1} + \dots + x'_{n_k}\| \geq ck^{1/q}.$$

For the proof of the following result, we refer the reader to Proposition 4.6 in [11], or Proposition 2.3 in [4] and references therein.

Proposition 3.4. *Let $q \in (1, \infty)$ and X be a Banach space. If X is q -AUC, then X has the q -co-Banach-Saks property.*

Using our Theorem 2.2 and adapting the arguments of Braga in [4] we obtain.

Theorem 3.5. *Let $1 < q < p$ in $(1, \infty)$. Assume that X is an infinite dimensional Banach space with the q -co-Banach-Saks property and Y is a p -AUS and quasi reflexive Banach space. Then X does not coarse Lipschitz embed into Y . More precisely, the compression exponent $\alpha_Y(X)$ of X into Y satisfies the following.*

- (i) *If X contains an isomorphic copy of ℓ_1 , then $\alpha_Y(X) \leq \frac{1}{p}$.*
- (ii) *Otherwise, $\alpha_Y(X) \leq \frac{q}{p}$.*

Proof. We follow the proof of Theorem 4.1 in [4].

Assume first that X contains an isomorphic copy of ℓ_1 and that $\alpha_Y(X) > \frac{1}{p}$. Then there exists $f : X \rightarrow Y$ and $\theta, A, B > 0$ and $\alpha > \frac{1}{p}$ such that

$$\forall x, x' \in X \quad \|x - x'\| \geq \theta \Rightarrow A\|x - x'\|^\alpha \leq \|f(x) - f(x')\| \leq B\|x - x'\|.$$

Let $(x_n)_{n=1}^\infty$ be a sequence in X which is equivalent to the canonical basis of ℓ_1 . We may assume, after a dilation, that there exists $K \geq 1$ such that

$$\forall a_1, \dots, a_k \in \mathbb{R}, \quad \theta \sum_{i=1}^k |a_i| \leq \left\| \sum_{i=1}^k a_i x_i \right\| \leq K\theta \sum_{i=1}^k |a_i|.$$

Then define $\varphi_k : (G_k(\mathbb{N}), d) \rightarrow X$ by

$$\varphi_k(\bar{n}) = x_{n_1} + \dots + x_{n_k}, \quad \text{for } \bar{n} = (n_1, \dots, n_k) \in G_k(\mathbb{N}).$$

We clearly have that $\text{Lip}(\varphi_k) \leq 2K\theta$. Moreover, for $\bar{n} \neq \bar{m} \in G_k(\mathbb{N})$ we have $\|\varphi_k(\bar{n}) - \varphi_k(\bar{m})\| \geq 2\theta$. We deduce that for all $k \in \mathbb{N}$, $\text{Lip}(f \circ \varphi_k) \leq 2\theta KB$. It now follows from Theorem 2.2 that there exists a constant $C > 0$ such that for all $k \in \mathbb{N}$, there is an infinite subset \mathbb{M}_k of \mathbb{N} so that

$$\forall(\bar{n}, \bar{m}) \in I_k(\mathbb{M}_k) \quad \|(f \circ \varphi_k)(\bar{n}) - (f \circ \varphi_k)(\bar{m})\| \leq Ck^{1/p}.$$

On the other hand, for all $(\bar{n}, \bar{m}) \in I_k(\mathbb{M}_k)$, $\|\varphi_k(\bar{n}) - \varphi_k(\bar{m})\| \geq 2k\theta$. Therefore

$$\forall(\bar{n}, \bar{m}) \in I_k(\mathbb{M}_k) \quad \|(f \circ \varphi_k)(\bar{n}) - (f \circ \varphi_k)(\bar{m})\| \geq A2^\alpha \theta^\alpha k^\alpha.$$

This yields a contradiction if k was chosen large enough.

Assume now that X does not contain an isomorphic copy of ℓ_1 and that $\alpha_Y(X) > \frac{q}{p}$. Then there exists $f : X \rightarrow Y$, $\alpha > \frac{q}{p}$ and $\theta, A, B > 0$ such that

$$\forall x, x' \in X \quad \|x - x'\| \geq \theta \Rightarrow A\|x - x'\|^\alpha \leq \|f(x) - f(x')\| \leq B\|x - x'\|.$$

Now, by Rosenthal's theorem, we can pick a normalized weakly null sequence $(x_n)_{n=1}^\infty$ in X . By extracting a subsequence, we may also assume that $(x_n)_{n=1}^\infty$ is a 2-basic sequence in X . We then define $\varphi_k : (G_k(\mathbb{N}), d) \rightarrow X$ as follows:

$$\varphi_k(\bar{n}) = 2\theta(x_{n_1} + \dots + x_{n_k}), \quad \text{for } \bar{n} = (n_1, \dots, n_k) \in G_k(\mathbb{N}).$$

Note that $\text{Lip}(\varphi_k) \leq 4\theta$. Since $(x_n)_{n=1}^\infty$ is 2-basic, we have that for all $\bar{n} \neq \bar{m} \in G_k(\mathbb{N})$, $\|\varphi_k(\bar{n}) - \varphi_k(\bar{m})\| \geq \theta$. Therefore $\text{Lip}(f \circ \varphi_k) \leq 4\theta B$, for all $k \in \mathbb{N}$. It now follows from Theorem 2.2 that, for any $k \in \mathbb{N}$, there exists an infinite subset \mathbb{M}_k of \mathbb{N} such that

$$(5) \quad \forall(\bar{n}, \bar{m}) \in I_k(\mathbb{M}_k) \quad \|(f \circ \varphi_k)(\bar{n}) - (f \circ \varphi_k)(\bar{m})\| \leq Ck^{1/p},$$

where $C > 0$ is a constant independent of k .

We can make sure in our construction that for all $k \in \mathbb{N}$, $\mathbb{M}_{k+1} \subset \mathbb{M}_k$. Let now \mathbb{M} be the diagonalization of the sequence $(\mathbb{M}_k)_{k=1}^\infty$ and enumerate $\mathbb{M} = \{r_1 < \dots < r_i < \dots\}$. Denote $z_n = x_{r_{2n}} - x_{r_{2n+1}}$. By applying the q -co-Banach-Saks property to the semi normalized weakly null sequence $(z_n)_{n=1}^\infty$, we can find $\lambda : \mathbb{N} \rightarrow \mathbb{N}$ increasing and a constant $d > 0$ such that for all $k \in \mathbb{N}$ and all $k \leq n_1 < \dots < n_k$:

$$\begin{aligned} \left\| \sum_{j=1}^k z_{\lambda(n_j)} \right\| &= \left\| \sum_{j=1}^k (x_{r_{2\lambda(n_j)}} - x_{r_{2\lambda(n_j)+1}}) \right\| \\ &= \|\varphi_k(r_{2\lambda(n_1)}, \dots, r_{2\lambda(n_k)}) - \varphi_k(r_{2\lambda(n_1)+1}, \dots, r_{2\lambda(n_k)+1})\| \geq dk^{1/q}. \end{aligned}$$

If k is chosen large enough so that $dk^{1/q} \geq \theta$, we get that for all $k \leq n_1 < \dots < n_k$:

$$\|(f \circ \varphi_k)(r_{2\lambda(n_1)}, \dots, r_{2\lambda(n_k)}) - (f \circ \varphi_k)(r_{2\lambda(n_1)+1}, \dots, r_{2\lambda(n_k)+1})\| \geq Ad^\alpha k^{\alpha/q}.$$

It is now important to note that, due to the diagonal construction of \mathbb{M} , we have that for all $k \leq n_1 < \dots < n_k$

$$((r_{2\lambda(n_1)}, \dots, r_{2\lambda(n_k)}), (r_{2\lambda(n_1)+1}, \dots, r_{2\lambda(n_k)+1})) \in I_k(\mathbb{M}_k).$$

Therefore, this yields a contradiction with (5) if k is chosen large enough. \square

The following result is now a direct consequence of Theorem 3.5 and Proposition 3.4.

Corollary 3.6. *Let $1 < q < p < \infty$. Assume that X is q -AUC and Y is a p -AUS and quasi reflexive Banach space. Then*

$$\alpha_Y(X) \leq \frac{q}{p}.$$

For our next application, we need to recall the definition of the Szlenk index. This ordinal index was first introduced by W. Szlenk [18], in a slightly different form, in order to prove that there is no separable reflexive Banach space universal for the class of all separable reflexive Banach spaces.

So, let X be a Banach space, K a weak*-compact subset of its dual X^* and $\varepsilon > 0$. Then we define

$s'_\varepsilon(K) = \{x^* \in K \text{ s.t. for any weak}^* \text{-neighborhood } U \text{ of } x^*, \text{diam}(K \cap U) \geq \varepsilon\}$ and inductively the sets $s_\varepsilon^\alpha(K)$ for α ordinal as follows: $s_\varepsilon^{\alpha+1}(K) = s'_\varepsilon(s_\varepsilon^\alpha(K))$ and $s_\varepsilon^\alpha(K) = \bigcap_{\beta < \alpha} s_\varepsilon^\beta(K)$ if α is a limit ordinal.

Then $Sz(K, \varepsilon) = \inf\{\alpha, s_\varepsilon^\alpha(K) = \emptyset\}$ if it exists and we denote $Sz(K, \varepsilon) = \infty$ otherwise. Next we define $Sz(K) = \sup_{\varepsilon > 0} Sz(K, \varepsilon)$. The Szlenk index of X

is $Sz(X) = Sz(B_{X^*})$. We also denote $Sz(X, \varepsilon) = Sz(B_{X^*}, \varepsilon)$.

We shall apply the following renorming theorem, which is proved in [7].

Theorem 3.7. *Let $q \in (1, \infty)$ and X be a Banach space. Assume that there exists $C > 0$ such that*

$$\forall \varepsilon > 0 \quad Sz(X, \varepsilon) \leq C\varepsilon^{-q}.$$

Then, for all $r \in (q, \infty)$, X admits an equivalent norm whose dual norm is r -weak-AUC.*

Since a dual norm which is r -weak*-AUC is also r -AUC, we obtain the following statement as an immediate consequence of our Theorem 3.7 and Corollary 3.6.

Corollary 3.8. *Let $1 < q < p < \infty$. Assume that Y is a p -AUS and quasi reflexive Banach space. Assume also that there exists $C > 0$ such that*

$$\forall \varepsilon > 0 \quad Sz(X, \varepsilon) \leq C\varepsilon^{-q}.$$

Then

$$\alpha_Y(X^*) \leq \frac{q}{p}.$$

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