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Abstract. We here address the question of restoration of uniqueness in mean-field games deriving from deterministic differential games with a large number of players. The general strategy for restoring uniqueness is inspired from earlier similar results on ordinary and stochastic differential equations. It consists in randomizing the equilibria through an external noise.

As a main feature, we choose the external noise as an infinite dimensional Ornstein-Uhlenbeck process. We first investigate existence and uniqueness of a solution to the noisy system made of the mean-field game forced by the Ornstein-Uhlenbeck process. We also show how such a noisy system can be interpreted as the limit version of a stochastic differential game with a large number of players.

1. Introduction

The theory of mean-field games has encountered a tremendous success since it was introduced in 2006 by two independent groups, Lasry and Lions [43, 44, 45] on the one hand and Huang, Caines and Malhamé [38, 39] on the other hand.

The purpose of mean-field games is to provide an asymptotic formulation for differential games involving a large number of players interacting with one another in a mean-field way. The standard writing of mean-field games consists in a forward-backward system involving a forward Fokker-Planck equation describing the state of the population in equilibrium and a backward Hamilton-Jacobi-Bellman describing the optimal cost to a typical player when the population is in equilibrium. This goes back to the earlier works of Lasry and Lions, see [43, 44, 45], and to the subsequent series of lectures by Lions at the Collège de France, see [46, 47] together with the lecture notes [7] of Cardaliaguet. This approach, referred to as "the PDE approach", fits both the cases when the underlying differential games are deterministic or stochastic; in the deterministic case, the PDEs involved in the representation are first-order PDEs, whilst they are second-order PDEs in the stochastic framework. As pointed out in several works by Carmona and Delarue, see [14, 15, 16, 17, 18], the problem may be reformulated in a purely Lagrangian form, using, instead of a forward-backward system of two PDEs, a forward-backward system of two ordinary or stochastic differential equations of the McKean-Vlasov type, the name "McKean-Vlasov" emphasizing the fact that the coefficients of the equations depend upon the statistical distribution of the solution. In that case, the differential equations appearing in the representation are ordinary or stochastic according to the deterministic or stochastic nature of the differential game; when the equations are ordinary, randomness manifests in the dynamics through the initial condition only.

Quite remarkably, the forward-backward structure is common to both formulations, the PDE one, in which equations are deterministic but set in infinite dimension, and the Lagrangian one, in which equations are finite dimensional but of the McKean-Vlasov type. The forward-backward nature of

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the problem is a crucial feature in the analysis of mean-field games since forward-backward systems are known to be hard to solve: Roughly speaking, Cauchy-Lipschitz theory for forward-backward systems of differential equations holds in small time only, even when the differential equations are finite dimensional. In arbitrary time, existence or uniqueness of solutions may fail, in which case the whole system is said to develop singularities in finite time. The typical example for such a phenomenon is provided by the inviscid one-dimensional backward Burgers equation: Solutions may be represented through characteristics that describe the motion of a representative particle. These characteristics solve the forward equation of the forward-backward system representing the Burgers equation; meanwhile, the backward equation describes the dynamics of the velocity of the particle, which remains constant along the motion of the particle. It is well known that, for some choices of the terminal condition, the forward paths may split, such a splitting phenomenon being usually referred to as a "shock". In this regard, one interesting question is to decide of the right continuation of the forward paths once singularities have emerged and uniqueness has been lost. Anyhow, and quite remarkably, the existence of shocks is deeply connected with the form of the terminal condition and, under an appropriate monotonicity assumption on the terminal condition, singularities cannot show up and existence and uniqueness hold true in arbitrary time.

The picture for solving mean-field games is quite similar. Sufficient conditions are known under which a solution (say for instance a solution to one of the two formulations) does exist in arbitrary time, but, except in small time, uniqueness may not be guaranteed in most of the cases. We refer to the original papers [43, 44, 45], to the video lectures [46], to the lecture notes [7] and to the two-volume book [16, 17] for a review on the general strategy used to solve a mean-field game. We also refer to the subsequent papers [8, 10, 11, 12] for other strategies, in connection with the theory of mean-field control problem, and to [33, 34, 35] for the analysis of more intricated cases. For the small time analysis, we also refer to [38] and to [16, Chapter 4] and [17, Chapter 5]. Existence of a solution to the Lagrangian formulation may be found in [13, 14, 18], see also [16, Chapter 4]. Regarding uniqueness in arbitrary time, things are as follows. Similar to the analysis of the Burgers equation, uniqueness is know to hold when the coefficients satisfy a suitable monotonicity condition with respect to the distribution of the population. The most popular monotonicity property used in this direction is due to Lasry and Lions, see once again [43, 44, 45], and is usually referred to as the Lasry-Lions monotonicity condition. However, as emphasized in [1] and in [16, Chapters 4 and 5], other forms of monotonicity may be used.

In analogy with our short description of the forward-backward system associated with Burgers' equation, the forward-backward system used for representing a mean-field game (whatever the formulation) reads as the system of characteristics of some partial differential equation. In the framework of mean-field games, this partial differential equation is called the "master equation" of the game, the word "master" emphasizing the fact that the equation encapsulates all the information that is necessary to describe the equilibria of the game. This equation was investigated first by Lions in his lectures at the Collège de France and then by Gangbo and Swiech [32] in small time, and by Chassagneux, Crisan and Delarue [21] and by Cardaliaguet, Delarue, Lasry and Lions [9] in arbitrary time. In the latter reference, it is shown to play a crucial role in the justification of the passage to the limit, from games with finitely many players to mean-field games. In arbitrary time, analysis of the equation is performed under the additional assumption that coefficients satisfy the Lasry-Lions monotonicity condition. We refer to [17, Chapters 5 and 6] for another point of view on the results contained in [21, 9] and to [3, 4, 5, 15, 36, 37, 40] for other and more heuristic approaches.

In the current paper, we consider the case when the Lasry-Lions monotonicity condition may fail, the question being to find a strategy to restore uniqueness. Pursuing the same parallel as before, we observe that, somehow, a similar program has been investigated for the Burgers equation: Adding a Laplace operator in front of the Burgers equation permits to restore the existence and uniqueness

of a classical solution in arbitrary time (as opposed to the inviscid case, for which the existence of a classical solution may fail). From the Lagrangian point of view, the additional Laplace operator reads as a Brownian motion that forces the motion of the underlying particle. Similar to the viscous version of the Burgers equation, the stochastically forced forward-backward system describing the "random characteristics" of the viscous Burgers equation is know to be uniquely solvable, see Delarue [24]. In a way, "noise restores uniqueness in the Lagrangian formulation". Our goal here is to adapt this strategy to mean-field games.

The idea of restoring uniqueness by means of a random forcing has been extensively studied in probability theory. It goes back to the earlier work of Zvonkin [55] on the solvability of one-dimensional stochastic differential equations driven by non-Lipschitz continuous drifts. Several people also contributed to the subject and investigated the higher dimensional framework, among which Veretennikov [50], Flandoli, Russo and Wolf [29, 30], Krylov and Röckner [41], Davie [23]... Similar questions have been also addressed in the framework of infinite dimensional stochastic differential equations, see for instance Flandoli, Gubinelli and Priola [28] and the monograph by Flandoli [27]. In any case, the idea is to force in a convenient way the Lagrangian dynamics in order to restore uniqueness of solutions. Transposed to mean-field games theory, the question is here to find a suitable randomly forced version of the original mean-field games in order to guarantee uniqueness of the equilibria.

Here is our main result: For a certain class of coefficients, we manage to restore uniqueness to mean-field games –deriving from a deterministic differential game– by means of a stochastic forcing. The stochastic forcing mostly consists in an infinite dimensional Ornstein-Uhlenbeck process. The reason why it is chosen of infinite dimension is well-understood. Roughly speaking, the stochastic forcing is indeed intended to act on the elements of the "infinite dimensional manifold" formed by the d-dimensional probability measures with a finite second-order moment, which is usually called "the Wasserstein space" (d is the state dimension of a typical player). Here, probability measures are used to describe the state of the population, whilst the limitation to probability measures with a finite second-order moment is a convenient assumption which permits to benefit from the Hilbertian structure of any L^2 space constructed above the Wasserstein space. Returning to the description of the forcing applied to the mean-field system, it is then well-understood that, in order to capture all the "possible tangent directions" to the manifold at any point of it, it is necessary to use a noise of infinite dimension. In order to bypass any description of the differential geometry on the space of probability measures, we use the approach introduced by Lions in his lectures: We lift equilibria from the space of probability measures to a well-chosen space of square-integrable random variables and then use, as we just alluded to, the Hilbertian structure of this L^2 space. Fortunately, the Lagrangian description of mean-field games gives a canonical way to realize such a lift. Our strategy then consists in forcing the dynamics of the random variables representing the equilibria. In other words, our goal is to force a differential equation defined on an L^2 space. A convenient way to do so is to force the modes of the solution along an orthonormal basis of L^2. For instance, when L^2 is chosen as the space $L^2(\mathbb{S}^1;\mathbb{R}^d)$ of square-integrable Borel mappings from $\mathbb{S}^1$ to $\mathbb{R}^d$, where $\mathbb{S}^1$ denotes the one-dimensional torus, it suffices to force the Fourier modes of square-integrable $\mathbb{R}^d$-valued functions defined on $\mathbb{S}^1$. It is then a standard fact from the theory of stochastic partial differential equations that the Ornstein-Ulhenbeck process has nice smoothing properties on $L^2(\mathbb{S}^1;\mathbb{R}^d)$, which is the key feature for restoring uniqueness.

In addition to proving existence and uniqueness of a solution to the noisy version of the original mean-field game, we also show that the randomly forced version may be interpreted as the limit of a game with a large number of agents. As a main feature, the finite game not only exhibits mean field interactions, which is well expected, but also local interactions to nearest neighbours, which is certainly a new point in the literature on mean-field games; from a mathematical point of view, local interactions arise from the discretization of the operator driving the additional infinite dimensional

Ornstein-Ulhenbeck process. The route we take to connect the finite and the infinite regimes is to prove that, from any equilibrium to the limiting problem, we can construct an approximate Nash equilibrium to the finite system. Although this way of doing is pretty standard in the theory of mean-field games, it turns out to be more challenging in our setting because of the additional local interactions. Of course, another route would consist in proving that any (say closed loop) equilibria to the finite player system do converge to the limiting equilibrium as the number of players grows up. It turns out to be a pretty difficult question in the framework of mean-field games; in this framework, the only generic approach that has been known to handle the convergence of closed loop equilibria is due to [9] and is based on the aforementioned master equation. We guess that a similar approach could be implemented here and we hope to address it in a future work. In fact, a form of master equation is already addressed in the paper: We prove that the equilibrium strategy (in the limiting regime) can be put in a feedback form and we show that the feedback function, which may be regarded as a function from $L^2(\mathbb{S}^1;\mathbb{R}^d)$ into itself, is a mild solution to a system of nonlinear equations on $L^2(\mathbb{S}^1;\mathbb{R}^d)$, driven by the second-order operator generated by the Ornstein-Ulhenbeck process inserted in the dynamics; the latter system reads as a kind of master equation for our problem. We just say a "kind of" because the usual master equation for mean field games is the equation satisfied by the value function and not by the feedback function. In the standard mean field game regime, both are explicitly connected since the feedback function is the derivative of the value function with respect to the so-called "private state variable". Things are slightly different in our setting and we prefer to work, in the noisy regime, with the feedback function directly. At the end of the day, our guess is that, to plug our own version of the master equation into the machinery developed by [9], we would need the feedback function to be more regular than what we show below. Once again, this question is deferred to another work.

Another interesting prospect that we would like to investigate is the zero noise limit: We guess that any limit of the solutions (to the noisy system), as the intensity of the forcing decreases to 0, should generate a randomized equilibrium to the original mean-field game. We are not aware of similar results in the theory of mean-field games, except maybe in the case investigated by Foguen [31]. There, restoration of uniqueness is investigated for linear-quadratic mean-field games. In comparison with the general case we handle here, linear-quadratic mean-field games present the main advantage to have parametrized solutions: Equilibria are Gaussian and are thus parametrized by their mean and variance and thus live in a finite-dimensional subspace of the space of probability measures. In this case, it suffices to use a finite dimensional noise to restore uniqueness, which is precisely what is done in [31]; then, it seems that, for some linear-quadratic mean-field games, zero noise limits could be addressed by using arguments similar to [2]. Once again, we hope to make this point clear in a future work in collaboration with Foguen.

Lastly, we emphasize the fact that all these questions should be revisited for mean-field games deriving from stochastic differential games with idiosyncratic noises. We believe that part of the technology developed in the paper could be recycled in this framework, except for the fact, due to the simultaneous presence of two sources of noise –the idiosyncratic one and the external one used to restore uniqueness–, the formulation of the randomized version of the game should require a modicum of care. We make this fact clear in the text.

The paper is organized as follows. We present in Section 2 the randomized version of the game. Main results are exposed in Section 3. The proof of existence and uniqueness of a solution to the randomized game is given in Section 4. Connection with finite games is addressed in Section 5.

2. Mollified/Randomized MFG

We first present the original Mean-Field Game (MFG for short) and then describe the "mollified" or "randomized" version that is expected to be uniquely solvable.

Throughout the article, d is an integer greater than 1 and $\mathcal{P}_2(\mathbb{R}^d)$ denotes the space of probability measures over $\mathbb{R}^d$. It is equipped with the 2-Wasserstein distance (see for instance [51, 52, 16]):

$$\forall \mu, \nu \in \mathcal{P}_2(\mathbb{R}), \quad W_2(\mu,\nu) = \inf_{\pi} \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\pi(x,y) \right)^{1/2},$$

where the infimum in the last line is taken over all the probability measures $\pi \in \mathcal{P}_2(\mathbb{R}^d \times \mathbb{R}^d)$ that have μ and ν as respective marginals.

2.1. **Original problem.** We start with a simple MFG consisting of the following matching problem:

(1) Given a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and a flow of probability measures $\boldsymbol{\mu} = (\mu_t)_{t\in[0,T]}$ on $\mathbb{R}^d$, consider the optimization problem

$$J^{\boldsymbol{\mu}}(\boldsymbol{\alpha}) = \mathbb{E}\Big[g\big(X_T^{\boldsymbol{\alpha}}, \mu_T\big) + \int_0^T \big(f\big(X_t^{\boldsymbol{\alpha}}, \mu_t\big) + \frac{1}{2}|\alpha_t|^2 \big) dt \Big],$$

over controlled dynamics of the form

$$dX_t^{\boldsymbol{\alpha}} = b(X_t^{\boldsymbol{\alpha}}, \mu_t) dt + \alpha_t dt, \tag{1}$$

with the initial condition $X_0^{\alpha} = X_0$, X_0 being a random variable from Ω to $\mathbb{R}^d$ with μ_0 as distribution.

(2) Find $(\mu_t)_{t\in[0,T]}$ in such a way that the flow of marginal measures of the optimal path $(X_t^\star)_{t\in[0,T]}$ in the above optimization problem satisfies

$$\mu_t = \mathcal{L}\big(X_t^\star\big), \quad t \in [0,T]. \tag{2}$$

Here, $\boldsymbol{\alpha}$ is called the control and is a jointly-measurable mapping

$$\boldsymbol{\alpha} : [0,T] \times \Omega \ni (t,\omega) \mapsto \alpha_t(\omega) \in \mathbb{R}^d,$$

satisfying

$$\mathbb{E} \int_0^T |\alpha_t|^2 dt < \infty.$$

The coefficient $b : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}^d$ is called the drift. It is assumed to be jointly Lipschitz continuous, so that (1) is uniquely solvable for any realization $\omega \in \Omega$ and the solution $\boldsymbol{X} : [0,T] \times \Omega \ni (t,\omega) \mapsto X_t(\omega) \in \mathbb{R}^d$ is also jointly-measurable. The coefficients $g : \mathbb{R}^d \to \mathbb{R}$ and $f : \mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d) \to \mathbb{R}$ are called cost functionals. They are assumed be jointly continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. Throughout the paper, we assume them to be at most of quadratic growth in the sense that, for some constant $C \geqslant 0$,

$$|f(x,\mu)| + |g(x,\mu)| \leqslant C\big(1 + |x|^2 + M_2(\mu)^2\big), \quad x \in \mathbb{R}^d, \quad \mu \in \mathcal{P}_2(\mathbb{R}^d),$$

where $M_2(\mu)^2 = \int_{\mathbb{R}^d} |x|^2 d\mu(x)$. In particular, it is well checked that the expectation in the definition of the cost J makes sense.

Remark 1. *All the coefficients are here assumed to be time homogeneous. This is for simplicity only and the results given below can be extended quite easily to the time-inhomogeneous framework. Similarly, the fact that f is a quadratic function of α is for convenience only; we could handle more general running costs of the form $f(x,\mu,\alpha)$ that are uniformly convex in α, see for instance* [16, Chapters 3 and 4]. *However, the fact that b is linear in α is really crucial for our purpose, at least if we want to make use, as we do below, of the sufficient version of the Pontryagin principle.*

Another possible generalization would be to insert a Brownian motion in the dynamics (1)*, in which case the mean-field game would be called "stochastic" or "second-order". However, the approach developed below for restoring uniqueness of solutions does not apply to that case, see Remark 5 below. We hope to address this question in a future work.*

Usually, solutions to the matching problem (2) may be characterized in two ways. The original one is to characterize the optimization problem in the first item above through a first order Hamilton-Jacobi-Bellman equation (HJB for short):

$$\partial_t u(t,x) + b(x,\mu_t)\cdot\partial_x u(t,x) + f(x,\mu_t) - \frac{1}{2}|\partial_x u(t,x)|^2 = 0, \tag{3}$$

for $(t,x) \in [0,T]\times\mathbb{R}^d$, with $u(T,x) = g(x,\mu_T)$ as boundary condition. Here, the function $u : [0,T]\times\mathbb{R}^d \to \mathbb{R}$ is understood as the value function of the optimization problem (in the environment $\boldsymbol{\mu} = (\mu_t)_{0\leqslant t\leqslant T}$). Given the value function, it is known that the optimal control process in the optimization problem reads (at least formally since the gradient below may not exist or may only exist as a multi-valued mapping):

$$\boldsymbol{\alpha}^\star = \big(\alpha_t^\star = -\partial_x u(t,X_t^\star)\big)_{0\leqslant t\leqslant T},$$

where $(X_t^\star)_{0\leqslant t\leqslant T}$ now denotes the solution of the ordinary differential equation:

$$dX_t^\star = \Big(b\big(X_t^\star,\mu_t\big) - \partial_x u\big(t,X_t^\star\big)\Big)dt, \quad t\in[0,T].$$

It is now easy to implement analytically the fixed point condition in the second item above. Under the identification $(\mu_t = \mathcal{L}(X_t^\star))_{0\leqslant t\leqslant T}$, the flow $\boldsymbol{\mu} = (\mu_t)_{0\leqslant t\leqslant T}$ must solve the nonlinear Fokker-Planck equation:

$$\partial_t\mu_t + \partial_x\Big(\big(b(x,\mu_t) - \partial_x u(t,x)\big)\mu_t\Big) = 0, \quad (t,x)\in[0,T]\times\mathbb{R}^d. \tag{4}$$

with the initial condition μ_0 for the population. The forward-backward system made of (3) and (4) is usually called the MFG system of PDEs. We refer to aforementioned references [7, 46] for further details.

Another strategy for characterizing the equilibria is to use the Pontryagin principle. Under appropriate conditions, we know that the optimal paths of the control problem $\inf_{\boldsymbol{\alpha}} J^{\boldsymbol{\mu}}(\boldsymbol{\alpha})$ in the first item of the above definition of an MFG equilibrium solve the forward-backward system of two ODEs:

$$\begin{aligned} dX_t^\star &= \Big(b\big(X_t^\star,\mu_t\big) - Y_t^\star\Big)dt,\\ dY_t^\star &= \Big(-\partial_x b\big(X_t^\star,\mu_t\big)Y_t^\star - \partial_x f\big(X_t^\star,\mu_t\big)\Big)dt, \end{aligned} \tag{5}$$

with the initial condition $X_0^\star = X_0$ and the terminal condition $Y_T = \partial_x g(X_T^\star,\mu_T)$. Implementing the matching condition (2) in the second item of the definition of an MFG equilibrium, we deduce that equilibria of the MFG must solve the forward-backward system of the McKean-Vlasov type:

$$\begin{aligned} dX_t^\star &= \Big(b\big(X_t^\star,\mathcal{L}(X_t^\star)\big) - Y_t^\star\Big)dt,\\ dY_t^\star &= \Big(-\partial_x b\big(X_t^\star,\mathcal{L}(X_t^\star)\big)Y_t^\star - \partial_x f\big(X_t^\star,\mathcal{L}(X_t^\star)\big)\Big)dt, \end{aligned} \tag{6}$$

with the terminal condition $Y_T = \partial_x g(X_T^\star,\mathcal{L}(X_T^\star))$. Under suitable convexity properties of the coefficients in the variable x, which we spell out in Subsection 2.6 below, the system (6) is not only a necessary condition satisfied by any equilibria of the mean-field game but is also a sufficient condition. In this framework, (6) characterizes the equilibria of the game. This is precisely this system that we force stochastically below.

Throughout the article, we focus on this specific convex regime when the Pontryagin principle is both a necessary and a sufficient condition of optimality. Although it demands strong assumptions on the structure of the coefficients in the spatial variable x, this so-called "convex regime" turns out to be especially useful for our purposes: It provides a sharp framework under which, for a given input $\boldsymbol{\mu} = (\mu_t)_{0\leqslant t\leqslant T}$, the system (5) is uniquely solvable for any initial condition and its

solution is stable under perturbation of the initial condition and perturbation of the input. It is worth mentioning that, even in this strong setting, it still makes sense to address the restoration of uniqueness for the mean-field game, since the McKean-Vlasov forward-backward system (6) may not be uniquely solvable. Clearly, we shall appreciate having a sharp framework for solving the control problem $\inf_{\boldsymbol{\alpha}} J^{\boldsymbol{\mu}}(\boldsymbol{\alpha})$ as it will permit to focus on the difficulties that are exclusively related with the non-uniqueness of the MFG equilibria.

2.2. **Reformulation.** In order to proceed, we first notice that $(\Omega, \mathcal{A}, \mathbb{P})$ may chosen as the probability space $(\mathbb{S}^1, \mathcal{B}(\mathbb{S}^1), \mathrm{Leb}_1)$, where Leb_1 is the Lebesgue measure. In this regard, we recall from [6] that there exists a measurable function $\Psi : \mathbb{S}^1 \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}^d$ such that, for every probability μ on $\mathbb{R}^d$, $[0,1] \ni u \mapsto \Psi(u,\mu)$ is a random variable with μ as distribution.

With such a convention, the control $\boldsymbol{\alpha}$ is understood as a jointly-measurable mapping

$$\boldsymbol{\alpha} : [0,T] \times \mathbb{S}^1 \ni (t,x) \mapsto \alpha_t(x) \in \mathbb{R}^d,$$

and the cost functional may be rewritten as

$$\begin{aligned} J^{\boldsymbol{\mu}}(\boldsymbol{\alpha}) = &\int_{\mathbb{S}^1} g\big(x, \mathcal{L}(X_T^{\boldsymbol{\alpha}})\big) d\mathcal{L}(X_T^{\boldsymbol{\alpha}})(x) \\ &+ \int_0^T \left[\int_{\mathbb{S}^1} f\big(x, \mathcal{L}(X_t^{\boldsymbol{\alpha}})\big) d\mathcal{L}(X_t^{\boldsymbol{\alpha}})(x) + \frac{1}{2} \int_{\mathbb{S}^1} |\alpha_t(x)|^2 dx \right] dt. \end{aligned} \tag{7}$$

With this reformulation, we introduce the L^2 spaces $L^2(\mathbb{S}^1) = L^2(\mathbb{S}^1, \mathcal{B}(\mathbb{S}^1), \mathrm{Leb}_1)$ and $L^2(\mathbb{S}^1; \mathbb{R}^d) \cong [L^2(\mathbb{S}^1)]^d$. A key fact is that the functions

$$e^0 : \mathbb{S}^1 \ni x \mapsto 1, \quad e^{n,+} : \mathbb{S}^1 \ni x \mapsto \sqrt{2}\cos\big(2\pi n x\big), \quad e^{n,-} : \mathbb{S}^1 \ni x \mapsto \sqrt{2}\sin\big(2\pi n x\big), \quad n \in \mathbb{N}^*,$$

form an orthonormal basis of $L^2(\mathbb{S}^1)$. In particular, for any element $\ell \in L^2(\mathbb{S}^1)$, we call ℓ^0, $\ell^{n,+}$ and $\ell^{n,-}$, $n \in \mathbb{N}^*$, the different weights of ℓ; we use the same notation when $\ell \in L^2(\mathbb{S}^1; \mathbb{R}^d)$, in which case ℓ^0, $\ell^{n,+}$ and $\ell^{n,-}$ are vectors of size d. Then, we may write

$$\int_{\mathbb{S}^1} |\alpha_t(x)|^2 dx = |\alpha_t^0|^2 + \sum_{n \in \mathbb{N}*} \big(|\alpha_t^{n,+}|^2 + |\alpha_t^{n,-}|^2 \big),$$

which we shall often summarize into

$$\int_{\mathbb{S}^1} |\alpha_t(x)|^2 dx = \sum_{n \in \mathbb{N}} |\alpha_t^{n,\pm}|^2,$$

with the convention that $\alpha^{0,+} = \alpha^0$ and $\alpha^{0,-} = 0$.

Moreover, given a mapping $h : \mathbb{R}^d \to \mathbb{R}$, at most of linear growth, we may consider the mapping

$$\mathfrak{h}_0 : L^2\big(\mathbb{S}^1; \mathbb{R}^d\big) \ni \ell \mapsto \mathfrak{h}_0(\ell) = \int_{\mathbb{S}^1} h\big(\ell(x)\big) dx.$$

Then, we observe that the cost functional $J^{\boldsymbol{\mu}}$ may be rewritten:

$$J^{\boldsymbol{\mu}}(\boldsymbol{\alpha}) = \mathfrak{g}_0\big(X_T^{\boldsymbol{\alpha}}(\cdot), \mu_T\big) + \int_0^T \Big\{ \mathfrak{f}_0\big(X_t^{\boldsymbol{\alpha}}(\cdot), \mu_t\big) + \frac{1}{2}\Big(|\alpha_t^0|^2 + \sum_{n \in \mathbb{N}} \big(|\alpha_t^{n,+}|^2 + |\alpha_t^{n,-}|^2 \big) \Big) \Big\} dt, \tag{8}$$

where $\boldsymbol{X}^{\boldsymbol{\alpha}}(\cdot) = (X_t^{\boldsymbol{\alpha}}(\cdot))_{0 \leqslant t \leqslant T}$ (pay attention to the dot we put in the notation to emphasize the fact that the path has functional values) is a path with values in $L^2(\mathbb{S}^1; \mathbb{R}^d)$ in such a way that $\boldsymbol{X}(\cdot) = \boldsymbol{X}^{\boldsymbol{\alpha}}(\cdot)$ (we get rid of the superscript $\boldsymbol{\alpha}$ to simplify the notations) satisfies

$$\dot{X}_t^{n,\pm} = \mathfrak{b}^{n,\pm}\big(X_t(\cdot), \mu_t\big) + \alpha_t^{n,\pm}, \quad t \in [0,T], \quad n \in \mathbb{N}, \tag{9}$$

where, for $\ell \in L^2(\mathbb{S}^1;\mathbb{R}^d)$ and $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\mathfrak{b}^{n,\pm}(\ell,\mu) = \int_{\mathbb{S}^1} b\big(\ell(x),\mu\big) e^{n,\pm}(x)dx, \quad n \in \mathbb{N},$$

denote the modes of $\mathfrak{b}(\ell,\mu)$.

2.3. **Enlarged problem.** A strategy for restoring uniqueness to mean-field games now consists in forcing the modes $(\boldsymbol{X}^{n,\pm})_{n\in\mathbb{N}}$ introduced in the previous paragraph. To do so, we need to disentangle the two sources of noise that will manifest in the construction of the new mean field game: On the one hand, the initial condition is still defined as a square-integrable random variable on the torus $\mathbb{S}^1$ (equipped with the collection $\mathcal{L}(\mathbb{S}^1)$ of Lebesgue sets); on the other hand, we need another space for carrying the random forcing acting on the nodes $(\boldsymbol{X}^{n,\pm})_{n\in\mathbb{N}}$.

Having this picture of our general strategy in mind, we now enlarge the probability space and consider $\Omega = \mathbb{S}^1 \times \Omega_0$, where $(\Omega_0, \mathcal{A}_0, \mathbb{F}_0 = (\mathcal{F}_{0,t})_{t\in[0,T]}, \mathbb{P}_0)$ is a complete filtered probability space equipped with a collection $(\boldsymbol{W}^0, (\boldsymbol{W}^{n,+}, \boldsymbol{W}^{n,-})_{n\in\mathbb{N}^*})$ of $\mathbb{F}_0$-Brownian motions of dimension d. The filtration $\mathbb{F}_0$ satisfies the usual conditions.

We then equip Ω with the completion $\mathcal{A}$ of $\mathcal{L}(\mathbb{S}^1)\otimes\mathcal{A}_0$ and with the completion $\mathbb{P}$ of $\mathrm{Leb}_1 \otimes \mathbb{P}_0$. We call $\mathbb{F}$ the completion of the filtration $(\mathcal{L}(\mathbb{S}^1)\otimes\mathcal{F}_{0,t})_{t\in[0,T]}$ and we denote by ξ the identity mapping on $\mathbb{S}^1$ (which is extended in a canonical way to Ω). Despite the fact that Ω has been enlarged, we keep the same notations as above for $\mathfrak{h}_0(\ell)$ and $\ell^{n,\pm}$ whenever ℓ is an element of $L^2(\mathbb{S}^1;\mathbb{R}^d)$. In particular, whenever X is a square-integrable random variable defined on Ω, we may consider, for $\mathbb{P}_0$-almost every $\omega_0 \in \Omega_0$, the random variable $X(\cdot,\omega_0)$ on $\mathbb{S}^1$ and then $\mathfrak{h}_0(X(\cdot,\omega_0))$ and $X^{n,\pm}(\cdot,\omega_0)$. Recall indeed from the version of Fubini's theorem for completion of product spaces that, for $\mathbb{P}_0$-almost every ω_0, $X(\cdot,\omega_0)$ is a square-integrable random variable on $(\mathbb{S}^1,\mathcal{L}(\mathbb{S}^1))$, see Lemma 2 for more details.

The question now is to explain how to use the collection $(\boldsymbol{W}^0, (\boldsymbol{W}^{n,\pm})_{n\in\mathbb{N}^*})$ in order to construct a uniquely solvable randomized mean field game. A naïve way would consist in forcing each mode process $\boldsymbol{X}^{n,\pm} = (X_t^{n,\pm})_{0\leqslant t\leqslant T}$ in (9), for $n \in \mathbb{N}$, by the corresponding Wiener process $\boldsymbol{W}^{n,\pm}$ (with the same convention as above that $\boldsymbol{X}^0$ and $\boldsymbol{W}^0$ are understood as $\boldsymbol{X}^{0,+}$ and $\boldsymbol{W}^{0,+}$). However, it is a well-known fact that the solution

$$X_t(\cdot,\omega_0) = \sum_{n\in\mathbb{N}} X_t^{n,\pm}(\cdot,\omega_0) e^{n,\pm}(\cdot), \quad t \in [0,T],$$

would not belong to $L^2(\mathbb{S}^1;\mathbb{R}^d)$.

In order to render the modes $((X_t^{n,\pm})_{0\leqslant t\leqslant T})_{n\in\mathbb{N}}$ square summable, we may force (9) by another $\mathbb{F}_0$-semi-martingale process $\boldsymbol{U}^{n,\pm}$ such that

$$\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\Big(\sum_{n\in\mathbb{N}} |U_t^{n,\pm}|^2\Big)\Big] < \infty, \tag{10}$$

namely

$$dX_t^{n,\pm} = \Big(\mathfrak{b}^{n,\pm}\big(X_t(\cdot),\mu_t\big) + \alpha_t^{n,\pm}\Big)dt + dU_t^{n,\pm}, \quad t \in [0,T], \quad n \in \mathbb{N}. \tag{11}$$

Assume for instance that

$$\int_0^T \sum_{n\in\mathbb{N}} \Big(\sup_{x\in\mathbb{S}^1} |\mathfrak{b}^{n,\pm}(x,\mu_t)|^2\Big)dt < \infty. \tag{12}$$

Then,

$$\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\Big(\sum_{n\in\mathbb{N}} |X_t^{n,\pm}|^2\Big)\Big] < \infty, \tag{13}$$

and we can regard

$$X_t(\cdot,\omega_0)=\sum_{n\in\mathbb{N}} X_t^{n,\pm}(\omega_0)e^{n,\pm}(\cdot),\quad t\in[0,T],$$

as a process with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$.

In this regard, the following lemma (see for instance [17, Chapter 2] for similar considerations) makes clear the connection between random variables from Ω into $\mathbb{R}$ and random variables from Ω_0 into $L^2(\mathbb{S}^1)$:

Lemma 2. *Assume that X is a square-integrable $\mathbb{R}^d$-valued random variable on Ω. Then, for $\mathbb{P}_0$ almost every $\omega_0\in\Omega_0$, $\mathbb{S}^1\ni x\mapsto X(x,\omega_0)\in L^2(\mathbb{S}^1;\mathbb{R}^d)$; moreover, we can construct a random variable $X(\cdot)$ on Ω_0 with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$, such that, for $\mathbb{P}_0$-almost every $\omega_0\in\Omega_0$, $\mathbb{S}^1\ni x\mapsto X(x,\omega_0)$ coincides in $L^2(\mathbb{S}^1;\mathbb{R}^d)$ with the realization of the variable $X(\cdot)$ at ω_0. Conversely, given a random variable $X(\cdot)$ from Ω_0 to $L^2(\mathbb{S}^1;\mathbb{R}^d)$, we can construct a random variable X on Ω such that, for $\mathbb{P}_0$-almost every $\omega_0\in\Omega_0$, $\mathbb{S}^1\ni x\mapsto X(x,\omega_0)$ coincides in $L^2(\mathbb{S}^1;\mathbb{R}^d)$ with the realization of the variable $X(\cdot)$ at ω_0.*

Proof. The proof is pretty straightforward. Given a square-integrable $\mathbb{R}^d$-valued random variable on Ω, Fubini's theorem for completion of product spaces says that, for $\mathbb{P}_0$-almost every ω_0, $\mathbb{S}^1\ni x\mapsto X(x,\omega_0)$ is a square-integrable random variable on $(\mathbb{S}^1,\mathcal{L}(\mathbb{S}^1))$. In particular, for $\mathbb{P}_0$-almost every ω_0, we can define $X^{n,\pm}(\omega_0)=\int_{\mathbb{S}^1}X(x,\omega_0)e^{n,\pm}(x)dx$. Each $X^{n,\pm}$ is a random variable (on Ω_0). We then let

$$X(\cdot)=\sum_{n\in\mathbb{N}}X^{n,\pm}e^{n,\pm}(\cdot).$$

Noticing that a mapping $\chi(\cdot)$ from Ω_0 into $L^2(\mathbb{S}^1;\mathbb{R}^d)$ is measurable with respect to a σ-field $\mathcal{G}$ if and only if its modes $(\chi^{n,\pm})_{n\in\mathbb{N}}$ are measurable with respect to $\mathcal{G}$, we deduce that $X(\cdot)$ is a random variable from Ω_0 to $L^2(\mathbb{S}^1;\mathbb{R}^d)$.

Conversely, if we are given a square integrable random variable $X(\cdot)$ from Ω_0 into $L^2(\mathbb{S}^1;\mathbb{R}^d)$, then we can define $(X^{n,\pm})_{n\in\mathbb{N}}$ as random variables with values in $\mathbb{R}^d$. We then let

$$X^n(x,\omega_0)=\sum_{k=0}^{n}X^{k,\pm}(\omega_0)e^{k,\pm}(x),\quad n\in\mathbb{N}.$$

Obviously, we can identify X^n (seen as a random variable on Ω with values in $\mathbb{R}^d$) with $X^n(\cdot)$ (seen as a random variable on Ω_0 with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$). It is clear that $X^n(\cdot)$ converges to $X(\cdot)$ in $L^2(\Omega_0,\mathcal{A}_0,\mathbb{P}_0;L^2(\mathbb{S}^1;\mathbb{R}^d))$ and X^n has a limit $\tilde{X}$ in $L^2(\Omega,\mathcal{A},\mathbb{P};\mathbb{R}^d)$. We then identify $\tilde{X}(\cdot)$ with $X(\cdot)$. ∎

Importantly, observe that we can proceed similarly with processes. For instance, we can associate, with any $\mathbb{F}$-progressively-measurable process with values in $\mathbb{R}^d$, an $\mathbb{F}_0$-progressively-measurable process with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$, and conversely. Indeed, if $\boldsymbol{X}=(X_t)_{0\leqslant t\leqslant T}$ is an $\mathbb{F}$-progressively-measurable $\mathbb{R}^d$-valued process on Ω satisfying $\mathbb{E}\int_0^T|X_t|^2dt<\infty$, then it can be approximated in $L^2([0,T]\times\Omega)$ by simple processes of the form

$$\left(X_t^n=\sum_{i=0}^{n-1}X^{n,i}\mathbf{1}_{(t_i,t_{i+1}]}(t)\right)_{0\leqslant t\leqslant T},\quad n\in\mathbb{N},$$

where $0=t_0<\cdots<t_n=T$ is a subdivision of $[0,T]$ and $X^{n,i}$, for each $i\in\{0,\cdots,n-1\}$, is $\mathcal{F}_{t_i}$ measurable. Then, by Lemma 2, we can associate with each $X^{n,i}$ an $\mathcal{F}_{0,t_i}$-measurable random

variable $X^{n,i}(\cdot)$ from Ω_0 into $\mathbb{R}^d$. Letting

$$\Big(X^n_t(\cdot)=\sum_{i=0}^{n-1}X^{n,i}(\cdot)\mathbf{1}_{(t_i,t_{i+1}]}(t)\Big)_{0\leqslant t\leqslant T},$$

the sequence $(\boldsymbol{X}^n(\cdot)=(X^n_t(\cdot))_{0\leqslant t\leqslant T})_{n\in\mathbb{N}}$ is Cauchy in $L^2([0,T]\times\Omega_0;L^2(\mathbb{S}^1;\mathbb{R}^d))$. The limit $\boldsymbol{X}(\cdot)=(X_t(\cdot))_{0\leqslant t\leqslant T}$ is $\mathbb{F}_0$-progressively-measurable and, for almost every $t\in[0,T]$, for almost every $\omega_0\in\Omega_0$, the realization of $X_t(\cdot)$ coincides with $\mathbb{S}^1\ni x\mapsto X_t(x,\omega_0)$.

Conversely, if we are given an $\mathbb{F}_0$-progressively-measurable $\boldsymbol{X}(\cdot)=(X_t(\cdot))_{0\leqslant t\leqslant T}$ from Ω_0 into $L^2(\mathbb{S}^1;\mathbb{R}^d)$ satisfying $\mathbb{E}_0\int_0^T\|X_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}dt<\infty$, then we can construct $\boldsymbol{X}=(X_t)_{0\leqslant t\leqslant T}$ as the limit in $L^2([0,T]\times\Omega;\mathbb{R}^d)$ of the sequence of processes

$$\Big(\Big((x,\omega_0)\mapsto\sum_{k=1}^n X^{k,\pm}_t(\omega_0)e^{k,\pm}(x)\Big)_{0\leqslant t\leqslant T}\Big)_{n\in\mathbb{N}}$$

Clearly, $\boldsymbol{X}=(X_t)_{0\leqslant t\leqslant T}$ is $\mathbb{F}$-progressively-measurable and, for almost every $t\in[0,T]$, for almost every $\omega_0\in\Omega_0$, the realization of $X_t(\cdot)$ coincides with $\mathbb{S}^1\ni x\mapsto X_t(x,\omega_0)$.

Given processes $\boldsymbol{X}$ and $\boldsymbol{X}(\cdot)$ as we just considered, we can define

$$\boldsymbol{\chi}=\Big(\chi_t=\int_0^t X_s ds\Big)_{0\leqslant t\leqslant T},\quad\text{and}\quad\boldsymbol{\chi}(\cdot)=\Big(\chi_t(\cdot)=\sum_{n\in\mathbb{N}}\int_0^t X^{n,\pm}_s e^{n,\pm}ds\Big)_{0\leqslant t\leqslant T}.$$

Then, it is pretty easy to check that, for almost every $\omega_0\in\Omega_0$, for all $t\in[0,T]$, the function $\mathbb{S}^1\ni x\mapsto\chi_t(x,\omega_0)$ coincides with the realization of $\chi_t(\cdot)$ at ω_0.

2.4. **Randomized MFG.** With the same assumption as in (10) for the collection of semi-martingales $(\boldsymbol{U}^{n,\pm}=(U^{n,\pm}_t)_{0\leqslant t\leqslant T})_{n\in\mathbb{N}}$, we consider the following (informally defined) *randomized* MFG in lieu of the original MFG presented in Subsection 2.1:

(1) Given an $\mathcal{F}_{0,0}$-measurable random variable $\mathscr{V}$ from Ω_0 into $\mathcal{P}_2(\mathbb{R}^d)$, with $\mathbb{E}_0[M_2(\mathscr{V})^2]<\infty$, and an $\mathbb{F}_0$-adapted flow of random measures $\boldsymbol{\mu}=(\mu_t)_{0\leqslant t\leqslant T}$ on $\mathbb{R}^d$ with continuous paths from $[0,T]$ into $\mathcal{P}_2(\mathbb{R}^d)$ such that $\mathbb{P}_0(\mu_0=\mathscr{V})=1$, consider the following cost functional

$$\begin{aligned}J^{\boldsymbol{\mu}}(\boldsymbol{\alpha})=\int_{\Omega_0}\Big[&\mathfrak{g}_0\big(X_T(\cdot,\omega_0),\mu_T(\omega_0)\big)\\&+\int_0^T\Big(\mathfrak{f}_0\big(X_t(\cdot,\omega_0),\mu_t(\omega_0)\big)+\frac{1}{2}\sum_{n\in\mathbb{N}}|\alpha^{n,\pm}_t(\omega_0)|^2\Big)dt\Big]d\mathbb{P}_0(\omega_0),\end{aligned}$$

over controlled dynamics of the form

$$dX^{n,\pm}_t=\Big(\mathfrak{b}^{n,\pm}\big(X_t(\cdot),\mu_t\big)+\alpha^{n,\pm}_t\Big)dt+dU^{n,\pm}_t,\quad t\in[0,T],\quad n\in\mathbb{N},\tag{14}$$

where $(X^{n,\pm}_0)_{n\in\mathbb{N}}$ denote the modes of a random variable $X_0(\cdot)$ with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$ such that, $\mathbb{P}_0$-almost everywhere, $\mathrm{Leb}_1\circ X_0(\cdot)^{-1}=\mathscr{V}$. Such a random variable exists: it suffices to take $X_0(\cdot):\Omega_0\ni\omega_0\mapsto\Psi(\xi,\mathscr{V}(\omega_0))\in L^2(\mathbb{S}^1;\mathbb{R}^d)$ (see the first lines of Subsection 2.2 for the definition of Ψ) and with the same convention as above that $\boldsymbol{X}^{0,-}$ is identically zero. Here the controls $((\alpha^{n,\pm}_t)_{0\leqslant t\leqslant T})_{n\in\mathbb{N}}$ are required to be progressively-measurable with respect to the filtration $\mathbb{F}_0$ and to satisfy:

$$\sum_{n\in\mathbb{N}}\mathbb{E}_0\int_0^T|\alpha^{n,\pm}_t|^2dt<\infty.\tag{15}$$

(2) Find $\boldsymbol{\mu} = (\mu_t : \Omega_0 \ni \omega_0 \mapsto \mu_t(\omega_0))_{t\in[0,T]}$ such that, with probability 1 under $\mathbb{P}_0$, for all $t \in [0,T]$,

$$\mu_t(\omega_0) = \mathrm{Leb}_1 \circ X_t^\star(\cdot,\omega_0)^{-1}, \tag{16}$$

where $\boldsymbol{X}^\star(\cdot) = (X_t^\star(\cdot))_{0\leqslant t\leqslant T}$ is the optimal path in the optimization problem $\inf_{\boldsymbol{\alpha}} J^{\boldsymbol{\mu}}(\boldsymbol{\alpha})$.

Recalling (12), observe that we can provide a simple assumption on $\mathfrak{b}$ such that, for X as in (14),

$$X_t(\cdot) = \sum_{n\in\mathbb{N}} X_t^{n,\pm} e^{n,\pm}(\cdot), \quad t \in [0,T],$$

makes sense as a process from Ω_0 into $L^2(\mathbb{S}^1;\mathbb{R}^d)$. In this regard, (14) just says that each Fourier mode of the state variable $X_t(\cdot)$ in the space $L^2(\mathbb{S}^1;\mathbb{R}^d)$ is forced by the corresponding $(\boldsymbol{U}^{n,\pm})_{n\in\mathbb{N}}$.

Of course, the choice of $(\boldsymbol{U}^{n,\pm})_{n\in\mathbb{N}}$ is the key point in our analysis. In full analogy with $\boldsymbol{\mu}$, we shall define it as the solution of a fixed point involving the optimal trajectory of the new optimization problem $\inf_{\boldsymbol{\alpha}} J^{\mu}(\boldsymbol{\alpha})$ introduced right above, namely we choose each $\boldsymbol{U}^{n,\pm} = (U_t^{n,\pm})_{0\leqslant t\leqslant T}$ as

$$U_t^{n,\pm} = -(2\pi n)^2 \int_0^t X_s^{\star n,\pm} ds + W_t^{n,\pm}, \quad t \in [0,T], \quad n \in \mathbb{N}. \tag{17}$$

Under this choice, the optimal trajectory of the optimization problem $\inf_{\boldsymbol{\alpha}} J^{\boldsymbol{\mu}}(\boldsymbol{\alpha})$ in environment $\boldsymbol{\mu}$ (as already explained, sufficient conditions will be given below so that an optimal path exists and is unique) takes the form:

$$dX_t^{\star n,\pm} = \Big(\mathfrak{b}^{n,\pm}\big(X_t(\cdot),\mu_t\big) + \alpha_t^{\star n,\pm} - (2\pi n)^2 X_t^{\star n,\pm}\Big)dt + dW_t^{n,\pm}, \quad t \in [0,T], \tag{18}$$

where $\boldsymbol{\alpha}^\star$ is the optimal control. Here the rationale for choosing the dissipative factor $-(2\pi n)^2$ in the dynamics is twofold. First, the fact that the series of the inverses of the factors, that is $\sum_{n\in\mathbb{N}^*}(2\pi n)^{-2}$, converge will permit us to prove, under suitable assumptions, that the modes of $\boldsymbol{X}^\star$ are square-summable. Second, the factors $-(2\pi n)^2$ appear in the formal computation:

$$\partial_x^2 X_t^\star(\cdot) = \sum_{n\in\mathbb{N}} X_t^{\star n,\pm} \partial_x^2 e_t^{n,\pm}(\cdot) = -\sum_{n\in\mathbb{N}^*} (2\pi n)^2 X_t^{\star n,\pm} e_t^{n,\pm}(\cdot),$$

where $X_t^\star(\cdot) = \sum_{n\in\mathbb{N}} X_t^{\star n,\pm} e^{n,\pm}(\cdot)$, which prompts us to reformulate (18) as the controlled SPDE:

$$\partial_t X_t^\star(x) = b(X_t(x),\mu_t) + \alpha_t^\star(x) + \partial_x^2 X_t^\star(x) + \dot{W}_t(x), \quad t \in [0,T], \quad x \in \mathbb{S}^1. \tag{19}$$

The notation $\dot{W}$ denotes a space-time white noise, namely

$$W_t(\cdot) = \sum_{n\in\mathbb{N}} W_t^{n,\pm} e^{n,\pm}(\cdot), \quad t \in [0,T],\ x \in \mathbb{S}^1, \tag{20}$$

is a cylindrical Wiener process with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$, meaning that, for any $f \in L^2(\mathbb{S}^1;\mathbb{R}^d)$, the process

$$\left(\int_{\mathbb{S}^1} f(x)\cdot W_t(dx) = \sum_{n\in\mathbb{N}} f^{n,\pm}\cdot W_t^{n,\pm}\right)_{t\in[0,T]}$$

is a Brownian motion with $\int_{\mathbb{S}^1} |f(x)|^2 dx$ as variance.

So, choosing $\boldsymbol{U}$ as in (17) is especially convenient for reformulating the dynamics of the equilibrium as the solution of an SPDE. In this regard, a crucial fact in the subsequent analysis will be played by the structure of the SPDE, which is close to that of an Ornstein-Ulhenbeck (OU) process with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$.

If the modes of $\boldsymbol{X}^\star(\cdot)$ satisfy

$$\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\Big(\sum_{n\in\mathbb{N}} |X_t^{\star n,\pm}|^2\Big)\Big] < \infty,$$

it is then obvious from (12), (15), (17) and (18) that $\boldsymbol{U}$ satisfy (10), which proves that (13) holds for any $\boldsymbol{\alpha}$.

In order to reconstruct the dynamics satisfied by $\boldsymbol{X}$ for any controlled $\boldsymbol{\alpha}$, we may focus on the difference $\boldsymbol{X}-\boldsymbol{U}$. Clearly, $\boldsymbol{X}-\boldsymbol{U}$ satisfies a controlled ODE with random coefficients:

$$d\big(X_t^{n,\pm}-U_t^{n,\pm}\big)=\big[\mathfrak{b}^{n,\pm}\big(X_t(\cdot),\mu_t\big)+\alpha_t^{n,\pm}\big]dt,\quad t\in[0,T],\quad n\in\mathbb{N},$$

so that

$$d\big(X_t-U_t\big)=\big[b(X_t,\mu_t)+\alpha_t\big]dt,\quad t\in[0,T],$$

with 0 as initial condition.

So, we end up with the following definition:

Definition 3. *Given a square integrable $\mathcal{F}_{0,0}$-measurable random variable $X_0(\cdot)$ from Ω_0 into $L^2(\mathbb{S}^1;\mathbb{R}^d)$, we call a solution of the randomized MFG a pair of $\mathbb{F}_0$-progressively measurable and $L^2(\mathbb{S}^1;\mathbb{R}^d)$-valued processes $\boldsymbol{X}^\star(\cdot)=(X_t^\star(\cdot))_{0\leqslant t\leqslant T}$, with $X_0^\star(\cdot)=X_0(\cdot)$ as initial condition, and $\boldsymbol{\alpha}^\star(\cdot)=(\alpha_t^\star(\cdot))_{0\leqslant t\leqslant T}$, satisfying the integrability conditions*

$$\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\|X_t^\star(\cdot)\|^2\Big]<\infty,$$
$$\mathbb{E}_0\Big[\int_0^T\|\alpha_t^\star(\cdot)\|^2dt\Big]<\infty,$$

and satisfying the system (18), such that, under the notations

$$\mu_t(\omega_0)=\mathrm{Leb}_1\circ X_t^\star(\cdot,\omega_0)^{-1},\quad \omega_0\in\Omega_0,$$
$$U_t^{n,\pm}=-(2\pi n)^2\int_0^t X_s^{\star n,\pm}ds+W_t^{n,\pm},\quad t\in[0,T],\quad n\in\mathbb{N},$$

the process $\boldsymbol{\alpha}^\star$ (regarded as an $\mathbb{F}$-progressively measurable process with values from Ω into $\mathbb{R}^d$) is an optimal control of the optimal control problem with random coefficients consisting in minimizing

$$\bar{J}^{\boldsymbol{\mu}}(\boldsymbol{\alpha})=\mathbb{E}\bigg[g\big(U_T+\bar{X}_T^{\boldsymbol{\alpha}},\mu_T\big)+\int_0^T\Big(f(U_t+\bar{X}_t^{\boldsymbol{\alpha}},\mu_t)+\frac{1}{2}|\alpha_t|^2\Big)dt\bigg],\tag{21}$$

over $\mathbb{F}$-progressively measurable processes $\boldsymbol{\alpha}$ satisfying

$$\mathbb{E}\int_0^T|\alpha_t|^2dt<\infty,$$

where $\bar{\boldsymbol{X}}^{\boldsymbol{\alpha}}$ solves

$$d\bar{X}_t^{\boldsymbol{\alpha}}=\Big(b\big(U_t+\bar{X}_t^{\boldsymbol{\alpha}},\mu_t\big)+\alpha_t\Big)dt,\quad t\in[0,T],\tag{22}$$

with $\bar{X}_0^{\boldsymbol{\alpha}}=X_0$ as initial condition (X_0 being regarded as an $\mathbb{R}^d$-valued random variable on Ω).

Remark 4. *Definition 3 provides another interpretation of the randomization of the equilibria. It says everything works as if we kept the same MFG as before, but with random coefficients obtained by an additive perturbation of the original ones.*

Remark 5. *The reader may now understand the reason why we have limited our result to the case of deterministic (instead of stochastic) differential equations. Our strategy is indeed clear: We enclose the private (or idiosyncratic) noise underpinning the initial condition of the representative player in the torus; the infinite dimensional noise $\boldsymbol{W}(\cdot)$ (which reads as a "common noise") then acts on the modes of the initial condition. If we had to do so with a stochastic differential game, we should enclose the whole private random signal (e.g., a Brownian motion) in the torus, but, then, adaptability conditions would be a delicate issue to handle. In fact, our guess is that, to*

respect the adaptability constraints, the forcing procedure has to be slightly different (and in fact less straightforward than it is here).

2.5. **Infinite dimensional McKV forward-backward system.** We now observe that, for a given $\mathbb{F}_0$-progressively measurable random flow $\boldsymbol{\mu} = (\mu_t)_{0\leqslant t\leqslant T}$ as in the first item of the randomized MFG problem defined in (14)–(16), the optimal paths (whenever they exist) should be given by the stochastic Pontryagin principle, see for instance [48, 49, 53], see also [16]. Here, the stochastic Pontryagin principle takes the form of the following forward-backward system of SDEs:

$$\begin{aligned} dX_t^{\star,n,\pm} &= \Big(\mathfrak{b}^{n,\pm}(X_t^\star(\cdot),\mu_t) - Y_t^{\star,n,\pm}\Big)dt + dU_t^{n,\pm}, \\ dY_t^{\star,n,\pm} &= \Big(-\sum_{k\in\mathbb{N}} D_{n,\pm}\mathfrak{b}^{k,\pm}(X_t^\star(\cdot),\mu_t)Y_t^{\star,k,\pm} - D_{n,\pm}\mathfrak{f}_0(X_t^\star(\cdot),\mu_t)\Big)dt + \sum_{k\in\mathbb{N}} Z_t^{\star,n,k,\pm}dW_t^{k,\pm}, \end{aligned} \tag{23}$$

for $t\in[0,T]$, with the terminal condition $Y_T^{\star,n,\pm} = D_{n,\pm}\mathfrak{g}_0(X_T^\star(\cdot),\mu_T)$, for all $n\in\mathbb{N}$. Above, $(X_t^{\star,n,\pm})_{0\leqslant t\leqslant T}$ and $(Y_t^{\star,n,\pm})_{0\leqslant t\leqslant T}$ take values in $\mathbb{R}^d$ and $(Z_t^{\star,n,\pm})_{0\leqslant t\leqslant T}$ takes values in $\mathbb{R}^{d\times d}$; also, we have denoted by D the Fréchet derivative on $L^2(\mathbb{S}^1;\mathbb{R}^d)$ and by $D_{n,\pm}\bullet = \langle e^{n,\pm}(\cdot), D\bullet\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}$ the d-dimensional derivative in the direction $e^{n,\pm}$. Of course, in the notation $D_{n,\pm}\mathfrak{h}(\ell,\mu)$, with $\mathfrak{h}$ matching $\mathfrak{b}^{k,\pm}$, $\mathfrak{f}_0$ or $\mathfrak{g}_0$, the operator D acts on the first coordinate depending on $\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)$. In the notation $D_{n,\pm}\mathfrak{b}^{k,\pm}(X_t^\star(\cdot),\mu_t)Y_t^{\star,k,\pm}$, $D_{n,\pm}\mathfrak{b}^{k,\pm}(X_t^\star(\cdot),\mu_t)$ is implicitly regarded as a square matrix with columns $(D_{n,\pm}\mathfrak{b}_j^{k,\pm}(X_t^\star(\cdot),\mu_t))_{1\leqslant j\leqslant d}$, so that the whole reads as $\sum_{j=1}^d D_{n,\pm}\mathfrak{b}_j^{k,\pm}(X_t^\star(\cdot),\mu_t)(Y_t^{\star,k,\pm})_j$. We shall check properly that all the derivatives make sense in our framework. Lastly, in (23), $X_t^\star(\cdot)$ is a shorten notation for the function in $L^2(\mathbb{S}^1;\mathbb{R}^d)$:

$$X_t^\star(\cdot) = \sum_{n\in\mathbb{N}} X_t^{\star,n,\pm}e^{n,\pm}(\cdot).$$

For the time being, we do not establish rigorously the derivation of the stochastic Pontryagin principle. We shall address this question in Proposition 9. Meanwhile, we observe that, inserting the fixed point condition (17), (23) may be rewritten as

$$\begin{aligned} dX_t^{\star,n,\pm} &= \Big(\mathfrak{b}^{n,\pm}\big(X_t^\star(\cdot),\mu_t\big) - Y_t^{\star,n,\pm} - (2\pi n)^2 X_t^{\star,n,\pm}\Big)dt + dW_t^{n,\pm}, \\ dY_t^{\star,n,\pm} &= \Big(-\sum_{k\in\mathbb{N}} D_{n,\pm}\mathfrak{b}^{k,\pm}\big(X_t^\star(\cdot),\mu_t\big)Y_t^{\star,k,\pm} - D_{n,\pm}\mathfrak{f}_0\big(X_t^\star(\cdot),\mu_t\big)\Big)dt + \sum_{k\in\mathbb{N}} Z_t^{\star,n,k,\pm}dW_t^{k,\pm}, \end{aligned} \tag{24}$$

for $t\in[0,T]$, with the terminal condition $Y_T^{\star,n,\pm} = D_{n,\pm}\mathfrak{g}_0(X_T^\star(\cdot),\mu_T)$, for all $n\in\mathbb{N}$.

Of course, nothing guarantees *a priori* that the modes in (24) are square summable. So, we impose, in the definition of a solution to (24), that the modes are indeed square summable.

Definition 6. *Given a square integrable $\mathcal{F}_{0,0}$-measurable random variable $X_0(\cdot)$ from Ω_0 into $L^2(\mathbb{S}^1;\mathbb{R}^d)$, we call a solution to* (24) *a countable collection of $\mathbb{F}_0$-progressively measurable processes $((X_t^{n,\pm})_{0\leqslant t\leqslant T})_{n\in\mathbb{N}}$, $((Y_t^{n,\pm})_{0\leqslant t\leqslant T})_{n\in\mathbb{N}}$, $((Z_t^{n,k,\pm})_{0\leqslant t\leqslant T})_{n,k\in\mathbb{N}}$, such that*

$$\sum_{n\in\mathbb{N}}\mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\big(|X_t^{n,\pm}|^2 + |Y_t^{n,\pm}|^2\big)\Big] + \mathbb{E}\Big[\sum_{k,n\in\mathbb{N}}\int_0^T |Z_t^{n,k,\pm}|^2dt\Big] < \infty,$$

satisfying, with probability 1, (24) *(and the associated terminal condition) with the initial condition $X_0^{n,\pm}$ for all $n\in\mathbb{N}$, as given by the modes of $X_0(\cdot)$.*

Then, we can define $\mathbb{F}_0$-adapted and continuous processes $(X_t(\cdot))_{0\leqslant t\leqslant T}$ and $(Y_t(\cdot))_{0\leqslant t\leqslant T}$ with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$ such that, with probability 1, for all $t\in[0,T]$,

$$X_t(\cdot) = \sum_{n\in\mathbb{N}} X_t^{n,\pm}e^{n,\pm}(\cdot), \quad Y_t(\cdot) = \sum_{n\in\mathbb{N}} Y_t^{n,\pm}e^{n,\pm}(\cdot).$$

Implementing the matching condition (16) in the formulation of the enlarged problem, we understand that, whenever they exist, fixed points should solve a McKean-Vlasov SDE of the conditional type. Similar to (24), this McKean-Vlasov SDE must be infinite dimensional. In analogy with (6) and with the same notation as in (16), it takes the form:

$$\begin{aligned} dX_t^{\star,n,\pm} &= \Big(\mathfrak{b}^{n,\pm}\big(X_t^\star(\cdot), \mathrm{Leb}_1 \circ (X_t^\star(\cdot))^{-1}\big) - Y_t^{\star,n,\pm} - (2\pi n)^2 X_t^{\star,n,\pm}\Big)dt + dW_t^{n,\pm}, \\ dY_t^{\star,n,\pm} &= \Big(-\sum_{k\in\mathbb{N}} D_{n,\pm}\mathfrak{b}^{k,\pm}\big(X_t^\star(\cdot), \mathrm{Leb}_1 \circ (X_t^\star(\cdot))^{-1}\big) Y_t^{\star,k,\pm} \\ &\qquad - D_{n,\pm}\mathfrak{f}_0\big(X_t^\star(\cdot), \mathrm{Leb}_1 \circ (X_t^\star(\cdot))^{-1}\big)\Big)dt \quad + \sum_{k\in\mathbb{N}} Z_t^{\star,n,k,\pm} dW_t^{k,\pm}, \end{aligned} \tag{25}$$

for $t \in [0,T]$, with the terminal condition $Y_T^{\star,n,\pm} = D_{n,\pm}\mathfrak{g}_0(X_T^\star(\cdot), \mathrm{Leb}_1 \circ (X_T^\star(\cdot))^{-1})$, for all $n \in \mathbb{N}$.

Letting

$$\begin{aligned} \mathfrak{B}(\ell) &= \mathfrak{b}\big(\ell, \mathrm{Leb}_1 \circ \ell^{-1}\big), \\ \mathfrak{H}(\ell, h) &= \sum_{k\in\mathbb{N}} \sum_{j=1}^{d} D\mathfrak{b}_j^{k,\pm}\big(\ell, \mathrm{Leb}_1 \circ \ell^{-1}\big)\big(h^{k,\pm}\big)^j + D\mathfrak{f}_0\big(\ell, \mathrm{Leb}_1 \circ \ell^{-1}\big), \\ \mathfrak{G}(\ell) &= D\mathfrak{g}_0\big(\ell, \mathrm{Leb}_1 \circ \ell^{-1}\big), \end{aligned} \tag{26}$$

for any two $\ell, h \in L^2(\mathbb{S}^1;\mathbb{R}^d)$, (25) may be written as

$$\begin{aligned} dX_t^{\star n,\pm} &= \Big(\mathfrak{B}^{n,\pm}\big(X_t^\star(\cdot)\big) - Y_t^{\star n,\pm} - (2\pi n)^2 X_t^{\star n,\pm}\Big)dt + dW_t^{n,\pm}, \quad n \in \mathbb{N}, \\ dY_t^{\star n,\pm} &= -\mathfrak{H}^{n,\pm}\big(X_t^\star(\cdot), Y_t^\star(\cdot)\big)dt + \sum_{k\in\mathbb{N}} Z_t^{\star n,k,\pm} dW_t^{k,\pm}, \end{aligned} \tag{27}$$

for $t \in [0,T]$, with the terminal condition $Y_T^{\star n,\pm} = \mathfrak{G}^{n,\pm}(X_T^\star(\cdot))$, for all $n \in \mathbb{N}$.

This permits to give a similar definition to Definition 6:

Definition 7. *Given a square integrable $\mathcal{F}_{0,0}$-measurable random variable $X_0(\cdot)$ from Ω_0 into $L^2(\mathbb{S}^1;\mathbb{R}^d)$, we call a solution to (27) (or (25)), a countable collection of $\mathbb{F}_0$-progressively measurable processes $((X_t^{n,\pm})_{0\leqslant t\leqslant T})_{n\in\mathbb{N}}$, $((Y_t^{n,\pm})_{0\leqslant t\leqslant T})_{n\in\mathbb{N}}$, $((Z_t^{n,k,\pm})_{0\leqslant t\leqslant T})_{n\in\mathbb{N},k\in\mathbb{N}}$, such that*

$$\sum_{n\in\mathbb{N}} \mathbb{E}\Big[\sup_{0\leqslant t\leqslant T}\big(|X_t^{n,\pm}|^2 + |Y_t^{n,\pm}|^2\big)\Big] + \mathbb{E}\Big[\sum_{k,n\in\mathbb{N}} \int_0^T |Z_t^{n,k,\pm}|^2 dt\Big] < \infty,$$

satisfying, with probability 1, (27) (and the associated terminal condition) with the initial condition $X^{\star n,\pm} = X_0^{n,\pm}$ for all $n \in \mathbb{N}$.

Then, we can define $\mathbb{F}_0$-adapted and continuous processes $(X_t)_{0\leqslant t\leqslant T}$ and $(Y_t)_{0\leqslant t\leqslant T}$ with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$ such that, with probability 1, for all $t \in [0,T]$,

$$X_t(\cdot) = \sum_{n\in\mathbb{N}} X_t^{n,\pm} e^{n,\pm}(\cdot), \quad Y_t(\cdot) = \sum_{n\in\mathbb{N}} Y_t^{n,\pm} e^{n,\pm}(\cdot).$$

2.6. **Standing assumptions.** Throughout the paper, we assume that

Assumption (A). The coefficient b is assumed to be independent of x and to be bounded and Lipschitz continuous on $\mathcal{P}_2(\mathbb{R}^d)$ –$\mathcal{P}_2(\mathbb{R}^d)$ being equipped with the 2-Wasserstein distance–. The coefficients f and g are differentiable in x, and $\partial_x f$ and $\partial_x g$ are bounded and Lipschitz continuous on $\mathbb{R}^d \times \mathcal{P}_2(\mathbb{R}^d)$. Moreover, for any $\mu \in \mathcal{P}_2(\mathbb{R}^d)$, the functions $\mathbb{R}^d \ni x \mapsto f(x,\mu)$ and $\mathbb{R}^d \ni x \mapsto g(x,\mu)$ are convex.

Importantly, notice that, under assumption **A**, the coefficients in (26) take the simplest form:

$$\begin{aligned}&\mathfrak{B}(\ell) = \mathfrak{b}_0(\ell)e_0(\cdot), \quad \text{with } \mathfrak{b}_0(\ell) = b\big(\mathrm{Leb}_1 \circ \ell^{-1}\big),\\ &\mathfrak{H}(\ell,h) = \mathfrak{F}(\ell), \quad \text{with } \mathfrak{F}(\ell) = D\mathfrak{f}_0\big(\ell, \mathrm{Leb}_1 \circ \ell^{-1}\big).\end{aligned} \tag{28}$$

In particular, the system (27) becomes (removing the stars in the labels):

$$\begin{aligned}dX_t^{n,\pm} &= \Big(\mathbf{1}_{(n,\pm)=(0,+)}\mathfrak{b}_0\big(X_t(\cdot)\big) - Y_t^{n,\pm} - (2\pi n)^2 X_t^{n,\pm}\Big)dt + dW_t^{n,\pm}, \quad n \in \mathbb{N},\\ dY_t^{n,\pm} &= -\mathfrak{F}^{n,\pm}\big(X_t(\cdot)\big)dt + \sum_{k\in\mathbb{N}} Z_t^{n,k,\pm} dW_t^{k,\pm},\end{aligned} \tag{29}$$

for $t \in [0,T]$, with the terminal condition $Y_T^{n,\pm} = \mathfrak{G}^{n,\pm}(X_T(\cdot))$, for all $n \in \mathbb{N}$.

In order to fully legitimate the existence of the Fréchet derivatives of $\mathfrak{f}_0$ and $\mathfrak{g}_0$ in the direction ℓ, we may invoke the following lemma, the proof of which is quite straightforward and is left to the reader:

Lemma 8. *For a continuously differentiable Lipschitz function $F : \mathbb{R}^d \to \mathbb{R}$ (so that F is at most of linear growth), define $\mathfrak{F}_0 : L^2(\mathbb{S}^1;\mathbb{R}^d) \ni \ell \mapsto \int_{\mathbb{S}^1} F(\ell(x))dx$. Then, $\mathfrak{F}_0$ is Fréchet differentiable and*

$$D\mathfrak{F}_0(\ell) = \nabla F \circ \ell.$$

In particular, we have the following expression for $\mathfrak{F}$ (and similarly for $\mathfrak{G}$):

$$\mathfrak{F} : L^2(\mathbb{S}^1;\mathbb{R}^d) \ni \ell \mapsto \big(\mathbb{S}^1 \ni x \mapsto \partial_x f\big(\ell(x), \mathrm{Leb}_1 \circ \ell^{-1}\big)\big),$$

and then

$$\mathfrak{F}^{n,\pm}(\ell) = \int_{\mathbb{S}^1} \partial_x f\big(\ell(x), \mathrm{Leb}_1 \circ \ell^{-1}\big)\big)e^{n,\pm}(x)dx.$$

The introduction of Assumption **(A)** –namely asking b to be independent of x and f and g to be convex in x– is fully justified by our desire to use the Pontryagin principle as a sufficient condition of optimality. Generally speaking, it requires the underlying Hamiltonian to be convex, which is indeed the case under Assumption **(A)** even though it could be slightly relaxed: We could certainly allow b to be linear in x; we could also think of allowing the derivatives of f and g to be at most of linear growth, but this seems a more challenging question. So, under Assumption **(A)**, the Pontryagin principle is not only a necessary but also a sufficient condition for the original control problem described in Subsection 2.1; in particular, the McKean-Vlasov equation (6) characterizes equilibria of the original (non-randomized) mean-field game. The following proposition is to check that this fact remains true in our randomized framework:

Proposition 9. *Given a square integrable $\mathcal{F}_{0,0}$-measurable random variable $X_0(\cdot)$ from Ω_0 into $L^2(\mathbb{S}^1;\mathbb{R}^d)$, any solution to (29) is a solution of the randomized matching problem defined in Definition 3. Conversely, any solution to the randomized matching problem provides a solution to (29).*

In particular, the randomized matching problem is uniquely solvable if and only if the McKean-Vlasov equation (29) is uniquely solvable.

Proof. First Step. Assume first that the McKean-Vlasov equation (29) has a solution, which we denote by $((X_t^{n,\pm})_{n\in\mathbb{N}}, (Y_t^{n,\pm})_{n\in\mathbb{N}}, (Z_t^{n,k,\pm})_{n,k\in\mathbb{N}})_{0\leqslant t\leqslant T}$. Denote by $(X_t(\cdot))_{0\leqslant t\leqslant T}$ and $(Y_t(\cdot))_{0\leqslant t\leqslant T}$ the associated $L^2(\mathbb{S}^1;\mathbb{R}^d)$-valued processes as in Definition 7 and let

$$\mu_t = \mathrm{Leb}_1 \circ X_t(\cdot)^{-1}, \quad t \in [0,T].$$

Since the mapping $L^2(\mathbb{S}^1;\mathbb{R}^d) \ni \ell \mapsto \mathrm{Leb}_1 \circ \ell^{-1} \in \mathcal{P}_2(\mathbb{R}^d)$ is continuous, each μ_t is a random variable with values in $\mathcal{P}_2(\mathbb{R}^d)$ and the process $(\mu_t)_{0\leqslant t\leqslant T}$ is $\mathbb{F}_0$-adapted. Following (17), we also let (pay

attention that we dropped the symbol $\star$ in the notation for the solution of the McKean-Vlasov equation):

$$U_t^{n,\pm} = -(2\pi n)^2 \int_0^t X_s^{n,\pm} ds + W_t^{n,\pm}, \quad t \in [0,T], \quad n \in \mathbb{N}.$$

Observe that $\boldsymbol{U}^{n,\pm}$ is also given by

$$U_t^{n,\pm} = X_t^{n,\pm} - X_0^{n,\pm} - \int_0^t \big[\mathbf{1}_{(n,\pm)=(0,+)} b\big(\mu_s\big) ds - Y_s^{n,\pm}\big] ds, \quad t \in [0,T], \quad n \in \mathbb{N},$$

from which we deduce that

$$\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T} \sum_{n\in\mathbb{N}} |U_t^{n,\pm}|^2\Big] < \infty.$$

Consider now an $\mathbb{R}^d$-valued control $\boldsymbol{\alpha} = (\alpha_t)_{0\leqslant t\leqslant T}$ as in (22) and denote by $(\bar{X}_t^{\boldsymbol{\alpha}})_{0\leqslant t\leqslant T}$ the solution to (22), namely

$$d\bar{X}_t^{\boldsymbol{\alpha}} = \big[b(\mu_t) + \alpha_t\big] dt, \quad t \in [0,T].$$

Thanks to Lemma 2, we can regard $\boldsymbol{\alpha}$ and $\bar{\boldsymbol{X}}^{\boldsymbol{\alpha}}$ as $\mathbb{F}_0$-progressively measurable processes $\boldsymbol{\alpha}(\cdot)$ and $\bar{\boldsymbol{X}}^{\boldsymbol{\alpha}}(\cdot)$ from Ω_0 to $L^2(\mathbb{S}^1)$. Since $\boldsymbol{\alpha}$ is fixed, we just note $\bar{\boldsymbol{X}}$ for $\bar{\boldsymbol{X}}^{\boldsymbol{\alpha}}$. Then, the modes of $\boldsymbol{X}(\cdot)$ satisfy:

$$d\bar{X}_t^{n,\pm} = \big(\mathbf{1}_{(n,\pm)=(0,+)} b(\mu_t) + \alpha_t^{n,\pm}\big) dt, \quad t \in [0,T],$$

where $(\alpha_t^{n,\pm})_{0\leqslant t\leqslant T}$ denotes the modes of $\boldsymbol{\alpha}(\cdot)$. Letting $(\hat{X}_t^{n,\pm} = \bar{X}_t^{n,\pm} + U_t^{n,\pm})_{0\leqslant t\leqslant T}$, we get

$$d\big(\hat{X}_t^{n,\pm} - X_t^{n,\pm}\big) = \big(\alpha_t^{n,\pm} + Y_t^{n,\pm}\big) dt, \quad t \in [0,T],$$

with $X_0^{\boldsymbol{\alpha},n,\pm} - X_0^{n,\pm} = 0$, for all $n \in \mathbb{N}$.

Now, using the notation "$\cdot$" for the inner product in $\mathbb{R}^d$,

$$\begin{aligned} d\Big[Y_t^{n,\pm} \cdot \big(\hat{X}_t^{n,\pm} - X_t^{n,\pm}\big)\Big] &= \big(\alpha_t^{n,\pm} + Y_t^{n,\pm}\big) \cdot Y_t^{n,\pm} dt \\ &\quad - D_{n,\pm}\mathfrak{f}_0\big(X_t(\cdot), \mu_t\big) \cdot \big(\hat{X}_t^{n,\pm} - X_t^{n,\pm}\big) dt + dM_t^{n,\pm}, \end{aligned}$$

where $(M_t^{n,\pm})_{0\leqslant t\leqslant T}$ is a square-integrable $\mathbb{F}_0$-martingale. Taking expectation, we deduce that

$$\begin{aligned} &\mathbb{E}_0\big[D_{n,\pm}\mathfrak{g}_0\big(X_T(\cdot), \mu_T\big) \cdot \big(\hat{X}_T^{n,\pm} - X_T^{n,\pm}\big)\big] \\ &\quad = \mathbb{E}_0 \int_0^T \Big[\big(\alpha_t^{n,\pm} + Y_t^{n,\pm}\big) \cdot Y_t^{n,\pm} - D_{n,\pm}\mathfrak{f}_0\big(X_t(\cdot), \mu_t\big) \cdot \big(\hat{X}_t^{n,\pm} - X_t^{n,\pm}\big)\Big] dt. \end{aligned}$$

Summing over $n \in \mathbb{N}$ (which is licit in our framework), we deduce that

$$\begin{aligned} &\mathbb{E}_0\big[\big\langle D\mathfrak{g}_0\big(X_T(\cdot), \mu_T\big), \big(\hat{X}_T(\cdot) - X_T(\cdot)\big)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big] \\ &= \mathbb{E}_0 \int_0^T \Big[\big\langle \big(\alpha_t(\cdot) + Y_t(\cdot)\big), Y_t(\cdot)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)} - \big\langle D\mathfrak{f}_0\big(X_t(\cdot), \mu_t\big), \big(X_t^{\boldsymbol{\alpha}}(\cdot) - X_t(\cdot)\big)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big] dt, \end{aligned}$$

where, as usual, we have let $\hat{X}_t(\cdot) = \sum_{n\in\mathbb{N}} \hat{X}_t^{n,\pm} e^{n,\pm}(\cdot)$. Observing that, for two random variables $\chi(\cdot)$ and $\chi'(\cdot)$ with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$, $\mathbb{E}_0[\langle \chi(\cdot), \chi'(\cdot)\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d}] = \mathbb{E}[\chi \cdot \chi']$, where, in the last term, χ and χ' are regarded as $\mathbb{R}^d$-valued random variables, we deduce from Lemma 8 that

$$\mathbb{E}\big[\partial_x g(X_T, \mu_T) \cdot \big(\hat{X}_T - X_T\big)\big] = \mathbb{E} \int_0^T \Big[\big(\alpha_t + Y_t\big) \cdot Y_t - \partial_x f(X_t, \mu_t) \cdot \big(\hat{X}_t - X_t\big)\Big] dt.$$

Therefore,

$$\bar{J}^{\boldsymbol{\mu}}(\boldsymbol{\alpha}) - \bar{J}^{\boldsymbol{\mu}}(-\boldsymbol{Y}) = \mathbb{E}_0\Big[g(\hat{X}_T, \mu_T) - g(X_T, \mu_T) - \partial_x g(X_T, \mu_T) \cdot \big(\hat{X}_T - X_T\big) + \int_0^T \Big(\frac{1}{2}\big|\alpha_t + Y_t\big|^2 + f(\hat{X}_t, \mu_t) - f(X_t, \mu_t) - \partial_x f(X_t, \mu_t) \cdot \big(\hat{X}_t - X_t\big) \Big) dt \Big].$$

Since g and f are convex, we deduce that the right-hand side above is non-negative, which shows that $-\boldsymbol{Y}$ is an optimal control for $\bar{J}^{\boldsymbol{\mu}}$, that is to say $\boldsymbol{X}$ and $-\boldsymbol{Y}$ form a randomized equilibrium.

Second Step. We now turn to the converse. Assume that a pair $(\boldsymbol{X}^\star(\cdot), \boldsymbol{\alpha}^\star(\cdot))$ satisfies Definition 3. Then, we regard the optimization problem $\inf_{\boldsymbol{\alpha}} \bar{J}^{\boldsymbol{\mu}}(\boldsymbol{\alpha})$ defined in (21)–(22) as a standard optimization problem in random environment. By the standard stochastic Pontryagin principle (up to a straightforward adaptation due to the fact that the noise is infinite dimensional), we know that a necessary condition of optimality for some control process $\boldsymbol{\alpha}$ –the corresponding path being denoted by $\bar{\boldsymbol{X}}^{\boldsymbol{\alpha}}$– is that the solution of the adjoint backward equation

$$d\bar{Y}_t = -\partial_x f(U_t + \bar{X}_t^{\boldsymbol{\alpha}}, \mu_t) dt + \sum_{n \in \mathbb{N}} Z_t^{n,\pm} dW_t^{n,\pm}, \quad t \in [0,T], \tag{30}$$

with $\bar{Y}_T = \partial_x g(U_T + \bar{X}_T^{\boldsymbol{\alpha}}, \mu_T)$ as terminal condition coincides with $-\boldsymbol{\alpha}$, namely

$$\bar{Y}_t = -\alpha_t, \quad t \in [0,T]. \tag{31}$$

Now, if, as required, we have a control process $\boldsymbol{\alpha}^\star(\cdot)$ (with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$) with $\boldsymbol{X}^\star(\cdot)$ as associated path (also with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$) such that $\boldsymbol{\alpha}^\star$ (when regarded as a process with values in $\mathbb{R}^d$, see Lemma 2) minimizes $\bar{J}^{\boldsymbol{\mu}}$ in (21) when $\boldsymbol{U}(\cdot)$ is given by (17) and $\boldsymbol{\mu}$ by (16), then, following the discussion right after Lemma 2, we can identify the path of $\boldsymbol{X}^\star - \boldsymbol{U}$ (seen as an $\mathbb{R}^d$-valued process on Ω) with the path of $\bar{\boldsymbol{X}}^{\boldsymbol{\alpha}^\star}$. Also, we can define $\boldsymbol{Y}^\star$ (also seen as an $\mathbb{R}^d$-valued process) through (31); it solves an equation of the same type as (30). Computing the modes of $\boldsymbol{X}^\star(\cdot)$ and $\boldsymbol{Y}^\star(\cdot)$, we get that $(\boldsymbol{X}^\star(\cdot), \boldsymbol{Y}^\star(\cdot))$ is a solution of the McKean-Vlaosv equation (29). If the latter one is at most uniquely solvable, this shows that there is at most one MFG equilibrium. ∎

3. Main results

We here expose the main results of the paper. Proofs will given next.

3.1. Existence and uniqueness.

The first main result of the paper (whose proof is deferred to Section 4) is

Theorem 10. *Under Assumption* **(A)**, (29) *is uniquely solvable for any initial condition in the form of a square-integrable $\mathcal{F}_{0,0}$-measurable random variable $X_0(\cdot)$ from Ω_0 to $L^2(\mathbb{S}^1;\mathbb{R}^d)$.*

Comparison with the case without noise. It is worth comparing Theorem 10 with solvability results for the original mean-field game. Existence of a solution under Assumption **(A)** to (6) was investigated by Carmona and Delarue [14], see also [16, Chapters 3 and 4], by adapting the analytical techniques developed by Lasry and Lions, see [43, 44, 45, 7]. Uniqueness is known to hold under the so-called monotonicity condition due to Lasry and Lions:

(1) b is independent of the measure argument μ; since b is here assumed to be independent of x, it is thus constant;
(2) for any two $\mu, \mu' \in \mathcal{P}_2(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} \big(f(x,\mu) - f(x,\mu')\big) d\big(\mu - \mu'\big)(x) \geqslant 0, \qquad \int_{\mathbb{R}^d} \big(g(x,\mu) - g(x,\mu')\big) d\big(\mu - \mu'\big)(x) \geqslant 0.$$

Conversely, we can provide explicit examples for which uniqueness fails under Assumption **(A)**. Choose for instance $d = 1$, $b \equiv 0$, $f \equiv 0$ and $g(x, \mu) = xg(\bar{\mu})$, where $\bar{\mu}$ is understood as the mean of μ when $\mu \in \mathcal{P}_2(\mathbb{R})$, with g being non-increasing. Then, taking the mean in (6), we get

$$d\mathbb{E}[X_t^\star] = -\mathbb{E}[Y_t^\star]dt,$$
$$d\mathbb{E}[Y_t^\star] = 0, \quad \mathbb{E}[Y_T^\star] = g\big(\mathbb{E}[X_T^\star]\big),$$

which coincides with the system of characteristics associated with the inviscid Burgers equation, which we alluded to in introduction:

$$\partial_x u(t,x) - u(t,x)\partial_x u(t,x) = 0, \quad u(T,x) = g(x), \quad x \in \mathbb{R}.$$

Choosing for instance $g(x) = -x$ for $|x| \leqslant 1$ and $g(x) = -\mathrm{sign}(x)$ for $|x| \geqslant 1$, we know that uniqueness fails to the above forward-backward system when $T > 1$ and $\mathbb{E}[X_0^\star] = 0$ (it is easily checked that $((\mathbb{E}[X_t^\star], \mathbb{E}[Y_t^\star]) = (0,0))_{0\leqslant t\leqslant T}$, $((\mathbb{E}[X_t^\star], \mathbb{E}[Y_t^\star]) = (t,-1))_{0\leqslant t\leqslant T}$, $((\mathbb{E}[X_t^\star], \mathbb{E}[Y_t^\star]) = (-t,1))_{0\leqslant t\leqslant T}$ are solutions). This shows that noise in the mollified version (29) indeed restores uniqueness.

3.2. **Master equation.** In our analysis, we shall use the fact that (29) is connected with some infinite dimensional PDE. Provided that existence and uniqueness hold true, the system (29) must admit a decoupling field $\mathcal{U} : [0,T] \times L^2(\mathbb{S}^1;\mathbb{R}^d) \to L^2(\mathbb{S}^1;\mathbb{R}^d)$ such that, with probability 1,

$$Y_t(\cdot) = \mathcal{U}\big(t, X_t(\cdot)\big), \quad t \in [0,T],$$

or, equivalently,

$$Y_t^{n,\pm} = \mathcal{U}^{n,\pm}\big(t, X_t(\cdot)\big), \quad t \in [0,T], \quad n \in \mathbb{N},$$

where $(\mathcal{U}^{n,\pm})_{n\in\mathbb{N}}$ denotes the Fourier modes of $\mathcal{U}$.

Construction of the decoupling field is a standard procedure in the theory of forward-backward processes. We provide a short account here and we refer to [16, Chapter 4] for further details. Given $t \in [0,T]$ and $\ell \in L^2(\mathbb{S}^1;\mathbb{R}^d)$, consider (29) but with $X_t = \ell$ as initial condition at time t (or equivalently $X_t^{n,\pm} = \ell^{n,\pm}$). Note the solution $((X_s^{n,\pm;t,\ell})_{n\in\mathbb{N}}, (Y_s^{n,\pm;t,\ell})_{n\in\mathbb{N}}, (Z_s^{n,k,\pm;t,\ell})_{n,k\in\mathbb{N}})_{0\leqslant t\leqslant T}$ and define accordingly the processes $(X_s^{t,\ell}, Y_s^{t,\ell})_{t\leqslant s\leqslant T}$ from Ω_0 into $L^2(\mathbb{S}^1;\mathbb{R}^d) \times L^2(\mathbb{S}^1;\mathbb{R}^d)$ as in the discussion right after Lemma 2. By changing the filtration $\mathbb{F}_0$ into the augmented filtration generated by $(W_s^{n,\pm} - W_t^{n,\pm})_{n\in\mathbb{N}, t\leqslant s\leqslant T}$, we deduce that $Y_t^{t,\ell}$ is almost surely deterministic, which permits to let

$$\mathcal{U}(t,\ell) = Y_t^{t,\ell}. \tag{32}$$

Given this definition, we prove next that

Lemma 11. *For any initial condition $X_0(\cdot) \in L^2(\Omega_0, \mathcal{F}_{0,0}, \mathbb{P}_0; L^2(\mathbb{S}^1;\mathbb{R}^d))$, it holds, with probability 1 under $\mathbb{P}_0$,*

$$Y_t(\cdot) = \mathcal{U}\big(t, X_t(\cdot)\big), \quad t \in [0,T]. \tag{33}$$

Provided that $\mathcal{U}$ is smooth enough, it must satisfy, by a formal application of Itô's formula

$$\begin{aligned} dY_t^{n,\pm} = \Big(&\partial_t \mathcal{U}^{n,\pm}\big(t, X_t(\cdot)\big) + \Big\langle D\mathcal{U}^{n,\pm}\big(t, X_t(\cdot)\big), \mathfrak{B}\big(X_t(\cdot)\big) - Y_t(\cdot) + \partial_x^2 X_t(\cdot)\Big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)} \\ &+ \frac{1}{2}\mathrm{Trace}\Big[D^2\mathcal{U}^{n,\pm}\big(t, X_t(\cdot)\big)\Big]\Big)dt \\ &+ \Big\langle D\mathcal{U}^{n,\pm}\big(t, X_t(\cdot)\big), dW_t(\cdot)\Big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}, \end{aligned}$$

where $\boldsymbol{W}(\cdot)$ denotes the white noise defined in (20).

Identifying with the backward equation in (29), we deduce that $\mathcal{U}$ should be a solution of the infinite dimensional system of infinite dimensional PDEs (on $L^2(\mathbb{S}^1;\mathbb{R}^d)$):

$$\begin{aligned}\partial_t \mathcal{U}^{n,\pm}(t,\ell) &+ \big\langle \partial_x^2 D\mathcal{U}^{n,\pm}(t,\ell),\ell\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)} + \frac{1}{2}\mathrm{Trace}\big[D^2\mathcal{U}^{n,\pm}(t,\ell)\big] \\ &+ \big\langle D\mathcal{U}^{n,\pm}(t,\ell),\mathfrak{B}(\ell)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)} - \big\langle \mathcal{U}(t,\ell), D\mathcal{U}^{n,\pm}(t,\ell)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)} + \mathfrak{F}^{n,\pm}\big(\ell,\mathcal{U}(t,\ell)\big) = 0,\end{aligned} \tag{34}$$

with $\mathcal{U}^{n,\pm}(T,\cdot) = \mathfrak{G}^{n,\pm}$. The operator

$$L\mathfrak{h}(\ell) = \big\langle \partial_x^2 D\mathfrak{h}(\ell),\ell\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)} + \frac{1}{2}\mathrm{Trace}\big[D^2\mathfrak{h}(\ell)\big], \quad \ell \in L^2(\mathbb{S}^1;\mathbb{R}^d),$$

is called the Ornstein-Uhlenbeck operator on $L^2(\mathbb{S}^1;\mathbb{R}^d)$ driven by the unbounded linear operator ∂_x^2 acting on $L^2(\mathbb{S}^1;\mathbb{R}^d)$. It is associated with the semi-group $(\mathcal{P}_t)_{t\geqslant 0}$ generated by the Ornstein-Ulhenbeck process on $L^2(\mathbb{S}^1;\mathbb{R}^d)$, namely, for a bounded measurable function $\mathcal{V}$ from $L^2(\mathbb{S}^1;\mathbb{R}^d)$ into $\mathbb{R}$, $\mathcal{P}_t\mathcal{V}$ maps $L^2(\mathbb{S}^1;\mathbb{R}^d)$ into $\mathbb{R}$:

$$\mathcal{P}_t\mathcal{V} : L^2(\mathbb{S}^1) \ni \ell \mapsto \mathbb{E}_0\big[\mathcal{V}(U_t^\ell)\big], \tag{35}$$

where, for $\ell \in L^2(\mathbb{S}^1;\mathbb{R}^d)$, $\boldsymbol{U}^\ell(\cdot) = (U_t^\ell(\cdot))_{0\leqslant t\leqslant T}$ is the solution of the OU equation on $L^2(\mathbb{S}^1;\mathbb{R}^d)$ (constructed on $(\Omega_0,\mathbb{F}_0,\mathbb{P}_0)$):

$$dU_t^\ell(\cdot) = \partial_x^2 U_t^\ell(\cdot)dt + dW_t(\cdot), \quad t\in[0,T]\ ; \quad U_0^\ell = \ell.$$

Although there exist several results on infinite dimensional nonlinear PDEs (see for instance [20, 26, 54]), it seems that systems of type (34) have not been considered so far. We thus prove in Section 4 the following tailored-made solvability result:

Theorem 12. *Under Assumption* **(A)**, *the decoupling field* $\mathcal{U}$ *of* (29) *is a mild solution of the system of PDEs* (34), *namely, for all* $n\in\mathbb{N}$:

$$\begin{aligned}\mathcal{U}^{n,\pm}(t,\cdot) &= \mathcal{P}_{T-t}\Big(D_{n,\pm}\mathfrak{g}_0(\cdot,\mathrm{Leb}_1\circ\cdot^{-1})\Big) \\ &\quad + \int_t^T \mathcal{P}_{s-t}\Big[D_{n,\pm}\mathfrak{f}_0(\cdot,\mathrm{Leb}_1\circ\cdot^{-1}) + \big\langle \mathfrak{B}(\cdot) - \mathcal{U}(s,\cdot), D\mathcal{U}^{n,\pm}(s,\cdot)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]ds.\end{aligned}$$

Moreover, the function $\mathcal{U}$ *is Lipschitz continuous in the direction* $\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)$, *uniformly in time* $t\in[0,T]$.

Comparison with the case without noise. Once again, it is worth comparing Theorem 12 with results obtained for the original mean-field game. Under the Lasry-Lions monotonicity condition (say with $b\equiv 0$) and appropriate regularity assumptions on the coefficients, it is proven in Chassagneux, Crisan and Delarue [21] (see also [9] for the periodic case and [17, Chapter 5] for another point of view on [21]) that there exists a function

$$V : [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)\to\mathbb{R},$$

such that the function

$$[0,T]\times\mathbb{R}\times L^2(\mathbb{S}^1;\mathbb{R}^d)\ni(t,x,\ell)\mapsto V\big(t,x,\mathrm{Leb}_1\circ\ell^{-1}\big)$$

is differentiable and satisfies the so-called master equation

$$\partial_t V(t,x,\mu) - \frac{1}{2}|\partial_x V(t,x,\mu)|^2 - \int_{\mathbb{R}}\partial_\mu V(t,x,\mu)(v)\partial_x V(t,v,\mu)d\mu(v) + f(x,\mu) = 0, \tag{36}$$

for $(t,x,\mu)\in[0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, with $V(T,x,\mu)=g(x,\mu)$, where $\partial_\mu V$ is understood as follows. The Fréchet derivative of $\ell\mapsto V(t,x,\mathrm{Leb}_1\circ\ell^{-1})$ in the direction ℓ takes the form

$$D\big[V\big(t,x,\mathrm{Leb}_1\circ\cdot^{-1}\big)\big]_{\cdot=\ell} = \partial_\mu V\big(t,x,\mathrm{Leb}_1\circ\ell^{-1}\big)(\ell(\cdot)), \tag{37}$$

for some function $\partial_\mu V(t,x,\mu)(\cdot) \in L^2(\mathbb{R}^d,\mu;\mathbb{R}^d)$ with $\mu = \mathrm{Leb}_1 \circ \ell^{-1}$. It is also shown in [21] that $\partial_x V$ and $\partial_\mu V$ are differentiable in x (provided that f and g are sufficiently smooth). Therefore,

$$\begin{aligned}&\partial_t\big(\partial_x V(t,x,\mu)\big) - \partial_x\big(\partial_x V(t,x,\mu)\big)\partial_x V(t,x,\mu)\\ &\quad - \int_{\mathbb{R}} \partial_x\partial_\mu V(t,x,\mu)(v)\partial_x V(t,v,\mu)d\mu(v) + \partial_x f(x,\mu) = 0,\end{aligned} \tag{38}$$

for $(t,x,\mu) \in [0,T]\times\mathbb{R}^d\times\mathcal{P}_2(\mathbb{R}^d)$, with $\partial_x V(T,x,\mu) = \partial_x g(x,\mu)$.

Define now

$$\mathcal{V} : [0,T]\times L^2(\mathbb{S}^1;\mathbb{R}^d) \ni (t,\ell) \mapsto \big(\mathbb{S}^1 \ni x \mapsto \partial_x V\big(t,\ell(x),\mathrm{Leb}\circ\ell^{-1}\big) \in \mathbb{R}^d\big) \in L^2(\mathbb{S}^1;\mathbb{R}^d).$$

Notice that the right-hand side indeed belongs to $L^2(\mathbb{S}^1;\mathbb{R}^d)$ if $\partial_x V$ is at most of linear growth in x, see the aforementioned references. On the model of (34), compute

$$D\mathcal{V}^{n,\pm}(t,\ell) = D\Big(L^2(\mathbb{S}^1;\mathbb{R}^d) \ni h \mapsto \int_{\mathbb{S}^1} \partial_x V\big(t,h(x),\mathrm{Leb}\circ h^{-1}\big)e^{n,\pm}(x)dx\Big)_{|h=\ell}.$$

By (37) and following Lemma 8 (provided again that we have enough regularity), we have

$$D\mathcal{V}^{n,\pm}(t,\ell)(x) = \partial_x^2 V\big(t,\ell(x),\mathrm{Leb}\circ\ell^{-1}\big)e^{n,\pm}(x) + \int_{\mathbb{S}^1} \partial_x\partial_\mu V\big(t,\ell(v),\mathrm{Leb}\circ\ell^{-1}\big)\big(\ell(x)\big)e^{n,\pm}(v)dv,$$

so that

$$\begin{aligned}&\big\langle \mathcal{V}(t,\ell), D\mathcal{V}^{n,\pm}(t,\ell)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\\ &= \int_{\mathbb{S}^1} \partial_x^2 V\big(t,\ell(x),\mathrm{Leb}\circ\ell^{-1}\big)\partial_x V\big(t,\ell(x),\mathrm{Leb}\circ\ell^{-1}\big)e^{n,\pm}(x)dx\\ &\quad + \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \partial_x\partial_\mu V\big(t,\ell(v),\mathrm{Leb}\circ\ell^{-1}\big)\big(\ell(x)\big)\partial_x V\big(t,\ell(x),\mathrm{Leb}\circ\ell^{-1}\big)e^{n,\pm}(v)dvdx.\end{aligned}$$

Going back to (38), changing x into $\ell(x)$ with $x\in\mathbb{S}^1$, choosing $\mu = \mathrm{Leb}\circ\ell^{-1}$, multiplying by $e^{n,\pm}(x)$ and taking the integral over $\mathbb{S}^1$, we can write

$$\partial_t\mathcal{V}^{n,\pm}(t,\ell) - \big\langle \mathcal{V}(t,\ell), D\mathcal{V}^{n,\pm}(t,\ell)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)} + \int_{\mathbb{S}^1} \partial_x f\big(\ell(x),\mathrm{Leb}_1\circ\ell^{-1}\big)e^{n,\pm}(x)dx = 0, \tag{39}$$

with $\mathcal{V}^{n,\pm}(T,\cdot) = \mathfrak{G}^{n,\pm}$, which is the inviscid analogue of (34). Put it differently, (34) reads as a second-order version of (39); equivalently, Theorems 10 and 12 read as a regularization result for the master equation *via* an infinite dimensional Ornstein-Ulhenbeck operator.

Remark 13. *The reader may wonder why, in the statement of Theorem 12, we focus on the equation satisfied by the feedback function and not on the equation satisfied by the value function. Indeed, it is worth noting that, in the standard theory of mean-field games, the so-called "master equation" is the equation for the value function, as exemplified in* (36) *(therein, V identifies with the value of the mean-field game).*

In fact, the main reason is that it looks simpler. Indeed, our analysis is based upon the auxiliary control problem (21)–(22)*, which is –and this is the key feature– driven by random coefficients (not only the measure-valued process $\boldsymbol{\mu}$ is random but also the process $\boldsymbol{U}$ depends on ω_0). In this framework, the Pontryagin principle provides a very robust approach: Except for the additional martingale term in the backward equation* (30) *in the proof of Proposition 9, it has a standard structure; and, in fact, the martingale structure plays almost no role in the overall discussion. This is the reason why we use this approach here; and, as a result, this explains why the master equation we get is an equation for the feedback function.*

Of course, once the feedback function is given, the value function is easily recovered. They are two strategies to do so. The first one is to regard the optimal cost $\bar{J}^{\boldsymbol{\mu}}(\boldsymbol{\alpha}^\star)$ in (21) when the initial condition $(t, X_t^\star(\cdot))$ varies in $[0,T]\times L^2(\mathbb{S}^1;\mathbb{R}^d)$; equivalently, this amounts to consider $\int_{\mathbb{R}^d} V(t,x,\mu)d\mu(x)$ in (36). Here the resulting function would satisfy a linear PDE on $[0,T]\times L^2(\mathbb{S}^1;\mathbb{R}^d)$, but the coefficients would depend on the feedback function. Pay attention that, as a mean-field game is not an optimization problem, this equation could not be regarded as an autonomous Hamilton-Jacobi-Bellman equation deriving from an optimal control problem in infinite dimension. Another strategy is to disentangle the initial state of $\bar{\boldsymbol{X}}^{\boldsymbol{\alpha}}$ in (22) from the initial condition $X_0(\cdot)\in L^2(\mathbb{S}^1;\mathbb{R}^d)$ for $\boldsymbol{X}^\star(\cdot)$, which is exactly what is done for standard mean-field games. In fact, by doing so, we first compute, with $\bar{X}_0^{\boldsymbol{\alpha}} = x\in\mathbb{R}^d$ as initial condition, the optimal value of the optimal control problem (21)–(22) in the random environment formed by $\boldsymbol{X}^\star(\cdot)$; since the environment is uniquely defined in terms of $X_0(\cdot)$ (this is Theorem 10), the optimal value is a mere function of x and $X_0(\cdot)$. Using the same notation as in (36), this should be "our" $V(0,x,X_0(\cdot))$ (here $t=0$ because (21)–(22) is initialized at time 0, but it is pretty easy to adapt the argument to any initial time t); then $\partial_x V(0,X_0(\cdot),X_0(\cdot))$ should coincide with $\mathcal{U}(0,X_0(\cdot))$.

It is worth noting that, following the usual approach to mean-field games based on the MFG PDE system, we could directly address the optimal value of the optimal control problem (21)–(22) in an arbitrary environment $\boldsymbol{X}^\star(\cdot)$ (before we know that it is an equilibrium) and then look for an equilibrium by solving a fixed point obtained by plugging the resulting optimal feedback in the dynamics of $\boldsymbol{X}^\star(\cdot)$. Basically, this would require to write down the stochastic Hamilton-Jacobi-Bellman equation associated with (21)–(22) in the arbitrary environment $\boldsymbol{X}^\star(\cdot)$; this is the point where we feel that using the Pontryagin principle is simpler.

3.3. **Interpretation as an asymptotic game.** Classical MFGs arise as asymptotic versions of games with a large number of players. Similarly, a natural question here is to address the interpretation of the randomized MFG defined above as the limiting version of a large game (with finitely many players). Generally speaking, there are two ways to make the connection between mean-field games and finite games: The first one is to prove that equilibria of the finite games (if they do exist) converge to a solution of the limiting mean-field game, see for instance [9] for the convergence of closed-loop equilibria and [42] for the convergence of open-loop equilibria; the second one is to prove that any solution to the limiting game induces a sequence of approximate Nash equilibria to the corresponding finite games, see for instance [7, 14, 38] for earlier references in that direction. It turns out that, for standard mean-field games, the second approach is (much) easier to implement than the first one; for that reason, this is that one that we try to adapt below, see however Remark 15 about the possible implementation of the first approach.

In comparison with the standard case, there are two main differences between our framework and the aforementioned references. The first one is that the limiting system is perturbed by an infinite dimensional noise, which should be called "an infinite dimensional common noise". This terminology is frequently used in the theory of MFGs to emphasize the fact that the law of the population feels the realization of the noise, as opposed to more standard cases where the law of the population is defined as the average over all the possible realizations of the noise, see for instance [9, 19] and the book [17]. The second feature is the presence of local interactions due to the Laplacian in the dynamics (29) (see also the SPDE (19)).

In order to describe the corresponding finite games, we proceed as follows. We consider NA_N particles (with state in $\mathbb{R}^d$) that are uniformly distributed all along the N roots of unity of order N, with exactly A_N particles per root, where $A_N\in\mathbb{N}^*$. States of the NA_N particles at time t are denoted by $(X_t^{k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}$. The index k is understood as a label for the position (or the site) of the particle (k,j) on the unit circle: it is located at point with angle $2\pi k/N$. In particular (and it is important for the sequel), the set of indices for the location of the site may be identified

with $\mathbb{Z}/N\mathbb{Z}$; sometimes, we thus use the notation $X_t^{k+\ell N,j}$ for $X_t^{k,j}$, for $k \in \{0,\dots,N-1\}$ and $\ell \in \mathbb{Z}$. In the notation $X_t^{k,j}$, j stands for the label of the particle at the site k, since that there are A_N particles at the site k.

The dynamics of each particle is controlled, each particle (k,j) having dynamics of the form

$$dX_t^{k,j} = \Big(b\big(\bar\mu_t^N\big) + \alpha_t^{k,j} + N^2\big(\bar X_t^{k+1} + \bar X_t^{k-1} - 2\bar X_t^k\big)\Big)dt + \sqrt{N}dB_t^k,$$

with

$$\bar X_t^k = \frac{1}{A_N}\sum_{j=1}^{A_N} X_t^{k,j},$$

and $X_0^{k,j} = \bar X_0^k$ for all $j \in \{1,\cdots,A_N\}$, where $(\bar X_0^k)_{k=0,\cdots,N-1}$ are given by the following finite volume approximation of $X_0(\cdot)$ (which is here assumed to be independent of ω_0):

$$\bar X_0^k = N\int_{k/N}^{(k+1)/N} X_0(x)dx, \quad k = 0,\cdots,N-1,$$

whilst the noises $(\boldsymbol{B}^k = (B_t^k)_{0\leqslant t\leqslant T})_{k=0,\cdots,N-1}$ are independent d-dimensional Brownian motions on the interval $[0,T]$ with the following definition:

$$B_t^k = \sqrt{N}\int_{k/N}^{(k+1)/N} W_t(dx).$$

The random variables $(\bar X_0^k)_{k=0,\cdots,N-1}$ are thus constructed on the space $(\mathbb{S}^1, \mathcal{L}(\mathbb{S}^1), \mathrm{Leb}_1)$ whilst the processes $(\boldsymbol{B}^k = (B_t^k)_{0\leqslant t\leqslant T})_{k=0,\cdots,N-1}$ are constructed on the space $(\Omega_0, \mathcal{A}_0, \mathbb{P}_0)$, as defined in Subsection 2.3.

Above $\bar\mu_t^N$ is the empirical distribution

$$\bar\mu_t^N = \frac{1}{NA_N}\sum_{k=0}^{N-1}\sum_{j=1}^{A_N}\delta_{X_t^{k,j}}.$$

Processes $(\boldsymbol{\alpha}^{k,j} = (\alpha_t^{k,j})_{0\leqslant t\leqslant T})_{k=0,\cdots,N-1;j=1,\cdots,A_N}$ are controls with values in $\mathbb{R}^d$; they are progressively-measurable with respect to the filtration generated by the cylindrical white noise $(W_t(\cdot))_{0\leqslant t\leqslant T}$. Controls are required to satisfy

$$\mathbb{E}\int_0^T |\alpha_t^{k,j}|^2 dt < \infty.$$

We assign to player (k,j) the following cost functional

$$J^{k,j}\big((\boldsymbol{\alpha}^{k',j'})_{k'=0,\cdots,N-1;j'=1,\cdots,A_N}\big) = \mathbb{E}\Big[g\big(X_T^{k,j},\bar\mu_T^N\big) + \int_0^T\Big(f\big(X_t^{k,j},\bar\mu_t^N\big) + \frac{1}{2}|\alpha_t^{k,j}|^2\Big)dt\Big].$$

Recall that we call an open-loop Nash equilibrium a tuple $(\boldsymbol{\alpha}^{\star k,j} = (\alpha_t^{\star k,j})_{0\leqslant t\leqslant T})_{k=0,\cdots,N-1;j=1,\cdots,A_N}$ such that, for any $(k_0,j_0) \in \{0,\cdots,N-1\}\times\{1,\cdots,A_N\}$, for any control $\boldsymbol{\alpha}^{k_0,j_0} = (\alpha_t^{k_0,j_0})_{0\leqslant t\leqslant T}$, $J^{k_0,j_0}((\boldsymbol{\beta}^{k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}) \geqslant J^{k_0,j_0}((\boldsymbol{\alpha}^{\star k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N})$, where $\boldsymbol{\beta}^{k,j} = \boldsymbol{\alpha}^{\star k,j}$ if $(k,j) \neq (k_0,j_0)$ and $\boldsymbol{\beta}^{k_0,j_0} = \boldsymbol{\alpha}^{k_0,j_0}$.

The following statement shows that we can construct an approximated Nash equilibrium from the solution to problem (24) (compare for instance with [7, 14, 38] and [17, Chapter 6]).

Theorem 14. *On top of Assumption* **(A)**, *assume that f and g are Lipschitz continuous in μ, uniformly in x. Assume also that the sequence $(A_N)_{N\in\mathbb{N}^*}$ tends to ∞ with N. For a (deterministic)*

initial condition $X_0(\cdot) \in L^2(\mathbb{S}^1;\mathbb{R}^d)$*, call* $(\boldsymbol{X}(\cdot),\boldsymbol{Y}(\cdot),\boldsymbol{Z}(\cdot))$ *the solution to* (29)*. Then, there exists a sequence of positive reals* $(\varepsilon_N)_{N\in\mathbb{N}*}$ *converging to* 0 *as* N *tends to* ∞ *such that, with*

$$\alpha_t^{\star k,j} = N\int_{(k-1)/N}^{k/N} Y_t(x)dx, \quad t\in[0,T],$$

for all $k \in \{0,\cdots,N-1\}$ *and* $j\in\{1,\cdots,A_N\}$*, it holds, for any* $k_0\in\{0,\cdots,N-1\}$ *and* $j_0\in\{1,\cdots,A_N\}$*, and for any control* $\boldsymbol{\alpha}^{k_0,j_0} = (\alpha_t^{k_0,j_0})_{0\leqslant t\leqslant T}$*,*

$$J^{k_0,j_0}\big((\boldsymbol{\beta}^{k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big) \geqslant J^{k_0,j_0}\big((\boldsymbol{\alpha}^{\star k,j}_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big) - \varepsilon_N,$$

where $\boldsymbol{\beta}^{k,j} = \boldsymbol{\alpha}^{\star k,j}$ *if* $(k,j)\neq(k_0,j_0)$ *and* $\boldsymbol{\beta}^{k_0,j_0} = \boldsymbol{\alpha}^{k_0,j_0}$*.*

Remark 15. *Theorem 14 must be regarded as a way to connect the problem* (24) *with a game of the same flavor as what appears in standard mean field game theory. In this regard, the assumption that* b*,* $f(0,\cdot)$ *and* $g(0,\cdot)$ *are at most of linear growth (with respect to* $M_2(\mu)$*) is mostly for convenience. Also, it must be emphasized that it is not the only way to make the connection. Another way would be to construct an approximate Nash equilibrium in a closed-loop form, as usually done in mean field games. We assert that it should be indeed possible provided that we let:*

$$\alpha_t^{\star k,j} = N\int_{(k-1)/N}^{k/N} \mathcal{U}\big(t,\bar{X}_t(\cdot)\big)(x)dx, \quad t\in[0,T],$$

with the notation

$$\bar{X}_t(\cdot) = \sum_{k=0}^{N-1}\bar{X}_t^k\mathbf{1}_{[k/N,(k+1)/N)}(\cdot) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=1}^{N}X_t^{k,j}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot), \quad t\in[0,T],$$

which means that

$$dX_t^{k,j} = \left(b\big(\bar{\mu}_t^N\big) + N\int_{(k-1)/N}^{k/N}\mathcal{U}\big(t,\bar{X}_t(\cdot)\big)(x)dx + N^2\big(\bar{X}_t^{k+1}+\bar{X}_t^{k-1}-2\bar{X}_t^k\big)\right)dt + \sqrt{N}dB_t^k.$$

As the paper is already quite long, we feel better to focus on the construction of an approximated Nash equilibrium over open-loop form controls only, which is in fact slightly simpler.

Another strategy would be to address the convergence of the Nash equilibria of the finite player game (if they do exist) to the solution of (24)*. Describing the dynamics of the equilibria to the finite player game by means of Pontryagin's principle and then using the master equation* (34)*, we could indeed implement the same strategy as that used in* [9] *for standard mean field games, but this would require first to improve Theorem 12 and to prove further regularity properties of* $\mathcal{U}$*. Again, we feel better to postpone this equation to further works.*

Last, we mention that the condition $A_N\to\infty$ *is absolutely crucial. It is must be regarded as a way to freeze the influence of the local interaction in the dynamics between the particles; this is the key fact to restore a mean field limit despite the local interactions.*

4. Proofs of Theorems 10 and 12

We now prove Theorems 10 and 12.

4.1. **Small time analysis.**

We start with the case when T is small enough.

Theorem 16. *There exists a constant* c*, only depending on the Lipschitz constant of the coefficients* $\mathfrak{b}_0$*,* $\mathfrak{F}=D\mathfrak{f}_0$ *and* $\mathfrak{G}=D\mathfrak{g}_0$ *such that, for* $T\leqslant c$*, the system* (29) *is uniquely solvable for any initial condition* $X_0(\cdot)\in L^2(\Omega_0,\mathcal{F}_{0,0},\mathbb{P}_0;L^2(\mathbb{S}^1;\mathbb{R}^d))$*. This permits to define the decoupling field* $\mathcal{U}$ *as in*

(32). *It maps $L^2(\mathbb{S}^1;\mathbb{R}^d)$ into itself. Then, there exists a constant Λ, only depending on the bound of the coefficients $\mathfrak{b}_0$, $\mathfrak{F} = D\mathfrak{f}_0$ such that, for $T \leqslant c$,*

$$\sup_{0\leqslant t\leqslant T}\sup_{\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)} \|\mathcal{U}(t,\ell)\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} \leqslant \sup_{\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)} \|\mathfrak{G}(\ell)\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} + \Lambda T^2.$$

Moreover, there exists a constant C, only depending on the Lipschitz constant of the coefficients $\mathfrak{b}_0$, $\mathfrak{F} = D\mathfrak{f}_0$ and $\mathfrak{G} = D\mathfrak{g}_0$ such that, for $T \leqslant c$, for any $t \in [0,T]$, $\mathcal{U}(t,\cdot)$ is C Lipschitz continuous. In particular, $\mathcal{U}$ satisfies Lemma 11.

Remark 17. *We let the reader check that the above result remains true if $\mathfrak{G}$ is not given as the gradient of $\mathfrak{g}_0$, but is a general bounded and Lipschitz continuous function from $L^2(\mathbb{S}^1;\mathbb{R}^d)$ into itself.*

Proof. The proof is quite standard in the finite dimensional framework. We give the sketch of it, insisting on the differences between the infinite-dimensional and finite-dimensional cases.

First step. Existence and uniqueness in small time follow from the application of Picard's fixed point theorem. We consider the space $\mathcal{S}$ of processes $(\boldsymbol{X}(\cdot),\boldsymbol{Y}(\cdot)) = (X_t(\cdot),Y_t(\cdot))_{0\leqslant t\leqslant T}$ with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)\times L^2(\mathbb{S}^1;\mathbb{R}^d)$, that are $\mathbb{F}_0$-adapted with continuous paths and that satisfy

$$\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\big(\|X_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)} + \|Y_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big)\Big] < \infty.$$

Given the initial condition $X_0(\cdot) \in L^2(\Omega_0,\mathcal{F}_{0,0},\mathbb{P}_0;L^2(\mathbb{S}^1;\mathbb{R}^d))$, we then call Φ the function that maps $(\boldsymbol{X}(\cdot),\boldsymbol{Y}(\cdot)) = (X_t(\cdot),Y_t(\cdot))_{0\leqslant t\leqslant T}$ onto the pair $(\tilde{\boldsymbol{X}}(\cdot),\tilde{\boldsymbol{Y}}(\cdot)) = (\tilde{X}_t(\cdot),\tilde{Y}_t(\cdot))_{0\leqslant t\leqslant T}$ satisfying

$$\begin{aligned}
d\tilde{X}^{n,\pm}_t &= \Big(\mathbf{1}_{(n,\pm)=(0,+)}\mathfrak{b}_0\big(\tilde{X}_t(\cdot)\big) - Y^{n,\pm}_t - (2\pi n)^2\tilde{X}^{n,\pm}_t\Big)dt + dW^{n,\pm}_t,\\
d\tilde{Y}^{n,\pm}_t &= -D_{n,\pm}\mathfrak{f}_0\big(X_t(\cdot),\mathrm{Leb}_1\circ X_t(\cdot)^{-1}\big)dt + \sum_{k\in\mathbb{N}}\tilde{Z}^{n,k,\pm}dW^{k,\pm}_t,
\end{aligned}$$

with the terminal condition $\tilde{Y}^{n,\pm}_T = D_{n,\pm}\mathfrak{g}_0(X_T(\cdot),\mathrm{Leb}_1\circ X_T(\cdot)^{-1})$. Obviously, the backward equation may be rewritten under the form:

$$\tilde{Y}^{n,\pm}_t = \mathbb{E}_0\Big[D_{n,\pm}\mathfrak{g}_0\big(X_T(\cdot),\mathrm{Leb}_1\circ X_T(\cdot)^{-1}\big) + \int_t^T D_{n,\pm}\mathfrak{f}_0\big(X_s(\cdot),\mathrm{Leb}_1\circ X_s(\cdot)^{-1}\big)ds\,\big|\,\mathcal{F}_{0,t}\Big].$$

Taking the square and summing over $n\in\mathbb{N}$, we deduce that

$$\begin{aligned}
\sum_{n\in\mathbb{N}}|\tilde{Y}^{n,\pm}_t|^2 \leqslant \sum_{n\in\mathbb{N}}\mathbb{E}_0\Big[&\big|D_{n,\pm}\mathfrak{g}_0\big(X_T(\cdot),\mathrm{Leb}_1\circ X_T(\cdot)^{-1}\big)\big|^2\\
&+ T\int_t^T\big|D_{n,\pm}\mathfrak{f}_0\big(X_s(\cdot),\mathrm{Leb}_1\circ X_s(\cdot)^{-1}\big)\big|^2 ds\,\big|\,\mathcal{F}_{0,t}\Big].
\end{aligned}$$

Since $D\mathfrak{f}_0(\cdot,\mathrm{Leb}_1\circ\cdot^{-1})$ and $D\mathfrak{g}_0(\cdot,\mathrm{Leb}_1\circ\cdot^{-1})$ are bounded, we deduce that

$$\sum_{n\in\mathbb{N}}|\tilde{Y}^{n,\pm}_t|^2 \leqslant \sup_{\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)}\|D\mathfrak{g}_0(\ell,\mathrm{Leb}_1\circ\ell^{-1})\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} + \Lambda T^2, \tag{40}$$

for some deterministic $\Lambda \geqslant 0$.

Consider now another input $(\boldsymbol{X}'(\cdot),\boldsymbol{Y}'(\cdot)) = (X'_t(\cdot),Y'_t(\cdot))_{0\leqslant t\leqslant T}$ in $\mathcal{S}$ and call $(\tilde{\boldsymbol{X}}'(\cdot),\tilde{\boldsymbol{Y}}'(\cdot)) = (\tilde{X}'_t(\cdot),\tilde{Y}'_t(\cdot))_{0\leqslant t\leqslant T}$ its image by Φ. By the same argument as above, using in addition Bürkholder-Davis-Gundy inequalities, we get

$$\begin{aligned}
&\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\|\tilde{Y}_t(\cdot)-\tilde{Y}'_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]\\
&\leqslant \mathbb{E}_0\Big[\big\|D\mathfrak{g}_0\big(X_T(\cdot),\mathrm{Leb}_1\circ X_T(\cdot)^{-1}\big)-D\mathfrak{g}_0\big(X'_T(\cdot),\mathrm{Leb}_1\circ X'_T(\cdot)^{-1}\big)\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]\\
&\quad+T\int_0^T\mathbb{E}_0\Big[\big\|D\mathfrak{f}_0\big(X_s(\cdot),\mathrm{Leb}_1\circ X_s(\cdot)^{-1}\big)-D\mathfrak{f}_0\big(X'_s(\cdot),\mathrm{Leb}_1\circ X'_s(\cdot)^{-1}\big)\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]ds.
\end{aligned}$$

Observe that $D\mathfrak{f}_0$ and $D\mathfrak{g}_0$ are Lipschitz continuous (from $L^2(\mathbb{S}^1;\mathbb{R}^d)$ into itself). Deduce that there exists a constant $C\geqslant 0$, only depending on the Lipschitz constants of the coefficients, such that, for $T\leqslant 1$,

$$\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\|\tilde{Y}_t(\cdot)-\tilde{Y}'_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]\leqslant C\sup_{0\leqslant t\leqslant T}\mathbb{E}_0\big[\|X_t(\cdot)-X'_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]. \tag{41}$$

Proceeding in a similar way with the forward equation and using the fact that the factor $(2\pi n)^2$ in the dynamics is affected with a sign minus (so that it is a friction term), we get

$$\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\|\tilde{X}_t(\cdot)-\tilde{X}'_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]\leqslant CT\sup_{0\leqslant t\leqslant T}\mathbb{E}_0\big[\|Y_t(\cdot)-Y'_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]. \tag{42}$$

We easily deduce that Φ is a contraction in small time, which shows the existence of a unique fixed point. This shows that the system (29) is uniquely solvable when $T\leqslant c$, for a constant c that only depends on the Lipschitz constants of the coefficients.

Second step. Now that existence and uniqueness are known to hold true, we can define the decoupling field $\mathcal{U}$ in a standard way. The key point is to observe that the system (29), when regarded under the initial condition $X_t=\ell$ at time $t\in[0,T]$ for some $\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)$, is also uniquely solvable when $T\leqslant c$ and that its solution, denoted by $((\boldsymbol{X}^{t,\ell,n,\pm}=(X_s^{t,\ell,n,\pm})_{t\leqslant s\leqslant T})_{n\in\mathbb{N}},(\boldsymbol{Y}^{t,\ell,n,\pm}=(Y_s^{t,\ell,n,\pm})_{t\leqslant s\leqslant T})_{n\in\mathbb{N}},(\boldsymbol{Z}^{t,\ell,n,\pm,k,\pm}=(Z_s^{t,\ell,n,\pm,k,\pm})_{t\leqslant s\leqslant T})_{n\in\mathbb{N},k\in\mathbb{N}})$ is adapted with respect to the completion of the filtration generated by the collection of Wiener processes $((W_s^0-W_t^0)_{t\leqslant s\leqslant T},((W_s^{n,\pm}-W_t^{n,\pm})_{t\leqslant s\leqslant T})_{n\in\mathbb{N}*})$. In particular, for each $n\in\mathbb{N}$, the random variable $Y_t^{n,\pm,t,\ell}$ is almost surely deterministic. We then let

$$\mathcal{U}^{n,\pm}(t,\ell)=Y_t^{t,\ell,n,\pm},$$

and

$$\mathcal{U}(t,\ell)=\sum_{n\in\mathbb{N}}\mathcal{U}^{n,\pm}(t,\ell)e^{n,\pm}(\cdot)\in L^2(\mathbb{S}^1;\mathbb{R}^d),\quad t\in[0,T],\quad \ell\in L^2(\mathbb{S}^1;\mathbb{R}^d).$$

The bound for $\mathcal{U}$ is a straightforward consequence of (40).

As for the Lispchitz constant of $\mathcal{U}$, it follows again from a straightforward adaptation of (41) and (42). Indeed, for any two solutions $(\boldsymbol{X}(\cdot),\boldsymbol{Y}(\cdot))$ and $(\boldsymbol{X}'(\cdot),\boldsymbol{Y}'(\cdot))$ to (29), we have

$$\begin{aligned}
&\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\|Y_t(\cdot)-Y'_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]\\
&\leqslant C\sup_{0\leqslant t\leqslant T}\mathbb{E}_0\big[\|X_t(\cdot)-X'_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\\
&\leqslant C\Big(\mathbb{E}_0\big[\|X_0(\cdot)-X'_0(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]+T\sup_{0\leqslant t\leqslant T}\mathbb{E}_0\big[\|Y_t(\cdot)-Y'_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\Big),
\end{aligned}$$

and then, for T small enough,

$$\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\|Y_t(\cdot)-Y'_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]\leqslant C\mathbb{E}_0\big[\|X_0(\cdot)-X'_0(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]. \tag{43}$$

By performing the analysis on the interval $[t,T]$ instead of $[0,T]$ and by choosing $\boldsymbol{X}(\cdot)=\boldsymbol{X}^{t,\ell}(\cdot)=\sum_{n\in\mathbb{N}}\boldsymbol{X}^{t,\ell,n,\pm}e^{n,\pm}(\cdot)$ and $\boldsymbol{X}'(\cdot)=\boldsymbol{X}^{t,\ell'}(\cdot)=\sum_{n\in\mathbb{N}}\boldsymbol{X}^{t,\ell',n,\pm}e^{n,\pm}(\cdot)$ for two $\ell,\ell'\in L^2(\mathbb{S}^1;\mathbb{R}^d)$, we deduce that $\mathcal{U}$ is Lipschitz continuous in the space variable.

It remains to check that Lemma 11 is satisfied. The argument is standard in the finite dimensional case, see for instance [24]; as for the infinite dimensional case, we refer to [17, Chapter 5]. So, we just provide a sketch of the proof. In fact, by regarding t in the formula (33) as the initial time of the forward process, it suffices to focus on the case $t=0$ and to prove that, for any $X_0(\cdot)\in L^2(\Omega,\mathcal{F}_{0,0},\mathbb{P};L^2(\mathbb{S}^1;\mathbb{R}^d))$, the unique solution $(\boldsymbol{X}(\cdot),\boldsymbol{Y}(\cdot))$ to (29) satisfies

$$Y_0(\cdot)=\mathcal{U}\big(0,X_0(\cdot)\big),$$

which is already known to be true when $X_0(\cdot)$ is deterministic, that is $X_0(\cdot)=\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)$. It is easily checked that it remains true when $X_0(\cdot)$ is a random variable of the form

$$X_0(\cdot)=\sum_{i=1}^n\mathbf{1}_{A_i}\ell_i, \tag{44}$$

with $A_i\in\mathcal{F}_{0,0}$ and $\ell_i\in L^2(\mathbb{S}^1;\mathbb{R}^d)$ for all $i\in\{1,\cdots,n\}$; indeed, in that case, $Y_0(\cdot)=\sum_{i=1}^n\mathbf{1}_{A_i}Y_0^{0,\ell_i}(\cdot)$. When the support of the law of $X_0(\cdot)$ is included in a compact subset of $L^2(\mathbb{S}^1;\mathbb{R}^d)$, we can approximate $X_0(\cdot)$ in $L^2(\Omega,\mathcal{F}_{0,0},\mathbb{P};L^2(\mathbb{S}^1;\mathbb{R}^d))$ by a sequence of random variables of the form (44). Using the fact that the representation formula (33) holds true along the approximation sequence and using the stability property (43), we deduce that the representation formula holds true when the law of $X_0(\cdot)$ is compactly supported. When $X_0(\cdot)$ is a general element in $L^2(\Omega,\mathcal{F}_{0,0},\mathbb{P};L^2(\mathbb{S}^1;\mathbb{R}^d))$, we can play the same game: We can approximate $X_0(\cdot)$ by a sequence of compactly supported initial conditions of the form $(\sum_{k=0}^n\vartheta_n(X_0^{k,\pm})e^{k,\pm})_{n\in\mathbb{N}}$, where $(\vartheta_n)_{n\in\mathbb{N}}$ is a sequence of cut-off functions from $\mathbb{R}^d$ into itself converging to the identity uniformly on compact sets. ∎

4.2. **Road map to existence and uniqueness in arbitrary time.** Our strategy for proving existence and uniqueness in arbitrary time is completely inspired from the finite dimensional case. The point is to apply iteratively Theorem 16 and to provide an *a priori* bound for the Lipschitz constant of the decoupling field $\mathcal{U}$ that holds true all along the induction. We refer to [24] for a complete description of the induction procedure in the finite dimensional case.

Change of measure. Below, we mostly focus on the derivation of the *a priori* bound for the Lipschitz constant of $\mathcal{U}$. We start with the following observation. For $T\leqslant c$ as in the statement of Theorem 16, we can define the probability $\tilde{\mathbb{P}}_0$ on Ω_0 by

$$\begin{aligned}\frac{d\tilde{\mathbb{P}}_0}{d\mathbb{P}_0}&=\exp\Big(-\int_0^T\big\langle\mathfrak{B}(X_t(\cdot))-Y_t(\cdot)\big),dW_t\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}-\frac{1}{2}\int_0^T\big\|\mathfrak{B}(X_t(\cdot))-Y_t(\cdot)\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}dt\Big)\\&=\exp\Big(-\sum_{n\in\mathbb{N}}\int_0^T\big(\mathbf{1}_{(n,\pm)=(0,+)}\mathfrak{b}_0(X_t(\cdot))-Y_t^{n,\pm}\big)\cdot dW_t^{n,\pm}\\&\qquad-\frac{1}{2}\sum_{n\in\mathbb{N}}\int_0^T\big|\mathbf{1}_{(n,\pm)=(0,+)}\mathfrak{b}_0(X_t(\cdot))-Y_t^{n,\pm}\big|^2dt\Big),\end{aligned}$$

where $\mathfrak{B}$ is as in (26). Since $\mathfrak{b}_0$ is bounded and $\boldsymbol{Y}$ satisfies (40), $\tilde{\mathbb{P}}_0$ is a probability measure equivalent to $\mathbb{P}_0$. Observe in particular that, for any $p\geqslant 1$,

$$\mathbb{E}_0\Big[\Big(\frac{d\tilde{\mathbb{P}}_0}{d\mathbb{P}_0}\Big)^p\Big]<\infty. \tag{45}$$

Of course, the bound (40) remains true under $\tilde{\mathbb{P}}_0$. Observe also from the identity

$$\sum_{k\in\mathbb{N}}\int_0^T Z_s^{n,k,\pm}dW_s^{k,\pm} = Y_T^{n,\pm} - Y_0^{n,\pm} + \int_0^T D\mathfrak{f}_0^{n,\pm}\big(X_t(\cdot),\mathrm{Leb}_1\circ X_t(\cdot)^{-1}\big)dt$$

that, for any $p\geqslant 1$,

$$\mathbb{E}_0\Big[\Big(\sum_{k\in\mathbb{N}}\int_0^T|Z_s^{n,k,\pm}|^2ds\Big)^p\Big]<\infty.$$

By (45), the same is true under $\tilde{\mathbb{P}}_0$, that is

$$\tilde{\mathbb{E}}_0\Big[\Big(\sum_{k\in\mathbb{N}}\int_0^T|Z_s^{n,k,\pm}|^2ds\Big)^p\Big]<\infty. \tag{46}$$

Now, we let

$$\tilde{W}_t^{n,\pm} = W_t^{n,\pm} + \int_0^t\big(\mathbf{1}_{(n,\pm)=(0,+)}\mathfrak{b}_0(X_s(\cdot)) - Y_s^{n,\pm}\big)ds,\quad t\in[0,T].$$

Under $\tilde{\mathbb{P}}_0$, the processes $((\tilde{W}_t^{n,\pm})_{0\leqslant t\leqslant T})_{n\in\mathbb{N}}$ are independent Brownian motions and the forward component of the solution to (29) satisfies

$$dX_t^{n,\pm} = -(2\pi n)^2X_t^{n,\pm}dt + d\tilde{W}_t^n,\quad t\in[0,T],$$

and is thus an Ornstein-Ulhenbeck process, with $X_0^{n,\pm}$ as initial condition. Also, under $\tilde{\mathbb{P}}_0$, the backward equation takes the form:

$$\begin{aligned}dY_t^{n,\pm} = &\Big[-D_{n,\pm}\mathfrak{f}_0\big(X_t(\cdot),\mathrm{Leb}_1\circ X_t(\cdot)^{-1}\big) - \sum_{k\in\mathbb{N}}Z_t^{n,k,\pm}\big(\mathbf{1}_{(k,\pm)=(0,+)}\mathfrak{b}_0\big(X_t(\cdot)\big) - Y_t^{k,\pm}\big)\Big]dt\\ &+\sum_{k\in\mathbb{N}}Z_t^{n,k,\pm}d\tilde{W}_t^{k,\pm}.\end{aligned} \tag{47}$$

By (46), the drift has finite moments of any order under $\tilde{\mathbb{P}}_0$.

According to the standard theory of backward SDEs (or, equivalently, by a formal application of Itô's formula), we expect

$$Z_t^{n,k,\pm} = D_{k,\pm}\mathcal{U}^{n,\pm}(t,X_t(\cdot))\quad \mathbb{P}_0 \text{ (or } \tilde{\mathbb{P}}_0\text{) almost everywhere.} \tag{48}$$

Initializing the process $(X_s)_{0\leqslant s\leqslant T}$ at some $\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)$ and at some $t\in[0,T]$ and taking the expectation in (47) under $\tilde{\mathbb{P}}_0$, we conjecture (and this in fact the purpose of Theorem 12 to make the statement clear) that:

$$\begin{aligned}\mathcal{U}^{n,\pm}(t,\cdot) = &\,\mathcal{P}_{T-t}\Big(D_{n,\pm}\mathfrak{g}_0(\cdot,\mathrm{Leb}_1\circ\cdot^{-1})\Big)\\ &+\int_t^T\mathcal{P}_{s-t}\Big[D_{n,\pm}\mathfrak{f}_0(\cdot,\mathrm{Leb}_1\circ\cdot^{-1}) + \big\langle D\mathcal{U}^{n,\pm}(s,\cdot),\mathfrak{B}(\cdot)-\mathcal{U}(s,\cdot)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]ds,\end{aligned} \tag{49}$$

where, differently from (47), we used the more compact notation $\mathfrak{B}$ for the drift coefficient. Here the notation $\langle D\mathcal{U}^{n,\pm}(s,\cdot),\mathfrak{B}(\cdot)-\mathcal{U}(s,\cdot)\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}$ may be slightly confusing and should be understood as a function from $L^2(\mathbb{S}^1;\mathbb{R}^d)$ into $\mathbb{R}^d$ defined by:

$$\begin{aligned}\big\langle D\mathcal{U}^{n,\pm}(s,\cdot),\mathfrak{B}(\cdot)-\mathcal{U}(s,\cdot)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)} : L^2(\mathbb{S}^1;\mathbb{R}^d)\ni\ell\mapsto&\big\langle D\mathcal{U}^{n,\pm}(s,\ell),\mathfrak{B}(\ell)-\mathcal{U}(s,\ell)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\\ =&\sum_{k\in\mathbb{N}}D_{k,\pm}\mathcal{U}^{n,\pm}(s,\ell)\big(\mathfrak{B}^{k,\pm}(\ell)-\mathcal{U}^{k,\pm}(s,\ell)\big),\end{aligned}$$

the summand in the right-hand side reading as the product of a matrix of size $d\times d$ by a vector of size d. Identity (49) is the cornerstone of the *a priori* bound on the Lipschitz constant of $\mathcal{U}$ (in space).

Galerkin approximation. The problem with the formula (49) is that we do not know yet whether $\mathcal{U}$ is Fréchet differentiable. In order to proceed, we take advantage of the stability properties of the solutions to (29) in small time, which can be shown by a mere variation of the arguments used in the proof of Theorem 16. Indeed, we can use a Galerkin approximation and approximate $\mathfrak{F} = D\mathfrak{f}_0$ and $\mathfrak{G} = D\mathfrak{g}_0$ by coefficients $\mathfrak{F}^{(N)}$ and $\mathfrak{G}^{(N)}$ with a truncated Fourier expansion, namely

$$\begin{aligned}
&\mathfrak{B}^{(N)}(\ell) = \mathfrak{B}\Big(\sum_{k=0}^{N} \ell^{k,\pm} e^{k,\pm}(\cdot)\Big) \\
&\mathfrak{F}^{(N),n,\pm}(\ell) = \mathfrak{F}^{n,\pm}\Big(\sum_{k=0}^{N} \ell^{k,\pm} e^{k,\pm}(\cdot)\Big)\mathbf{1}_{\{n\leqslant N\}}, \quad \mathfrak{G}^{(N),n,\pm}(\ell) = \mathfrak{G}^{n,\pm}\Big(\sum_{k=0}^{N} \ell^{k,\pm} e^{k,\pm}(\cdot)\Big)\mathbf{1}_{\{n\leqslant N\}},
\end{aligned} \tag{50}$$

for $n \in \mathbb{N}$, where we refer to (28) for the definitions of $\mathfrak{F}$ and $\mathfrak{G}$. It is clear that $\mathfrak{F}^{(N)}$ and $\mathfrak{G}^{(N)}$ are bounded by the same constants as $\mathfrak{F}$ and $\mathfrak{G}$ and satisfy the same Lipschitz property. Therefore, we can solve, for $T \leqslant c$ with the same c as in Theorem 16, the forward backward system

$$\begin{aligned}
dX_t^{(N),n,\pm} &= \Big(\mathfrak{B}^{(N),n,\pm}\Big(\sum_{k=0}^{N} X_t^{(N),k,\pm} e^{k,\pm}(\cdot)\Big) - Y_t^{(N),n,\pm} - (2\pi n)^2 X_t^{(N),n,\pm}\Big)dt + dW_t^{n,\pm}, \\
dY_t^{(N),n,\pm} &= -\mathfrak{F}^{(N),n,\pm}\Big(\sum_{k=0}^{N} X_t^{(N),k,\pm} e^{k,\pm}(\cdot)\Big)dt + \sum_{k\in\mathbb{N}} Z_t^{(N),n,k,\pm} dW_t^{k,\pm}, \quad n \in \mathbb{N},
\end{aligned} \tag{51}$$

with $X_0^{(N),n,\pm} = X_0^{n,\pm}$ as initial condition and $Y_T^{(N),n,\pm} = \mathfrak{G}^{(N),n,\pm}(\sum_{k=0}^{N} X_T^{(N),k,\pm} e^{k,\pm}(\cdot))$ as terminal condition. Observe in particular that $\boldsymbol{Y}^{(N),n,\pm}$ and $\boldsymbol{Z}^{(N),n,\pm}$ are null for $n > N$. Denoting by $\mathcal{U}^{(N)}$ the corresponding decoupling field, it is then well-checked that $\mathcal{U}^{(N)}(t,\ell)$, for $\ell \in L^2(\mathbb{S}^1;\mathbb{R}^d)$, is a function of $(\ell^{n,\pm})_{0\leqslant n\leqslant N}$ only, meaning that

$$\mathcal{U}^{(N)}(t,\ell) = \mathcal{U}^{(N)}\Big(t, \sum_{k=0}^{N} \ell^{k,\pm} e^{k,\pm}(\cdot)\Big). \tag{52}$$

Also, $\mathcal{U}^{(N),n,\pm}$ is zero when $n > N$.

In words, the system (51) reduces to a finite dimensional system of $2N+1$ equations (i.e. up to the order $n = N$) on $\mathbb{R}^{(2N+1)d}$. By standard results for non-degenerate forward-backward equations, see for instance [25] (in order to fit the framework of the latter paper, notice that the linear term $-(2\pi n)^2 X_t^{(N),n,\pm}$ can be easily removed by considering $\exp((2\pi n)^2 t) X_t^{(N),n,\pm}$ instead of $X_t^{(N),n,\pm}$), we know that $\mathcal{U}^{(N)}(t,\cdot)$ is differentiable in $(\ell^{n,\pm})_{0\leqslant n\leqslant N}$ for $t < T$ and that, for $n \leqslant N$, (48) holds true with $Z^{n,k,\pm}$ replaced by $Z^{(N),n,k,\pm}$ and $\mathcal{U}^{n,\pm}$ replaced by $\mathcal{U}^{(N),n,\pm}$.

By stability in small time of the solutions to (29) (the proof of which works on the model of the proof of Theorem 16), we can check that, for $T \leqslant c$,

$$\begin{aligned}
&\mathbb{E}_0\Big[\sup_{0\leqslant t\leqslant T}\Big(\|X_t(\cdot) - X_t^{(N)}(\cdot)\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}^2 + \|Y_t(\cdot) - Y_t^{(N)}(\cdot)\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}^2\Big)\Big] \\
&\qquad + \mathbb{E}_0\Big[\sum_{n,k\in\mathbb{N}} \int_0^T |Z_t^{n,k,\pm} - Z_t^{(N),n,k,\pm}|^2 dt\Big] \\
&\leqslant \mathbb{E}_0\Big[\big\|\big(\mathfrak{G}^{(N)} - \mathfrak{G}\big)\big(X_T(\cdot)\big)\big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}^2 \\
&\qquad + \int_0^T \Big(\big\|\big(\mathfrak{B}^{(N)} - \mathfrak{B}\big)\big(X_t(\cdot)\big)\big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}^2 + \big\|\big(\mathfrak{F}^{(N)} - \mathfrak{F}\big)\big(X_t(\cdot)\big)\big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}^2\Big)dt\Big].
\end{aligned} \tag{53}$$

Observe now that, for all $\ell \in L^2(\mathbb{S}^1;\mathbb{R}^d)$,

$$\begin{aligned}\|\mathfrak{F}^{(N)}(\ell)-\mathfrak{F}(\ell)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)} &= \sum_{n=0}^{N}\Big|\mathfrak{F}^{n,\pm}\Big(\sum_{n=0}^{N}\ell^{n,\pm}e^{n,\pm}(\cdot)\Big)-\mathfrak{F}^{n,\pm}(\ell)\Big|^2+\sum_{n\geqslant N+1}\big|\mathfrak{F}^{n,\pm}(\ell)\big|^2\\ &\leqslant \Big\|\mathfrak{F}\Big(\sum_{n=0}^{N}\ell^{n,\pm}e^{n,\pm}(\cdot)\Big)-\mathfrak{F}(\ell)\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}+\sum_{n\geqslant N+1}\big|\mathfrak{F}^{n,\pm}(\ell)\big|^2 \qquad (54)\\ &\leqslant C\sum_{n\geqslant N+1}|\ell^{n,\pm}|^2+\sum_{n\geqslant N+1}\big|\mathfrak{F}^{n,\pm}(\ell)\big|^2,\end{aligned}$$

from which we get that the left-hand side tends to 0. Proceeding in the same way with $\mathfrak{B}^{(N)}-\mathfrak{B}$ and $\mathfrak{G}^{(N)}-\mathfrak{G}$ and combining with Lebesgue's dominated convergence theorem, we deduce that the right-hand side in (53) tends to 0 as N tends to ∞. We deduce that the left-hand side also tends to 0. And then,

$$\lim_{N\to+\infty}\mathcal{U}^{(N)}(t,\ell)=\mathcal{U}(t,\ell),\quad t\in[0,T],\ \ell\in L^2(\mathbb{S}^1;\mathbb{R}^d),$$

and, for a given initial condition ℓ in (51),

$$\lim_{N\to+\infty}\mathbb{E}_0\sum_{n,k\in\mathbb{N}}\int_0^T|Z_s^{\ell;(N),n,k,\pm}-Z_s^{\ell;n,k,\pm}|^2ds=0,\quad t\in[0,T],\ \ell\in L^2(\mathbb{S}^1;\mathbb{R}^d),$$

where we added the superscript ℓ in the notations to emphasize the fact that $X_0^{(N)}(\cdot)$ and $X_0(\cdot)$ were both equal to ℓ. This says that, to prove (48) and the statement of Theorem 12, we can focus first on the Galerkin approximation and then pass to the limit as N tends to $+\infty$. We shall come back to this point later on.

Smoothing estimates for the OU semi-group. The long time analysis relies on the smoothing properties of the OU semi-group $(\mathcal{P}_t)_{t\geqslant 0}$ we introduced earlier, see (35).

The following lemma is standard in the literature, see for instance [54, Section 5], see also [20]. It will play a key role in the proof of Theorem 10.

Lemma 18. *Let $\mathcal{V}$ be a bounded and measurable function from $L^2(\mathbb{S}^1;\mathbb{R}^d)$ into $\mathbb{R}$. Then, for any $t\in(0,T]$, $\mathcal{P}_t\mathcal{V}$ is Fréchet differentiable and, for all $\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)$,*

$$\big\|D\mathcal{P}_t\mathcal{V}(\ell)\big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant Ct^{-1/2}\mathbb{E}_0\big[|\mathcal{V}(U_t^\ell)|^2\big]^{1/2}\leqslant Ct^{-1/2}\|\mathcal{V}\|_\infty,$$

for a constant C independent of $t\in(0,T]$. If $\mathcal{V}$ is Lipschitz continuous on $L^2(\mathbb{S}^1;\mathbb{R}^d)$, then, for any $t\in(0,T]$ and any $\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)$,

$$\big\|D\mathcal{P}_t\mathcal{V}(\ell)\big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant \mathrm{Lip}(\mathcal{V}),$$

where $\mathrm{Lip}(\mathcal{V})$ is the Lipschitz constant of $\mathcal{V}$.

The second inequality in the statement is just a consequence of the fact that the function $L^2(\mathbb{S}^1;\mathbb{R}^d)\ni\ell\mapsto\mathbb{E}[\mathcal{V}(U_t^\ell)]$ is $\mathrm{Lip}(\mathcal{V})$-Lipschitz continuous.

4.3. **Analysis of the Galerkin approximation.** For a given fixed $T>0$, we consider the Galerkin approximation of the coefficients, as defined in (50), together with the corresponding Galerkin approximation of the forward-backward system, as defined in (51).

As we already explained, the system (51) is already known to be uniquely solvable, for any given initial condition for $\boldsymbol{X}^{(N)}$, whatever the time duration T is. Also, we know from [25] that the decoupling field $\mathcal{U}^{(N)}$, when regarded as a function from $[0,T]\times\mathbb{R}^{(2N+1)d}$ into $\mathbb{R}^{(2N+1)d}$ satisfies a

system of $(2N+1)$ PDEs in dimension $(2N+1)d$. By identifying the Fréchet derivative $D\mathcal{U}^{(N)}$ of $\mathcal{U}^{(N)}$ with the derivatives in $\mathbb{R}^{2N+1}$ through the formula:

$$D\mathcal{U}^{(N)}(t,\ell) = \sum_{n=0}^{N} \partial_{\ell^{n,\pm}} \mathcal{U}^{(N)}\Big(t, \sum_{k=0}^{N} \ell^{k,\pm} e^{k,\pm}(\cdot)\Big) e^{n,\pm}(\cdot),$$

the system of PDEs satisfied by the decoupling field of (51) coincides, in the mild form, with (49), but with $D\mathfrak{f}_0$ and $D\mathfrak{g}_0$ and replaced by $\mathfrak{F}^{(N)}$ and $\mathfrak{G}^{(N)}$. Namely, we have:

$$\begin{aligned}\mathcal{U}^{(N),n,\pm}(t,\cdot) &= \mathcal{P}_{T-t}\big(\mathfrak{G}^{(N),n,\pm}\big) \\ &\quad + \int_t^T \mathcal{P}_{s-t}\Big[\mathfrak{F}^{(N),n,\pm}(\cdot) + \big\langle D\mathcal{U}^{(N),n,\pm}(s,\cdot), \mathfrak{B}^{(N)}(\cdot) - \mathcal{U}^{(N)}(s,\cdot)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big] ds,\end{aligned} \tag{55}$$

the identity holding true in $L^2(\mathbb{S}^1;\mathbb{R}^d)$, for any $t \in [0,T]$.

Following (40), we claim first:

Lemma 19. *There exists a constant C such that, for all $N \in \mathbb{N}^*$,*

$$\sup_{t\in[0,T]} \sup_{\ell \in L^2(\mathbb{S}^1;\mathbb{R}^d)} \|\mathcal{U}^{(N)}(t,\ell)\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} \leqslant C.$$

The following lemma provides a uniform bound for the Fréchet derivative of the Galerkin approximation:

Lemma 20. *There exists a constant C independent of N such that, for all $t \in [0,T)$ and all $N \in \mathbb{N}^*$,*

$$\sup_{\ell \in L^2(\mathbb{S}^1;\mathbb{R}^d)} \|\!| D\mathcal{U}^{(N)}(t,\ell) |\!\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)\times L^2(\mathbb{S}^1;\mathbb{R}^d)} \leqslant C,$$

where

$$\|\!| D\mathcal{U}^{(N)}(t,\ell) |\!\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)\times L^2(\mathbb{S}^1;\mathbb{R}^d)} = \sup_{h\in L^2(\mathbb{S}^1;\mathbb{R}^d):\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant 1} \Big\| D\big[\langle \mathcal{U}^{(N)}(t,\cdot), h\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]_{|\cdot=\ell} \Big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)},$$

the notation $D[\varphi(\cdot)]_{|\cdot=\ell}$ indicating the fact that the differential is computed with respect to the argument $\cdot$ and then taken at point ℓ.

Proof. We start from (55). For $h \in L^2(\mathbb{S}^1)$,

$$\begin{aligned}\big\langle \mathcal{U}^{(N)}(t,\cdot), h\big\rangle &= \mathcal{P}_{T-t}\Big[\big\langle \mathfrak{G}^{(N)}(\cdot), h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big] + \int_t^T \mathcal{P}_{s-t}\Big[\big\langle \mathfrak{F}^{(N)}(\cdot), h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big] ds \\ &\quad + \int_t^T \mathcal{P}_{s-t}\Big[\mathfrak{b}_0^{(N)}(\cdot)\cdot D_0\langle \mathcal{U}^{(N)}(s,\cdot), h\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big] ds \\ &\quad - \int_t^T \mathcal{P}_{s-t}\Big[\big\langle D\langle \mathcal{U}^{(N)}(s,\cdot), h\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}, \mathcal{U}^{(N)}(s,\cdot)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big] ds \\ &= T_1 + T_2 + T_3.\end{aligned} \tag{56}$$

Apply now Lemma 18 when $\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} \leqslant 1$. Deduce that

$$\begin{aligned}
&\sup_{\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)} \sup_{\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant 1} \big\| \big[D\big\langle \mathcal{U}^{(N)}(t,\cdot), h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]_{|\cdot=\ell} \big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} \\
&\leqslant C\Bigg\{ \sup_{\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant 1} \mathrm{Lip}\Big(\big\langle \mathfrak{G}^{(N)}(\cdot), h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big) \\
&\quad + \int_t^T \frac{1}{\sqrt{s-t}} \sup_{\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)} \sup_{\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant 1} \Big| \mathfrak{b}_0^{(N)}(\ell)\cdot \big[D_0\big\langle \mathcal{U}^{(N)}(s,\cdot), h\big\rangle_{L^2(\mathbb{S}^1)}\big]_{|\cdot=\ell}\Big| ds \\
&\quad + \int_t^T \frac{1}{\sqrt{s-t}} \sup_{\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)} \sup_{\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant 1} \Big| \big\langle D\big[\big\langle \mathcal{U}^{(N)}(s,\cdot), h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]_{|\cdot=\ell}, \mathcal{U}^{(N)}(s,\ell)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big| ds \\
&\quad + \int_t^T \frac{1}{\sqrt{s-t}} \sup_{\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)} \big\| \mathfrak{F}^{(N)}(\ell)\big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} ds\Bigg\},
\end{aligned}$$

for a constant C whose value may change from line to line. Recall now that

$$\sup_{\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant 1} \mathrm{Lip}\Big(\big\langle \mathfrak{G}^{(N)}(\cdot), h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big) \leqslant C,$$

and that

$$\sup_{\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)} \Big\{ \big\| \mathfrak{F}^{(N)}(\ell)\big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}, \sup_{t\in[0,T]} \big\| \mathcal{U}^{(N)}(t,\ell)\big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big\} \leqslant C.$$

We deduce that

$$\begin{aligned}
&\sup_{\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)} \sup_{\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant 1} \big\| D\big[\big\langle \mathcal{U}^{(N)}(t,\cdot), h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]_{|\cdot=\ell}\big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} \\
&\leqslant C + \int_t^T \frac{C}{\sqrt{s-t}} \sup_{\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)} \sup_{\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant 1} \Big\| D\big[\big\langle \mathcal{U}^{(N)}(s,\cdot), h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]_{|\cdot=\ell}\Big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} ds.
\end{aligned}$$

By a variant of Gronwall's lemma, see Lemma 23 right below, we complete the proof. ∎

Using a similar argument, we claim:

Lemma 21. *For any compact subset $\mathcal{K}\subset L^2(\mathbb{S}^1;\mathbb{R}^d)$, there exist a constant C and real $\varepsilon>0$, such that, for all $t\in[0,T]$ and all $N,M\in\mathbb{N}^*$,*

$$\begin{aligned}
&\sup_{\ell\in\mathcal{K}} \| D\mathcal{U}^{(N)}(t,\ell) - D\mathcal{U}^{(M)}(t,\ell)\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)\times L^2(\mathbb{S}^1;\mathbb{R}^d)} \\
&\leqslant \frac{C}{\sqrt{T-t}}\Bigg[\Big(\sup_{\ell\in\mathcal{K}} \sum_{n>N\wedge M} |\ell^{n,\pm}|^2 + \sup_{h\in\mathcal{K}^\varepsilon} \sum_{n>N\wedge M} |\mathfrak{F}^{n,\pm}(h)|^2 + \sup_{h\in\mathcal{K}^\varepsilon} \sum_{n>N\wedge M} |\mathfrak{G}^{n,\pm}(h)|^2 \\
&\qquad\qquad + \sup_{s\in[0,T]} \sup_{h\in\mathcal{K}^\varepsilon} \big\| \big(\mathcal{U}^{(N)} - \mathcal{U}^{(M)}\big)(s,h)\big\|^2_{L^2(\mathbb{S}^1)} + \sup_{\ell\in\mathcal{K}} \sup_{r\in[0,T]} \mathbb{P}\big(U_r^\ell \notin \mathcal{K}^\varepsilon\big)\Big)^{1/2}\Bigg].
\end{aligned}$$

where

$$\begin{aligned}
&\| D\mathcal{U}^{(N)}(t,\ell) - D\mathcal{U}^{(M)}(t,\ell)\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)\times L^2(\mathbb{S}^1;\mathbb{R}^d)} \\
&= \sup_{h\in L^2(\mathbb{S}^1;\mathbb{R}^d):\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant 1} \big\| D\big[\big\langle \mathcal{U}^{(N)}(t,\cdot) - \mathcal{U}^{(M)}(t,\cdot), h\big\rangle\big]_{\cdot=\ell}\big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)},
\end{aligned}$$

and

$$\sup_{\ell\in\mathcal{K}} \sup_{r\in[0,T]} \mathbb{P}_0\big(U_r^\ell \notin \mathcal{K}^\varepsilon\big)\Big) \leqslant \varepsilon.$$

Proof. Throughout the proof, we use the fact that $\mathcal{K}$ is compact in $L^2(\mathbb{S}^1;\mathbb{R}^d)$ if and only if $\mathcal{K}$ is closed and, for any $\epsilon>0$, there exists $n\in\mathbb{N}$, such that for all $h\in\mathcal{K}$, $\sum_{k\geqslant n}|h^{k,\pm}|^2\leqslant\varepsilon$.

First step. Also, we recall that $\mathfrak{G}$ and $\mathfrak{F}$ are continuous from $L^2(\mathbb{S}^1;\mathbb{R}^d)$ into itself. Hence, $\mathfrak{G}(\mathcal{K})$ and $\mathfrak{F}(\mathcal{K})$ are compact subsets of $L^2(\mathbb{S}^1;\mathbb{R}^d)$. In particular, for all $\varepsilon>0$, there exists $n\in\mathbb{N}^*$, such that for all $h\in\mathcal{K}$,

$$\sum_{k\geqslant n}|\mathfrak{F}^{k,\pm}(h)|^2\leqslant\varepsilon,\quad \sum_{k\geqslant n}|\mathfrak{G}^{k,\pm}(h)|^2\leqslant\varepsilon.$$

Also, we observe that that, for any compact subset $\mathcal{K}$ and any $\varepsilon>0$, there exists another compact subset $\mathcal{K}_\varepsilon$ such that, for all $\ell\in\mathcal{K}$, for all $t\in[0,T]$,

$$\mathbb{P}_0\big[U^\ell_t\in\mathcal{K}^\varepsilon\big]\geqslant 1-\varepsilon. \tag{57}$$

The proof is quite straightforward. We give it for the sake of completeness. Indeed, we recall that:

$$U^\ell_t=\sum_{n\in\mathbb{N}}\Big(e^{-(2\pi n)^2t}\ell^{n,\pm}+\int_0^t e^{-(2\pi n)^2(t-s)}dW^{n,\pm}_s\Big)e^{n,\pm}(\cdot). \tag{58}$$

Obviously, we have, for any $n\in\mathbb{N}$,

$$\sum_{k\geqslant n}\big|e^{-(2\pi k)^2t}\ell^{k,\pm}\big|^2\leqslant\sum_{k\geqslant n}\big|\ell^{k,\pm}\big|^2, \tag{59}$$

which can be made as small as desired by choosing n large enough, uniformly in $\ell\in\mathcal{K}$. Also, for any $n\in\mathbb{N}$,

$$\sum_{k\geqslant n}\mathbb{E}_0\bigg[\bigg|\int_0^t e^{-(2\pi k)^2(t-s)}dW^{k,\pm}_s\bigg|^2\bigg]=\sum_{k\geqslant n}\int_0^t e^{-2(2\pi k)^2(t-s)}ds\leqslant\sum_{k\geqslant n}\frac{1}{2(2\pi k)^2}.$$

In particular, we can find a universal constant $c>0$ such that:

$$\sum_{k\geqslant n}\mathbb{E}_0\bigg[\bigg|\int_0^t e^{-(2\pi k)^2(t-s)}dW^{k,\pm}_s\bigg|^2\bigg]\leqslant\frac{c}{n}. \tag{60}$$

We deduce that

$$\mathbb{P}_0\bigg[\sum_{k\geqslant n^3}\bigg|\int_0^t e^{-(2\pi k)^2(t-s)}dW^{k,\pm}_s\bigg|^2\geqslant\frac{1}{n}\bigg]\leqslant\frac{c}{n^2},$$

and then, by Borel-Cantelli's Lemma, we obtain:

$$\begin{aligned}&\lim_{p\to\infty}\mathbb{P}_0\bigg(\bigcap_{n\geqslant p}\bigg\{\sum_{k\geqslant n^3}\bigg|\int_0^t e^{-(2\pi k)^2(t-s)}dW^{k,\pm}_s\bigg|^2\leqslant\frac{1}{n}\bigg\}\bigg)\\&=\mathbb{P}_0\bigg(\bigcup_{p\geqslant1}\bigcap_{n\geqslant p}\bigg\{\sum_{k\geqslant n^3}\bigg|\int_0^t e^{-(2\pi k)^2(t-s)}\big)dW^{k,\pm}_s\bigg|^2\leqslant\frac{1}{n}\bigg\}\bigg)=1.\end{aligned}$$

It remains to observe that, for any $p\geqslant1$, the set

$$\bigcap_{n\geqslant p}\Big\{h\in L^2(\mathbb{S}^1;\mathbb{R}^d):\sum_{k\geqslant n^3}|h^{k,\pm}|^2\leqslant\frac{1}{n}\Big\}$$

is compact in $L^2(\mathbb{S}^1;\mathbb{R}^d)$.

Second step. Following (54), we observe that there exists a constant $C\geqslant0$ such that, for all $N\in\mathbb{N}^*$, $t\in[0,T]$ and $\ell\in\mathcal{K}$,

$$\mathbb{E}_0\Big[\big\|\mathfrak{G}(U^\ell_{T-t})-\mathfrak{G}^{(N)}(U^\ell_{T-t})\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]\leqslant C\mathbb{E}_0\Big[\sum_{n>N}\big|(U^\ell_{T-t})^{n,\pm}\big|^2\Big]+C\mathbb{E}_0\Big[\sum_{n>N}|\mathfrak{G}^{n,\pm}(U^\ell_{T-t})|^2\Big]. \tag{61}$$

By (58), (59) and (60), we have

$$\mathbb{E}_0\Big[\sum_{n>N}\big|(U^\ell_{T-t})^{n,\pm}\big|^2\Big]\leqslant\sum_{n>N}|\ell^{n,\pm}|^2+\frac{c}{N}.$$

Also, using the same notation $\mathcal{K}^\varepsilon$ as in (57), we have, for any $\varepsilon>0$ and for all $N\in\mathbb{N}^*$, $t\in[0,T]$ and $\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)$:

$$\begin{aligned}\mathbb{E}_0\Big[\sum_{n>N}|\mathfrak{G}^{n,\pm}(U^\ell_{T-t})|^2\Big]&\leqslant\mathbb{E}_0\Big[\mathbf{1}_{\{U^\ell_{T-t}\in\mathcal{K}^\varepsilon\}}\sum_{n>N}|\mathfrak{G}^{n,\pm}(U^\ell_{T-t})|^2\Big]+C\mathbb{P}_0\big(U^\ell_{T-t}\notin\mathcal{K}^\varepsilon\big)\\&\leqslant\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N}|\mathfrak{G}^{n,\pm}(l)|^2+C\mathbb{P}_0\big(U^\ell_{T-t}\notin\mathcal{K}^\varepsilon\big),\end{aligned}\tag{62}$$

where we used the fact that $\mathfrak{G}$ is bounded and where we allowed the constant C to increase from line to line.

Therefore, (61) yields

$$\begin{aligned}&\mathbb{E}_0\Big[\big\|\mathfrak{G}(U^\ell_{T-t})-\mathfrak{G}^{(N)}(U^\ell_{T-t})\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]\\&\quad\leqslant C\Big(\sup_{l\in\mathcal{K}}\sum_{n>N}|l^{n,\pm}|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N}|\mathfrak{G}^{n,\pm}(l)|^2+\mathbb{P}_0\big(U^\ell_{T-t}\notin\mathcal{K}^\varepsilon\big)+\frac{1}{N}\Big).\end{aligned}\tag{63}$$

Similarly,

$$\begin{aligned}&\sup_{s\in[0,T-t]}\mathbb{E}_0\Big[\big\|\mathfrak{F}(U^\ell_s)-\mathfrak{F}^{(N)}(U^\ell_s)\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]\\&\quad\leqslant C\Big(\sup_{l\in\mathcal{K}}\sum_{n>N}|l^{n,\pm}|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N}|\mathfrak{F}^{n,\pm}(l)|^2+\sup_{s\in[0,T]}\mathbb{P}_0\big(U^\ell_s\notin\mathcal{K}^\varepsilon\big)+\frac{1}{N}\Big).\end{aligned}\tag{64}$$

Obviously, the same bound holds true when replacing $\mathfrak{F}$ by $\mathfrak{B}$. We now return to (56) and we write:

$$\begin{aligned}&\big\langle\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(t,\cdot),h\big\rangle\\&=\mathcal{P}_{T-t}\Big[\big\langle\big(\mathfrak{G}^{(N)}-\mathfrak{G}^{(M)}\big)(\cdot),h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]+\int_t^T\mathcal{P}_{s-t}\Big[\big\langle\big(\mathfrak{F}^{(N)}-\mathfrak{F}^{(M)}\big)(\cdot),h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]ds\\&\quad+\int_t^T\mathcal{P}_{s-t}\Big[\big(\mathfrak{b}_0^{(N)}-\mathfrak{b}_0^{(M)}\big)(\cdot)\cdot D_0\langle\mathcal{U}^{(N)}(s,\cdot),h\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]ds\\&\quad+\int_t^T\mathcal{P}_{s-t}\Big[\mathfrak{b}_0^{(M)}(\cdot)\cdot D_0\big\langle\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,\cdot),h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]ds\\&\quad-\int_t^T\mathcal{P}_{s-t}\Big[\big\langle D\langle(\mathcal{U}^{(N)}-\mathcal{U}^{(M)})(s,\cdot),h\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)},\mathcal{U}^{(N)}(s,\cdot)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]ds\\&\quad-\int_t^T\mathcal{P}_{s-t}\Big[\big\langle D\langle\mathcal{U}^{(M)}(s,\cdot),h\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)},\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,\cdot)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]ds.\end{aligned}\tag{65}$$

We then make use of Lemma 18. We can find a constant C such that, for all $N,M\geqslant1$, $\ell\in\mathcal{K}$, $h\in L^2(\mathbb{S}^1;\mathbb{R}^d)$ with $\|h\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant1$, and $t\in[0,T]$,

$$\begin{aligned}&\Big\|D\Big[\mathcal{P}_{T-t}\big[\big\langle\big(\mathfrak{G}^{(N)}-\mathfrak{G}^{(M)}\big)(\cdot),h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\Big]_{|\cdot=\ell}\Big\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\\&\quad\leqslant\frac{C}{\sqrt{T-t}}\mathbb{E}_0\Big[\big\|\big(\mathfrak{G}^{(N)}-\mathfrak{G}^{(M)}\big)(U^\ell_{T-t})\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2},\end{aligned}$$

where we used the fact that $\mathbb{E}_0[|\langle(\mathfrak{G}^{(N)}-\mathfrak{G}^{(M)})(U^\ell_{T-t}),h\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}|^2]^{1/2}$ is less than $\mathbb{E}_0[\|(\mathfrak{G}^{(N)}-\mathfrak{G}^{(M)})(U^\ell_{T-t})\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}]^{1/2}$. If, instead of ℓ, we choose the realization of the random variable $U^\ell_{t-t_0}$,

for $t_0 \in [0,t]$, we get by the flow property of the Ornstein-Uhlenbeck process:

$$\begin{aligned}&\mathbb{E}_0\Big[\Big\|D\Big[\mathcal{P}_{T-t}\big[\big\langle\big(\mathfrak{G}^{(N)}-\mathfrak{G}^{(M)}\big)(\cdot),h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\Big]_{\cdot=U_{t-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}\\&\leqslant\frac{C}{\sqrt{T-t}}\mathbb{E}_0\Big[\big\|\big(\mathfrak{G}^{(N)}-\mathfrak{G}^{(M)}\big)(U^\ell_{T-t_0})\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}.\end{aligned}\tag{66}$$

By (63) and (57), we obtain:

$$\begin{aligned}&\mathbb{E}_0\Big[\big\|\big(\mathfrak{G}^{(N)}-\mathfrak{G}^{(M)}\big)(U^\ell_{T-t_0})\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]\\&\quad\leqslant C\Big(\sup_{l\in\mathcal{K}}\sum_{n>N\wedge M}|l^{n,\pm}|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N\wedge M}|\mathfrak{G}^{n,\pm}(l)|^2+\sup_{l\in\mathcal{K}}\sup_{r\in[0,T]}\mathbb{P}_0\big(U^l_r\notin\mathcal{K}^\varepsilon\big)+\frac{1}{N\wedge M}\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}_0\Big[\Big\|D\Big[\mathcal{P}_{T-t}\big[\big\langle\big(\mathfrak{G}^{(N)}-\mathfrak{G}^{(M)}\big)(\cdot),h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\Big]_{|\cdot=U^\ell_{t-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}\\&\leqslant\frac{C}{\sqrt{T-t}}\Big(\sup_{l\in\mathcal{K}}\sum_{n>N\wedge M}|l^{n,\pm}|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N\wedge M}|\mathfrak{G}^{n,\pm}(l)|^2+\sup_{l\in\mathcal{K}}\sup_{r\in[0,T]}\mathbb{P}_0\big(U^l_r\notin\mathcal{K}^\varepsilon\big)+\frac{1}{N\wedge M}\Big)^{1/2}.\end{aligned}$$

By the same argument,

$$\begin{aligned}&\mathbb{E}_0\Big[\Big\|D\Big[\mathcal{P}_{s-t}\big[\big\langle\big(\mathfrak{F}^{(N)}-\mathfrak{F}^{(M)}\big)(\cdot),h\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\Big]_{|\cdot=U^\ell_{t-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}\\&\leqslant\frac{C}{\sqrt{s-t}}\Big(\sup_{l\in\mathcal{K}}\sum_{n>N\wedge M}|l^{n,\pm}|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N\wedge M}|\mathfrak{F}^{n,\pm}(l)|^2+\sup_{l\in\mathcal{K}}\sup_{r\in[0,T]}\mathbb{P}_0\big(U^l_r\notin\mathcal{K}^\varepsilon\big)+\frac{1}{N\wedge M}\Big)^{1/2}.\end{aligned}$$

Similarly, using Lemma 20, it holds that

$$\begin{aligned}&\mathbb{E}_0\Big[\Big\|D\Big[\mathcal{P}_{s-t}\big[\big(\mathfrak{b}^{(N)}_0-\mathfrak{b}^{(M)}_0\big)(\cdot)\cdot D_0\langle\mathcal{U}^{(N)}(s,\cdot),h\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\Big]_{|\cdot=U^\ell_{t-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}\\&\leqslant\frac{C}{\sqrt{s-t}}\Big(\sup_{l\in\mathcal{K}}\sum_{n>N\wedge M}|l^{n,\pm}|^2+\sup_{l\in\mathcal{K}}\sup_{r\in[0,T]}\mathbb{P}_0\big(U^l_r\notin\mathcal{K}^\varepsilon\big)+\frac{1}{N\wedge M}\Big)^{1/2}.\end{aligned}$$

We now turn to the term on the third line in (65). Following (66), we have

$$\begin{aligned}&\mathbb{E}_0\Big[\Big\|D\Big[\mathcal{P}_{s-t}\big[\mathfrak{b}^{(M)}_0(\cdot)\cdot D_0\langle\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,\cdot),h\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\Big]_{|\cdot=U^\ell_{t-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}\\&\leqslant\frac{C}{\sqrt{s-t}}\mathbb{E}_0\Big[\Big\|D\big[\big\langle\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,\cdot),h\big\rangle\big]_{|\cdot=U^\ell_{s-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}.\end{aligned}$$

Obviously, the same holds for the term on the fourth line in (65).

$$\begin{aligned}&\mathbb{E}_0\Big[\Big\|D\Big[\mathcal{P}_{s-t}\big[\big\langle D\langle(\mathcal{U}^{(N)}-\mathcal{U}^{(M)})(s,\cdot),h\rangle_{L^2(\mathbb{S}^1)},\mathcal{U}^{(N)}(s,\cdot)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\Big]_{|\cdot=U^\ell_{t-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}\\&\leqslant\frac{C}{\sqrt{s-t}}\mathbb{E}_0\Big[\Big\|D\big[\big\langle\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,\cdot),h\big\rangle\big]_{|\cdot=U^\ell_{s-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}.\end{aligned}$$

Finally,

$$\mathbb{E}_0\Big[\Big\|D\Big[\mathcal{P}_{s-t}\big[\big\langle D\langle \mathcal{U}^{(M)}(s,\cdot),h\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)},\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,\cdot)\big\rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\Big]_{|\cdot=U^\ell_{t-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}$$
$$\leqslant \frac{C}{\sqrt{s-t}}\mathbb{E}_0\Big[\big\|\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,U^\ell_{s-t_0})\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}.$$

Collecting all the bounds and plugging them into (65), we get

$$\begin{aligned}
&\mathbb{E}_0\Big[\Big\|D\big[\big\langle\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(t,\cdot),h\big\rangle\big]_{|\cdot=U^\ell_{t-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}\\
&\leqslant \frac{C}{\sqrt{T-t}}\Big(\sup_{l\in\mathcal{K}}\sum_{n>N\wedge M}|l^{n,\pm}|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N\wedge M}|\mathfrak{F}^{n,\pm}(l)|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N\wedge M}|\mathfrak{G}^{n,\pm}(l)|^2\\
&\qquad\qquad+\sup_{l\in\mathcal{K}}\sup_{r\in[0,T]}\mathbb{P}_0\big(U^l_r\notin\mathcal{K}^\varepsilon\big)+\frac{1}{N\wedge M}\Big)^{1/2}\\
&\quad+\int_t^T\frac{C}{\sqrt{s-t}}\mathbb{E}_0\Big[\Big\|D\big[\big\langle\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,\cdot),h\big\rangle\big]_{|\cdot=U^\ell_{s-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}ds\\
&\quad+\int_t^T\frac{C}{\sqrt{s-t}}\mathbb{E}_0\Big[\big\|\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,U^\ell_{s-t_0})\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}ds.
\end{aligned}$$

By Lemma 24 below, we get

$$\begin{aligned}
&\mathbb{E}_0\Big[\Big\|D\big[\big\langle\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(t,\cdot),h\big\rangle\big]_{|\cdot=U^\ell_{t-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}\\
&\leqslant \frac{C}{\sqrt{T-t}}\Big[\Big(\sup_{l\in\mathcal{K}}\sum_{n>N\wedge M}|l^{n,\pm}|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N\wedge M}|\mathfrak{F}^{n,\pm}(l)|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N\wedge M}|\mathfrak{G}^{n,\pm}(l)|^2\\
&\qquad\qquad+\sup_{l\in\mathcal{K}}\sup_{r\in[0,T]}\mathbb{P}_0\big(U^l_r\notin\mathcal{K}^\varepsilon\big)+\frac{1}{N\wedge M}\Big)^{1/2}\\
&\qquad+\sup_{s\in[t_0,T]}\mathbb{E}_0\Big[\big\|\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,U^\ell_{s-t_0})\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}\Big].
\end{aligned}$$

And then, using the boundedness of $\mathcal{U}^{(N)}$ (and $\mathcal{U}^{(M)}$), we obtain

$$\begin{aligned}
&\mathbb{E}_0\Big[\Big\|D\big[\big\langle\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(t,\cdot),\ell\big\rangle\big]_{|\cdot=U^h_{t-t_0}}\Big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\Big]^{1/2}\\
&\leqslant \frac{C}{\sqrt{T-t}}\Big[\Big(\sup_{l\in\mathcal{K}}\sum_{n>N\wedge M}|l^{n,\pm}|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N\wedge M}|\mathfrak{F}^{n,\pm}(l)|^2+\sup_{l\in\mathcal{K}^\varepsilon}\sum_{n>N\wedge M}|\mathfrak{G}^{n,\pm}(l)|^2\\
&\qquad+\sup_{s\in[0,T]}\sup_{l\in\mathcal{K}^\varepsilon}\big\|\big(\mathcal{U}^{(N)}-\mathcal{U}^{(M)}\big)(s,l)\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}+\sup_{l\in\mathcal{K}}\sup_{r\in[0,T]}\mathbb{P}_0\big(U^l_r\notin\mathcal{K}^\varepsilon\big)+\frac{1}{N\wedge M}\Big)^{1/2}\Big],
\end{aligned}$$

which completes the proof by taking $t=t_0$. ∎

Corollary 22. *For any compact subset $\mathcal{K}\subset L^2(\mathbb{S}^1;\mathbb{R}^d)$, there exists a function $w:\mathbb{R}_+\to\mathbb{R}_+$ satisfying $\lim_{\delta\searrow 0}w(\delta)=0$ such that, for any $N\in\mathbb{N}^*$, any $s,t\in[0,T]$ and any $\ell\in L^2(\mathbb{S}^1;\mathbb{R}^d)$,*

$$\|\mathcal{U}^{(N)}(t,\ell)-\mathcal{U}^{(N)}(s,\ell)\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\leqslant Cw\big(|s-t|\big).$$

Proof. Without any loss of generality, we can assume $t < s$. We then consider the solution $(X_r^{(N),t,\ell}(\cdot), Y_r^{(N),t,\ell}(\cdot), Z_r^{(N),t,\ell}(\cdot))_{t\leqslant r\leqslant T}$ of the forward-backward system (51) on the interval $[t,T]$ with $X_t^{(N),t,\ell}(\cdot) = \ell$ as initial condition.

We then have:

$$\mathcal{U}^{(N)}(t,\ell) = \mathbb{E}_0\Big[\mathcal{U}^{(N)}\big(s, X_s^{(N),t,\ell}(\cdot)\big) + \int_t^s \mathfrak{F}^{(N)}\big(X_r^{(N),t,\ell}(\cdot)\big)dr\Big],$$

so that

$$\mathcal{U}^{(N)}(t,\ell) - \mathcal{U}^{(N)}(s,\ell) = \mathbb{E}_0\Big[\Big(\mathcal{U}^{(N)}\big(s, X_s^{(N),t,\ell}(\cdot)\big) - \mathcal{U}^{(N)}(s,\ell)\Big) + \int_t^s \mathfrak{F}^{(N)}\big(X_r^{(N),t,\ell}(\cdot)\big)dr\Big].$$

Recalling that the functions $(\mathfrak{F}^{(N)})_{N\in\mathbb{N}*}$ are bounded, uniformly in $N \in \mathbb{N}^*$, and invoking Lemma 20, we deduce that there exists a constant C such that, for any $N \in \mathbb{N}^*$, any $t \in [0,T]$ and any $\ell \in L^2(\mathbb{S}^1;\mathbb{R}^d)$,

$$\|\mathcal{U}^{(N)}(t,\ell) - \mathcal{U}^{(N)}(s,\ell)\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} \leqslant C\Big(|s-t| + \mathbb{E}_0\big[\|X_s^{(N),t,\ell} - \ell\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}\big]\Big). \tag{67}$$

We now recall the Fourier expansion of the forward equation in (51):

$$dX_r^{(N),n,\pm} = \Big(\mathbf{1}_{(n,\pm)=(0,+)}\mathfrak{B}^{(N),0,+}\big(X_r^{(N)}(\cdot)\big) - \mathcal{U}^{(N),n,\pm}\big(r, X_r^{(N)}(\cdot)\big) - (2\pi n)^2 X_r^{(N),n,\pm}\Big)ds + dW_r^{n,\pm},$$

for $r \in [t,T]$, where, for the sake of simplicity, we omitted the indices (t,ℓ) in the notation and we just indicated the mode indices. We get:

$$\begin{aligned} X_s^{(N),n,\pm} = e^{-(2\pi n)^2(s-t)}\ell^{n,\pm} &+ \int_t^s e^{(2\pi n)^2(r-s)}\mathbf{1}_{(n,\pm)=(0,+)}\mathfrak{B}^{(N),0,+}\big(X_r^{(N)}(\cdot)\big)dr \\ &- \int_t^s e^{(2\pi n)^2(r-s)}\mathcal{U}^{(N),n,\pm}\big(r, X_r^{(N)}(\cdot)\big)dr + \int_t^s e^{(2\pi n)^2(r-s)}dW_r^{n,\pm}. \end{aligned} \tag{68}$$

Since the functions $\mathfrak{B}^{(N)}$ and $\mathcal{U}^{(N)}$ can be bounded independently of N, we deduce that:

$$\mathbb{E}_0\big[\|X_s^{(N)} - \ell\|_{L^2}^2\big] \leqslant C\Big[|s-t|^2 + \sum_{n\in\mathbb{N}}|\ell^{n,\pm}|^2\Big(e^{-(2\pi n)^2(s-t)} - 1\Big)^2 + \sum_{n\in\mathbb{N}}\int_t^s e^{2(2\pi n)^2(r-s)}dr\Big]. \tag{69}$$

Now,

$$\begin{aligned} \sum_{n\in\mathbb{N}}|\ell^{n,\pm}|^2\Big(e^{-(2\pi n)^2(s-t)} - 1\Big)^2 &\leqslant C\sum_{n\in\mathbb{N}}|\ell^{n,\pm}|^2\big[1\wedge\big(n^2(s-t)\big)\big]^2 \\ &\leqslant C\Big[|s-t|\sum_{n\in\mathbb{N}}|\ell^{n,\pm}|^2 + \sum_{n\geqslant(s-t)^{-1/4}}|\ell^{n,\pm}|^2\Big]. \end{aligned} \tag{70}$$

Also, allowing the constant C to change from line to line, we get

$$\sum_{n\in\mathbb{N}}\int_t^s e^{-2(2\pi n)^2(s-r)}dr \leqslant C(s-t) + C\int_0^\infty\int_t^s e^{-2(2\pi x)^2(s-r)}drdx \leqslant C(s-t)^{1/2}. \tag{71}$$

Collecting (67), (69), (70) and (71), we finally obtain:

$$\|\mathcal{U}^{(N)}(t,\ell) - \mathcal{U}^{(N)}(s,\ell)\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)} \leqslant C\big(1 + \sup_{l\in\mathcal{K}}\|l\|_{L^2(\mathbb{S}^1;\mathbb{R}^d)}^2\big)\Big(|s-t|^{1/4} + \sup_{l\in\mathcal{K}}\sum_{n\geqslant(s-t)^{-1/4}}|l^{n,\pm}|^2\Big),$$

which completes the proof. ∎

Here are now the two variants of Gronwall's lemma we appealed to right above.

Lemma 23. *Consider two bounded measurable functions $g_1, g_2 : [0,T] \to \mathbb{R}_+$ such that*

$$g_1(t) \leqslant C_1 + C_2 \int_t^T \frac{g_2(s)}{\sqrt{s-t}} ds, \tag{72}$$

for some constants $C_1, C_2 \geqslant 0$. Then there exist $\lambda, \mu > 0$, depending on C_2 and T only, such that

$$\begin{aligned} &\int_0^T g_1(t) \exp(\lambda t) dt \leqslant \mu C_1 + \frac{1}{2} \int_0^T g_2(t) \exp(\lambda t) dt, \\ &\sup_{0 \leqslant t \leqslant T} \big[g_1(t) \big] \leqslant \mu C_1 + 2C_2^2 \int_0^T g_2(t) dt + \frac{1}{2} \sup_{0 \leqslant t \leqslant T} \big[g_2(t) \big]. \end{aligned} \tag{73}$$

In particular, if $g_1 = g_2$, then g_1 is bounded by $\mu' C_1$, for a constant μ' depending on C_2 and T only.

Lemma 24. *Consider two bounded measurable functions $g_1, g_2 : [0,T] \to \mathbb{R}_+$ such that*

$$g_1(t) \leqslant \frac{C_1}{\sqrt{T-t}} + C_2 \int_t^T \frac{g_2(s)}{\sqrt{s-t}} ds, \tag{74}$$

for some constants $C_1, C_2 \geqslant 0$. Then there exist $\lambda, \mu > 0$, depending on C_2 and T only, such that

$$\begin{aligned} &\int_0^T g_1(t) \exp(\lambda t) dt \leqslant \mu C_1 + \frac{1}{2} \int_0^T g_2(t) \exp(\lambda t) dt, \\ &\sup_{0 \leqslant t \leqslant T} \big[\sqrt{T-t}\, g_1(t) \big] \leqslant \mu C_1 + \mu \int_0^T g_2(t) dt + \frac{1}{2} \sup_{0 \leqslant t \leqslant T} \big[\sqrt{T-t}\, g_2(t) \big]. \end{aligned} \tag{75}$$

In particular, there exists a constant μ' depending on C_2 and T only such that, whenever $g_1 = g_2$,

$$\sup_{0 \leqslant t \leqslant T} \big[\sqrt{T-t}\, g_1(t) \big] \leqslant \mu' C_1.$$

We just prove the second statement. The proof of the first one may be found in [22, Lemma 2.13].

Proof. The first part of Lemma 24 may be proved as in [22, Lemma 2.13]. So, we focus on the second inequality. For any $\varepsilon > 0$, (74) yields

$$\begin{aligned} &(T-t)^{1/2} g_1(t) \\ &\leqslant C_1 + C_2 \int_t^{(t+\varepsilon)\wedge T} \frac{(T-t)^{1/2}}{(s-t)^{1/2}(T-s)^{1/2}} (T-s)^{1/2} g_2(s) ds + C_2 \varepsilon^{-1/2} \int_{(t+\varepsilon)\wedge T}^T g_2(s) ds \\ &\leqslant C_1 + C_2 \varepsilon^{-1/2} \int_0^T g_2(s) ds + C_2 \sup_{0 \leqslant s \leqslant T} \big[(T-s)^{1/2} g_2(s) \big] \int_t^{(t+\varepsilon)\wedge T} \frac{(T-t)^{1/2}}{(s-t)^{1/2}(T-s)^{1/2}} ds. \end{aligned}$$

Now,

$$\begin{aligned} \int_t^{(t+\varepsilon)\wedge T} \frac{(T-t)^{1/2}}{(s-t)^{1/2}(T-s)^{1/2}} ds &= \int_0^{\varepsilon \wedge (T-t)} \frac{(T-t)^{1/2}}{s^{1/2}(T-t-s)^{1/2}} ds \\ &= (T-t)^{1/2} \int_0^{1 \wedge [\varepsilon/(T-t)]} \frac{1}{s^{1/2}(1-s)^{1/2}} ds \end{aligned}$$

If $\varepsilon^{1/2} \leqslant T-t$, then

$$\int_t^{(t+\varepsilon)\wedge T} \frac{(T-t)^{1/2}}{(s-t)^{1/2}(T-s)^{1/2}} ds \leqslant T^{1/2} \int_0^{1 \wedge \varepsilon^{1/2}} \frac{1}{s^{1/2}(1-s)^{1/2}} ds.$$

Otherwise, $T-t \leqslant \varepsilon^{1/2}$ and

$$\int_t^{(t+\varepsilon)\wedge T} \frac{(T-t)^{1/2}}{(s-t)^{1/2}(T-s)^{1/2}} ds \leqslant \varepsilon^{1/4} \int_0^1 \frac{1}{s^{1/2}(1-s)^{1/2}} ds.$$

So, we can find a function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ converging to 0 in 0 such that

$$(T-t)^{1/2} g_1(t) \leqslant C_1 + C_2 \varepsilon^{-1/2} \int_0^T g_2(s) ds + C_2 \delta(\varepsilon) \sup_{0 \leqslant s \leqslant T} \big[(T-s)^{1/2} g_2(s) \big].$$

The proof of the second claim is easily completed. Whenever $g_1 = g_2$,

$$\int_0^T g_1(t) \exp(\lambda t) dt \leqslant 2 C_1 \mu,$$

and then, choosing ε small enough in the second claim, we get by the first part of the statement:

$$\begin{aligned} \sup_{0 \leqslant t \leqslant T} \big[\sqrt{T-t}\, g_1(t) \big] &\leqslant 2\mu C_1 + 2\mu \int_0^T g_1(t) dt \\ &\leqslant 2\mu C_1 + 2\mu \int_0^T g_1(t) \exp(\lambda t) dt \leqslant 2\mu C_1 + 4 C_1 \mu^2, \end{aligned}$$

which completes the proof. ∎

4.4. **End of the proof of Theorem 10.** We now turn to the proof of Theorem 10. To this end, we recall the constant C from Lemma 20. Without any loss of generality, we assume that the Lipschitz constants of the coefficients $\mathfrak{b}_0$, $\mathfrak{F}$ and $\mathfrak{G}$ are less than the same constant C. We then call c the constant in the statement of Theorem 16 when the Lipschitz constant of the coefficients is less than C.

We let $N = \lceil T/c \rceil$ and $\tau_n = T - (N-n)c$ for $n \in \{1, \dots, N\}$ and $\tau_0 = 0$. We know from Theorem 16 that, for any square-integrable $\mathcal{F}_{0,\tau_{N-1}}$-measurable initial condition $X^{(N-1)}(\cdot)$ with values in $L^2(\mathbb{S}^1; \mathbb{R}^d)$, the forward-backward system (29) is uniquely solvable. Following Lemma 11, this permits to define the decoupling field $\mathcal{U}$ on $[\tau_{N-1}, T] \times L^2(\mathbb{S}^1; \mathbb{R}^d)$. By (53), we know that, for any $(t, \ell) \in [\tau_{N-1}, T] \times L^2(\mathbb{S}^1; \mathbb{R}^d)$, the sequence $(\mathcal{U}^{(N)}(t, \ell))_{N \in \mathbb{N}^*}$, defined as the sequence of decoupling fields of the systems (51), converges to $\mathcal{U}(t, \ell)$. In particular, we deduce from Lemma 20 that $\mathcal{U}$ is C-Lipschitz in the space variable on $[\tau_{N-1}, T] \times L^2(\mathbb{S}^1; \mathbb{R}^d)$.

Since $\mathcal{U}(\tau_{N-1}, \cdot)$ is C-Lipschitz, we can iterate the argument and apply Theorem 16 on the interval $[\tau_{N-2}, \tau_{N-1}]$. This permits to extend the definition of the decoupling field $\mathcal{U}$ to the set $[\tau_{N-2}, \tau_{N-1}] \times L^2(\mathbb{S}^1; \mathbb{R}^d)$. By invoking (53) once again but on $[\tau_{N-2}, \tau_{N-1}]$, we deduce that, for any $(t, \ell) \in [\tau_{N-2}, \tau_{N-1}] \times L^2(\mathbb{S}^1; \mathbb{R}^d)$, the sequence $(\mathcal{U}^{(N)}(t, \ell))_{N \in \mathbb{N}^*}$ converges to $\mathcal{U}(t, \ell)$, which permits to iterate the argument and, in the end, to construct a candidate $\mathcal{U}$ for being the decoupling field on the entire $[0, T] \times L^2(\mathbb{S}^1; \mathbb{R}^d)$. Once $\mathcal{U}$ has been constructed, the proof is completed as in the finite dimensional case, see for instance [24] and [16, Chapter 4].

4.5. **Proof of Theorem 12.** *First Step.* As a by-product of the analysis achieved in the previous subsection to complete the proof of Theorem 10, we claim that, for any $(t, \ell) \in [0, T] \times L^2(\mathbb{S}^1; \mathbb{R}^d)$,

$$\lim_{M, N \to \infty} \| (\mathcal{U}^{(N)} - \mathcal{U}^{(M)})(t, \ell) \|_{L^2(\mathbb{S}^1; \mathbb{R}^d)} = 0.$$

Recall from Lemmas 19 and 20 and Corollary 22 that the mappings $(\mathcal{U}^{(N)})_{N \in \mathbb{N}^*}$ are uniformly bounded and uniformly continuous on any compact subset of $L^2(\mathbb{S}^1; \mathbb{R}^d)$. Hence, we have:

$$\lim_{M, N \to \infty} \sup_{t \in [0,T]} \sup_{\ell \in \mathcal{K}} \| (\mathcal{U}^{(N)} - \mathcal{U}^{(M)})(t, \ell) \|_{L^2(\mathbb{S}^1; \mathbb{R}^d)} = 0.$$

We now invoke Lemma 21, from which we deduce that for any compact subset of $[0,T)\times L^2(\mathbb{S}^1;\mathbb{R}^d)$, il holds that:

$$\lim_{M,N\to\infty}\sup_{(t,\ell)\in\mathcal{K}}\bigl|\!\bigl|\!\bigl|(D\mathcal{U}^{(N)}-D\mathcal{U}^{(M)})(t,\ell)\bigr|\!\bigr|\!\bigr|_{L^2(\mathbb{S}^1;\mathbb{R}^d)\times L^2(\mathbb{S}^1;\mathbb{R}^d)}=0,$$

which shows that the sequence $(D\mathcal{U}^{(N)})_{N\in\mathbb{N}*}$ converges, uniformly on compact subsets of $[0,T)\times L^2(\mathbb{S}^1;\mathbb{R}^d)$. Since each $D\mathcal{U}^{(N)}$ is continuous on $[0,T)\times L^2(\mathbb{S}^1;\mathbb{R}^d)$, we deduce that the limit, denoted by $D\mathcal{U}$ is continuous and is the Fréchet derivative of $\mathcal{U}$ in the space variable. Of course, $D\mathcal{U}$ satisfies Lemma 20. Passing to the limit in (55), we deduce that $\mathcal{U}$ is a mild solution of the system of PDEs (34), as formulated in the statement of Theorem 12.

5. Construction of an approximated Nash equilibrium

The purpose of this section is to prove Theorem 14. To do so, we use the same setting as in Subsection 3.3, a short reminder of which is recalled below.

The game consists of NA_N particles that are uniformly distributed along the points (which we call roots) $(e^{i2\pi k/N})_{k=0,\cdots,N-1}$ of the unit circle, with $i^2=-1$ and with exactly A_N particles per root, where $A_N\in\mathbb{N}^*$. States of the particles at time t are denoted by $(X_t^{k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}$, where k stands for the index of the root occupied by the particle and j for its label among the collection of particles located at the same site. As already explained in Subsection 3.3, we put $X_t^{k+\ell N,j}=X_t^{k,j}$, for $k\in\{0,\ldots,N-1\}$ and $\ell\in\mathbb{Z}$.

Each particle (k,j) has dynamics of the following form:

$$dX_t^{k,j}=\Bigl\{b\bigl(\bar\mu_t^N\bigr)+\alpha_t^{k,j}+N\sum_{l=1}^{A_N}\bigl(X_t^{k+1,l+1}+X_t^{k+1,l-1}-2X_t^{k+1,l}\bigr)\Bigr\}dt+\sqrt{N}dB_t^k, \tag{76}$$

for $t\in[0,T]$, with the initial condition $X_0^{k,j}=\bar X_0^k$, where $(\bar X_0^k)_{k=0,\cdots,N-1}$ are given by:

$$\bar X_0^k=N\int_{k/N}^{(k+1)/N}X_0(x)dx,\quad k=0,\cdots,N-1, \tag{77}$$

whilst the noises $(\boldsymbol{B}^k=(B_t^k)_{0\leqslant t\leqslant T})_{k=0,\cdots,N-1}$ are independent d-dimensional Brownian motions on the interval $[0,T]$ with the following definition:

$$B_t^k=\sqrt{N}\int_{k/N}^{(k+1)/N}W_t(dx).$$

We recall that $\bar\mu_t^N$ denotes the empirical distribution:

$$\bar\mu_t^N=\frac{1}{NA_N}\sum_{k=0}^{N-1}\sum_{j=1}^{A_N}\delta_{X_t^{k,j}}.$$

The processes $(\boldsymbol{\alpha}^{k,j}=(\alpha_t^{k,j})_{0\leqslant t\leqslant T})_{k=0,\cdots,N-1;j=1,\cdots,A_N}$ are constructed on $(\Omega_0,\mathcal{A}_0,\mathbb{P}_0)$ and are $\mathbb{R}^d$-valued progressively-measurable controls with respect to the filtration generated by the cylindrical white noise $(W_t(\cdot))_{0\leqslant t\leqslant T}$ satisfying the condition:

$$\mathbb{E}_0\int_0^T|\alpha_t^{k,j}|^2dt<\infty.$$

The cost functional to player (k,j) is then given by:

$$J^{k,j}\bigl((\boldsymbol{\alpha}^{k',j'})_{k'=0,\cdots,N-1;j'=1,\cdots,A_N}\bigr)=\mathbb{E}_0\biggl[g\bigl(X_T^{k,j},\bar\mu_T^N\bigr)+\int_0^T\Bigl(f\bigl(X_t^{k,j},\bar\mu_t^N\bigr)+\frac{1}{2}|\alpha_t^{k,j}|^2\Bigr)dt\biggr].$$

Following the statement of Theorem 14, we introduce the collection of controls:

$$\alpha_t^{\star k,j} = \bar{Y}_t^k, \quad \bar{Y}_t^k = N \int_{(k-1)/N}^{k/N} Y_t(x)dx, \quad t \in [0,T], \tag{78}$$

for all $k \in \{0, \cdots, N-1\}$ and $j \in \{1, \cdots, A_N\}$. Then, for some $k_0 \in \{0, \cdots, N-1\}$ and $j_0 \in \{1, \cdots, A_N\}$ and for some $\mathbb{R}^d$-valued process $\boldsymbol{\gamma} = (\gamma_t)_{0 \leqslant t \leqslant T}$ that is progressively-measurable with respect to the filtration generated by the cylindrical white noise $\boldsymbol{W}(\cdot) = (W_t(\cdot))_{0 \leqslant t \leqslant T}$ (that is, the filtration generated by the processes $(\langle W_t(\cdot), h \rangle_{L^2(\mathbb{S}^1;\mathbb{R}^d)})_{0 \leqslant t \leqslant T}$ for $h \in L^2(\mathbb{S}^1;\mathbb{R}^d)$) and that satisfies the condition

$$\mathbb{E}_0 \int_0^T |\gamma_t|^2 dt < \infty,$$

we let $\boldsymbol{\beta}^{\star k,j} = \boldsymbol{\alpha}^{\star k,j}$, for $k \in \{0, \cdots, N-1\}$ and $j \in \{1, \cdots, A_N\}$, with $(k,j) \neq (k_0, j_0)$. When $k = k_0$ and $j = j_0$, we let $\boldsymbol{\beta}^{\star k_0, j_0} = \boldsymbol{\gamma}$.

The goal of this section is to prove that there exists a sequence of positive reals $(\varepsilon_N)_{N \in \mathbb{N}*}$, converging to 0, independent of $\boldsymbol{\gamma}$, k_0 and j_0, such that

$$J^{k_0,j_0}\big((\boldsymbol{\beta}^{\star k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big) \geqslant J^{k_0,j_0}\big((\boldsymbol{\alpha}^{\star k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big) - \varepsilon_N.$$

Throughout the analysis, we assume that, on top of Assumption **(A)**, f and g are Lipschitz continuous in μ, uniformly in x. In particular, f and g are Lipschitz in (x, μ).

5.1. **Distance between discrete and continuous systems.** Most of the proof relies on a stability property under discretization for SPDEs of the form:

$$\partial_t X_t(x) = \alpha_t(x) + \Delta X_t(x) + \dot{W}_t(x), \quad (t,x) \in [0,T] \times \mathbb{S}^1, \tag{79}$$

with some initial condition $X_0(\cdot) \in L^2(\mathbb{S}^1;\mathbb{R}^d)$. Above, the process $\boldsymbol{\alpha}(\cdot) = (\alpha_t(\cdot))_{0 \leqslant t \leqslant T}$ is an $L^2(\mathbb{S}^1;\mathbb{R}^d)$-valued progressively-measurable process with respect to the filtration generated by $(W_t(\cdot))_{0 \leqslant t \leqslant T}$. We assume it to satisfy

$$\mathbb{E}_0 \int_0^T \|\alpha_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)} dt < \infty.$$

The solution to (79) will be denoted $(X_t^{(\boldsymbol{\alpha})}(\cdot))_{0 \leqslant t \leqslant T}$. For another $L^2(\mathbb{S}^1;\mathbb{R}^d)$-valued progressively-measurable process $\boldsymbol{\beta}(\cdot) = (\beta_t(\cdot))_{0 \leqslant t \leqslant T}$ satisfying

$$\mathbb{E}_0 \int_0^T \|\beta_t(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)} dt < \infty,$$

we let

$$\bar{\beta}_t^k = N \int_{k/N}^{(k+1)/N} \beta_t(x)dx, \quad t \in [0,T], \quad k \in \{0, \cdots, N-1\}, \tag{80}$$

and we consider the discretized version

$$d\bar{X}_t^k = \bar{\beta}_t^k dt + N^2\big(\bar{X}_t^{k+1} + \bar{X}_t^{k-1} - 2\bar{X}_t^k\big)dt + \sqrt{N} dB_t^k, \tag{81}$$

for $t \in [0,T]$ and $k \in \{0, \cdots, N-1\}$, with the same convention as before that $\bar{X}_t^{-1} = \bar{X}_t^{N-1}$ and $\bar{X}_t^N = \bar{X}_t^0$. Above the initial condition is given by the same approximation as in (77). The solution to (81) will be denoted $((\bar{X}_t^{(\boldsymbol{\beta}),k})_{k=0,\cdots,N-1})_{0 \leqslant t \leqslant T}$. With this solution, we associate the periodic function

$$\bar{X}_t^{(\boldsymbol{\beta})}(\cdot) = \sum_{k=0}^{N-1} \bar{X}_t^{(\boldsymbol{\beta}),k} \mathbf{1}_{[k/N,(k+1)/N)+\mathbb{Z}}(\cdot), \quad t \in [0,T].$$

Notice that (and this is the key point of the proof) the equation (81) is just indexed by the label k of the root (and not by the label j we used before to denote a particle).

Mild solution of the discrete equation. Equation (81) forms a system of stochastic differential equations, the solution of which may be put under a discrete mild form, the mild formulation being based upon the following operator:

$$\Delta^{(N)}\Big(\sum_{k=0}^{N-1}\bar\lambda^k\mathbf{1}_{[k/N,(k+1)/N)+\mathbb{Z}}(\cdot)\Big)=\sum_{k=0}^{N-1}N^2\big(\bar\lambda^{k+1}+\bar\lambda^{k-1}-2\bar\lambda^k\big)\mathbf{1}_{[k/N,(k+1)/N)+\mathbb{Z}}(\cdot),$$

for any sequence $(\bar\lambda^k)_{k=0,\cdots,N-1}$. Obviously, $\Delta^{(N)}$ is acting on piecewise constant functions from the torus $\mathbb{S}^1$ into $\mathbb{R}$ (or, more generally, into $\mathbb{R}^d$) with $(k/N+\mathbb{Z})_{k=0,\cdots,N-1}$ as mesh. We often identify these functions with piecewise constant functions from $[0,1)$ into $\mathbb{R}$ (or $\mathbb{R}^d$) with $(k/N+\mathbb{Z})_{k=0,\cdots,N-1}$ as mesh, in which case the above identity becomes (with a slight abuse of notation):

$$\Delta^{(N)}\Big(\sum_{k=0}^{N-1}\bar\lambda^k\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\Big)=\sum_{k=0}^{N-1}N^2\big(\bar\lambda^{k+1}+\bar\lambda^{k-1}-2\bar\lambda^k\big)\mathbf{1}_{[k/N,(k+1)/N)}(\cdot),$$

Throughout the analysis, we shall use the following convention. For a collection of weights $(\bar\lambda^k)_{k=0,\cdots,N-1}$ (with values in $\mathbb{R}$ or in $\mathbb{R}^d$), we call

$$\bar\lambda(\cdot)=\sum_{k=0}^{N-1}\bar\lambda^k\mathbf{1}_{[k/N,(k+1)/N)+\mathbb{Z}}(\cdot)\tag{82}$$

the corresponding piecewise constant step functions on the torus. Observe that, for the sake of convenience, we omitted to specify the dependence of the function $\bar\lambda(\cdot)$ upon the discretization parameter N. Also, according to the previous convention, we shall identify the function $\bar\lambda(\cdot)$ with the function $\sum_{k=0}^{N-1}\bar\lambda^k\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)$ from $[0,1)$ into $\mathbb{R}$. With this convention of notation, the solution to (81) may be written under the form:

$$\bar X_t^{(\boldsymbol\beta)}(\cdot)=e^{t\Delta^{(N)}}\bar X_0(\cdot)+\int_0^t e^{(t-s)\Delta^{(N)}}\bar\beta_s(\cdot)ds+\int_0^t e^{(t-s)\Delta^{(N)}}\Big(\sum_{n\in\mathbb{N}}\bar e^{n,\pm}(\cdot)dW_s^{n,\pm}\Big),\tag{83}$$

with the same convention as before for the notation $\bar e^{n,\pm}(\cdot)$, namely:

$$\bar e^{n,\pm}(\cdot)=\sum_{k=0}^{N-1}\bar e^{n,\pm,k}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot),\quad\text{with}\quad\Big(\bar e^{n,\pm,k}=N\int_{k/N}^{(k+1)/N}e^{n,\pm}(x)dx\Big)_{k=0,\cdots,N-1},$$

which is to say that $\bar e^{n,\pm}(\cdot)$ is the piecewise constant step function associated with the family of weights $(\bar e^{n,k,\pm})_{k=0,\cdots,N-1}$.

The above writing of the stochastic integral is justified by the fact that

$$\sqrt{N}\Big(\sum_{k=0}^{N-1}B_t^k\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\Big)=\sum_{n\in\mathbb{N}}W_t^{n,\pm}\Big[\sum_{k=0}^{N-1}\bar e^{n,\pm,k}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\Big],$$

which follows from a straightforward application of the decomposition of W in Fourier modes, namely

$$\sqrt{N}B_t^k=N\int_{k/N}^{(k+1)/N}W_t(dx)=\sum_{n\in\mathbb{N}}W_t^{n,\pm}\bar e^{n,\pm,k}.$$

Distance between $X^{(\boldsymbol\alpha)}$ and $\bar X^{(\boldsymbol\beta)}$. For the sake of completeness, we recall the mild formulation of the SPDE (79):

$$X_t^{(\boldsymbol\alpha)}(\cdot)=e^{t\Delta}X_0(\cdot)+\int_0^t e^{(t-s)\Delta}\alpha_s(\cdot)ds+\int_0^t e^{(t-s)\Delta}\Big(\sum_{n\in\mathbb{N}}e^{n,\pm}(\cdot)dW_s^{n,\pm}\Big),\quad t\in[0,T].\tag{84}$$

Here is the main statement of this subsection.

Theorem 25. *There exist a constant C together with a sequence $(\varepsilon_N)_{N\in\mathbb{N}^*}$, converging to 0 as N tends to ∞, such that for any initial condition $X_0(\cdot) \in L^2(\mathbb{S}^1;\mathbb{R}^d)$, any two square-integrable progressively-measurable process $(\alpha_t(\cdot))_{0\leqslant t\leqslant T}$ and $(\beta_t(\cdot))_{0\leqslant t\leqslant T}$ with values in $L^2(\mathbb{S}^1;\mathbb{R}^d)$ and any integer $N \in \mathbb{N}^*$, it holds*

$$\begin{aligned}\sup_{x\in\mathbb{S}} \mathbb{E}_0\big[|\bar{X}_t^{(\boldsymbol{\beta})}(x) - X_t^{(\boldsymbol{\alpha})}(x)|^2\big]\Big\} \leqslant C\bigg(1 + \frac{1}{t^{3/4}}\|X_0(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)} + \mathbb{E}_0\int_0^t \|\alpha_s(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}ds\bigg)\varepsilon_N \\ + C\mathbb{E}_0\int_0^t \big\|\big(\alpha_s - \beta_s\big)(\cdot)\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}ds,\end{aligned} \tag{85}$$

for all $t \in (0,T]$.

Proof. The proof is split in several steps. The goal is to compare (83) and (84). Basically, each step of the proof corresponds to the comparison of a pair of terms in the right-hand sides of (83) and (84).

Preliminary Step. As a preliminary step, we have the following two standard results, the proofs of which are postponed to the end of the subsection.

The first identity is

$$\bar{e}^n(\cdot) = e^{i\pi\frac{n}{N}}\frac{\sin(\pi n/N)}{\pi n/N}\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot), \quad \text{with } \bar{e}^n(\cdot) = \frac{\bar{e}^{n,+} + i\bar{e}^{n,-}}{\sqrt{2}}(\cdot), \tag{86}$$

and $i^2 = -1$. The second one is

$$\Delta^{(N)}\bigg[\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\bigg] = -2N^2\big[1-\cos\big(\frac{2\pi n}{N}\big)\big]\bigg[\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\bigg], \tag{87}$$

which shows that the function $\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)$ is an eigenvector of $\Delta^{(N)}$. In particular, we have

$$e^{(t-s)\Delta^{(N)}}\bigg[\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\bigg] = e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bigg[\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\bigg],$$

for any $s,t \in [0,T]$, with $0 \leqslant s \leqslant t$. Combining with the first identity (86), we get:

$$\begin{aligned}e^{(t-s)\Delta^{(N)}}\bar{e}^n(\cdot) &= e^{i\pi\frac{n}{N}}\frac{\sin(\pi n/N)}{\pi n/N}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bigg[\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\bigg] \\ &= e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^n(\cdot),\end{aligned}$$

which shows that $\bar{e}^n(\cdot)$ is also an eigenvector of $\Delta^{(N)}$. Taking the real and imaginary parts, the same holds for $\bar{e}^{n,+}(\cdot)$ and $\bar{e}^{n,-}(\cdot)$.

Second Step. We now compare the martingale terms in (83) and (84). We start with (83). Thanks to the preliminary step, it may be rewritten under the form:

$$\int_0^t e^{(t-s)\Delta^{(N)}}\Big(\sum_{n\in\mathbb{N}}\bar{e}^{n,\pm}(\cdot)dW_s^{n,\pm}\Big) = \sum_{n\in\mathbb{N}}\int_0^t e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(\cdot)dW_s^{n,\pm}.$$

We then observe that there exists a universal constant C such that

$$\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\bigg[\bigg|\sum_{n\geqslant N^{1/4}}\int_0^t e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(x)dW_s^{n,\pm}\bigg|^2\bigg] \leqslant \frac{C}{N^{1/4}}.$$

Indeed, the left hand side is equal to

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Big[\Big|\sum_{n\geqslant N^{1/4}}\int_0^t e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(x)dW_s^{n,\pm}\Big|^2\Big]\\
&=\sup_{x\in\mathbb{S}^1}\sum_{n\geqslant N^{1/4}}\int_0^t e^{-4N^2[1-\cos(2\pi n/N)](t-s)}|\bar{e}^{n,\pm}(x)|^2ds\\
&=\sum_{n\geqslant N^{1/4}}\frac{\sin^2(\pi n/N)}{(\pi n/N)^2}\int_0^t e^{-4N^2[1-\cos(2\pi n/N)](t-s)}ds,
\end{aligned}\tag{88}$$

so that

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Big[\Big|\sum_{n\geqslant N^{1/4}}\int_0^t e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(x)dW_s^{n,\pm}\Big|^2\Big]\\
&\leqslant\sum_{|n|\geqslant N^{1/4}}\frac{\sin^2(\pi n/N)}{(\pi n/N)^2}\frac{1}{4N^2[1-\cos(2\pi n/N)]}.
\end{aligned}\tag{89}$$

We then observe that the function $\varrho:\mathbb{S}^1\ni x\mapsto\varrho(x)=\sin^2(x)/(1-\cos(2x))$ is equal to $1/2$ as $\cos(2x)=2\cos^2(x)-1=1-2\sin^2(x)$. So, the above ratio $\sin^2(\pi n/N)/[1-\cos(2\pi n/N)]$ is bounded by a universal constant c. In the sequel, this constant c may vary from line to line as long as it remains universal. Then,

$$\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Big[\Big|\sum_{n\geqslant N^{1/4}}\int_0^t e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(x)dW_s^{n,\pm}\Big|^2\Big]\leqslant c\sum_{n\geqslant N^{1/4}}\frac{1}{n^2}\leqslant\frac{c}{N^{1/4}}.\tag{90}$$

Actually, the same bound holds for the solution of the SPDE, namely:

$$\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Big[\Big|\int_0^t e^{(t-s)\Delta}\Big(\sum_{n\geqslant N^{1/4}}e^{n,\pm}(\cdot)dW_s^{n,\pm}\Big)(x)\Big|^2\Big]\leqslant\frac{c}{N^{1/4}},$$

which may be proved in the same way by recalling that $e^{(t-s)\Delta}e^{n,\pm}=-(2\pi n)^2e^{n,\pm}$, for all $n\in\mathbb{N}$.

We now handle the difference

$$\sum_{0\leqslant n<N^{1/4}}\Big(\int_0^t e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(x)dW_s^{n,\pm}-\int_0^t e^{-(2\pi n)^2(t-s)}e^{n,\pm}(x)dW_s^{n,\pm}\Big).$$

Taking the L^2 norm of the modulus, we obtain:

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1}\mathbb{E}\Big[\Big|\sum_{0\leqslant n<N^{1/4}}\Big(\int_0^t e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(x)dW_s^{n,\pm}-\int_0^t e^{-(2\pi n)^2(t-s)}e^{n,\pm}(x)dW_s^{n,\pm}\Big)\Big|^2\Big]\\
&=\sup_{x\in\mathbb{S}^1}\sum_{0\leqslant n<N^{1/4}}\int_0^t\Big|e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(x)-e^{-(2\pi n)^2(t-s)}e^{n,\pm}(x)\Big|^2ds\\
&\leqslant 4\sum_{0\leqslant n<N^{1/4}}\int_0^t\Big|e^{-2N^2[1-\cos(2\pi n/N)](t-s)}-e^{-(2\pi n)^2(t-s)}\Big|^2ds\\
&\quad+2\sup_{x\in\mathbb{S}^1}\sum_{0\leqslant n<N^{1/4}}\int_0^t e^{-2(2\pi n)^2(t-s)}\big|\bar{e}^{n,\pm}(x)-e^{n,\pm}(x)\big|^2ds\\
&=(i)+(ii).
\end{aligned}\tag{91}$$

As for the first term (i), we proceed as follows. We use the following two facts. First, we observe that, for $0 \leqslant n \leqslant N^{1/4}$,

$$N^2\big[1-\cos\big(\frac{2\pi n}{N}\big)\big] = N^2\big[\frac{1}{2}\big(\frac{2\pi n}{N}\big)^2 + O\big(\frac{n^4}{N^4}\big)\big] = \frac{(2\pi n)^2}{2} + O\big(\frac{1}{N}\big).$$

Therefore, for any $0 \leqslant s \leqslant t \leqslant T$,

$$\exp\Big(-2N^2\big[1-\cos\big(\frac{2\pi n}{N}\big)\big](t-s)\Big) = \exp\big(-(2\pi n)^2(t-s)\big)\big(1+O\big(\frac{1}{N}\big)\big),$$

where the Landau symbol is uniform in $s,t\in[0,T]$, with $s\leqslant t$, and in $0\leqslant n\leqslant N^{1/4}$. Therefore,

$$|(i)| \leqslant \frac{c}{N}\sum_{0\leqslant n\leqslant N^{1/4}}\int_0^t \exp\big(-(2\pi n)^2(t-s)\big)ds \leqslant \frac{c}{N}\Big(1+\sum_{n\in\mathbb{N}^*}\frac{1}{n^2}\Big),$$

which is less than c/N.

In order to handle (ii), we notice that, for $0\leqslant n\leqslant N^{1/4}$,

$$\sup_{x\in\mathbb{S}^1}|e^{n,\pm}(x)-\bar{e}^{n,\pm}(x)| \leqslant \frac{cn}{N} \leqslant \frac{c}{N^{1/4}}.$$

We easily deduce that $|(ii)|$ is less than $c/N^{1/4}$.

So, the conclusion of this second step is that there exists a sequence $(\varepsilon_N)_{N\in\mathbb{N}*}$, independent of the data, converging to 0 as N tends to ∞, such that

$$\begin{aligned}
&\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Big[\Big|\int_0^t e^{(t-s)\Delta^{(N)}}\Big(\sum_{n\in\mathbb{N}}\bar{e}^{n,\pm}(\cdot)dW_s^{n,\pm}\Big)(x)-\int_0^t e^{(t-s)\Delta}\Big(\sum_{n\in\mathbb{N}}e^{n,\pm}(\cdot)dW_s^{n,\pm}\Big)(x)\Big|^2\Big]\\
&=\sup_{0\leqslant t\leqslant T}\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Big[\Big|\sum_{n\in\mathbb{N}}\Big(\int_0^t e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(x)dW_s^{n,\pm}-\int_0^t e^{-(2\pi n)^2(t-s)}e^{n,\pm}(x)dW_s^{n,\pm}\Big)\Big|^2\Big]\\
&\leqslant \varepsilon_N,
\end{aligned}$$

which proves that the two martingale terms in (83) and (84) get closer as N tends to ∞, uniformly in time (and in the data).

Third Step. We now provide a similar analysis but for the control terms in (83) and (84). We start with the case when $\boldsymbol{\alpha}(\cdot)=\boldsymbol{\beta}(\cdot)$. To do so, we call $(\alpha_t^{n,\pm})_{n\in\mathbb{N}}$ the sequence of Fourier coefficients of each $\alpha_t(\cdot)$, seen as a (random) element of $L^2(\mathbb{S}^1;\mathbb{R}^d)$. Similar to (80), we also define the sequence $((\bar{\alpha}_t^k)_{0\leqslant t\leqslant T})_{k=0,\cdots,N-1}$:

$$\bar{\alpha}_t^k = N\int_{k/N}^{(k+1)/N}\alpha_t(x)dx, \quad t\in[0,T], \quad k\in\{0,\cdots,N-1\},$$

and we define $(\bar{\alpha}_t(\cdot))_{0\leqslant t\leqslant T}$ accordingly, see (82), namely

$$\bar{\alpha}_t(\cdot) = \sum_{k=0}^{N-1}\bar{\alpha}_t^k\mathbf{1}_{[k/N,(k+1)/N)}(\cdot).$$

With this notation, we have the following identity:

$$\bar{\alpha}_t(\cdot) = \sum_{n\in\mathbb{N}}\alpha_t^{n,\pm}\Big[\sum_{k=0}^{N-1}\Big(N\int_{k/N}^{(k+1)/N}e^{n,\pm}(x)dx\Big)\mathbf{1}_{[k/n,(k+1)/N)}(\cdot)\Big] = \sum_{n\in\mathbb{N}}\alpha_t^{n,\pm}\bar{e}^{n,\pm}(\cdot).$$

So, using the preliminary step, we deduce that, for any $s,t\in[0,T]$ with $s\leqslant t$,

$$e^{(t-s)\Delta^{(N)}}\bar{\alpha}_s(\cdot) = e^{(t-s)\Delta^{(N)}}\Big[\sum_{n\in\mathbb{N}}\alpha_s^{n,\pm}\bar{e}^{n,\pm}(\cdot)\Big] = \sum_{n\in\mathbb{N}}\alpha_s^{n,\pm}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(\cdot),$$

and then

$$\int_0^t e^{(t-s)\Delta^{(N)}}\bar{\alpha}_s(\cdot)ds=\sum_{n\in\mathbb{N}}\Big(\int_0^t\alpha_s^{n,\pm}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Big)\bar{e}^{n,\pm}(\cdot).$$

Proceeding as in the second step, we first focus on

$$\sum_{n\geqslant N^{1/4}}\Big(\int_0^t\alpha_s^{n,\pm}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Big)\bar{e}^{n,\pm}(\cdot).$$

By Cauchy Schwartz inequality, we have

$$\begin{aligned}&\sup_{x\in\mathbb{S}^1}\Big|\sum_{n\geqslant N^{1/4}}\Big(\int_0^t\alpha_s^{n,\pm}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Big)\bar{e}^{n,\pm}(x)\Big|^2\\&\leqslant\Big(\sum_{n\geqslant N^{1/4}}\int_0^t|\alpha_s^{n,\pm}|^2ds\Big)\Big(\sup_{x\in\mathbb{S}^1}\sum_{n\geqslant N^{1/4}}\int_0^t e^{-4N^2[1-\cos(2\pi n/N)](t-s)}|\bar{e}^{n,\pm}(x)|^2ds\Big).\end{aligned}$$

Take now expectation and deduce that:

$$\begin{aligned}&\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Big[\Big|\sum_{n\geqslant N^{1/4}}\Big(\int_0^t\alpha_s^{n,\pm}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Big)\bar{e}^{n,\pm}(x)\Big|^2\Big]\\&\leqslant\mathbb{E}_0\Big[\sum_{n\geqslant N^{1/4}}\int_0^t|\alpha_s^{n,\pm}|^2ds\Big]\Big(\sup_{x\in\mathbb{S}^1}\sum_{n\geqslant N^{1/4}}\int_0^t e^{-4N^2[1-\cos(2\pi n/N)](t-s)}|\bar{e}^{n,\pm}(x)|^2ds\Big).\end{aligned}$$

By Parseval's identity, the first term is bounded by $\mathbb{E}_0\int_0^t\|\alpha_s(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}ds$. The second one may be handled as in (88) and (90). We deduce that:

$$\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Big[\Big|\sum_{n\geqslant N^{1/4}}\Big(\int_0^t\alpha_s^{n,\pm}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Big)\bar{e}^{n,\pm}(x)\Big|^2\Big]\leqslant\frac{c}{N^{1/4}}\mathbb{E}_0\int_0^t\|\alpha_s(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}ds.$$

Similarly, we have

$$\int_0^t e^{(t-s)\Delta}\alpha_s(\cdot)ds=\sum_{n\in\mathbb{N}}\int_0^t\alpha_s^{n,\pm}e^{-(2\pi n)^2(t-s)}e^{n,\pm}(\cdot)ds,$$

and then,

$$\begin{aligned}&\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Big[\Big|\sum_{n\geqslant N^{1/4}}\int_0^t\alpha_s^{n,\pm}e^{-(2\pi n)^2(t-s)}e^{n,\pm}(x)ds\Big|^2\Big]\\&\leqslant\mathbb{E}_0\Big[\sum_{|n|\geqslant N^{1/4}}\int_0^t|\alpha_s^{n,\pm}|^2ds\Big]\Big(\sum_{n\geqslant N^{1/4}}\int_0^t e^{-2(2\pi n)^2(t-s)}ds\Big),\end{aligned}$$

and again, it is less than $(c/N^{1/4})\mathbb{E}_0\int_0^t\|\alpha_s(\cdot)\|^2_{L^2(\mathbb{S}^1)}ds$. We now handle the difference

$$\begin{aligned}&\sum_{0\leqslant n<N^{1/4}}\Big(\int_0^t\alpha_s^{n,\pm}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Big)\bar{e}^{n,\pm}(\cdot)-\sum_{0\leqslant n<N^{1/4}}\Big(\int_0^t\alpha_s^{n,\pm}e^{-(2\pi n)^2(t-s)}ds\Big)e^{n,\pm}(\cdot)\\&=\sum_{|n|<N^{1/4}}\int_0^t\alpha_s^{n,\pm}\Big(e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(\cdot)-e^{-(2\pi n)^2(t-s)}e^{n,\pm}(\cdot)\Big)ds.\end{aligned}$$

By Cauchy-Schwarz inequality,

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Bigg[\Bigg|\sum_{0\leqslant n<N^{1/4}}\int_0^t \alpha_s^{n,\pm}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(x)ds \\
&\qquad\qquad - \sum_{0\leqslant n<N^{1/4}}\int_0^t \alpha_s^{n,\pm}e^{-(2\pi n)^2(t-s)}e^{n,\pm}(x)ds\Bigg|^2\Bigg] \\
&\leqslant \mathbb{E}_0\Bigg[\sum_{0\leqslant n<N^{1/4}}\int_0^t |\alpha_s^{n,\pm}|^2ds\Bigg] \\
&\quad\times \sup_{x\in\mathbb{S}^1}\sum_{0\leqslant n<N^{1/4}}\int_0^t |e^{-2N^2[1-\cos(2\pi n/N)](t-s)}\bar{e}^{n,\pm}(x) - e^{-(2\pi n)^2(t-s)}e^{n,\pm}(x)|^2ds.
\end{aligned}$$

We then follow (91). We deduce that there exist a constant C and a sequence $(\varepsilon_N)_{N\in\mathbb{N}*}$, independent of the data, the sequence $(\varepsilon_N)_{N\in\mathbb{N}*}$ converging to 0 as N tends to ∞, such that, for all $t\in[0,T]$,

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Bigg[\Bigg|\sum_{n\in\mathbb{N}}\Bigg(\int_0^t \alpha_s^{n,\pm}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Bigg)\bar{e}^{n,\pm}(x) - \sum_{n\in\mathbb{N}}\Bigg(\int_0^t \alpha_s^{n,\pm}e^{-(2\pi n)^2(t-s)}ds\Bigg)e^{n,\pm}(x)\Bigg|^2\Bigg] \\
&\quad\leqslant C\Bigg(\mathbb{E}_0\int_0^t \|\alpha_s(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}ds\Bigg)\varepsilon_N, \qquad (92)
\end{aligned}$$

which proves in particular that, whenever $\boldsymbol{\alpha}(\cdot)=\boldsymbol{\beta}(\cdot)$, the control terms in (83) and (84) get closer as N tends to ∞, uniformly in time.

Now, in order to handle the general case when $\boldsymbol{\alpha}(\cdot)\neq\boldsymbol{\beta}(\cdot)$, it suffices to handle the term:

$$\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Bigg[\Bigg|\sum_{n\in\mathbb{N}}\Bigg(\int_0^t \big(\alpha_s^{n,\pm}-\beta_s^{n,\pm}\big)e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Bigg)\bar{e}^{n,\pm}(x)\Bigg|^2\Bigg].$$

By Cauchy-Schwarz' inequality and then by Parseval's identity, it is less than

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Bigg[\Bigg|\sum_{n\in\mathbb{N}}\Bigg(\int_0^t \big(\alpha_s^{n,\pm}-\beta_s^{n,\pm}\big)e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Bigg)\bar{e}^{n,\pm}(x)\Bigg|^2\Bigg] \\
&\leqslant \sup_{x\in\mathbb{S}^1}\Bigg\{\mathbb{E}_0\Bigg[\sum_{n\in\mathbb{N}}\int_0^t |\alpha_s^{n,\pm}-\beta_s^{n,\pm}|^2ds\Bigg]\Bigg[\sum_{n\in\mathbb{N}}\int_0^t e^{-4N^2[1-\cos(2\pi n/N)](t-s)}|\bar{e}^{n,\pm}(x)|^2ds\Bigg]\Bigg\} \\
&\leqslant \mathbb{E}_0\Bigg[\int_0^t \big\|\big(\alpha_s-\beta_s\big)(\cdot)\big\|^2_{L^2(\mathbb{S}^1)}ds\Bigg]\Bigg[\sup_{x\in\mathbb{S}^1}\sum_{n\in\mathbb{N}}\int_0^t e^{-4N^2[1-\cos(2\pi n/N)](t-s)}|\bar{e}^{n,\pm}(x)|^2ds\Bigg].
\end{aligned}$$

Following (88) and (90), we can easily bound the second factor. We deduce that

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Bigg[\Bigg|\sum_{n\in\mathbb{N}}\Bigg(\int_0^t \big(\alpha_s^{n,\pm}-\beta_s^{n,\pm}\big)e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Bigg)\bar{e}^{n,\pm}(x)\Bigg|^2\Bigg] \\
&\qquad\leqslant C\mathbb{E}_0\Bigg[\int_0^t \big\|\big(\alpha_s-\beta_s\big)(\cdot)\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}ds\Bigg].
\end{aligned}$$

And then, combining with (92),

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\Bigg[\sum_{n\in\mathbb{N}}\Bigg(\int_0^t \beta_s^{n,\pm}e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\Bigg)\bar{e}^{n,\pm}(x) - \sum_{n\in\mathbb{N}}\Bigg(\int_0^t \alpha_s^{n,\pm}e^{-(2\pi n)^2(t-s)}ds\Bigg)e^{n,\pm}(x)\Bigg|^2\Bigg] \\
&\quad\leqslant C\varepsilon_N\mathbb{E}_0\int_0^t \|\alpha_s(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}ds + C\mathbb{E}_0\Bigg[\int_0^t \big\|\big(\alpha_s-\beta_s\big)(\cdot)\big\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}ds\Bigg].
\end{aligned}$$

Fourth Step. We now handle the initial condition on the same principle. As before, we denote by $(X_0^{n,\pm})_{n\in\mathbb{N}}$ the Fourier coefficients of $X_0(\cdot)$. Then, we let

$$\bar{X}_0(\cdot) = \sum_{k=0}^{N} \left(N \int_{k/N}^{(k+1)/N} X_0(x)dx \right) \mathbf{1}_{[k/N,(k+1)/N)}(\cdot) = \sum_{n\in\mathbb{N}} X_0^{n,\pm} \bar{e}^{n,\pm}(\cdot).$$

Therefore,

$$e^{t\Delta^{(N)}} \bar{X}_0(\cdot) = \sum_{n\in\mathbb{N}} X_0^{n,\pm} e^{-2N^2[1-\cos(2\pi n/N)]t} \bar{e}^{n,\pm}(\cdot).$$

Proceeding as above,

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1} \left| \sum_{n\geqslant N^{1/4}} X_0^{n,\pm} e^{-2N^2[1-\cos(2\pi n/N)]t} \bar{e}^{n,\pm}(x) \right|^2 \\
&\qquad \leqslant 2 \sum_{n\in\mathbb{N}} |X_0^{n,\pm}|^2 \times \sum_{n\geqslant N^{1/4}} \frac{\sin^2(\pi n/N)}{(\pi n/N)^2} e^{-2N^2[1-\cos(2\pi n/N)]t},
\end{aligned}$$

which yields to a somewhat different bound from what we obtained in the two previous steps. In order to recover the same kind of bounds, we use the following trick:

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1} \left| \sum_{n\geqslant N^{1/4}} X_0^{n,\pm} e^{-2N^2[1-\cos(2\pi n/N)]t} \bar{e}^{n,\pm}(x) \right|^2 \\
&\leqslant \frac{1}{t^{3/4}} \sum_{n\in\mathbb{N}} |X_0^{n,\pm}|^2 \cdot \sum_{n\geqslant N^{1/4}} \frac{\sin^2(\pi n/N)}{(\pi n/N)^2} t^{3/4} e^{-2N^2[1-\cos(2\pi n/N)]t} \\
&\leqslant \frac{c}{t^{3/4}} \sum_{n\in\mathbb{N}} |X_0^{n,\pm}|^2 \cdot \sum_{n\geqslant N^{1/4}} \frac{\sin^2(\pi n/N)}{(\pi n/N)^2} \frac{1}{(N^2[1-\cos(2\pi n/N)])^{3/4}},
\end{aligned} \tag{93}$$

for a new value of the universal constant c. Recalling that the function $\mathbb{R} \ni x \mapsto \sin(x)/x$ is bounded by 1, we deduce that

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1} \left| \sum_{n\geqslant N^{1/4}} X_0^{n,\pm} e^{-2N^2[1-\cos(2\pi n/N)]t} \bar{e}^{n,\pm}(x) \right|^2 \\
&\leqslant \frac{c}{t^{3/4}} \sum_{n\in\mathbb{N}} |X_0^{n,\pm}|^2 \cdot \sum_{n\geqslant N^{1/4}} \left(\frac{\sin^2(\pi n/N)}{(\pi n/N)^2} \right)^{3/4} \frac{1}{(N^2[1-\cos(2\pi n/N)])^{3/4}},
\end{aligned}$$

and then following the argument used to pass from (89) to (90), we deduce that:

$$\sup_{x\in\mathbb{S}^1} \left| \sum_{n\geqslant N^{1/4}} X_0^{n} e^{-2N^2[1-\cos(2\pi n/N)]t} \bar{e}^{n,\pm}(x) \right|^2 \leqslant \frac{c}{t^{3/4}} \sum_{n\in\mathbb{N}} |X_0^{n,\pm}|^2 \cdot \sum_{n\geqslant N^{1/4}} \frac{1}{n^{3/2}} \leqslant \frac{c}{t^{3/4} N^{1/8}}.$$

It is well-checked that a similar bound holds true for

$$\sup_{x\in\mathbb{S}^1} \left| \sum_{n\geqslant N^{1/4}} X_0^{n,\pm} e^{-(2\pi n)^2 t} e^{n,\pm}(x) \right|^2.$$

So, in order to compare $e^{t\Delta} X_0$ and $e^{t\Delta^{(N)}} \bar{X}_0$, see (83) and (84), it remains to handle the difference

$$\sum_{0\leqslant n<N^{1/4}} \left(X_0^{n,\pm} e^{-N^2[1-\cos(2\pi n/N)]t} \bar{e}^{n,\pm}(\cdot) - X_0^{n,\pm} e^{-(2\pi n)^2 t} e^{n,\pm}(\cdot) \right).$$

By Cauchy-Schwarz inequality, we have the following bound.

$$\sup_{x\in\mathbb{S}^1}\Big|\sum_{0\leqslant n<N^{1/4}} X_0^{n,\pm}e^{-N^2[1-\cos(2\pi n/N)]t}\bar{e}^{n,\pm}(x) - X_0^{n,\pm}e^{-(2\pi n)^2 t}e^{n,\pm}(x)\Big|^2$$
$$\leqslant \Big[\sum_{n\in\mathbb{N}}|X_0^n|^2\Big]\sup_{x\in\mathbb{S}^1}\Big[\sum_{0\leqslant n<N^{1/4}}\big|e^{-N^2[1-\cos(2\pi n/N)]t}\bar{e}^{n,\pm}(x) - e^{-(2\pi n)^2 t}e^{n,\pm}(x)\big|^2\Big].$$

Following the analysis of (91) and using the same trick as in (93), we deduce that there exist a constant C and a sequence $(\varepsilon_N)_{N\in\mathbb{N}*}$ converging to 0 as N tends to ∞, both the constant and the sequence being independent of the data, such that

$$\sup_{x\in\mathbb{S}^1}\big|\big(e^{t\Delta^{(N)}}\bar{X}_0 - e^{t\Delta}X_0\big)(x)\big|^2 \leqslant \frac{\varepsilon_N}{t^{3/4}}\|X_0(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}.$$

Fifth Step. By combining the three previous steps, we easily deduce (85). ∎

Proof of the two auxiliary identities (86) *and* (87). We now prove the identity (86). We start with

$$\begin{aligned}\bar{e}^n(\cdot) &= \sum_{k=0}^{N-1}\Big(N\int_{k/N}^{(k+1)/N} e^{i2\pi nx}dx\Big)\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\\ &= \Big(N\int_0^{1/N} e^{i2\pi nx}dx\Big)\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\\ &= e^{i\pi\frac{n}{N}}\frac{\sin(\pi n/N)}{\pi n/N}\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot).\end{aligned}$$

We now check the second identity (87). Implementing the definition of $\Delta^{(N)}$, we get:

$$\begin{aligned}\Delta^{(N)}\Big[\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\Big] &= N^2\sum_{k=0}^{N-1}\big(e^{i2\pi\frac{(k+1)n}{N}} + e^{i2\pi\frac{(k-1)n}{N}} - 2e^{i2\pi\frac{kn}{N}}\big)\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\\ &= -2N^2\big[1-\cos\big(\frac{2\pi n}{N}\big)\big]\Big[\sum_{k=0}^{N-1} e^{i2\pi\frac{kn}{N}}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot)\Big],\end{aligned}$$

5.2. **Application to games.** We now return to (76) with $(\boldsymbol{\alpha}^{k,j} = \boldsymbol{\alpha}^{\star k,j} = -\bar{\boldsymbol{Y}}^k)_{k=0,\cdots,N-1;j=1,\cdots,A_N}$ as defined in (78) where $(\boldsymbol{X}(\cdot),\boldsymbol{Y}(\cdot),\boldsymbol{Z}(\cdot))$ now denotes the solution to (29). We denote the corresponding solution by $(\boldsymbol{X}^{\star,k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}$. Since $\boldsymbol{\alpha}^{\star,k,j}$ does not depend on j, we have $\boldsymbol{X}^{\star,k,j} = \bar{\boldsymbol{X}}^{\star,k}$ for any $k\in\{0,\cdots,N-1\}$, with $\bar{\boldsymbol{X}}^{\star,k} = 1/A_N\sum_{j=1}^{A_N}\bar{\boldsymbol{X}}^{\star,k,j}$.

Also, we notice that $(\bar{X}_t^{\star,0},\cdots,\bar{X}_t^{\star,N-1})_{0\leqslant t\leqslant T}$ solves the system of SDEs:

$$d\bar{X}_t^{\star,k} = \Big\{b\big(\bar{\mu}_t^{\star,N}\big) - \bar{Y}_t^k + N^2\big(\bar{X}_t^{\star,k+1} + \bar{X}_t^{\star,k-1} - 2\bar{X}_t^{\star,k}\big)\Big\}dt + \sqrt{N}dB_t^k,$$

for $t\in[0,T]$, with the same initial condition $\bar{X}_0^k$ as before and for $k\in\{0,\cdots,N-1\}$. The above system fits the form of (81). To make it clear, we use the following notations:

$$\bar{X}_t^{\star}(\cdot) = \sum_{k=0}^{N-1}\bar{X}_t^{\star,k}\mathbf{1}_{[k/N,(k+1)/N)}(\cdot),\quad\text{and}\quad \bar{\mu}_t^{\star,N} = \frac{1}{N}\sum_{k=0}^{N-1}\delta_{\bar{X}_t^{\star,k}},$$

We then apply Theorem 25 with $\boldsymbol{\alpha}^{\star}(\cdot) = b(\mathrm{Leb}_{\mathbb{S}^1}\circ\boldsymbol{X}^{-1}(\cdot)) - \boldsymbol{Y}(\cdot)$ and $\boldsymbol{\beta}^{\star} = b(\bar{\boldsymbol{\mu}}^{\star,N}) - \boldsymbol{Y}(\cdot)$ and thus $(\boldsymbol{\beta}^{\star,k} = b(\bar{\boldsymbol{\mu}}^{\star,N}) - \bar{\boldsymbol{Y}}^k)_{k=0,\cdots,N-1}$. Then, the SPDE (79) takes the form:

$$\partial_t X_t(x) = b\big(\mathrm{Leb}_{\mathbb{S}^1}\circ(X_t(\cdot))^{-1}\big) - Y_t(x) + \Delta X_t(x) + \dot{W}_t(x),\quad (t,x)\in[0,T]\times\mathbb{S}^1,$$

with the same $X_0(\cdot)$ as before as initial condition. Then, by Theorem 25, we get for all $t \in (0,T]$,

$$\begin{aligned}
&\sup_{x\in\mathbb{S}^1} \mathbb{E}_0\big[|X_t(x) - \bar{X}^\star_t(x)|^2\big] \\
&\leqslant C\varepsilon_N\Big(1 + \frac{1}{t^{3/4}}\Big) + C\mathbb{E}_0 \int_0^t \big|b\big(\mathrm{Leb}_{\mathbb{S}^1} \circ (X_s(\cdot))^{-1}\big) - b(\bar{\mu}^{\star,N}_s)\big|^2 ds \\
&\leqslant C\varepsilon_N\Big(1 + \frac{1}{t^{3/4}}\Big) + C\mathbb{E}_0 \int_0^t \int_{\mathbb{S}^1} |\bar{X}^\star_s(x) - X_s(x)|^2 dx ds + C\mathbb{E}_0 \int_0^t W_2\big(\bar{\mu}^{\star,N}_s, \mathrm{Leb}_{\mathbb{S}^1} \circ (\bar{X}^\star_s(\cdot))^{-1}\big)^2 ds,
\end{aligned}$$

where C now depends upon $\mathbb{E}_0 \int_0^T \|Y_s(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)} ds$ and $\|X_0(\cdot)\|^2_{L^2(\mathbb{S}^1;\mathbb{R}^d)}$. Since $\bar{\mu}^{\star,N}_s$ coincides with $\mathrm{Leb}_{\mathbb{S}^1} \circ (\bar{X}^\star_s(\cdot))^{-1}$, the last term in the above inequality is 0. Therefore, by the general version of Gronwall's lemma, we get, for any $t \in (0,T]$,

$$\sup_{x\in\mathbb{S}^1} \mathbb{E}_0\big[|X_t(x) - \bar{X}^\star_t(x)|^2\big] \leqslant C\varepsilon_N\Big(1 + \frac{1}{t^{3/4}}\Big). \tag{94}$$

In order to show that we have constructed an approximate Nash equilibria, we apply a variant of the sufficiency proof in the Pontryagin principle.

Particle system associated with $\boldsymbol{\beta}$. Recall the definition of $\boldsymbol{\beta}$ from the introduction of Section 5: Fix a pair $(k_0, j_0) \in \{0, \cdots, N-1\} \times \{1, \cdots, A_N\}$ and let $\boldsymbol{\beta}^{\star k,j} = \boldsymbol{\alpha}^{\star k,j}$, for $k \in \{0, \cdots, N-1\}$ and $j \in \{1, \cdots, A_N\}$, with $(k,j) \neq (k_0, j_0)$; when $k = k_0$ and $j = j_0$, let $\boldsymbol{\beta}^{\star k_0,j_0} = \boldsymbol{\gamma}$, for some $\mathbb{R}^d$-valued process $\boldsymbol{\gamma} = (\gamma_t)_{0\leqslant t\leqslant T}$ that is progressively-measurable with respect to the filtration generated by the cylindrical white noise $\boldsymbol{W}(\cdot) = (W_t(\cdot))_{0\leqslant t\leqslant T}$. Then, we call $((\chi^{k,j}_t)_{0\leqslant t\leqslant T})_{k=0,\cdots,N-1;j=1,\cdots,A_N}$ the system of particles:

$$d\chi^{k,j}_t = \Big\{b\big(\bar{\nu}^N_t\big) + \beta^{k,j}_t + N^2\big(\bar{\chi}^{k+1}_t + \bar{\chi}^{k-1}_t - 2\bar{\chi}^k_t\big)\Big\}dt + \sqrt{N}dB^k_t, \quad t \in [0,T],$$

for $k \in \{0, \cdot, N-1\}$ and $j \in \{1, \cdots, A_N\}$, with the initial condition $\chi^{k,j}_0 = \bar{X}^k_0$, and with

$$\bar{\nu}^N_t = \frac{1}{NA_N} \sum_{k=0}^{N-1} \sum_{j=1}^{A_N} \delta_{\chi^{k,j}_t},$$

and

$$\bar{\chi}^k_t = \frac{1}{A_N} \sum_{j=1}^{A_N} \chi^{k,j}_t, \quad t \in [0,T].$$

As usual, we let $\bar{\chi}_t(x) = \sum_{k=0}^{N-1} \bar{\chi}^k_t \mathbf{1}_{[k/N,(k+1)/N)}(x)$.

Pontryagin principle. For (k_0, j_0) as above, we compute

$$\begin{aligned}
&d\Big[(\chi^{k_0,j_0}_t - \bar{X}^{\star,k_0}_t) \cdot \bar{Y}^{k_0}_t\Big] \\
&= \Bigg[\Big(b\big(\bar{\nu}^N_t\big) - b\big(\bar{\mu}^{\star,N}_t\big) + \beta^{k_0,j_0}_t + \bar{Y}^{k_0}_t \\
&\qquad + N^2\big(\bar{\chi}^{k_0+1}_t + \bar{\chi}^{k_0-1}_t - 2\bar{\chi}^{k_0}_t - \bar{X}^{\star,k_0+1}_t - \bar{X}^{\star,k_0-1}_t + 2\bar{X}^{\star,k_0}_t\big)\Big) \cdot \bar{Y}^{k_0}_t \\
&\qquad - N\int_{k_0/N}^{(k_0+1)/N} \Big[\partial_x f\big(X_t(x), \mathrm{Leb}_{\mathbb{S}^1} \circ (X_t(\cdot))^{-1}\big) \cdot \big(\chi^{k_0,j_0}_t - \bar{X}^{\star,k_0}_t\big)\Big]dx\Bigg]dt \\
&\quad + dm_t,
\end{aligned}$$

where $(m_t)_{0\leqslant t\leqslant T}$ is a square integrable martingale. Therefore,

$$
\begin{aligned}
&d\Big[\big[(\chi_t^{k_0,j_0}-\bar X_t^{\star,k_0})\cdot\bar Y_t^{k_0}\big]+\int_0^t\Big(f\big(\chi_s^{k_0,j_0},\bar\nu_s^N\big)-f\big(\bar X_s^{\star,k_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_s^\star(\cdot))^{-1}\big)\Big)ds\\
&\qquad+\frac12\Big(\int_0^t|\beta_s^{k_0,j_0}|^2-|\bar Y_s^{k_0}|^2\Big)ds\Big]\\
&=\Big[\frac12\big|\beta_t^{k_0,j_0}+\bar Y_t^{k_0}\big|^2+\delta_t^N\\
&\quad+\Big[f\big(\chi_t^{k_0,j_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_t^\star(\cdot))^{-1}\big)-f\big(\bar X_t^{\star,k_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_t^\star(\cdot))^{-1}\big)\\
&\qquad-\partial_x f\big(\bar X_t^{\star,k_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_t^\star(\cdot))^{-1}\big)\cdot\big(\chi_t^{k_0,j_0}-\bar X_t^{\star,k_0}\big)\Big]\Big]dt\\
&\quad+dm_t,
\end{aligned}
\tag{95}
$$

where we have let

$$
\begin{aligned}
\delta_t^N&=f\big(\chi_t^{k_0,j_0},\bar\nu_t^N\big)-f\big(\chi_t^{k_0,j_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_t^\star(\cdot))^{-1}\big)\\
&\quad+\Big(b\big(\bar\nu_t^N\big)-b\big(\bar\mu_t^{\star,N}\big)+N^2\big(\bar\chi_t^{k_0+1}+\bar\chi_t^{k_0-1}-2\bar\chi_t^{k_0}-\bar X_t^{\star,k_0+1}-\bar X_t^{\star,k_0-1}+2\bar X_t^{\star,k_0}\big)\Big)\cdot\bar Y_t^{k_0}\\
&\quad-N\int_{k_0/N}^{(k_0+1)/N}\Big[\Big(\partial_x f\big(X_t(x),\mathrm{Leb}_{\mathbb{S}^1}\circ(X_t(\cdot))^{-1}\big)\\
&\qquad\qquad-\partial_x f\big(\bar X_t^{\star,k_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_t^\star(\cdot))^{-1}\big)\Big)\cdot\big(\chi_t^{k_0,j_0}-\bar X_t^{\star,k_0}\big)\Big]dx.
\end{aligned}
$$

Hence, taking the expectation in (95), using the convexity of f and inserting the terminal costs, we get:

$$
\begin{aligned}
&\mathbb{E}_0\Big[g\big(\chi_T^{k_0,j_0},\bar\nu_T^N\big)+\int_0^T\Big(f\big(\chi_t^{k_0,j_0},\bar\nu_t^N\big)+\frac12|\beta_t^{k_0,j_0}|^2\Big)dt\Big]\\
&\geqslant\mathbb{E}_0\Big[g\big(\bar X_T^{\star,k_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_T^\star(\cdot))^{-1}\big)+\int_0^T\Big(f\big(\bar X_t^{\star,k_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_t^\star(\cdot))^{-1}\big)+\frac12|\bar Y_t^{\star,k_0}|^2\Big)dt\Big]\\
&\quad+\mathbb{E}_0\Big[g\big(\chi_T^{k_0,j_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_T^\star(\cdot))^{-1}\big)-g\big(\bar X_T^{\star,k_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_T^\star(\cdot))^{-1}\big)\\
&\qquad\qquad-\partial_x g\big(\bar X_T^{\star,k_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_T^\star(\cdot))^{-1}\big)\cdot\big(\chi_T^{k_0,j_0}-\bar X_T^{\star,k_0}\big)\Big]\\
&\quad+\frac12\mathbb{E}_0\int_0^T\big|\beta_t^{k_0,j_0}+\bar Y_t^{k_0}\big|^2dt+\mathbb{E}_0\int_0^T\delta_t^N dt+\mathbb{E}_0\delta'_N,
\end{aligned}
$$

where we have let

$$
\begin{aligned}
\delta'_N&=g\big(\chi_T^{k_0,j_0},\bar\nu_T^N\big)-g\big(\chi_T^{k_0,j_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_T^\star(\cdot))^{-1}\big)\\
&\quad-N\int_{k_0/N}^{(k_0+1)/N}\Big[\Big(\partial_x g\big(X_T(x),\mathrm{Leb}_{\mathbb{S}^1}\circ(X_T(\cdot))^{-1}\big)\\
&\qquad\qquad-\partial_x g\big(\bar X_T^{\star,k_0},\mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_T^\star(\cdot))^{-1}\big)\Big)\cdot\big(\chi_T^{k_0,j_0}-\bar X_T^{\star,k_0}\big)\Big]dx.
\end{aligned}
$$

By convexity of g and from the identity $(\bar\mu_t^{\star,N} = \mathrm{Leb}_{\mathbb{S}^1} \circ (\bar X_t^\star(\cdot))^{-1})_{0\leqslant t\leqslant T}$, we end-up with:

$$J^{k_0,j_0}\big((\boldsymbol{\beta}^{k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big) \geqslant J^{k_0,j_0}\big((\boldsymbol{\alpha}^{\star k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big) + \frac{1}{2}\mathbb{E}_0\int_0^T \big|\beta_t^{k_0,j_0} + \bar Y_t^{k_0}\big|^2 dt + \mathbb{E}_0\int_0^T \delta_t^N dt + \mathbb{E}_0\delta'_N. \tag{96}$$

Proving the convergence of the remainder. We now investigate the two sequences $(\delta'_N)_{N\geqslant 1}$ and $(\int_0^T \delta_t^N dt)_{N\in\mathbb{N}*}$. Using once again the identity $(\bar\mu_t^{\star,N} = \mathrm{Leb}_{\mathbb{S}^1} \circ (\bar X_t^\star(\cdot))^{-1})_{0\leqslant t\leqslant T}$ together with the regularity properties of the coefficients, we have

$$\begin{aligned}
&\mathbb{E}_0\big[|\delta'_N|\big] + \mathbb{E}_0\int_0^T |\delta_t^N| dt \\
&\leqslant C\bigg(\mathbb{E}_0\big[W_2\big(\bar\nu_T^N, \mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_T^\star(\cdot))^{-1}\big)\big] + \int_0^T \mathbb{E}_0\big[W_2\big(\bar\nu_t^N, \mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_t^\star(\cdot))^{-1}\big)\big]dt\bigg) \\
&\quad + C\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\big[|X_T(x) - \bar X_T^\star(x)|^2\big]^{1/2}\Big(1 + \mathbb{E}_0\big[|\chi_T^{k_0,j_0} - \bar X_T^{\star,k_0}|^2\big]^{1/2}\Big) \\
&\quad + C\bigg(\int_0^T \sup_{x\in\mathbb{S}^1}\mathbb{E}_0\big[|X_t(x) - \bar X_t^\star(x)|^2\big]dt\bigg)^{1/2}\bigg[1 + \bigg(\int_0^T \mathbb{E}_0\big[|\chi_t^{k_0,j_0} - \bar X_t^{\star,k_0}|^2\big]dt\bigg)^{1/2}\bigg] \\
&\quad + C\,\mathbb{E}_0\int_0^T \sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}(\bar\chi_t - \bar X_t^\star)(x)\big|dt,
\end{aligned}$$

where, in the last line, we used the fact that the process $(\bar Y_t^{k_0})_{0\leqslant t\leqslant T}$ is bounded independently of k_0, see for instance Lemma 19.

Observe from (94) that we can find a sequence $(\varepsilon_N)_{N\in\mathbb{N}*}$, converging to 0 as N tends to ∞, such that

$$\sup_{x\in\mathbb{S}^1}\mathbb{E}_0\big[|X_T(x) - \bar X_T^\star(x)|^2\big]^{1/2} + \bigg(\int_0^T \sup_{x\in\mathbb{S}^1}\mathbb{E}_0\big[|X_t(x) - \bar X_t^\star(x)|^2\big]dt\bigg)^{1/2} \leqslant \varepsilon_N.$$

Now, for any $t\in[0,T]$,

$$W_2\big(\bar\nu_t^N, \mathrm{Leb}_{\mathbb{S}^1}\circ(\bar X_t^\star(\cdot))^{-1}\big) \leqslant \bigg(\frac{1}{N}\frac{1}{A_N}\sum_{k=0}^{N-1}\sum_{j=1}^{N}|\chi_t^{k,j} - \bar X_t^{\star,k}|^2\bigg)^{1/2}.$$

So, we end up with:

$$\begin{aligned}
&\mathbb{E}\big[|\delta'_N|\big] + \mathbb{E}\int_0^T |\delta_t^N| dt \\
&\quad\leqslant \varepsilon_N\Big(1 + \sup_{0\leqslant t\leqslant T}\mathbb{E}\big[|\chi_t^{k_0,j_0} - \bar X_t^{\star,k_0}|^2\big]^{1/2}\Big) \\
&\qquad + C\sup_{0\leqslant t\leqslant T}\mathbb{E}_0\bigg[\frac{1}{N}\frac{1}{A_N}\sum_{k=0}^{N-1}\sum_{j=1}^{N}|\chi_t^{k,j} - \bar X_t^{\star,k}|^2\bigg]^{1/2} + C\,\mathbb{E}_0\int_0^T \sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}(\bar\chi_t - \bar X_t^\star)(x)\big|dt.
\end{aligned} \tag{97}$$

Now, for any $t\in[0,T]$,

$$\begin{aligned}
\frac{1}{N}\frac{1}{A_N}\sum_{k=0}^{N-1}\sum_{j=1}^{N}|\chi_t^{k,j} - \bar X_t^{\star,k}|^2 &\leqslant C\int_0^t \frac{1}{N}\frac{1}{A_N}\sum_{k=0}^{N-1}\sum_{j=1}^{N}|\chi_s^{k,j} - \bar X_s^{\star,k}|^2 ds \\
&\quad + \frac{C}{NA_N}\int_0^T |\gamma_s + \bar Y_s^{k_0}|^2 ds + C\bigg(\int_0^T \sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}\big(\bar X_s^\star - \bar\chi_s\big)(x)\big|ds\bigg)^2,
\end{aligned}$$

so that, by Gronwall's lemma,

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\frac{1}{N}\frac{1}{A_N}\sum_{k=0}^{N-1}\sum_{j=1}^{N}|\chi_t^{k,j}-\bar X_t^{\star,k}|^2\\ \leqslant \frac{C}{NA_N}\int_0^T|\gamma_s+\bar Y_s^{k_0}|^2ds+C\bigg(\int_0^T\sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}\big(\bar X_s^\star-\bar\chi_s\big)(x)\big|ds\bigg)^2.\end{aligned}\tag{98}$$

We then claim from Proposition 26 below that there exists a constant c, only depending on T, such that

$$\mathbb{E}\bigg[\bigg(\int_0^T\sup_{x\in\mathbb{S}^1}|\Delta^{(N)}(\bar X_t^\star-\bar\chi_t)(x)|dt\bigg)^2\bigg]\leqslant\frac{c}{A_N^2}\mathbb{E}_0\int_0^T|\gamma_t+\bar Y_t^{k_0}|^2dt,$$

from which we deduce that

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}_0\bigg[\bigg(\frac{1}{N}\frac{1}{A_N}\sum_{k=0}^{N-1}\sum_{j=1}^{N}|\chi_t^{k,j}-\bar X_t^{\star,k}|\bigg)^2\bigg]\leqslant\frac{C}{\min(N,A_N)^2}\mathbb{E}_0\int_0^T|\gamma_t+\bar Y_t^{k_0}|^2dt,$$

the constant C being allowed to vary from line to line. By a similar argument, but without averaging, we obtain

$$\sup_{0\leqslant t\leqslant T}\mathbb{E}_0\Big[|\chi_t^{k_0,j_0}-\bar X_t^{\star,k_0}|^2\Big]\leqslant C\mathbb{E}_0\int_0^T|\gamma_t+\bar Y_t^{k_0}|^2dt.$$

Returning to (97), this yields to

$$\mathbb{E}_0\big[|\delta_N'|\big]+\mathbb{E}_0\int_0^T|\delta_t^N|dt\leqslant\varepsilon_N+C\varepsilon_N\bigg(\mathbb{E}_0\int_0^T|\gamma_t+\bar Y_t^{k_0}|^2dt\bigg)^{1/2},$$

and then, inserting into (96), we get:

$$\begin{aligned}J^{k_0,j_0}\big((\boldsymbol{\beta}^{k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big)\geqslant J^{k_0,j_0}\big((\boldsymbol{\alpha}^{\star k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big)+\frac{1}{2}\mathbb{E}_0\int_0^T\big|\gamma_t+\bar Y_t^{k_0}\big|^2dt\\ -\varepsilon_N\bigg[1+\bigg(\mathbb{E}_0\int_0^T|\gamma_s+\bar Y_s^{k_0}|^2ds\bigg)^{1/2}\bigg],\end{aligned}\tag{99}$$

the sequence $(\varepsilon_N)_{N\in\mathbb{N}^*}$ being now allowed to depend upon $(A_N)_{N\in\mathbb{N}^*}$.

Obviously, we can a find constant $a>0$, independent of N, such that the sum of the last two terms in the right-hand side is positive whenever $\mathbb{E}_0\int_0^T|\gamma_s|^2ds$ is greater than a. In such a case, we have

$$J^{k_0,j_0}\big((\boldsymbol{\beta}^{k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big)\geqslant J^{k_0,j_0}\big((\boldsymbol{\alpha}^{\star k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big),$$

which is the required inequality.

Now, if $\mathbb{E}_0\int_0^T|\gamma_s|^2ds\leqslant a$, (99) yields

$$J^{k_0,j_0}\big((\boldsymbol{\beta}^{k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big)\geqslant J^{k_0,j_0}\big((\boldsymbol{\alpha}^{\star k,j})_{k=0,\cdots,N-1;j=1,\cdots,A_N}\big)-\varepsilon_N\big(1+a^{1/2}\big),$$

and the result follows easily.

Stability of the interaction. In order to complete the proof, it remains to evaluate the distance between $\Delta^{(N)}\bar{\boldsymbol{X}}^\star(\cdot)$ and $\Delta^{(N)}\bar{\boldsymbol{\chi}}(\cdot)$, which is the purpose of the next statement.

Proposition 26. *There exists a constant C, only depending on T, such that, with the same notations as before,*

$$\mathbb{E}_0\bigg[\bigg(\int_0^T\sup_{x\in\mathbb{S}^1}|\Delta^{(N)}(\bar X_t^\star-\bar\chi_t)(x)|dt\bigg)^2\bigg]\leqslant\frac{C}{A_N^2}\mathbb{E}_0\int_0^T|\gamma_t+\bar Y_t^{k_0}|^2dt.$$

Proof. By (84), we notice that, for any $t \in [0,T]$,

$$\begin{aligned}\bar{X}_t(\cdot) - \bar{\chi}_t(\cdot) &= \int_0^t e^{(t-s)\Delta^{(N)}}\big(b(\bar{\mu}_s^{\star,N}) - b(\bar{\nu}_s^N)\big)ds + \int_0^t e^{(t-s)\Delta^{(N)}}\big(\bar{\alpha}_s^\star(\cdot) - \bar{\beta}_s(\cdot)\big)ds \\ &= (i)_t + (ii)_t,\end{aligned} \tag{100}$$

where we have let

$$\bar{\alpha}_t^\star(\cdot) = \sum_{k=0}^{N-1} \alpha_t^\star \mathbf{1}_{[k/N,(k+1)/N)}(\cdot), \quad \bar{\beta}_t(\cdot) = \sum_{k=0}^{N-1}\bigg(\frac{1}{A_N}\sum_{j=1}^{A_N}\beta_t^{k,j}\bigg)\mathbf{1}_{[k/N,(k+1)/N)}(\cdot).$$

As for (i), using the fact that both $b(\bar{\mu}_s^{\star,N})$ and $b(\bar{\nu}_s^N)$ are constant functions of $x \in \mathbb{S}^1$ for each $s \in [0,T]$, it is absolutely obvious that

$$(i)_t = \int_0^t \big(b(\bar{\mu}_s^{\star,N}) - b(\bar{\nu}_s^N)\big)ds,$$

and then $\Delta^{(N)}(i)_t = 0$. Returning to (100), it suffices to focus on $(ii)_t$.

Letting $(\bar{\varrho}_t(\cdot) = (\bar{\alpha}_t^\star - \bar{\beta}_t)(\cdot))_{0\leqslant t\leqslant T}$ and following the third step in the proof of Theorem 25, we have

$$\int_0^t e^{(t-s)\Delta^{(N)}}\big(\bar{\alpha}_s^\star - \bar{\beta}_s\big)(\cdot)ds = \sum_{n\in\mathbb{N}}\bigg(\int_0^t \bar{\varrho}_s^{n,\pm} e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\bigg)\bar{e}^{n,\pm}(\cdot),$$

where $(\bar{\varrho}_s^{n,\pm})_{n\in\mathbb{N}}$ denote the Fourier coefficients of the function $\bar{\varrho}_s \in L^2(\mathbb{S}^1;\mathbb{R}^d)$. Here we used the identity

$$\bar{\varrho}_s(\cdot) = \sum_{n\in\mathbb{N}}\sum_{k=0}^{N-1}\bigg(N\int_{k/N}^{(k+1)/N}\bar{\varrho}_s(x)dx\bigg)\mathbf{1}_{[k/N,(k+1)/N)}(\cdot) = \sum_{n\in\mathbb{N}}\bar{\varrho}_s^{n,\pm}\bar{e}^{n,\pm}(\cdot),$$

which follows from the fact that $\bar{\varrho}_s(\cdot)$ is constant on each $[k/N,(k+1)/N)$. Then,

$$\Delta^{(N)}(ii)_t = \sum_{n\in\mathbb{N}}\bigg(\int_0^t \bar{\varrho}_s^{n,\pm}\big(-2N^2[1-\cos(2\pi n/N)]\big)e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\bigg)\bar{e}^{n,\pm}(\cdot).$$

We deduce that

$$\sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}(ii)_t(x)\big| \leqslant 2\sum_{n\in\mathbb{N}}\int_0^t |\bar{\varrho}_s^{n,\pm}|\frac{|\sin(\pi n/N)|}{\pi n/N}\big(2N^2[1-\cos(2\pi n/N)]\big)e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds,$$

which we rewrite

$$\sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}(ii)_t(x)\big| \leqslant \sum_{n\in\mathbb{N}}\int_0^t \bar{\varrho}_s^{n,\pm}\cdot h_s^{n,\pm}ds,$$

where we have let

$$h_s^{n,\pm} = \mathrm{sign}\big(\bar{\varrho}_s^{n,\pm}\big)\big(2N^2[1-\cos(2\pi n/N)]\big)\frac{|\sin(\pi n/N)|}{\pi n/N}e^{-2N^2[1-\cos(2\pi n/N)](t-s)},$$

where $\mathrm{sign}(x)$ is understood as $(\mathrm{sign}(x_1),\cdots,\mathrm{sign}(x_d))$ for $x \in \mathbb{R}^d$. Obviously,

$$\sup_{0\leqslant s\leqslant t}\sum_{n\in\mathbb{N}}|h_s^{n,\pm}|^2 < \infty,$$

so that, by Parseval's identity,

$$\sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}(ii)_t(x)\big| \leqslant \int_0^t\int_{\mathbb{S}^1}\bar{\varrho}_s(x)\cdot h_s(x)dx,$$

with

$$h_s(\cdot) = \sum_{n\in\mathbb{N}} h_s^{n,\pm} e^{n,\pm}(\cdot).$$

In fact $\bar{\varrho}_s(\cdot) = [(\gamma_s + \bar{Y}_s^{k_0})/A_N]\mathbf{1}_{[k_0/N,(k_0+1)/N)}(\cdot)$. Hence, we have

$$\sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}(ii)_t(x)\big| \leqslant \frac{1}{A_N}\bigg|\int_0^t \big(\gamma_s + \bar{Y}_s^{k_0}\big)\cdot\bigg(\int_{k_0/N}^{(k_0+1)/N} h_s(x)dx\bigg)ds\bigg|.$$

Clearly, by (86),

$$\bigg|\int_{k_0/N}^{(k_0+1)/N} h_s(x)dx\bigg| = \bigg|\sum_{n\in\mathbb{N}} h_s^{n,\pm}\int_{k_0/N}^{(k_0+1)/N} e^{n,\pm}(x)dx\bigg| \leqslant 2\sum_{n\in\mathbb{N}}|h_s^{n,\pm}|\frac{|\sin(\pi n/N)|}{\pi n}.$$

Then,

$$\begin{aligned}
&\int_0^T \sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}(ii)_t(x)\big|dt\\
&\leqslant \frac{4}{A_N}\sum_{n\in\mathbb{N}}\int_0^T\int_0^t |\gamma_s + \bar{Y}_s^{k_0}|\frac{\sin^2(\pi n/N)}{(\pi n)^2}\big(N^3[1-\cos(2\pi n/N)]\big)e^{-2N^2[1-\cos(2\pi n/N)](t-s)}ds\,dt\\
&= \frac{4}{A_N}\sum_{n\in\mathbb{N}}\int_0^T |\gamma_s + \bar{Y}_s^{k_0}|\frac{\sin^2(\pi n/N)}{(\pi n)^2}\big(N^3[1-\cos(2\pi n/N)]\big)\bigg(\int_s^T e^{-2N^2[1-\cos(2\pi n/N)](t-s)}dt\bigg)ds.
\end{aligned}$$

We thus have

$$\begin{aligned}
\int_0^T \sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}(ii)_t(x)\big|dt &\leqslant \frac{2}{A_N}\bigg(\int_0^T |\gamma_s + \bar{Y}_s^{k_0}|ds\bigg)\bigg(N\sum_{n\in\mathbb{N}}\frac{\sin^2(\pi n/N)}{(\pi n)^2}\bigg)\\
&\leqslant \frac{2}{A_N}\bigg(\int_0^T |\gamma_s + \bar{Y}_s^{k_0}|ds\bigg)\bigg(\frac{1}{N}\sum_{n=0}^{N}\frac{\sin^2(\pi n/N)}{(\pi n/N)^2} + N\sum_{n>N}\frac{1}{n^2}\bigg).
\end{aligned}$$

So, there exists a constant C, only depending on T, such that

$$\mathbb{E}_0\bigg[\bigg|\int_0^T \sup_{x\in\mathbb{S}^1}\big|\Delta^{(N)}(ii)_t(x)\big|dt\bigg|^2\bigg] \leqslant \frac{C}{A_N^2}\mathbb{E}_0\int_0^T |\gamma_t + \bar{Y}_t^{k_0}|^2 dt,$$

which completes the proof. ∎

References

[1] S. Ahuja. Wellposedness of mean field games with common noise under a weak monotonicity condition. Technical report, `http://arxiv.org/pdf/1406.7028v2.pdf`, 2015.

[2] R. Bafico and P. Baldi. Small random perturbations of Peano phenomena. Stochastics, 6:279–292, 1982.

[3] A. Bensoussan, J. Frehse, and P. Yam. Mean Field Games and Mean Field Type Control Theory. Springer Briefs in Mathematics. Springer Verlag, 2013.

[4] A. Bensoussan, J. Frehse, and S. C. P. Yam. The master equation in mean field theory. Journal de Mathématiques Pures et Appliquées, 2014.

[5] A. Bensoussan, J. Frehse, and S. C. P. Yam. On the interpretation of the master equation. Technical report, 2015.

[6] David Blackwell and Lester E. Dubins. An extension of Skorohod's almost sure representation theorem. Proc. Amer. Math. Soc., 89(4):691–692, 1983.

[7] P. Cardaliaguet. Notes from P.L. Lions' lectures at the Collège de France. Technical report, `https://www.ceremade.dauphine.fr/~cardalia/MFG100629.pdf`, 2012.

[8] P. Cardaliaguet. Weak solutions for first order mean field games with local coupling. Analysis and Geometry in Control Theory and its Applications. Springer INdAM Series, 11:111–158, 2015.

[9] P. Cardaliaguet, F. Delarue, J.-M. Lasry, and P.-L. Lions. The master equation and the convergence problem in mean field games. Technical report, 2015.

[10] P. Cardaliaguet and J. Graber. Mean field games systems of first order. ESAIM: Control, Optimisation and Calculus of Variations, 21:690–722, 2015.
[11] P. Cardaliaguet and J. Graber. Second order mean field games with degenerate diffusion and local coupling. NoDEA, 22:1287–1317, 2015.
[12] P. Cardaliaguet, A. R. Mészáros, and F. Santambrogio. First order mean field games with density constraints: Pressure equals price. Technical report, http://arxiv.org/abs/1507.02019.
[13] R. Carmona and F. Delarue. Mean field forward-backward stochastic differential equations. Electronic Communications in Probability.
[14] R. Carmona and F. Delarue. Probabilistic analysis of mean field games. SIAM Journal on Control and Optimization, 51:2705–2734, 2013.
[15] R. Carmona and F. Delarue. The master equation for large population equilibriums. In Stochastic Analysis and Applications, pages 77 – 128. Springer Verlag, 2014.
[16] R. Carmona and F. Delarue. Probabilistic Theory of Mean Field Games: Vol. I, Mean Field FBSDEs, Control, and Games. Stochastic Analysis and Applications. Springer Verlag, 2018.
[17] R. Carmona and F. Delarue. Probabilistic Theory of Mean Field Games: Vol. II, Mean Field Games with Common Noise and Master Equations. Stochastic Analysis and Applications. Springer Verlag, 2018.
[18] R. Carmona, F. Delarue, and A. Lachapelle. Control of McKean-Vlasov versus Mean Field Games. Mathematics and Financial Economics, 7:131–166, 2013.
[19] R. Carmona, F. Delarue, and D. Lacker. Probabilistic analysis of mean field games with a common noise. Annals of Probability, 44:3740–3803, 2016.
[20] Sandra Cerrai. Second order PDE's in finite and infinite dimension, volume 1762 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 2001. A probabilistic approach.
[21] J.F. Chassagneux, D. Crisan, and F. Delarue. Mckean-vlasov fbsdes and related master equation. Technical report, 2015.
[22] Dan Crisan and François Delarue. Sharp derivative bounds for solutions of degenerate semi-linear partial differential equations. J. Funct. Anal., 263(10):3024–3101, 2012.
[23] A.M. Davie. Uniqueness of solutions of stochastic differential equations. Int. Math. Res. Not. IMRN, 24:Art. ID rnm124, 26 pp., 2007.
[24] F. Delarue. On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case. Stochastic Processes and Applications, 99:209–286, 2002.
[25] F. Delarue and G. Guatteri. Weak existence and uniqueness for fbsdes. Stochastic Processes and Applications, 116:1712–1742, 2006.
[26] Giorgio Fabbri, Fausto Gozzi, and Andrzej Swiech. Stochastic optimal control in infinite dimension, volume 82 of Probability Theory and Stochastic Modelling. Springer, Cham, 2017. Dynamic programming and HJB equations, With a contribution by Marco Fuhrman and Gianmario Tessitore.
[27] F. Flandoli. Random Perturbation of PDEs and Fluid Dynamics: Ecole d'été de probabilités de Saint-Flour XL. Lecture Notes in Mathematics. Springer Verlag, 2011.
[28] F. Flandoli, M. Gubinelli, and E. Priola. Well posedness of the transport equation by stochastic perturbation. Invent. Math., 180:153, 2010.
[29] F. Flandoli, F. Russo, and J. Wolf. Some sdes with distributional drift. i. general calculus. Osaka J. Math., 40:493–542, 2003.
[30] F. Flandoli, F. Russo, and J. Wolf. Some sdes with distributional drift. ii. lyons-zheng structure, it's formula and semimartingale characterization. Random Oper. Stochastic Equations, 2:145–184, 2004.
[31] R. Foguen Tchuendom. Restoration of uniqueness of nash equilibria for a class of linear-quadratic mean field games with common noise. Technical report, 2015.
[32] W. Gangbo and A. Swiech. Existence of a solution to an equation arising from the theory of mean field games. Technical report, To appear.
[33] D.A. Gomes and E. Pimentel. Time-dependent mean-field games with logarithmic nonlinearities. SIAM J. Math. Anal., 47:37983812, 2015.
[34] D.A. Gomes, E. Pimentel, and H. Sánchez-Morgado. Time-dependent mean-field games in the sub- quadratic case. Comm. Partial Differential Equations, 40:40–76, 2015.
[35] D.A. Gomes, E. Pimentel, and H. Sánchez-Morgado. Time-dependent mean-field games in the superquadratic case. ESAIM: Control, Optimisation and Calculus of Variations, 22:562580, 2016.
[36] D.A. Gomes and J. Saude. Mean field games models - a brief survey. Technical report, 2014.
[37] D.A. Gomes and V. Voskanyan. Extended mean field games-formulation, existence, uniqueness and examples. Technical report, 2016.

[38] M. Huang, P.E. Caines, and R.P. Malhamé. Large population stochastic dynamic games: closed-loop McKean-Vlasov systems and the Nash certainty equivalence principle. Communications in Information and Systems, 6:221–252, 2006.
[39] M. Huang, P.E. Caines, and R.P. Malhamé. Large population cost coupled LQG problems with nonuniform agents: individual mass behavior and decentralized ϵ-Nash equilibria. IEEE Transactions on Automatic Control, 52:1560–1571, 2007.
[40] V.N. Kolokolstov and M. Troeva. On the mean field games with common noise and the McKean-Vlasov SPDEs. Technical report, `http://arxiv.org/abs/1506.04594`, 2015.
[41] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. Probab. Theory Related Fields, 131:154–196, 2005.
[42] D. Lacker. A general characterization of the mean field limit for stochastic differential games. Probab. Theory Related Fields, 165:581648, 2016.
[43] J.M. Lasry and P.L. Lions. Jeux à champ moyen i. le cas stationnaire. Comptes Rendus de l'Académie des Sciences de Paris, ser. A, 343(9), 2006.
[44] J.M. Lasry and P.L. Lions. Jeux à champ moyen ii. horizon fini et contrôle optimal. Comptes Rendus de l'Académie des Sciences de Paris, ser. A, 343(10), 2006.
[45] J.M. Lasry and P.L. Lions. Mean field games. Japanese Journal of Mathematics, 2(1):229–260, 2007.
[46] P.L. Lions. Théorie des jeux à champs moyen et applications. Lectures at the Collège de France. http://www.college-de-france.fr/default/EN/all/equ_der/cours_et_seminaires.htm, 2007-2008.
[47] P.L. Lions. Estimées nouvelles pour les équations quasilinéaires. Seminar in Applied Mathematics at the Collège de France. http://www.college-de-france.fr/site/pierre-louis-lions/seminar-2014-11-14-11h15.htm, 2014.
[48] S. Peng. A general stochastic maximum principle for optimal control problems. SIAM Journal on Control and Optimization, 2(4):966–979, 1990.
[49] H. Pham. Continuous-time Stochastic Control and Optimization with Financial Applications. Stochastic Modelling and Applied Probability. Springer Verlag, 2009.
[50] A. Y. Veretennikov. Strong solutions and explicit formulas for solutions of stochastic integral equations. Mat. Sb., 111:434–452, 1980.
[51] C. Villani. Topics in optimal transportation, volume 58 of Graduate Studies in Mathematics. American Mathematical Society, 2003.
[52] C. Villani. Optimal Transport, Old and New. Springer Verlag, 2009.
[53] J. Yong and X. Zhou. Stochastic Controls: Hamiltonian Systems and HJB Equations. Springer Verlag, 1999.
[54] J. Zabczyk. Parabolic equations on Hilbert spaces. In Stochastic PDE's and Kolmogorov equations in infinite dimensions (Cetraro, 1998), volume 1715 of Lecture Notes in Math., pages 117–213. Springer, Berlin, 1999.
[55] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. Mat. Sb., 93:129–149, 1974.