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Asymptotic limit of linear parabolic equations with spatio-temporal degenerated potentials

Pablo Àlvarez-Caudevilla, Matthieu Bonnivard, Antoine Lemenant

April 4, 2018

Abstract

In this paper, we observe how the heat equation in a non-cylindrical domain can arise as the asymptotic limit of a parabolic problem in a cylindrical domain, by adding a potential that vanishes outside the limit domain. This can be seen as a parabolic version of a previous work by the first and last authors, concerning the stationary case [3]. We provide a strong convergence result for the solution by use of energetic methods and Γ-convergence technics. Then, we establish an exponential decay estimate coming from an adaptation of an argument by B. Simon.

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1 Introduction

For $\Omega \subset \mathbb{R}^N$ open and $T > 0$, we define the cylinder $Q_T = \Omega \times (0, T)$. Let $\lambda > 0$ be a positive real parameter. For $f_\lambda \in L^2(Q_T)$, $g_\lambda \in H^1_0(\Omega)$ and $a : Q_T \to \mathbb{R}^+$ a bounded measurable function, we consider the solution $u_\lambda$ of the parabolic problem

$$\begin{cases}
    \partial_t u - \Delta u + \lambda a(x, t)u = f_\lambda \\
    u = 0 \\
    u(x, 0) = g_\lambda(x)
  \end{cases} \quad \text{in } Q_T, \quad \text{on } \partial \Omega \times (0, T), \quad \text{in } \Omega.
$$

Since $(P_\lambda)$ is a classical parabolic problem, existence and regularity of solutions follow from a standard theory well developed in the literature (see Section 3). In particular, under our assumptions, $u \in L^2(0, T; H^1_0(\Omega))$ is continuous in time (thus the initial condition $u(x, 0) = g_\lambda(x)$ is well defined in $L^2(\Omega)$) and the equation is satisfied in a weak sense (see Section 3 for an exact formulation).

In this paper we are interested in the limit of $u_\lambda$ when $\lambda \to +\infty$. In particular, we assume spatial and temporal degeneracies for the potential $a$, which means that

$$O_a := \text{Int}(\{(x, t) \in Q_T : a(x, t) = 0\}) \neq \emptyset. \quad (1.1)$$

We also assume that $\partial O_a$ has zero Lebesgue measure.

In order to describe the results of this paper, let us start with elementary observations. Assume that, when $\lambda$ goes to $+\infty$, $f_\lambda$ converges to $f$ and $g_\lambda$ converges to $g$, for instance in $L^2$. Assume also that $u_\lambda$ converges weakly in $L^2(Q_T)$ to some $u \in L^2(Q_T)$.

Under those assumptions it is not very difficult to get the following a priori bound using the equation in $(P_\lambda)$ (see Lemma 3)

$$\lambda \int_{Q_T} a u_\lambda^2 \, dx dt \leq C. \quad (1.2)$$

This shows that $u_\lambda$ converges strongly to 0 in any set of the form $\{a(x, t) > \varepsilon\}$, for any $\varepsilon > 0$.

Then, multiplying the equation in $(P_\lambda)$ by any $\varphi \in C_0^\infty(O_a)$ we get, after some integration by parts (in this paper we shall denote $\nabla$ for $\nabla_x$, i.e. the gradient in space),

$$\int_{Q_T} u_\lambda \partial_t \varphi - \int_{Q_T} u_\lambda \Delta \varphi = \int_{Q_T} f_\lambda \varphi.$$

Passing to the limit, we obtain that $\partial_t u - \Delta u = f$ in $\mathcal{D}'(O_a)$. Under some suitable extra assumptions on the potential $a$, we will actually be able to prove that the limit $u$ satisfies the following more precise problem:

$$\begin{cases}
    u \in L^2(0, T; H^1_0(\Omega)), & u' \in L^2(0, T; L^2(\Omega)) \\
    u = 0 \text{ a.e. in } Q_T \setminus O_a \\
    \int_{Q_T} (u' v + \nabla u \nabla v) = \int_{Q_T} f_\lambda v, & \text{for all } v \in L^2(0, T; H^1_0(\Omega)) \text{ s.t. } v = 0 \text{ a.e. in } Q_T \setminus O_a \\
    u(x, 0) = g(x) \text{ in } \Omega.
  \end{cases}
$$
Problem \((P_\infty)\), which arises here naturally as the limit problem associated with the family of problems \((P_\lambda)\), is a non standard heat equation since \(O_a\) may, in general, not be cylindrical. This type of heat equation in a noncylindrical domain appears in many applications, and different approaches have been developed recently to solve problems related to \((P_\infty)\) (see for e.g. [6, 8, 9, 10, 14, 18] and the references therein). As a byproduct of our work, we have obtained an existence and uniqueness result for the problem \((P_\infty)\) (see Corollary 1).

In this paper, we study in more detail the convergence of \(u_\lambda\), when \(\lambda\) goes to infinity. Our first result gives a sufficient condition on the potential \(a\), for which the convergence of \(u_\lambda\) to \(u\) is stronger than a weak \(L^2\) convergence. Indeed, assuming a monotonicity condition on the potential \(a\), and using purely energetic and variational methods, we obtain that the convergence holds strongly in \(L^2(0,T;H^1(\Omega))\); see Section 5. Our approach can be seen as the continuation of a previous work [3], where the stationary problem has been studied using the theory of \(\Gamma\)-convergence.

Here is our first main result.

**Theorem 1.** For all \(\lambda > 0\), let \(u_\lambda\) be the solution of \((P_\lambda)\) with \(f_\lambda \in L^2(\Omega \times (0,T))\) and \(g_\lambda \in H^1_0(\Omega)\). Assume that \(a : \Omega \times [0,T] \to \mathbb{R}^+\) is a Lipschitz function which satisfies

\[
\partial_t a(x,t) \leq 0 \text{ a.e. in } Q_T. \tag{1.3}
\]

Assume also that the initial condition \(g_\lambda\) satisfies

\[
\sup_{\lambda > 0} \left( \lambda \int_\Omega a(x,0) g_\lambda(x)^2 dx \right) < +\infty,
\]

converges weakly to \(g\) in \(L^2(\Omega)\), and that \(f_\lambda\) converges weakly to \(f\) in \(L^2(Q_T)\).

Then \(u_\lambda\) converges strongly to \(u\) in \(L^2(0,T;H^1(\Omega))\), where \(u\) is the unique solution of \((P_\infty)\).

**Remark 1.** In particular, condition (1.3) implies that the family of sets \(\Omega_a(t) \subset \Omega\), defined for \(t > 0\) by \(\Omega_a(t) := \{ x \in \Omega, (x,t) \in O_a \}\), is increasing in time for the inclusion. In that case, by a slight abuse of terminology, we will often write simply that \(O_a\) is increasing in time (for the inclusion).

Our second result is a quantitative convergence of \(u_\lambda\) to 0, outside \(O_a\) (in other words, away from the vanishing region), with very general assumptions on \(a\) (only continuous and \(O_a \neq \emptyset\)), but in the special case when \(f_\lambda = 0\) in \(Q_T \setminus O_a\). This is obtained using an adaptation of an argument due to Simon [19], and proves that \(u_\lambda\) decays exponentially fast to 0 in the region \(Q_T \setminus O_a\). Compared to the standard bound (1.2), this results expresses that \(u_\lambda\) goes to 0 much faster than one could expect. We also take the opportunity of this paper to write a similar estimate for the stationary problem (see Lemma 5 in Section 6).

**Theorem 2.** For all \(\lambda > 0\), let \(u_\lambda\) be the solution of \((P_\lambda)\) with \(f_\lambda \in L^2(Q_T)\) and \(g_\lambda \in H^1_0(O_a \cap \{t = 0\})\). Assume that \(f_\lambda = 0\) in \(Q_T \setminus O_a\). Let \(a : \overline{\Omega} \times [0,T] \to \mathbb{R}^+\)
be a continuous function for which \( O_a \) is non empty. For every \( \varepsilon > 0 \), define \( A_\varepsilon := \{(x,t); \text{dist}((x,t),O_a) > \varepsilon\} \). Then, for every \( \varepsilon > 0 \), there exists a constant \( C > 0 \) such that
\[
\sup_{\lambda > 0} \left( \lambda \varepsilon c_\varepsilon \sqrt{\lambda} \int_{A_\varepsilon} u_\lambda^2 \, dx \right) \leq C,
\]
where \( c_\varepsilon := \varepsilon \min_{(x,t) \in A_\varepsilon} a(x,t) \).

The convergence of weak solutions of \((P_\lambda)\) was already observed in [11] as a starting point for a more detailed analysis about the associated semigroup. This was then used in [11] to analyse the asymptotic behaviour of a non linear periodic-parabolic problem of logistic type, where the equation is the following, also considered before in [14],
\[
\partial_t u - \Delta u = \mu u - a(x,t)u^p,
\]
(1.4)
used in some models of population dynamics. A possible link between our Problem \((P_\lambda)\) and the non-linear equation (1.4) is coming from the fact that asymptotic limit of the principal eigenvalue for the linear parabolic operator \( \partial_t - \Delta + \lambda a(x,t) \) plays a role in the dynamical behaviour of non-linear logistic equation (cf. [5, 11, 14]). We thus believe that the results and technics developed in the present paper could be possibly used in the study of more general equations such as (1.4).

Furthermore, another possible application of our results could be for numerical purposes. Indeed, for the ones who would be interested by computing a numerical solution of the non-cylindrical limiting problem \((P_\infty)\), one could use the cylindrical problem \((P_\lambda)\) for a large \( \lambda \), much easier to compute via standard methods. The strong convergence stated in Theorem 1 together with the exponential rate of convergence stated in Theorem 2 give some good estimates about the difference between those two different solutions.

## 2 The stationary problem

This section concerns only the stationary problem. In particular, throughout the section, all functions \( u, a, f, \text{ etc.} \), will be functions of \( x \in \Omega \) (and independent of \( t \)).

We assume \( \Omega \subset \mathbb{R}^N \) to be an open and bounded domain and \( a : \overline{\Omega} \to \mathbb{R}^+ \) be a measurable and bounded non-negative function. We suppose that
\[
K_a := \{x \in \overline{\Omega}; a(x) = 0\} \subset \Omega \text{ is a closed set in } \mathbb{R}^N.
\]
(2.1)
Moreover, we assume that
\[
\Omega_a := \text{Int}(K_a) \neq \emptyset.
\]
(2.2)
Under hypothesis (2.1) we know that
\[
H_0^1(K_a) := H^1(\mathbb{R}^N) \cap \{u = 0 \text{ q.e. in } \mathbb{R}^N \setminus K_a\} = H^1(\mathbb{R}^N) \cap \{u = 0 \text{ a.e. in } \mathbb{R}^N \setminus K_a\},
\]
and hypothesis (2.2) implies that
\[
H_0^1(K_a) \neq \{0\}.
\]
Notice that we are working with a functional space of the form \( H_0^1(A) \), where \( A \) is a closed subspace of \( \mathbb{R}^N \). Therefore, we do not claim that \( H_0^1(A) = H_0^1(\text{Int}(A)) \), which is true only under more regularity assumptions on the set \( A \).

Furthermore, we define the functionals \( E_\lambda \) and \( E \) on \( L^2(\Omega) \) as follows.

\[
E_\lambda(u) = \begin{cases} 
\frac{1}{2} \int_\Omega |\nabla u|^2 + \lambda au^2 \, dx & \text{if } u \in H_0^1(\Omega) \\
+\infty & \text{otherwise.}
\end{cases}
\tag{2.3}
\]

\[
E(u) = \begin{cases} 
\frac{1}{2} \int_\Omega |\nabla u|^2 \, dx & \text{if } u \in H_0^1(K_a) \\
+\infty & \text{otherwise.}
\end{cases}
\]

The following result was already stated and used in [3]. For the sake of completeness, we reproduce the proof here and refer the reader to [3] for the connection of this result with \( \Gamma \)-convergence and several examples.

**Proposition 1.** Let \( f_\lambda \in L^2(\Omega) \) be a family of functions indexed by some real parameter \( \lambda > 0 \) and uniformly bounded in \( L^2(\Omega) \). Moreover, assume that \( f_\lambda \) converges to a function \( f \in L^2(\Omega) \) in the weak topology of \( L^2(\Omega) \), when \( \lambda \) tends to \( +\infty \). Then the unique solution of the problem

\[
(P_{s\lambda}^\infty) \begin{cases} 
-\Delta u + \lambda au = f_\lambda \\
u \in H_0^1(\Omega),
\end{cases}
\]

converges strongly in \( H^1(\Omega) \), when \( \lambda \to +\infty \), to the unique solution of the problem

\[
(P_s^\infty) \begin{cases} 
-\Delta u = f \\
u \in H_0^1(K_a).
\end{cases}
\]

**Proof.** This is a standard consequence of the \( \Gamma \)-convergence of energies \( E_\lambda \), which relies on the fact that \( u_\lambda \) is the unique minimizer in \( H_0^1(\Omega) \) for

\[
v \mapsto E_\lambda(v) - 2 \int_\Omega f_\lambda v,
\]

whereas \( u \) is the unique minimizer in \( H_0^1(K_a) \) for

\[
v \mapsto E(v) - 2 \int_\Omega f v.
\]

Let us write the full details of the proof. For any \( \lambda > 0 \), let \( u_\lambda \) be the solution of \( (P_{s\lambda}^\infty) \). We first prove that \( \{u_\lambda\}_{\lambda > 0} \) is compact in \( L^2(\Omega) \). This comes from the energy equality

\[
\int_\Omega (|\nabla u_\lambda|^2 + \lambda au_\lambda^2) \, dx = \int_\Omega f_\lambda u_\lambda \, dx,
\]

which implies

\[
\int_\Omega |\nabla u_\lambda|^2 \leq \|f_\lambda\|_{L^2(\Omega)} \|u_\lambda\|_{L^2(\Omega)} \leq C \|u_\lambda\|_{L^2(\Omega)} \quad \text{with } C \text{ a positive constant.}
\]
Thanks to Poincaré’s inequality we also have that
\[ \|u_\lambda\|_{L^2(\Omega)}^2 \leq C(\Omega) \int_{\Omega} |\nabla u_\lambda|^2 \, dx, \]
which finally proves that \( u_\lambda \) is uniformly bounded in \( H^1_0(\Omega) \).

Now let \( w \) be any point in the \( L^2 \)-adherence of the family \( \{u_\lambda\}_{\lambda > 0} \). In other words, there exists a subsequence, still denoted by \( u_\lambda \), converging strongly in \( L^2 \) to \( w \). Since \( u_\lambda \) is bounded in \( H^1(\Omega) \), we can assume, up to extracting a further subsequence, that \( u_\lambda \) converges weakly in \( H^1(\Omega) \) to a function that must necessarily be \( w \). Now let \( u \) be the solution of the limit problem \( (P^\infty_s) \). Since \( u_\lambda \) is a minimizer of
\[ u \mapsto E_\lambda(u) - 2 \int_{\Omega} f_\lambda u \, dx, \tag{2.4} \]
and since \( au = 0 \) we have
\[ E_\lambda(u_\lambda) - 2 \int_{\Omega} f_\lambda u_\lambda \, dx \leq E_\lambda(u) - 2 \int_{\Omega} f_\lambda u \, dx = E(u) - 2 \int_{\Omega} f_\lambda u \, dx. \]
Hence, passing to the limsup in the previous inequality, it follows that
\[ E(w) - 2 \int_{\Omega} f_\lambda u_\lambda \, dx \leq \liminf_{\lambda} \left( E_\lambda(u_\lambda) - 2 \int_{\Omega} f_\lambda u_\lambda \, dx \right) \]
\[ \leq \limsup_{\lambda} \left( E_\lambda(u_\lambda) - 2 \int_{\Omega} f_\lambda u_\lambda \, dx \right) \]
\[ \leq E(u) - 2 \int_{\Omega} f_\lambda u \, dx, \tag{2.5} \]
which shows that \( w \) is a minimizer, and thus \( w = u \). By uniqueness of the adherence point, we infer that the whole sequence \( u_\lambda \) converges strongly in \( L^2 \) to \( u \) (and weakly in \( H^1 \)).

It remains to prove the strong convergence in \( H^1 \). To do so, it is enough to prove
\[ \|\nabla u_\lambda\|_{L^2(\Omega)} \to \|\nabla u\|_{L^2(\Omega)}. \]
Due to the weak convergence in \( H^1(\Omega) \) (up to subsequences) we already have
\[ \|\nabla u\|_{L^2(\Omega)} \leq \liminf_{\lambda} \|\nabla u_\lambda\|_{L^2(\Omega)}, \]
and going back to (2.5) we get the reverse inequality, with a limsup.

The proof of convergence of the whole sequence follows by uniqueness of the adherent point in \( H^1(\Omega) \).

\[ \square \]

**Remark 2.** Notice that when \( u \) is a solution of \( (P^\infty_s) \), then \( -\Delta u = f \) only in \( \text{Int}(K_a) \) and \( -\Delta u = 0 \) in \( K_a^c \). However, in general \( -\Delta u \) has a singular part on \( \partial K_a \). Typically, if \( K_a \) is for instance a set of finite perimeter, then in the distributional sense in \( \Omega \),
\[ -\Delta u = f1_{K_a} + \frac{\partial u}{\partial \nu} \mathcal{H}^{N-1}|_{\partial K_a}, \]
where \( \nu \) is the outer normal on \( \partial K_a \) and \( \mathcal{H}^{N-1} \) is the \( N-1 \) dimensional Hausdorff measure.
As a consequence of Proposition 1, we easily obtain the following result.

**Proposition 2.** Assume that \( f_\lambda \) converges weakly to a function \( f \) in \( L^2(\Omega) \). For any \( \lambda > 0 \), let \( u_\lambda \) be the solution of Problem \((P^*_\lambda)\). Then, when \( \lambda \to \infty \),

\[
\lambda \int_\Omega au_\lambda^2 \, dx \to 0,
\]

\[
\lambda au_\lambda \to f \mathbf{1}_{\Omega \setminus K_n} + (\Delta u)|_{\partial K_n} \text{ in } \mathcal{D}'(\Omega),
\]

where \( u \) is the solution of \((P^*_\infty)\). Moreover, the convergence in (2.7) holds in the weak-* topology of \( H^{-1} \).

**Proof.** Due to Proposition 1 we know that \( u_\lambda \) converges strongly in \( H^1(\Omega) \) to \( u \), the solution of Problem \((P^*_\infty)\). In particular, from the fact that

\[
\int_\Omega |\nabla u_\lambda|^2 \, dx \to \int_\Omega |\nabla u|^2 \, dx = \int_\Omega u f \, dx,
\]

and

\[
\int_\Omega u_\lambda f_\lambda \, dx \to \int_\Omega u f \, dx,
\]

passing to the limit in the following energy equality

\[
\int_\Omega |\nabla u_\lambda|^2 \, dx + \lambda \int_\Omega au_\lambda^2 \, dx = \int_\Omega u_\lambda f_\lambda \, dx,
\]

we obtain (2.6). Next, let us now prove (2.7). Thus, since \( u_\lambda \) is a solution of \((P^*_\lambda)\) then, for every test function \( \psi \in C_c^\infty(\Omega) \), after integrating by parts in \( \Omega \) we arrive at

\[
\int_\Omega u_\lambda (-\Delta \psi) \, dx + \lambda \int_\Omega au_\lambda \psi \, dx = \int_\Omega f_\lambda \psi \, dx.
\]

Passing to the limit we obtain that \( \lambda au_\lambda \to f + \Delta u \) in \( \mathcal{D}'(\Omega) \). Now returning to (2.8), we can write, for every \( \psi \) satisfying \( \|\psi\|_{H^1(\Omega)} \leq 1 \),

\[
\left| \lambda \int_\Omega au_\lambda \psi \, dx \right| \leq \|f_\lambda\|_2 + \|\nabla u_\lambda\|_2 \leq C.
\]

Taking the supremum in \( \psi \) we get

\[
\|\lambda au_\lambda\|_{H^{-1}} \leq C.
\]

Therefore, \( \lambda au_\lambda \) is weakly-* sequentially compact in \( H^{-1} \) and we obtain the convergence by uniqueness of the limit in the distributional sense. \( \square \)
3  Existence and regularity of solutions for \((P_\lambda)\)

In order to define properly a solution for \((P_\lambda)\), we first recall the definition of the spaces \(L^p(0,T;X)\), with \(X\) a Banach space, which consist of all (strongly) measurable functions (see [13, Appendix E.5]) \(u : [0,T] \to X\) such that

\[
\|u\|_{L^p(0,T;X)} = \left( \int_0^T \|u(t)\|^p_X dt \right)^{1/p} < +\infty,
\]

for \(1 \leq p < +\infty\), and

\[
\|u\|_{L^\infty(0,T;X)} = \operatorname{ess} \sup_{t \in (0,T)} \|u(t)\|_X < +\infty.
\]

For simplicity we will sometimes use the following notation for \(p = 2\) and \(X = L^2(\Omega)\) :

\[
\| \cdot \|_2 \equiv \| \cdot \|_{L^2(0,T;L^2(\Omega))}.
\]

We will also use the notation \(u(x,t) = u(t)(x)\) for \((x,t) \in \Omega \times (0,T)\).

Next, we will denote by \(u'\) the derivative of \(u\) in the \(t\) variable, intended in the following weak sense: we say that \(u' = v\) if

\[
\int_0^T \varphi'(t)u(t)dt = -\int_0^T \varphi(t)v(t)dt
\]

for all scalar test functions \(\varphi \in C_0^\infty(0,T)\). The space \(H^1(0,T;X)\) consists of all functions \(u \in L^2(0,T;X)\) such that \(u' \in L^2(0,T;X)\).

We will often use the following remark.

**Remark 3.** By [13, Theorem 3 page 303], if \(u \in L^2(0,T;H^1_0(\Omega))\) and \(u' \in L^2(0,T;H^{-1}(\Omega))\), then \(u \in C([0,T],L^2(U))\) (after possibly being redefined on a set of measure zero). Moreover the mapping \(t \mapsto \|u(t)\|^2_{L^2(\Omega)}\) is absolutely continuous and \(\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} = 2\langle u'(t),u(t)\rangle_{L^2(\Omega)}\).

In this section we collect some useful information about the solution \(u_\lambda\) coming from the classical theory of parabolic problems that can be directly found in the literature.

Firstly, the existence and uniqueness of a weak solution \(u\) for the problem \((P_\lambda)\) follows from the standard Galerkin method see [13, Theorems 3 and 4, Section 7.1]. According to this theory, a weak solution means that:

\[
(P_\lambda) \left\{ \begin{aligned}
\int_{Q_T} (u'v + \nabla u \cdot \nabla v + \lambda a u v) &= \int_{Q_T} f_\lambda v \\
\text{for all } v \in L^2(0,T;H^1_0(\Omega)), \\
u(x,0) &= g_\lambda(x).
\end{aligned} \right.
\]

Remember that by Remark 3 above, such weak solution \(u\) is continuous in time so that the initial condition makes sense. In the rest of the paper, \((P_\lambda)\) will always refer to the above precise formulation of the problem that was first stated in the Introduction.

Next, thanks to [13, Theorem 5, Section 7.1], by considering \(\lambda a u\) as a right hand side term (in \(L^2(\Omega \times (0,T))\)), we have the following.
Lemma 1. Let $\lambda > 0$, $g_\lambda \in H^1_0(\Omega)$, $f_\lambda \in L^2(0, T; L^2(\Omega))$, and let $u_\lambda$ be the weak solution to $(P_\lambda)$. Then,

$$u_\lambda \in L^2(0, T; H^2(\Omega)) \cap L^\infty(0, T; H^1_0(\Omega)), \quad u_\lambda' \in L^2(0, T; L^2(\Omega)),$$

and $u_\lambda$ satisfies the following estimate:

$$\sup_{0 \leq t \leq T} \|u_\lambda(t)\|_{H^1_0(\Omega)} + \|u_\lambda\|_{L^2(0, T; H^2(\Omega))} + \|u_\lambda'\|_2 \leq C \left( \lambda \|au_\lambda\|_2 + \|f_\lambda\|_2 + \|g_\lambda\|_{H^1_0(\Omega)} \right),$$

(3.1)

where the constant $C$ depends only on $\Omega$ and $T$.

Remark 4. Notice that the bound (3.1) is not very useful when $\lambda \to +\infty$ since what we usually control is $\sqrt{\lambda} \|au\|_2$ (shown below in Lemma 3) but not $\lambda \|au\|_2$ thus the right hand side blows-up a priori.

Finally, we recall that assuming more regularity on $g_\lambda$, $f_\lambda$ and $a$ yields more regularity on $u_\lambda$ (but we shall not use it in this paper).

Lemma 2. Assume that

$$a', \ f'_\lambda \in L^2(0, T; L^2(\Omega)) \quad \text{and} \quad g_\lambda \in H^2(\Omega).$$

Let $u_\lambda$ be the weak solution of problem $(P_\lambda)$. Then

$$u_\lambda \in L^\infty(0, T; H^2(\Omega)), \quad u_\lambda' \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1_0(\Omega)), \quad u_\lambda'' \in L^2(0, T; H^{-1}(\Omega)),$$

and the quantity

$$\sup_{0 \leq t \leq T} \left( \|u_\lambda(t)\|_{H^2(\Omega)} + \|u_\lambda'(t)\|_{L^2(\Omega)} + \|u_\lambda''\|_{L^2(0, T; H^1_0(\Omega))} + \|u_\lambda''\|_{L^2(0, T; H^{-1}(\Omega))} \right)$$

is bounded by

$$C\left( \|f_\lambda\|_{H^1(0, T; L^2(\Omega))} + \|\lambda au_\lambda\|_{H^1(0, T; L^2(\Omega))} + \|g_\lambda\|_{H^2(\Omega)} \right),$$

where the constant $C$ depends only on $\Omega$ and $T$.

Remark 5. Notice again that this bound is not very useful when $\lambda \to +\infty$. However, if $\lambda$ is fixed, we observe at least that $u_\lambda''$ is in $L^2(0, T; H^{-1}(\Omega))$. 

9
4 Uniqueness of solution for \((P_\infty)\)

In this section we focus on the following problem that will arise as the limit of \((P_\lambda)\). Our notion of solution for the problem \(\partial_t u - \Delta u = f\) in \(O_a\) will precisely be the following:

\[
(P_\infty) \begin{cases} 
    u \in L^2(0,T;H^1_0(\Omega)), & u' \in L^2(0,T;L^2(\Omega)) \\
    u = 0 \text{ a.e. in } Q_T \setminus O_a \\
    \int_{Q_T}(u'v + \nabla u \cdot \nabla v) = \int_{Q_T}fv, & \text{for all } v \in L^2(0,T;H^1_0(\Omega)) \text{ s.t. } v = 0 \text{ a.e. in } Q_T \setminus O_a \\
    u(x,0) = g(x) \text{ in } \Omega.
\end{cases}
\]

As a byproduct of Section 5 we will prove the existence of a solution for the problem \((P_\infty)\), as a limit of solutions for \((P_\lambda)\). In this section, we prove the uniqueness which follows from a simple energy bound. Notice that a solution \(u\) to \((P_\infty)\) is continuous in time (see Remark 3) thus the initial condition \(u(x,0) = g(x)\) makes sense in \(L^2(\Omega)\).

**Proposition 3.** Any solution \(u\) of \((P_\infty)\) satisfies the following energy bound

\[
\frac{1}{4} \sup_{t \in (0,T)} \|u(t)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq \frac{1}{2} \|g\|^2_{L^2(\Omega)} + T\|f\|_2. \tag{4.1}
\]

Consequently, there exists at most one solution to problem \((P_\infty)\).

**Proof.** Let \(u\) be a solution to \((P_\infty)\), and \(s \in (0,T)\). Choosing \(v = u 1_{(0,s)}\) (where \(1_{(0,s)}\) is the characteristic function of \((0,s)\)) in the weak formulation of the problem, we deduce that

\[
\int_0^s \int_\Omega u'u \, dxdt + \int_0^s \int_\Omega |\nabla u|^2 \, dxdt = \int_0^s \int_\Omega fu \, dxdt. \tag{4.2}
\]

Now applying Remark 3 and using the fact that \(u \in L^2(0,T;H^1_0(\Omega))\) and \(u' \in L^2(0,T;L^2(\Omega))\) we obtain that \(t \mapsto \|u(t)\|^2_{L^2(\Omega)}\) is absolutely continuous, and for a.e. \(t\), there holds

\[
\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} = 2\langle u'(t), u(t)\rangle_{L^2(\Omega)}. \tag{4.3}
\]

Returning to (4.2) we get

\[
\frac{1}{2} \|u(s)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u(0)\|^2_{L^2(\Omega)} + \int_0^s \int_\Omega |\nabla u|^2 \, dxdt = \int_0^s \int_\Omega fu \, dxdt. \tag{4.4}
\]

By Young’s inequality we have

\[
|\int_0^s \int_\Omega fu \, dxdt| \leq \frac{\alpha}{2} \|f\|^2_{L^2(\Omega \times (0,s))} + \frac{1}{2\alpha} \|u\|^2_{L^2(\Omega \times (0,s))} \\
\leq \frac{\alpha}{2} \|f\|^2_{L^2(\Omega \times (0,s))} + \frac{T}{2\alpha} \sup_{t \in (0,T)} \|u(t)\|^2_{L^2(\Omega)}. \tag{4.4}
\]
Setting $\alpha = 2T$, estimating (5.2) by (4.4) and passing to the supremum in $s \in (0, T)$ finally gives
\[
\frac{1}{4} \sup_{t \in (0, T)} \|u(t)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_2 \leq \frac{1}{2} \|g\|^2_{L^2(\Omega)} + T\|f\|^2_2,
\]
as desired.

Now assume that $u_1$ and $u_2$ are two solutions of $(P_\infty)$, and set $w := u_1 - u_2$. Then $w$ is a solution of $(P_\infty)$ with $f = 0$ and $g = 0$. Therefore, applying (4.1) to $w$ automatically gives $w = 0$, which proves the uniqueness of the solution of $(P_\infty)$.

5 Convergence of $u_\lambda$

We now analyse the convergence of $u_\lambda$, which will follow from energy bounds for $u_\lambda$ and $u'_\lambda$. As already mentioned before, the standard energy bound for the solutions of $(P_\lambda)$ that is stated in Lemma 1, blows up a priori when $\lambda$ goes to $+\infty$. Our goal in the sequel is to get better estimates, uniform in $\lambda$. The price to pay is the condition $\partial_t a \leq 0$ which implies that $O_a$ is nondecreasing in time (for the set inclusion).

5.1 First energy bound

Lemma 3. Assume that $g_\lambda \in H_0^1(\Omega)$ and $f_\lambda \in L^2(0, T; L^2(\Omega))$, and let $u_\lambda$ be the weak solution of problem $(P_\lambda)$. Then,
\[
\frac{1}{4} \sup_{t \in (0, T)} \|u_\lambda(t)\|^2_{L^2(\Omega)} + \|\nabla u_\lambda\|^2_2 + \lambda \int_0^T \int_{\Omega} a u_\lambda^2 \, dx dt \leq \|g_\lambda\|^2_{L^2(\Omega)} + T\|f_\lambda\|^2_2. \tag{5.1}
\]

Proof. Let $u_\lambda$ be a solution of $(P_\lambda)$ and $s \in (0, T)$. Testing with $v = u \mathbf{1}_{[0, s]}$ in the weak formulation of $(P_\lambda)$, we obtain
\[
\frac{1}{2} \|u_\lambda(s)\|^2_{L^2(\Omega)} - \frac{1}{2} \|u_\lambda(0)\|^2_{L^2(\Omega)} + \int_0^s \int_{\Omega} |\nabla u_\lambda|^2 \, dx dt + \lambda \int_0^s \int_{\Omega} a u_\lambda^2 \, dx dt = \int_0^s \int_{\Omega} f_\lambda u_\lambda \, dx dt. \tag{5.2}
\]

Arguing as in the proof of Proposition 3, we obtain for any $\alpha > 0$,
\[
\frac{1}{2} \|u_\lambda(s)\|^2_{L^2(\Omega)} + \|\nabla u\|^2_2 + \lambda \int_0^s \int_{\Omega} a u_\lambda^2 \, dx dt \leq \|g_\lambda\|^2_{L^2(\Omega)} + |\langle f_\lambda, u_\lambda \rangle_{L^2(\Omega \times (0, s))}| \\
\leq \|g_\lambda\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|f_\lambda\|^2_2 + \frac{1}{2\alpha} \|u_\lambda\|^2_{L^2(\Omega \times (0, s))}, \\
\leq \|g_\lambda\|^2_{L^2(\Omega)} + \frac{\alpha}{2} \|f_\lambda\|^2_2 + \sup_{t \in (0, T)} \frac{T}{2\alpha} \|u_\lambda(t)\|^2_{L^2(\Omega)}.
\]
Inequality (5.1) follows by taking the supremum in $s \in (0, T)$ and fixing $\alpha := 2T$. \qed
5.2 Second energy bound

We now derive a uniform bound on $\|u_\lambda'(t)\|_2$. To this end, we will assume a time-monotonicity condition on $a$.

**Definition 1** (Assumptions (A)). We say that Assumptions (A) hold if $a : Q_T \to \mathbb{R}^+$ is Lipschitz and

$$\partial_t a(x,t) \leq 0 \text{ for a.e. } (x,t) \in Q_T.$$  \hspace{1cm} (5.3)

**Lemma 4.** We suppose that Assumptions (A) hold. Then, the solution $u_\lambda$ of $(P_\lambda)$ satisfies the estimate:

$$\int_0^T \int_\Omega (u_\lambda')^2 \, dxdt + \sup_{s \in (0,T)} \left( \int_\Omega |\nabla u_\lambda(s)|^2 \, dx \right) \leq \int_0^T \int_\Omega f_\lambda^2 \, dxdt + \int_\Omega |\nabla u_\lambda(0)|^2 \, dx + \lambda \int_\Omega a(0)g_\lambda^2 \, dx. \hspace{1cm} (5.4)$$

**Proof.** Thanks to Lemma 1, we know that $u_\lambda' \in L^2(0,T; H_0^1(\Omega))$. Consequently, for every $s \in (0,T)$, the function $v := u_\lambda' 1_{(0,s)}$ is an admissible test function in the weak formulation of $(P_\lambda)$. Hence, we obtain the identity

$$\int_0^s \int_\Omega (u_\lambda')^2 \, dxdt + \int_0^s \int_\Omega \nabla u_\lambda \cdot \nabla v \, dxdt + \lambda \int_0^s \int_\Omega au_\lambda v \, dxdt = \int_0^s \int_\Omega f_\lambda v \, dxdt,$$

or written differently (applying Remark 3),

$$\int_0^s \int_\Omega |u_\lambda'|^2 \, dxdt + \int_0^s \left( \frac{1}{2} \int_\Omega |\nabla u_\lambda|^2 \, dx \right) \, dt + \lambda \int_0^s \left[ \left( \frac{1}{2} \int_\Omega a u_\lambda^2 \, dx \right)' - \frac{1}{2} \int_\Omega a u_\lambda^2 \, dx \right] \, dt = \int_0^s \int_\Omega f_\lambda u_\lambda' \, dxdt.$$

This yields

$$\int_0^s \int_\Omega (u_\lambda')^2 \, dxdt + \frac{1}{2} \int_\Omega |\nabla u_\lambda(s)|^2 \, dx + \lambda \int_\Omega a(s)u_\lambda(s)^2 \, dx \leq \frac{\lambda}{2} \int_0^s \int_\Omega a'u_\lambda^2 \, dxdt$$

$$= \int_0^s \int_\Omega f_\lambda u_\lambda' \, dxdt + \frac{1}{2} \int_\Omega |\nabla u_\lambda(0)|^2 \, dx + \lambda \int_\Omega a(0)u_\lambda(0)^2 \, dx.$$

By Young’s inequality,

$$\int_0^s \int_\Omega f_\lambda u_\lambda' \, dxdt \leq \frac{1}{2} \int_0^s \int_\Omega f_\lambda^2 \, dxdt + \frac{1}{2} \int_0^s \int_\Omega (u_\lambda')^2 \, dxdt,$$

so that we obtain

$$\int_0^s \int_\Omega (u_\lambda')^2 \, dxdt + \frac{1}{2} \int_\Omega |\nabla u_\lambda(s)|^2 \, dx + \lambda \int_\Omega a(s)u_\lambda(s)^2 \, dx \leq \frac{\lambda}{2} \int_0^s \int_\Omega a'u_\lambda^2 \, dxdt$$

$$\leq \int_0^s \int_\Omega |f_\lambda|^2 \, dxdt + \int_\Omega |\nabla u_\lambda(0)|^2 \, dx + \lambda \int_\Omega a(0)u_\lambda(0)^2 \, dx.$$

Finally, using Assumption (A), the initial condition on $u_\lambda(0)$ and passing to the supremum in $s$, we conclude that estimate (5.4) holds. \hfill \Box
5.3 Weak convergence of solutions

Using the previous energy estimates, we first establish the weak convergence of \( u_\lambda \) to the solution \( u \) of Problem \((P_\infty)\), under assumption (A), and supposing certain bounds on the right hand side \( f_\lambda \) and on the initial data \( g_\lambda \).

**Proposition 4.** Assume that \( a \) satisfies assumption (A). Let \((f_\lambda)\) be a bounded sequence in \( L^2(Q_T) \) and \((g_\lambda)\) be a bounded sequence in \( H^1_0(\Omega) \), satisfying

\[
\sup_{\lambda} \left( \lambda \int_{\Omega} a(0) g^2_\lambda \, dx \right) < \infty. \tag{5.5}
\]

Up to extracting subsequences, we can assume that \( f_\lambda \) converges weakly to a function \( f \) in \( L^2(Q_T) \), and \( g_\lambda \) converges weakly to a function \( g \in H^1_0(\Omega) \).

Let \( u_\lambda \) be the solution of \((P_\lambda)\). Then \( u_\lambda \) converges weakly in \( L^2(Q_T) \) to the unique solution \( u \) of problem \((P_\infty)\).

**Proof.** We know by Lemma 3 that \( u_\lambda \) is uniformly bounded in \( L^2(0,T;H^1_0(\Omega)) \), thus converges weakly (up to extracting a subsequence) to some function \( u \in L^2(0,T;H^1_0(\Omega)) \). Under assumption (A), we also know, thanks to Lemma 4, that

\[
\|u_\lambda\|_{L^2(Q_T)} \leq C,
\]

so that, \( u_\lambda' \) converges also weakly in \( L^2(Q_T) \) (up to extract a further subsequence) to some limit \( w \in L^2(Q_T) \), which must be equal to \( u' \) by uniqueness of the limit in \( D'(Q_T) \). This shows that \( u' \in L^2(Q_T) \).

Next, from (5.1) we know that

\[
\sup_{\lambda} \left( \lambda \int_{0}^{T} \int_{\Omega} a u_\lambda^2 \, dx \, dt \right) \leq C,
\]

which implies that \( u \) must be equal to zero a.e. on any set of the form \( \{a > \varepsilon\} \), with \( \varepsilon > 0 \). By considering the union for \( n \in \mathbb{N}^* \) of those sets with \( \varepsilon = 1/n \), we obtain that \( u = 0 \) a.e. on \( Q_T \setminus O_a \).

Now let us check that \( u \) satisfies the equation in the weak sense. Let \( v \) be any test function in \( L^2(0,T;H^1_0(\Omega)) \) such that \( v = 0 \) a.e. in \( Q_T \setminus O_a \). Then \( au_\lambda v = 0 \) a.e. in \( Q_T \), and using the fact that \( u_\lambda \) is a solution of \((P_\lambda)\), we can write

\[
\langle u_\lambda', v \rangle_{L^2(Q_T)} + \langle \nabla u_\lambda, \nabla v \rangle_{L^2(Q_T)} = \langle f_\lambda, v \rangle_{L^2(Q_T)}.
\]

Thus passing to the (weak) limit in \( u_\lambda, u_\lambda' \) and \( f_\lambda \) we get

\[
\langle u', v \rangle_{L^2(Q_T)} + \langle \nabla u, \nabla v \rangle_{L^2(Q_T)} = \langle f, v \rangle_{L^2(Q_T)}.
\]

To conclude that \( u \) is a solution of \((P_\infty)\) it remains to prove that \( u(x,0) = g(x) \) for a.e. \( x \in \Omega \). For this purpose, we let \( v \in C^1([0,T],H^1_0(\Omega)) \) be any function satisfying \( v(T) = 0 \).
Testing the equation with this \( v \), using that \( u_\lambda(0) = g_\lambda \) and integrating by parts with respect to \( t \) we obtain

\[
- \langle g_\lambda, v(0) \rangle_{L^2(\Omega)} - \int_0^T \langle u_\lambda', v \rangle_{L^2(\Omega)} + \int_0^T \langle \nabla u_\lambda, \nabla v \rangle_{L^2(\Omega)} = \int_0^T \langle f_\lambda, v \rangle.
\]

Passing to the limit in \( \lambda \) and using the weak convergence of \( g_\lambda \) to \( g \), we get

\[
- \langle g, v(0) \rangle_{L^2(\Omega)} - \int_0^T \langle u, v' \rangle_{L^2(\Omega)} + \int_0^T \langle \nabla u, \nabla v \rangle_{L^2(\Omega)} = \int_0^T \langle f, v \rangle_{L^2(\Omega)}.
\]

Integrating back again by parts on \( u \) yields

\[
\langle g, v(0) \rangle_{L^2(\Omega)} = \langle u(0), v(0) \rangle_{L^2(\Omega)},
\]

and since \( v(0) \) is arbitrary, we deduce that \( u(0) = g \) in \( L^2(\Omega) \).

Finally, the convergence of \( u_\lambda \) to \( u \) holds a priori up to a subsequence, but by uniqueness of the solution for the problem \((P_\infty)\) (see Proposition 3), the convergence holds for the whole sequence.

**Corollary 1.** Let \( O_a \subset Q_T \) be open and increasing in time (in the sense of Remark 1), and let \( f \in L^2(\Omega \times (0, T)) \) and \( g \in H^1_0(\Omega) \). Then there exists a (unique) solution for \((P_\infty)\).

**Proof.** It suffices to apply Theorem 3 with, for instance \( a(x,t) := \text{dist}((x,t), O_a) \), \( f_\lambda = f \) and \( g_\lambda = g \).

**Remark 6.** (Convergence in \( \mathcal{D}'(\Omega \times (0, T)) \)). Under assumptions (A), letting \( u \) being the weak limit of \( u_\lambda \) in \( L^2(Q_T) \), we already know that

\[
u = 0 \quad \text{a.e. in } Q_T \setminus O_a.
\]

Then

\[
f_\lambda + \Delta u_\lambda - u_\lambda' \longrightarrow f + \Delta u - u' \quad \text{in } \mathcal{D}'(\Omega \times (0, T)),
\]

which implies that

\[
\lambda au_\lambda \longrightarrow h \quad \text{in } \mathcal{D}'(\Omega \times (0, T)),
\]

for some distribution \( h = f + \Delta u - u' \in \mathcal{D}'(\Omega \times (0, T)) \), supported in \( O_a^c \). Actually, since \( u = 0 \) in \( O_a^c \), we have

\[
\Delta u = 0 \quad \text{and} \quad u' = 0 \quad \text{in } \mathcal{D}'(\overline{O_a^c}).
\]

This means that

\[
h = 0 \quad \text{in } \mathcal{D}'(O_a) \quad \text{and} \quad h = f \quad \text{in } \mathcal{D}'(\overline{O_a^c}).
\]

Notice that, a priori, \( h \) could have a singular part supported on \( \partial O_a \). We finally deduce that

\[
\lambda au_\lambda \xrightarrow{\lambda \to +\infty} f \mathbf{1}_{O_a^c} + (\Delta u - u')|_{\partial O_a} \quad \text{in } \mathcal{D}'(\Omega \times (0, T)).
\]
5.4 Strong convergence in $L^2(0,T;H^1(\Omega))$

We now go further using the same argument as for the stationary problem, and prove a stronger convergence which is one of our main results.

**Theorem 3.** Under the same hypotheses as in Proposition 4, denote by $u_\lambda$ the solution of $(P_\lambda)$. Then, $u_\lambda$ converges strongly in $L^2(0,T;H^1_0(\Omega))$ to the solution $u$ of problem $(P_\infty)$.

**Proof.** We have the bound

$$\|u_\lambda\|_{L^2(0,T;H^1(\Omega))} \leq C,$$

and we already know (by Proposition 4) that $u_\lambda$ converges weakly in $L^2(0,T;H^1(\Omega))$ to $u$, the unique solution of problem $(P_\infty)$.

Moreover, by the lower semicontinuity of the norm with respect to the weak convergence, there holds

$$\|u\|_{L^2(0,T;H^1(\Omega))} \leq \liminf \|u_\lambda\|_{L^2(0,T;H^1(\Omega))}.$$

Hence, to prove the strong convergence we only need to prove the reverse inequality, with a limsup. For this purpose we use the fact that $u(t)$ is a competitor for $u_\lambda(t)$ in the minimization problem solved by $u_\lambda$ at $t$ fixed. Indeed, for a.e. $t$ fixed, $u_\lambda$ solves

$$-\Delta u_\lambda + \lambda au_\lambda = f_\lambda - u'_\lambda,$$

thus, $u_\lambda$ is a minimizer in $H^1_0(\Omega)$ for the energy

$$v \mapsto E_\lambda(v) - 2 \int_\Omega v(f_\lambda - u'_\lambda),$$

where $E_\lambda$ is defined by (2.3). Furthermore, due to the bound (5.4) obtained in Lemma 4, since $f_\lambda$ is bounded in $L^2(Q_T)$ and $g_\lambda$ is bounded in $L^2(\Omega)$ and satisfies (5.5), we know that $u'_\lambda$ is bounded in $L^2(Q_T)$, and

$$u'_\lambda \to u' \quad \text{in weak } \quad L^2(Q_T).$$

We also know that, up to a subsequence, $u_\lambda \to u$ strongly in $L^2(Q_T)$ (because it is bounded in $H^1(Q_T)$).

Now, using that $u$ is a competitor for $u_\lambda$ (for a.e. $t$ fixed), we obtain

$$\int_\Omega |\nabla u_\lambda|^2 dx - 2 \int_\Omega u_\lambda(f_\lambda - u'_\lambda) \leq E_\lambda(u_\lambda) - 2 \int_\Omega u_\lambda(f_\lambda - u'_\lambda) \leq E_\lambda(u) - 2 \int_\Omega u(f_\lambda - u'_\lambda) \leq \int_\Omega |\nabla u|^2 dx - 2 \int_\Omega u(f_\lambda - u'_\lambda).$$

Integrating in $t \in [0,T]$ and passing to the limsup in $\lambda$, we get the desired inequality, which achieves the proof. \qed
6 Simon’s exponential estimate

6.1 The stationary case

Following a similar argument to [19, Theorem 4.1] we ascertain some strong convergence far from the set \( \Omega_a := \text{Int}(K_a) \), where \( K_a \) is defined by \( K_a := \{ x \in \Omega; a(x) = 0 \} \).

**Lemma 5.** Let \( a : \overline{\Omega} \to \mathbb{R}^+ \) be a continuous non-negative potential and \( u_\lambda \) be the unique weak solution in \( H_0^1(\Omega) \) of \( -\Delta u_\lambda + \lambda a u_\lambda = f_\lambda \) in \( \Omega \). Introduce \( K_a := \{ x \in \overline{\Omega}; a(x) = 0 \} \) and assume that \( \Omega_a := \text{Int}\{a(x) = 0\} \) is non empty (hypothesis (2.2)). Let \( \varepsilon > 0 \) be fixed, and define

\[
\Omega_\varepsilon := \{ x \in \Omega; \text{dist}(x, \Omega_a) > \varepsilon \} \quad \text{and} \quad \delta := \min_{x \in \Omega_\varepsilon} a(x) > 0.
\]

Then, there exists a constant \( C > 0 \) such that for all \( \lambda > 0 \) and for all Lipschitz function \( \eta : \Omega \to \mathbb{R} \) that is equal to 1 in \( \Omega_{2\varepsilon} \) and to 0 outside \( \Omega_\varepsilon \), we have

\[
\int_{\Omega_\varepsilon} e^{\frac{\lambda \varepsilon}{2}} \frac{1}{\text{dist}(x, \Omega_a)} \eta^2 u_\lambda(x) \left( \frac{\lambda \delta}{2} u_\lambda - f_\lambda \right) dx \leq C. \tag{6.1}
\]

**Proof.** To lighten the notation, in this proof we will denote by \( \langle \cdot, \cdot \rangle \) the scalar product in \( L^2(\Omega) \).

Let \( \varepsilon > 0 \) be fixed. For any function \( \psi \) supported in \( \Omega_\varepsilon \) and for any Lipschitz function \( \rho : \Omega \to \mathbb{R} \) satisfying \( |\nabla \rho|^2 \leq \delta/2 \), a direct calculation shows the following identities

\[
\Delta(e^{-\sqrt{\lambda} \rho} \psi) = -\sqrt{\lambda} e^{-\sqrt{\lambda} \rho} \Delta \rho + \lambda |\nabla \rho|^2 e^{-\sqrt{\lambda} \rho},
\]

\[
\Delta(e^{-\sqrt{\lambda} \rho} \psi) = \psi \Delta(e^{-\sqrt{\lambda} \rho} \psi) + 2 \nabla(e^{-\sqrt{\lambda} \rho} \psi) \cdot \nabla \psi,
\]

from which we obtain that

\[
\langle e^{\sqrt{\lambda} \rho} \psi, (-\Delta + \lambda a) e^{-\sqrt{\lambda} \rho} \psi \rangle = \langle \psi, (\sqrt{\lambda} \Delta \rho - \lambda |\nabla \rho|^2 + 2 \sqrt{\lambda} \nabla \rho \cdot \nabla - \Delta + \lambda a) \psi \rangle.
\]

Then, expanding \( \Delta(\rho \psi) \) and rearranging terms as follows:

\[
\Delta(\rho \psi) = \Delta \rho \psi + 2 \nabla \rho \nabla \psi + \Delta \psi \rho = \Delta \rho \psi + \nabla \rho \nabla \psi + \nabla(\rho \nabla \psi),
\]

it is not difficult to see that

\[
\langle \psi, (\Delta \rho + 2 \nabla \rho \cdot \nabla) \psi \rangle = \langle \psi, \Delta(\rho \psi) \rangle - \langle \psi, \rho \Delta \psi \rangle = 0,
\]

since \( \psi \) is compactly supported in \( \Omega \). Therefore, by positivity of \( -\Delta \) and using the assumption \( |\nabla \rho|^2 \leq \delta/2 \) we get

\[
\langle e^{\sqrt{\lambda} \rho} \psi, (-\Delta + \lambda a) e^{-\sqrt{\lambda} \rho} \psi \rangle \geq \langle \psi, [\lambda (a - |\nabla \rho|^2)] \psi \rangle \geq \frac{\lambda \delta}{2} \langle \psi, \psi \rangle = \frac{\lambda \delta}{2} \| \psi \|^2_2. \tag{6.2}
\]
Next, we apply (6.2) with the choice $\psi = e^{\sqrt{\lambda} \rho} \eta u_\lambda(x)$, where $\eta$ is a function equal to 1 in $\Omega_{2\varepsilon}$ and 0 outside $\Omega_{\varepsilon}$. Since by construction $\|u_\lambda\|_2 + \|\nabla u_\lambda\|_2 \leq C \|f_\lambda\|_2$, (6.2) implies

$$\frac{\lambda \delta}{2} \int_{\Omega_{2\varepsilon}} e^{2\sqrt{\lambda} \rho(x)} \eta^2 u_\lambda^2 dx \leq \frac{\lambda \delta}{2} \|\psi\|_2^2 \leq \int_{\Omega} e^{2\sqrt{\lambda} \rho} \eta u_\lambda (-\Delta + \lambda a)(\eta u_\lambda) dx$$

$$\leq \int_{\Omega_{\varepsilon} \setminus \Omega_{2\varepsilon}} e^{2\sqrt{\lambda} \rho} u_\lambda (-\Delta \eta - 2\nabla \eta \cdot \nabla u_\lambda) dx + \int_{\Omega} e^{2\sqrt{\lambda} \rho} \eta^2 u_\lambda (-\Delta u_\lambda + \lambda a u_\lambda) dx$$

$$\leq \int_{\Omega_{\varepsilon} \setminus \Omega_{2\varepsilon}} e^{2\sqrt{\lambda} \rho} u_\lambda (-\Delta \eta - 2\nabla \eta \cdot \nabla u_\lambda) dx + \int_{\Omega} e^{2\sqrt{\lambda} \rho} \eta^2 u_\lambda f_\lambda dx$$

$$\leq C(\|f_\lambda\|_2) e^{2\sqrt{\lambda} M} + \int_{\Omega} e^{2\sqrt{\lambda} \rho_\varepsilon} \eta^2 u_\lambda f_\lambda dx,$$

(6.3)

where

$$M := \sup_{x \in \Omega \setminus \Omega_{2\varepsilon}} \rho(x),$$

and the constant $C$ in (6.3) depends on the derivatives of $\eta$ and $\varepsilon$. Now we take the particular choice

$$\rho(x) := \sqrt{\frac{\delta}{2}} \mathbf{1}_{\Omega_{2\varepsilon}} \text{dist}(x, \Omega_{2\varepsilon}^c),$$

which satisfies all our needed assumptions (i.e. $\rho$ is Lipschitz with $|\nabla \rho|^2 \leq \delta/2$ and supported in $\Omega_{\varepsilon}$). In this case $M = 0$ thus (6.3) simply implies

$$\frac{\lambda \delta}{2} \int_{\Omega_{2\varepsilon}} e^{2\sqrt{\lambda} \rho(x)} \eta^2 u_\lambda^2 dx \leq C + \int_{\Omega} e^{2\sqrt{\lambda} \rho_\varepsilon} \eta^2 u_\lambda f_\lambda dx,$$

or differently,

$$\int_{\Omega_{\varepsilon}} e^{2\sqrt{\lambda} \rho_\varepsilon} u_\lambda(x) \left( \frac{\lambda \delta}{2} u_\lambda - f_\lambda \right) dx \leq C,$$

which ends the proof.

**Remark 7.** The previous lemma can be used for instance in the following two particular cases: first in the particular case when $f = 0$ in $\Omega \setminus \Omega_a$. Thus, we get the useful rate of convergence of $u_\lambda \to 0$ as $\lambda \to 0$ far from $\Omega_a$:

$$\int_{\Omega_{2\varepsilon}} \lambda e^{2\sqrt{\lambda} \rho} \eta^2 u_\lambda^2 dx \leq C.$$ 

This is much better compared to the usual and simple energy bound:

$$\lambda \int_{\Omega} |u_\lambda|^2 \leq C.$$
Another application is when \( u_\lambda \) is an eigenfunction (this is actually the original framework of Simon [19]), i.e., when \( f_\lambda = \sigma(\lambda) u_\lambda \) and with \( \sigma(\lambda) \) standing for the first eigenvalue associated with \( u_\lambda \). In this case, since we are assuming that the potential \( a \) might vanish in a subdomain, we have that \( \sigma(\lambda) \) is bounded (cf. [2] for further details). Consequently, thanks to this bound for \( \lambda \) large enough \( \frac{\lambda^2}{2} - \sigma(\lambda) \geq 1 \) which implies

\[
\int_{\Omega_{2\epsilon}} e^{2\sqrt{\lambda^2 + \text{dist}(x,\Omega_{\lambda}^c)}} u_\lambda^2 dx \leq C.
\]

### 6.2 The parabolic case

We now extend the previous decay estimate to the parabolic problem.

**Lemma 6.** Let \( a : Q_T \rightarrow \mathbb{R}^+ \) be a continuous non-negative potential such that \( O_a \) is non empty, \( f_\lambda \in L^2(Q_T) \), \( g_\lambda \in H^1_0(\overline{Q}_a \cap \{ t = 0 \}) \) and let \( u_\lambda \) be the solution of (\( P_\lambda \)).

For every \( \epsilon > 0 \), we define

\[
A_\epsilon := \{(x,t) \in \overline{Q}_T; \text{dist}(x,t,O_a) > \epsilon \} \quad \text{and} \quad \delta := \min_{(x,t) \in A_\epsilon} a(x,t) > 0.
\]  

Then, for any \( \lambda \geq 4 \), and for any Lipschitz function \( \eta : Q_T \rightarrow \mathbb{R} \) equal to 1 in \( A_{2\epsilon} \) and 0 outside \( A_\epsilon \), there exists a constant \( C > 0 \) such that

\[
\int_{A_\epsilon} e^{2\sqrt{\lambda^2 + \text{dist}(x,A_{\epsilon})}} \eta^2 u_\lambda(x) \left( \frac{\lambda \delta}{2} u_\lambda - f_\lambda \right) dx \leq C.
\]

**Proof.** Let \( \epsilon > 0 \) be fixed. We consider any function \( \psi \) supported in \( A_\epsilon \) and any Lipschitz function \( \rho : Q_T \rightarrow \mathbb{R} \) satisfying \( |\partial_t \rho| + |\nabla \rho| \leq \delta/2 \). Then we notice that for any fixed \( t \), the functions \( x \mapsto \rho(x,t), \ x \mapsto \psi(x,t) \) and \( x \mapsto a(x,t) \) satisfy all the conditions required to prove the key estimate (6.2) in the stationary case. Thus, for any fixed \( t \), there holds

\[
\langle e^{\sqrt{\lambda^2}} \psi, (-\Delta + \lambda a) e^{\sqrt{\lambda^2}} \psi \rangle_{L^2(Q_T)} \geq \frac{\lambda \delta}{2} \| \psi \|^2_{L^2(Q_T)}.
\]

Integrating this inequality over \( t \in (0,T) \) yields

\[
\langle e^{\sqrt{\lambda^2}} \psi, (-\Delta + \lambda a) e^{\sqrt{\lambda^2}} \psi \rangle_{L^2(Q_T)} \geq \frac{\lambda \delta}{2} \| \psi \|^2_{L^2(Q_T)}.
\]

Next we also compute

\[
\partial_t (e^{-\sqrt{\lambda^2}} \psi) = -\sqrt{\lambda} e^{-\sqrt{\lambda^2}} \psi \partial_t \rho + e^{-\sqrt{\lambda^2}} \partial_t \psi,
\]

so that,

\[
\langle e^{\sqrt{\lambda^2}} \psi, \partial_t (e^{-\sqrt{\lambda^2}} \psi) \rangle_{L^2(Q_T)} = -\sqrt{\lambda} \langle \psi, \psi \partial_t \rho \rangle_{L^2(Q_T)} + \langle \psi, \partial_t \psi \rangle_{L^2(Q_T)}
\]

\[
= -\sqrt{\lambda} \langle \psi, \psi \partial_t \rho \rangle_{L^2(Q_T)} + \frac{1}{2} \| \psi(T) \|^2_{L^2(Q_T)} - \frac{1}{2} \| \psi(0) \|^2_{L^2(Q_T)}
\]

\[
\geq -\sqrt{\lambda \delta} \| \psi \|^2_{L^2(Q_T)} - \frac{1}{2} \| \psi(0) \|^2_{L^2(Q_T)}.
\]
Gathering the previous estimates, we obtain

\[
\langle e^{\sqrt{\lambda} \rho} \phi, (\partial_t - \Delta + \lambda a) e^{-\sqrt{\lambda} \rho} \phi \rangle_{L^2(Q_T)} \geq \frac{(\lambda - \sqrt{\lambda}) \delta}{2} \| \phi \|_2^2 - \frac{1}{2} \| \phi(0) \|_{L^2(\Omega)}^2.
\] (6.5)

Next, we apply (6.5) with the choice \( \psi = e^{\sqrt{\lambda} \rho} \eta u_\lambda(x) \), where \( \eta \) is a function equal to 1 in \( A_{2\varepsilon} \) and 0 outside \( A_\varepsilon \). We assume that \( \lambda \geq 4 \) so that \( \lambda - \sqrt{\lambda} \geq \lambda/2 \). Due to the assumptions, \( g \in H^1_0(\overline{O_a} \cap \{ t = 0 \}) \) and, then, \( \| \psi(0) \|_{L^2(\Omega)} = 0 \). Thus, (6.5) implies

\[
\frac{\lambda \delta}{4} \int_{A_\varepsilon} e^{2 \sqrt{\lambda} \rho} \eta^2 u_\lambda^2 dx = \frac{\lambda \delta}{4} \| \psi \|_2^2 \leq \int_{\Omega \times (0,T)} e^{2 \sqrt{\lambda} \rho} \eta u_\lambda (\partial_t - \Delta + \lambda a) (\eta u_\lambda) dx
\]
\[
\leq \int_{A_\varepsilon \setminus A_{2\varepsilon}} e^{2 \sqrt{\lambda} \rho} u_\lambda (\partial_t \eta - u_\lambda \Delta \eta - 2 \nabla \eta \cdot \nabla u_\lambda) dx
\]
\[
+ \int_{\Omega \times (0,T)} e^{2 \sqrt{\lambda} \rho} \eta^2 u_\lambda (\partial_t u_\lambda - \Delta u_\lambda + \lambda u_\lambda) dx
\]
\[
= \int_{A_\varepsilon \setminus A_{2\varepsilon}} e^{2 \sqrt{\lambda} \rho} u_\lambda (\partial_t \eta - u_\lambda \Delta \eta - 2 \nabla \eta \cdot \nabla u_\lambda) dx
\]
\[
+ \int_{\Omega \times (0,T)} e^{2 \sqrt{\lambda} \rho} \eta^2 u_\lambda f_\lambda dx
\]
\[
\leq C(\| u_\lambda \|_2^2) e^{2 \sqrt{\lambda} M} + \int_{A_\varepsilon} e^{2 \sqrt{\lambda} \rho} \eta^2 u_\lambda f_\lambda dx,
\] (6.6)

where

\[ M := \sup_{x \in Q_T \setminus \Omega_{2\varepsilon}} \rho(x), \]

and the constant \( C \) in (6.6) depends on the derivatives of \( \eta \) and on \( \varepsilon \). Now we take the particular choice

\[ \rho(x, t) := \sqrt{\frac{\delta}{2}} \mathbf{1}_{A_2 \varepsilon}(x, A_{2\varepsilon}), \]

which satisfies all our needed assumptions (i.e. \( \rho \) is Lipschitz with \( |\partial_t \rho| + |\nabla \rho| \leq \delta/2 \) and supported in \( A_\varepsilon \)). In this case, \( M = 0 \) so that that (6.6) reduces to

\[
\frac{\lambda \delta}{4} \int_{A_\varepsilon} e^{2 \sqrt{\lambda} \rho} \eta^2 u_\lambda^2 dx \leq C + \int_{A_\varepsilon} e^{2 \sqrt{\lambda} \rho} \eta^2 u_\lambda f_\lambda dx,
\]

and hence

\[
\int_{A_\varepsilon} e^{2 \sqrt{\lambda} \rho} \eta^2 u_\lambda(x) \left( \frac{\lambda \delta}{4} u_\lambda - f_\lambda \right) dx \leq C.
\]

We end this section by noticing that Theorem 2 follows directly from Lemma 6.

**Corollary 2.** In the particular case when \( f = 0 \) in \( Q_T \setminus O_a \) we get the useful rate of convergence of \( u_\lambda \to 0 \) as \( \lambda \to 0 \) far from \( O_a \):

\[
\lambda e^{4e \sqrt{\lambda}^2} \int_{A_{2\varepsilon}} u_\lambda^2 dx \leq C.
\]
References


