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Growth patterns of US professional services firms.

Edouard Ribes^{*†}

August 17, 2018

Abstract

This paper investigates the growth of professional services firms [PSFs] in the US. It first demonstrates based on a 20 year longitudinal dataset, that Gibrat's law of firm growth doesn't hold for PSFs as the growth rate of their establishment is age and size dependent. While this behavior is shared across the US economy and the notable reference point of US manufacturing firms, US PSFs exhibit a specificity in the development of their firms that exhibit persistence in growth and in their development pattern that relies on the creation of small establishments (≤ 50 employees) in a dynamic of market expansion.

Finally, this paper drills down on the growth pattern of the 25% biggest PSFs for which the explanation of geographical expansion can be challenged. For those mature firms, growth is assumed to be spurred by investments in information and communication technology (ICT) as well as research and development (R&D) in order to improve labor productivity. The model is calibrated to US cross sectional data and shows that under this hypothesis, an arms race exists between the top firms.

Keywords and phrases. Firm growth Professional Services Firms Firm performance Mean Field Games

Economic Classification [JEL]. E17; L16; L25; L84; D23.

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1 Introduction

This work investigates the growth of US professional services firms. This is addressed both theoretically and empirically using both longitudinal and cross sectional datasets made available by the US Census bureau. The analysis of the PSFs sector shows that firm age and size impact PSFs growth in a fashion similar to other firms in the US economy, notably in the US manufacturing. However it shows that PSFs, to the difference of US manufacturing firms, develops themselves by creating new establishment, which reveals a question of market access and business development. If this may hold for the main body of PSFs, those patterns can be challenged for large PSFs. I therefore build upon the Hopenhayn framework to show that assuming that PSFs invests in programs to increase their labor productivity can explain the observed growth patterns.

This work contributes to two main strands of the related literature. First this work builds upon the studies that stress the effects of size and age on firms growth. Since the pioneering work of Gibrat [20] that stated that the relationship between firm growth and its initial size can be described by a constant growth rate and a random element that is normally distributed, it has been empirically shown that firm growth is industry specific and exhibits power law patterns ([28], [36]). However size is not the only relevant variable to describe firms growth dynamics. Starting with [17], authors have started to complement the classical firms growth dependency in size by observing a dependency in age. The relationship between firm development and its age has spurred an increased interest over the last years ([9],[23]). This has motivated numerous applications on firms entry and exit rates [1],[38], firms internationalization strategies [21] and resilience to extreme random events [12]. However most of the growth study focuses manufacturing and little has been done on the service industry [29]. To my knowledge there has been no study on professional services firm industry (see [16] for a definition), which therefore constitutes one of the main contribution of this paper.

Second this study goes beyond the descriptive mechanisms related to firm growth to explore its drivers. The initial explanations in terms productivity and technology shocks of [24] gave birth to numerous discussion between product and firm life cycle [10], [3]. This has caught up with the evolutionary literature focusing on age as an important determinant by providing some evidence of the learning behavior [2] of firms towards their customer preferences [23]. Growing beyond considerations on demand, the field has had a special interest in financial considerations associated to growth and transformation. The main areas of interest have lately revolved around convex entry costs [13] and capital management [11], [8]. With respect to the firm growth driver literature, this paper provides a discussion around the Hopenhayn framework by transforming the assumption of random exogenous shocks in productivity in an assumption of deterministic endogenous productivity investments. This is shown to depict properly the growth patterns of the largest US PSFs and can be used to infer that large US PSFs face an arms race with respect to their investments.

From a technical standpoint, this work builds upon the mean field game literature [26] (referred to as MFG thereafter). This type of structure indeed mixes population dynamics through Kolmogorov forward equations (see [18] for a review in economy and [27] for an example on firm growth) with profit optimization expressed as Bellman equations (see [30] for an example). If this serves well mechanistic description, it also leads to interesting results associated to costs optimization (see [14] for an example). If the MFG related literature is growing, there has been numerous discussion about its inherent general complexity. However when paired with economic concepts that are technically related to power laws [34], the MFG frameworks can be simplified to yield closed formulas that can easily be interpreted. This is applied in this paper to demonstrate how sectoral firm considerations can be used to estimate productivity investments. Note that one of the key features of the MFG

apparatus lies in its capacity to indirectly tackle competition questions that appear to have yet eluded the evolutionary field in economic with respect to firm growth. MFGs indeed address actors behavior in a given environment. The MFG application developed in this paper therefore opens an interesting doorway for further research.

The remaining of this paper is organized as follows. Section 2 provides a descriptive analysis of the growth patterns of US PSFs sector and shows that PSFs growth is notably spurred by the creation of new establishments in a dynamic of market expansion. Section 3 further explore the growth of top 25% largest US PSFs, where market expansion may no longer be relevant and proposes an explanation of the observed pattern through the notion of labor productivity improvement investments. Section 4 concludes.

Important Legal Remarks. The findings and opinions expressed in this paper are those of the authors and do not reflect any positions from any company or institution.

2 US PSFs growth patterns.

To describe US PSFs growth patterns, a longitudinal database made available by the US Census bureau is used. This database, which features are depicted in 2.1, covers 25 years of history and the overall US economy. It is used to discuss the application of Gibrat’s law to PSFs in 2.2 at establishment level, prior to a discussion of the growth dynamics in the field in 2.3.

2.1 1975 -2000 Longitudinal data description.

To analyze the growth of PSFs, this study leverages the US synthetic longitudinal database ([37]) (synLDB). The synLDB covers the entire US economy over a 25 years period starting 1975 and provides establishment level records of employment. If the database leverages the old standard industrial classification (SIC) system, mapping tables are provided by the US bureau of labor statistics to link the data to the recent NAICS system. Therefore, describing PSFs through the NAICS code 54 is similar to describing PSFs through the SIC code 731 (Advertising), 733 (Computer Programming, Data Processing, And Other Computer Related Services), 811 (Legal services) and 871 (Engineering, Architectural, And Surveying), 872 (Accounting, Auditing, And Bookkeeping Services), 873 (Research, Development, And Testing Services) and 874 (Management And Public Relations Services).

Between 1975 and 2000, the PSFs establishments landscape has stayed relatively unchanged from a distribution standpoint (see figure (1)) with an average of 8 to 9 employees per PSFs establishment. However its has grown steadily over time displaying an average 5.4% growth in the number of establishments (see figure (2)). Interestingly the growth in the number of establishments appears to decrease over time.

2.2 PSFs age and size growth patterns.

2.2.1 PSFs establishment growth.

From a growth perspective, the US establishments PSFs growth changes with size. This means that for small firms with one or two establishments (i.e firms below 100 employees as per table (9), Gibrat’s law won’t apply. Establishments indeed have their growth rate decreasing and stabilizing as they become

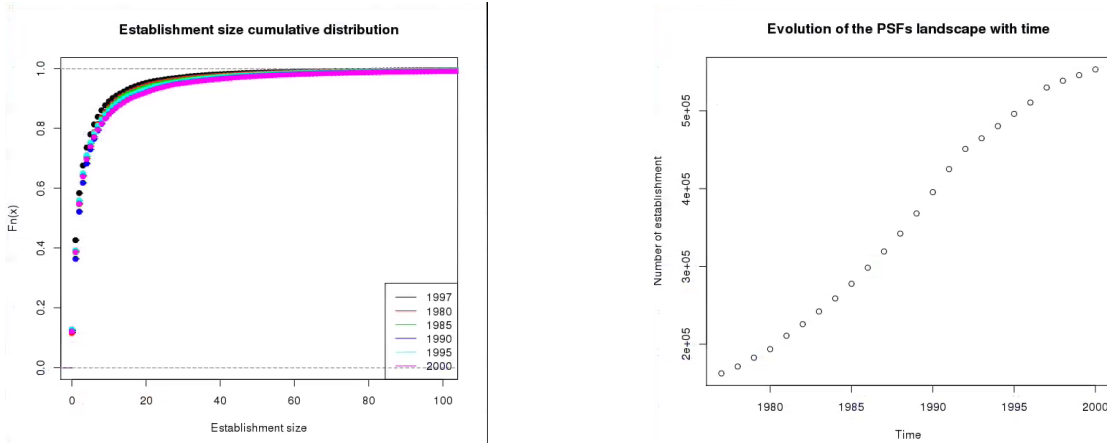


Figure 1: US PSFs establishment size distribution. Figure 2: US PSFs establishment number growth.

bigger. Interestingly, large PSFs growth do appear to follow Gibrat’s law. This seems to stem from the fact that they are composed of numerous establishments which growth rate has stabilized, so that the impact of new establishment launch is mitigated. As seen on tables (5) and (5), growth rates yet highly fluctuates on a year on a year basis. Establishments have different dynamics though depending on whether they belong to a firm with multiple establishments or they are the firms themselves. The main difference is that when a new establishment is not the first one in a firm, its initial growth speed is much faster (i.e the sourcing of the first 20 employees).

From an age standpoint, professional services establishments growth rates decrease with age as seen on tables (7) and (6). Additionally establishments which are newly created within an existing firm have a higher growth rate than the one of a new established firm.

Those findings are similar to the one on the US manufacturing sector, where it was found that size does not effect growth rate for large firms (i.e. Gibrat’s law holds) but that growth rates slightly decrease with size for small firms [22]. Additionally, the growth rate in manufacturing decreases with age [17].

2.2.2 PSFs growth persistence.

Over the 25 years period covered in the US SynLDB, PSFs mono (resp. multi) establishments have grown by 16.1% (resp. 23.9%) on average (see figure (3)). When testing the PSFs growth rate time series for auto and cross correlation, it came that the growth rate of mono establishments doesn’t present autocorrelation nor cross correlation effects with the multi establishment time series. However the multi establishments time series presents strong autocorrelation patterns as seen in figure (4). As seen on the previous subsection, Gibrat’s law doesn’t hold for professional services establishments as the growth rate changes with the establishment size and age. Additionally, the analysis of the aggregated sector growth rate hints at a persistence in growth. To further investigate this characteristic, Gibrat’s law was tested for growth persistence over time in a manner similar to [7] by running the following regression at establishment level from 1979 to 2000.

$$z_{i,t}^u = z_{i,t-1}^u \cdot \beta^u + \gamma^u \cdot z_{i,t-2}^u + \epsilon_{i,t}^u$$

where t is an index for time, i is an index for the establishments, u an index differentiating mono versus multiple firm establishments, and $z_{t,i}$ is the deviation of the logarithm of the size of company i at time

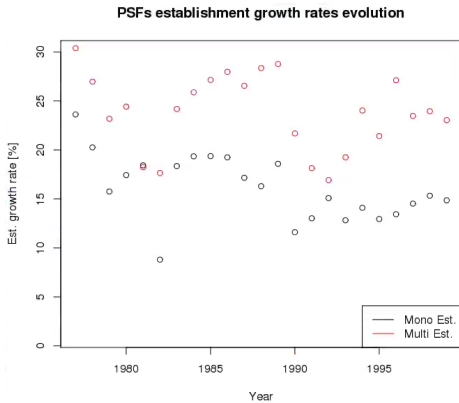


Figure 3: US Establishments growth rate evolution.

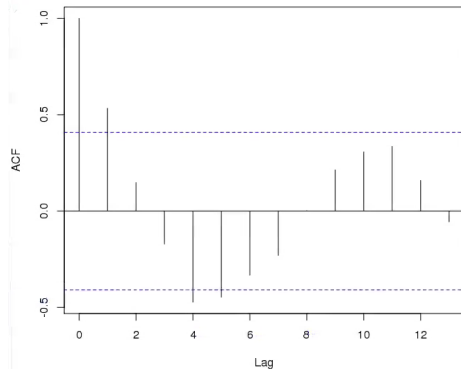


Figure 4: US multi establishments growth rates autocorrelation.

t from the mean of the logarithms of the sizes of companies at time t. With this test, if $(\beta; \gamma) = (1, 0)$ then Gibrat's law is considered valid, otherwise, if $\beta < 1$ (resp. $\beta > 1$) large establishments are expected to grow more slowly (resp. rapidly) than their smaller counterparts. The regression was performed on a sample of establishments who were still operating to avoid sampling bias as suggested by [28]. The estimation results are displayed in table (11).

There are three main results on this analysis. First, whatever their type, large establishments grow slower than smaller ones as $\beta^0 < 1$ and $\beta^1 < 1$. As PSFs are a collection of establishments, this means that smaller firms grow faster than their larger counterparts. Second, as suggested in the aggregated analysis, establishments that belong to a firm that has multiple locations display a positive persistence in growth, which means they grow faster in the second period if they have grown faster in the past. However, this is not something that holds for firms with only one establishment. Third, Gibrat's law validity appears to fluctuate with the year, which could suggest that PSFs establishments are subject to a business cycle.

From a comparison standpoint, the observed US PSFs persistence pattern in growth slightly differs from the current findings on US manufacturing. First of all, as discussed in [5], US manufacturing establishments do not overall exhibit persistence in their growth patterns. However, when focusing on the largest firms with heavy investments in R&D, persistence in growth has been empirically demonstrated for US manufacturing firms [4]. This could mean that the persistence in growth between the two industries is triggered by different mechanisms. While in the PSFs case, persistence could be linked to geographical expansion, persistence in manufacturing growth could be linked to heavy capital investments. This is consistent with macro level findings that state that at an aggregated U.S. economy level, business growth is the sum of a cyclic component and a deterministic trend [31].

2.3 PSFs growth through geographical expansion.

2.3.1 US PSFs growth dynamics at an aggregated level.

When looking at the 2007 Statistics of US businesses (SUBS), it has to be noted that large firms have many establishments and that those establishments are larger than the ones of smaller firms. As seen in table (9) PSFs have only one establishment until they reach 20 employees, then additional locations appear as the firms become bigger. On average, it seems that establishments grow until about 50 to 60 people at which point other branches are appearing. Note that on average in the US, PSFs do have

1 establishment and employ about 9 to 10 persons.

The professional services sector appears to have an establishment composition with firm size that differs from the overall economy and the manufacturing sector (see table (9)). Compared to the manufacturing sector, PSFs have much more establishments with suggest a geographical expansion associated to firm growth. However this growth pattern is not as strong in the PSFs than in the overall US economy. This would suggest that PSFs have to trade off between consolidating knowledge in hubs to enjoy returns to scale and winning local markets. Large PSFs indeed have a maximum number of establishments of 12 to 48 when their size grows above 2500 employees, while manufacturing have a maximum of 5 to 10 establishments. On the other hand, large firms in the overall US economy have between 60 to 400 locations.

PSFs do not concentrate employment in a specific location to the difference of manufacturing. As seen in table (10), PSFs have an employment footprint per establishment that is similar to the overall US economy. To this extent, large firms with more than 2000 people do employ an average of 65 persons per establishment, while the manufacturing sector concentrates between 180 and 280 persons at a same location. Therefore if PSFs growth is generated by the growth of its existing establishments, growth pattern at firms level may also be impacted by the development of a firm's establishment network. This may especially be true for small and medium firms (i.e < 250 employees). An additional analysis would be required here that is out of scope of this paper.

2.3.2 PSFs establishments: creation and destruction patterns.

To dig deeper in the geographical expansion of PSFs, it is important to look at the entry and exiting dynamics of PSFs at establishment level. From an entry standpoint, there a two main cases: either PSFs are created by single individuals from scratch or establishments are created as part of an existing firm with an average of 12 employees. The first option is more likely as seen in table (12) and over the 1976 to 2000 period there as been an average there is a 12% of newly created PSFs with on single employee. On the other hand, there is chance for a firm to open a new establishment 9%. Note that while the entry of single employee firm has been quite stable proportion wise, the market expansion through the opening of new establishment appears to fluctuate. Note that on average, the entry rate of new establishments has lowered over time as market expansion has slowed for existing firms.

From an exit pattern standpoint, PSFs establishments that disappear have 2 to 3 employees when belonging to single establishment firm or about 12 when belonging to an existing firm with multiple establishment. This suggests that PSFs cease their activity is they fail to launch. Overall about 7% of the PSFs establishments close on a year on year basis. Establishments belonging a a multi establishment firms are more robust as their exiting probability has been of 4.5% on average for the period, versus a 7.4% chance for the mono establishment counterparts. Interestingly, most of variance in the exit patterns is associated to the multi establishment firms. It therefore seems that the risk of a successful market expansion even though limited compared to the creation of a firm from scratch may be fluctuating because of markets evolutions.

Finally, from a dynamic point of view, both size and age have a clear effect on the PSFs landscape establishment exiting probability, as seen on table (8).The observed entry and exit patterns with respect to age, size and the difference between mono and multi establishments are consistent with the overall findings on US manufacturing [15] and the rest of the US [33] and developed economy in general [19]. The interesting element at stake is that when looking at entry and exit patterns of US manufacturing, one of the main elements is to consider whether or not the plants are being invested in and are diversifying, which spurs a reduction in exit rates. For PSFs, the driver of exit rate reduction

appears slightly different as PSFs that expand geographically are more likely to survive. The firm growth patterns investigated in 2 depict most of the PSFs in the US, which are small to medium firms (i.e. 250 employees) composed of 1 to 4 establishments on average that expand geographically. However as those findings can be challenged for large firms as they do exhibit as they are composed of multiple established locations. Therefore geographical expansion and the addition of another 10 to 20 employees, may not be the main drivers at stake.

3 Large US PSFs growth.

To discuss what could be happening for large firms, an option is to leverage the traditionally notion of firm productivity, which has been used in the literature to explain firms growth patterns. In section (3.1), the Hopenhayn's framework [24] is revisited by replacing the hypothesis of exogenous random productivity shocks by endogenous productivity investment. The US QWI and US BLS dataset are then used to calibrate US PSFs productivity improvement investments in (3.2). Finally, in (3.3), productivity investment are tied to the distribution of PSFs in the case of large and high productive firms (i.e. > 250 employees) to discuss the implications of the framework in terms of growth speed with age and size, with the help of a firm level cross sectional dataset made available by the US census.

3.1 A toy model of firm growth

Similarly to [24], assume that PSFs supply an output $q(\phi, p, \omega)$ (i.e. charged hours) to their clients, where ϕ is the PSF productivity (i.e utilization), ω the price of labor and p the price of the output (i.e the hourly rate). On the other hand, PSFs demand labor up to a quantity $n(\phi, p, \omega)$ (i.e. number of employees), so that PSFs profit π can be expressed as:

$$\pi(\phi, p, \omega) = p.q(\phi, p, \omega) - \omega.n(\phi, p, \omega)$$

Assume that PSFs are price taking, so that p_t, ω_t are known and driven by the overall market demand. The problem for the firm is to change its productivity ϕ_t at a speed $\rho(t)$ ($d\phi_t = \rho(\phi)dt$)

$$v(t, \phi_t) = \max_{\rho(\cdot)} \int_t^{t+T} e^{-\beta \cdot x} (\pi(\phi_x, p_x, \omega_x) - c(\rho(x))) dx \quad (1)$$

where $c(\rho(x))$ represents the investment costs to change the firm productivity. According to ([25]), those investments in the case of professional services are mainly of two nature: investments in information and communication technology (ICT) and human capital. Assume that the price for the output is driven by the overall market demand $p = D(Q)$ where $Q = \int q(\phi, p, \omega).f(\phi)d\phi$ and that the price for the overall labor supply is driven by the firms demand $\omega = W(N)$ where $N = \int n(\phi, p, \omega)f(\phi)d\phi$. Following the US bureau of labor statistics (BLS) methodology ([6]), productivity in professional services is defined as the amount of output divided by the amount of labor required to deliver this output. In the case of services, the BLS recommendation is to measure output is as the dollar amount of sold of services adjusted for price changes in these over time. In the case of PSFs, the productivity can therefore be interpreted as number of charged hours (i.e. output) sold at a given price p that have been generated by a hour of work (i.e labor input). With respect to the previously developed notation, this means that:

$$\phi_t = \frac{q(\phi_t, p_t, \omega_t)}{n(\phi_t, p_t, \omega_t)}$$

Under this framework, competition between firms is in essence a competition in terms of productivity. Intuitively at equal level of supply (i.e equal number of employees), PSFs with the highest productivity will be able to generate more profit and to increase their size faster than their competitors. This will be shown in the case of highly productive firms in (3.3).

Note that under this view of productivity, comparing the productivity from one firm to another in a given year is not that straight forward. Sales and labor are easily obtained at firm level, but the output in terms of charged hours or the firm levels differences in terms of hourly rates are not accessible. Therefore in this particular study, the output in terms of charged hours is defined at a constant market price p which is shared across all firms. So that if firm A and firm B are selling the same amount of charged hours and have the same amount of workers, if firm A is selling services at a price above the price of firm B, its productivity will be higher.

3.2 Insights from US macro level data

To estimate the PSFs demand function W in workers, the US quarterly workforce indicators (QWI) provided by the US Census were used over the last 10 years (i.e 2007 to 2017). N was taken as the total number of US PSFs employees (where PSFs are defined according to the NAICS code 54), while labor prices ω have been taken as the average monthly wage in the sector. On the other hand, the estimation of the PSFs supply function D was achieved by leveraging data from the quarterly service survey (QSS) provided by the US census. Price p where estimated as the total PSFs revenue divided by the total number of worked hours, while the output Q was associated to the number of worked hours. Results are displayed in figures (5) and (6).

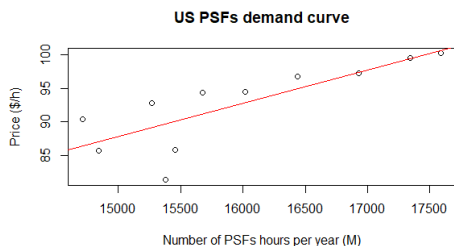


Figure 5: US PSFs demand curve.

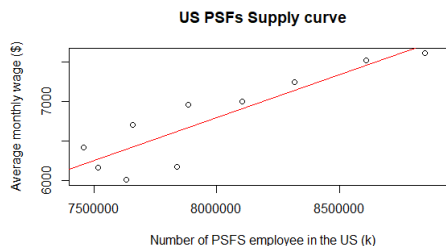


Figure 6: US PSFs supply curve.

From the overall US level yearly time series provided by the US census, demand and supply curves can be modeled in a linear fashion for the sake of simplicity (see table (1)):

$$\omega = W(N) = a.N + b \quad p = D(Q) = h.Q + k$$

	Slope	Intercept	R^2
Demand W	a=0.0011 (0.0002)	b=-1938 (1552)	77.3%
Supply D	h=0.005 (0.001)	k=12.90 (19.91)	64.2%

Table 1: US PSFs Demand and Supply Curve estimates

Assuming that $q(\phi, 0, \omega) = 0$ and $n(\phi, p, 0) = 0$ and leveraging the previous definition of productivity

as $q/n = \phi$, the linear models imply that:

$$n(\phi, p, \omega) = A(\phi) \cdot \omega \quad q(\phi, p, \omega) = A(\phi) \cdot \phi \cdot p \quad \int A(\phi) \cdot (\phi - 1) f(\phi) d\phi = 0$$

Firm (resp. PSFs sector) level profit π (resp. Π) therefore can be simplified as:

$$\pi = A(\phi) \cdot (\phi \cdot p^2 - \omega^2) \quad \Pi = \left(\int A(\phi) f(\phi) d\phi \right) \cdot (p^2 - \omega^2)$$

The optimization program (see eq. (1)) of the firm can therefore be expressed as:

$$\partial_t v + \max_{\rho} (\rho \cdot \partial_{\phi} v - \beta \cdot v + \pi(\phi, p, \omega) - c(\rho)) = 0 \quad (2)$$

This leads to $\partial_{\phi} v = \partial_{\rho} c$. As we assume that ρ is dependent in ϕ , it is possible to look for a form on productivity improvement investment $\hat{c}(\phi) = c(\rho(\phi))$. Further differentiating (2) towards ϕ yields:

$$\partial_{\phi} \hat{c} - \frac{\beta}{\rho} \partial_{\phi} \hat{c} = \frac{\partial_{\phi}(\pi)}{\rho \cdot \partial_{\rho} \phi} \quad (3)$$

3.3 Link to the overall firm size - age growth patterns of high productive large PSFs?

Leveraging cross sectional data for the 2007 SBO public micro sample from the US Census, it is easy to access the distribution $f(\phi)$ (resp. the exiting firm distribution $T(\phi)$). Interestingly figure (7) shows that new entrants have a lower productivity than existing firms. However this also shows that firms that cease operations have the same productivity than the existing ones. Although this may not be a consistent pattern over the year, this could suggest that the exit criteria based on productivity shocks detailed by [24] may not be the drivers for PSFs.

When zooming on the top 25% most productive PSFs, extreme value theory yields that those firm are distributed in productivity according to a power law which exponent can be estimated according to the log rank technique of [18]. Results and graphical representations are displayed in table (2) and figures (8) and (9). The main characteristic of those top firms is that they generate more than 107k\$ per year and per employee, so that with an overall market price of 81.34\$/h as per the 2007 US QSS, those firms are actually able to charge 1320 hours per employee per year on average. Knowing that an average year of work represents 260 days of 8 hours each, this means that top PSFs firms have on average a productivity/utilization above 50%.

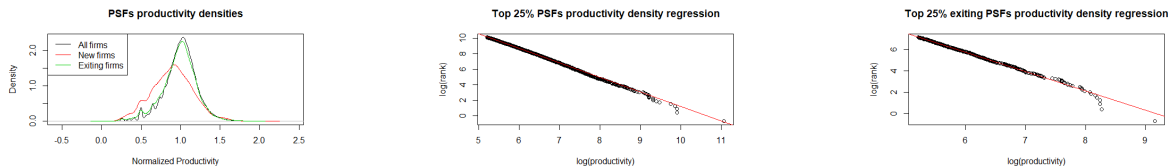


Figure 7: US PSFs productivity densities. Figure 8: Top productive PSFs - density estimation. Figure 9: Top productive exiting PSFs - density estimation.

Taking a population dynamic point of view, and assuming that the PSFs distribution is time invariant, it comes that:

$$\partial_{\phi}(\rho \cdot f) = T(\phi)$$

	$f(s)$	$T(s)$
Pareto Exponent	-1.86 (0.0004)	-1.81 (0.003)
R^2	99.9%	99.6%

Table 2: Highly productive PSFs - Pareto exponent Estimation.

As the top productive firms are distributed and exiting according to a power law, it comes that the productivity growth speed follows $\rho(\phi) = \rho_0 \cdot \phi^\gamma$, where $\gamma \approx 0.95$ according on the estimates displayed in table (2). When looking at the PSFs landscape dynamics and further restricting the scope to the high productive firm that employs more than 250 employees and investigating the problem under a firm size lens n , it comes that under the assumptions of time invariance:

$$\partial_n(\mu \cdot m) = \theta(n)$$

Where m (resp. θ) represents the distribution of existing (resp. entry) firm according to their size and μ is the growth speed of a firm (i.e $dn_t = \mu(n_t)dt$). Restricting the analysis to the top productive large firms (i.e the distribution tail) leads to a power law form for m , θ and therefore $\mu = \mu_0 \cdot n^\Psi$. The power exponents can be estimated from the SBO PUMS sample, which features 760 large high productive high firms (i.e about 0.7% of the overall SBO sample), using [18]. The estimation results shown in table (3) yield a value for $\Psi \approx 1$. Interestingly this means, that for the subset of large PSFs with high productivity, Gibrat's law could hold.

	$m(s)$	$\theta(s)$
Pareto Exponent	-2.35 (0.015)	-2.43 (0.25)
R^2	96.9%	90.8%

Table 3: Highly productive large PSFs - Pareto exponent Estimation.

For a given firm, assuming the market is at the equilibrium ($\omega_t = Cstte$), the framework developed in the previous section entails that:

$$dn_t = d\phi_t A'(\phi) \cdot \omega \cdot dt = \mu(n_t)dt \leftrightarrow \rho(\phi) \cdot A'(\phi) \cdot \omega = \mu(A(\phi) \cdot \omega)$$

On the high end (i.e. large ϕ , large n), this means that:

$$\frac{dA(\phi)}{A(\phi)} = \left(\frac{\mu_0 \cdot \omega}{\rho_0 \cdot \phi^\gamma} \right) d\phi \leftrightarrow A(\phi) = A(\phi_{min}) \cdot e^{\left(\left[\frac{\mu_0 \cdot \omega}{\rho_0 \cdot (1-\gamma)} \phi^{1-\gamma} \right]_{\phi_{min}}^\phi \right)}$$

From a firm growth standpoint; for high productive large PSFs, as $d\phi = \rho_0 \cdot \phi^\gamma \cdot dt$ and $dn = \mu_0 \cdot n \cdot dt$, it comes that:

$$\phi(t) = \left((1-\gamma) \cdot \rho_0 \cdot t + \phi_0^{1-\gamma} \right)^{\frac{1}{1-\gamma}} \quad n(t) = n_0 \cdot e^{\mu_0 \cdot t}$$

This complements the empirical findings of the synLBD, where establishment growth speed decrease with age and size. The rate of change in growth speed observed in the tables (4) and (6) indeed show that for old and large PSFs, the growth rate may not change much. It is then possible to conclude

the analysis by pushing the framework with respect to productivity investments. As $\rho = \rho_0 \cdot \phi^\gamma \leftrightarrow \phi = (\frac{\rho}{\rho_0})^{\frac{1}{\gamma}}$, the equation (3) becomes:

$$\partial_{\phi\phi}\hat{c} - \frac{\beta}{\rho_0 \cdot \phi^\gamma} \partial_{\phi}\hat{c} = \gamma \cdot \frac{A'(\phi)(\phi \cdot p^2 - \omega^2) + A(\phi)p^2}{\phi}$$

Assume for the sake of simplicity that $\beta = 0$ (i.e. large firms do not discount their future cash flows). Asymptotically, for highly productive large firms $A(\phi) = \hat{A}e^{\phi^{1-\gamma}}$. Under this approximation, it comes that:

$$\partial_{\phi\phi}\hat{c} = \gamma \cdot (\hat{A} \cdot (1 - \gamma) \cdot \phi^{-\gamma} \cdot e^{\phi^{1-\gamma}} (\phi \cdot p^2 - \omega^2) + \hat{A} \cdot e^{\phi^{1-\gamma}} \cdot p^2) > 0$$

As the higher the productivity, the higher the value of the firm $\partial_{\phi}v = \partial_{\phi}c > 0$, it comes that $\partial_{\phi}c$ has the same sign as $\gamma > 0$. This means that for large highly productive PSFs, productivity investments are increasing with productivity to stay competitive and that their followers tend to invest relatively less and mimic the leader, so that there is a form of arms race competition.

4 Conclusion.

In this paper, both a longitudinal empirical dataset ranging from 1977 to 2000 and a cross-sectional dataset from 2007 from the US were used to show that within the professional services sector Gibrat's law doesn't hold. PSFs growth is indeed slowing both at equal size with age and with size. Directionally those findings are similar to the US manufacturing sector, which constitutes the main reference in the study of US firms growth. If the size and age patterns disappear for large PSFs, growth becomes persistent and exhibits 3 to 4 years cycles, which appears different from the manufacturing field. Additionally, while manufacturing firms concentrate their growth on a handful of localization, PSFs growth, similarly to the average US firm growth appears driven by geographical expansion.

As for large PSFs, the pattern of geographical expansion can be challenged, this paper proposes a variant to the classical Hopenhayn framework. It notably shows that, in the context of large and highly productive US PSFs, growth patterns can be explained by investment policies in labor productivity improvement programs. Those investments come as a form of arms race between the top competitors. Finally it could be interesting to further detail the developed considerations in terms of productivity. It could be for example worth to investigate what is happening for firms with a low productivity and perhaps to challenge their goals as profit maximizing organization by leveraging some of considerations of [35]. Additionally, productivity investments from firms do not yield deterministic results. The framework proposed in section (3.1) could be easily randomized and numerical simulations performed. Eventually, the main next step of this analysis would be to drill down at a firm level to investigate the main productivity levers a PSF can pull (e.g. developing a multi local model and/or investing in automation, increasing its market access capabilities through its quality perception/reputation (see [32])).

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5 Appendix.

Year/ Size	0	1	2	[3;5]	[6;10]	[11;20]	20+	Total
1977	-	9.42	10.0	2.89	2.18	1.03	0.77	
1982	-	4.69	2.66	2.41	2.05	1.79	1.28	
1987	-	4.94	3.67	2.94	2.44	1.77	1.60	
1992	-	5.02	2.95	1.92	1.89	1.33	1.03	
1997	-	1.65	1.66	1.32	1.20	1.15	0.96	

Table 4: US PS multi establishments Growth rate with size between 1977 & 2000 [%].

Year/ Size	0	1	2	[3;5]	[6;10]	[11;20]	20+	Total
1977	-	3.03	2.66	2.12	0.46	0.71	0.12	
1982	-	2.32	1.70	1.06	0.97	1.07	1.05	
1987	-	2.58	1.36	1.04	1.09	0.77	0.94	
1992	-	2.43	1.24	1.16	0.94	0.89	0.66	
1997	-	1.13	1.02	1.05	1.05	1.09	1.07	

Table 5: US PS mono establishments Growth rate with size between 1977 & 2000 [%].

Size / Age	0	1	2	3	[4;5]	[6;7]	[8;11]	[12;14]	[15;20]	21+	Total
0	-	-	-	-	-	-	-	-	-	-	-
1	1.78	1.89	1.88	1.84	1.82	1.92	2.04	2.19	1.80	2.04	1.89
2	1.60	1.62	1.61	1.57	1.54	1.55	1.68	1.77	1.61	1.77	1.62
[3;5]	1.50	1.49	1.44	1.39	1.40	1.49	1.47	1.56	1.48	1.40	1.46
[6;10]	1.34	1.32	1.31	1.30	1.25	1.29	1.32	1.34	1.30	1.27	1.31
[11;20]	1.24	1.25	1.25	1.25	1.22	1.21	1.22	1.20	1.14	1.16	1.21
20+	1.02	1.01	1.03	1.03	1.01	1.02	1.03	1.03	1.03	1.00	1.02
Total	1.34	1.31	1.30	1.28	1.24	1.24	1.23	1.24	1.17	1.14	-

Table 6: Average US PSFs multi establishments 1998 to 1999 Growth rate with size and age [%].

Size / Age	0	1	2	3	[4;5]	[6;7]	[8;11]	[12;14]	[15;20]	21+	Total
0	-	-	-	-	-	-	-	-	-	-	-
1	1.29	1.26	1.26	1.25	1.25	1.24	1.24	1.23	1.23	1.22	1.25
2	1.11	1.10	1.11	1.10	1.09	1.09	1.09	1.09	1.07	1.06	1.09
[3;5]	1.06	1.06	1.06	1.06	1.05	1.05	1.05	1.04	1.03	1.04	1.05
[6;10]	1.03	1.02	1.03	1.05	1.04	1.03	1.03	1.01	1.01	1.01	1.03
[11;20]	1.09	1.07	1.06	1.06	1.06	1.05	1.04	1.02	1.01	1.00	1.03
20+	1.16	1.21	1.22	1.17	1.16	1.10	1.06	1.06	1.03	1.02	1.08
Total	1.20	1.18	1.18	1.18	1.17	1.15	1.14	1.13	1.12	1.11	-

Table 7: Average US PSFs mono establishments 1998 to 1999 Growth rate with size and age [%].

Size / Age	0	1	2	3	[4;5]	[6;7]	[8;11]	[12;14]	[15;20]	21+	Total
0	9.43	26.36	20.86	20.32	19.32	16.44	15.22	13.13	11.98	6.57	9.15
1	4.11	12.41	18.43	17.64	16.50	13.95	12.80	11.01	10.04	5.52	9.71
2	3.92	11.49	16.31	14.96	13.55	11.71	10.22	8.84	8.04	4.37	7.49
[3;5]	3.48	10.89	15.28	12.87	12.01	10.38	8.86	7.46	6.97	4.05	6.42
[6;10]	3.07	9.76	12.94	10.98	10.17	8.16	7.46	6.82	6.10	3.51	4.91
[11;20]	2.74	7.79	9.76	8.60	8.19	6.30	5.90	5.35	4.44	2.87	3.18
20+	2.20	5.66	7.57	8.33	8.12	5.99	5.17	4.87	4.28	2.14	2.50
Total	6.04	14.54	14.40	12.63	12.87	10.76	9.54	8.27	7.52	4.04	

Table 8: Average US PSFs establishments exit probability [%] between 1975 and 2000.

Firm Size	Establishment(s) per PSFs	Establishment(s) per Manufacturing firm	Establishments(s) per firm in the US
0-4	1	1	1
5-9	1	1	1,01
10-14	1,03	1	1,08
15-19	1,06	1	1,04
20-24	1,11	1	1,13
25-29	1,20	1,03	1,18
30-34	1,23	1,04	1,25
35-39	1,28	1,05	1,31
40-44	1,31	1,08	1,35
45-49	1,44	1,07	1,45
50-74	1,56	1,11	1,61
75-99	1,82	1,19	2,03
100-149	2,19	1,27	2,59
150-199	2,86	1,44	3,57
200-299	3,28	1,67	4,94
300-399	4,03	1,91	6,81
400-499	4,73	1,98	9,26
500-749	5,25	2,42	11,76
750-999	6	3,26	15,48
1,000-1,499	6,03	3,58	20,33
1,500-1,999	6,78	3,62	29,97
2,000-2,499	9,09	5,41	38,62
2,500-4,999	12,21	5,59	61,61
5,000 +	48,07	9,39	405,24
Total	1,1	1,21	1,27

Table 9: Number of US establishments with firm size according to 2007 SUBS in professional services, manufacturing and the overall economy.

Firm Size	Employment per PSFs establishment	Employment per Manufacturing establishment	Employment per US establishment
0-4	1,50	1,85	1,65
5-9	6,49	6,68	6,49
10-14	11,36	11,68	11,20
15-19	15,67	16,56	15,45
20-24	19,44	21,35	19,19
25-29	22,05	25,88	22,56
30-34	25,34	30,13	25,45
35-39	28,20	34,51	27,90
40-44	31,06	38,46	30,87
45-49	31,78	42,75	32,12
50-74	36,81	52,45	37,14
75-99	43,78	67,4	42,04
100-149	49,11	85,64	46,49
150-199	49,80	97,78	48,12
200-299	57,51	115,88	49,01
300-399	62,90	127,54	50,58
400-499	63,23	141,25	48,12
500-749	66,45	143,95	51,54
750-999	71,64	157,28	55,7
1,000-1,499	93,63	162,92	59,89
1,500-1,999	85,78	171,48	57,64
2,000-2,499	66,85	167,52	57,56
2,500-4,999	63,67	176,52	56,53
5,000 +	53,24	269,88	50,55
Total	9,48	40,19	15,65

Table 10: Employment per establishment with firm size according to 2007 SUBS in professional services, manufacturing and the overall economy.

Period	β^0	γ^0	β^1	γ^1
1977 - 1979	0.76 (0.05)	0.14 (0.05)	0.90 (0.04)	0.03 (0.05)
1978 - 1980	0.94 (0.05)*	0.04 (0.05)*	0.85 (0.05)	0.11 (0.05)
1979 - 1981	0.93 (0.05)	-0.03 (0.06)	0.83 (0.06)	0.10 (0.06)
1980 - 1982	0.80 (0.05)	0.03 (0.05)	0.92 (0.04)	0.06 (0.04)
1981 - 1983	0.94 (0.05)	0.06 (0.04)	0.88 (0.05)	0.11 (0.06)
1982 - 1984	0.81 (0.05)	0.08 (0.06)	0.95 (0.05)*	0.05 (0.05)*
1983 - 1985	0.87 (0.05)	0.05 (0.05)	0.89 (0.04)	0.07 (0.05)
1984 - 1986	0.93 (0.05)	-0.01 (0.05)	0.83 (0.05)	0.16 (0.05)
1985 - 1987	0.80 (0.05)	0.09 (0.05)	0.76 (0.05)	0.18 (0.06)
1986 - 1988	0.77 (0.05)	0.09 (0.05)	0.90 (0.05)	0.11 (0.05)
1987 - 1989	0.80 (0.05)	0.13 (0.05)	0.88 (0.05)	0.12 (0.05)
1988 - 1990	0.87 (0.05)	0.00 (0.05)	0.95 (0.05)*	0.02 (0.05)*
1989 - 1991	0.87 (0.05)	0.00 (0.05)	0.91 (0.05)	0.04 (0.005)
1990 - 1992	0.86 (0.05)	0.09 (0.05)	0.80 (0.05)	0.15 (0.05)
1991 - 1993	0.87 (0.05)	-0.02 (0.05)	0.96 (0.04)*	0.00 (0.04)*
1992 - 1994	0.82 (0.05)	0.08 (0.05)	0.94 (0.05)*	0.04 (0.05)*
1993 - 1995	0.80 (0.05)	0.16 (0.05)	0.95 (0.05)*	-0.01 (0.05)*
1994 - 1996	0.88 (0.05)	0.02 (0.05)	0.89 (0.04)	0.08 (0.04)
1995 - 1997	0.78 (0.05)	0.12 (0.05)	0.89 (0.06)	0.07 (0.06)
1996 - 1998	0.87 (0.05)	0.12 (0.05)	0.91 (0.04)	0.04 (0.04)
1997 - 1999	0.93 (0.05)	-0.01 (0.05)	0.98 (0.05)*	-0.02 (0.05)*
1998 - 2000	0.89 (0.05)	0.07 (0.05)	0.89 (0.05)	0.07(0.05)

Table 11: Gibrat Law - Persistence test at establishment level.

Year	N_t^0	N_t^1	E_t^0	E_t^1	$N_{e,t}^0$	$N_{e,t}^1$	$E_{e,t}^0$	$E_{e,t}^1$	$N_{c,t}^0$	$N_{c,t}^1$	$E_{c,t}^0$	$E_{c,t}^1$
1977	134926	15302	5.14	35.23	18116	3595	0.81	8.51	11678	633	2.40	15.86
1978	143406	16048	4.32	36.93	18857	1235	0.88	14.52	11317	567	2.61	10.76
1979	153002	16805	4.54	38.63	20155	1447	0.95	7.46	12140	698	2.65	11.7
1980	163153	17938	4.46	38.82	20900	1547	0.86	8.81	12205	427	2.69	12.51
1981	177449	18809	4.45	38.73	25771	2070	0.97	10.52	13420	1207	2.89	8.14
1982	191575	20930	4.42	36.81	24985	2659	1.12	11.48	12696	590	2.57	16.81
1983	206718	22143	4.77	37.23	26635	1768	0.80	10.85	12637	565	2.56	11.63
1984	221733	23236	5.01	37.14	26859	2011	0.86	8.63	12925	927	2.7	13.38
1985	239627	24866	4.69	35.66	29405	2040	0.84	7.8	12825	426	2.84	14.42
1986	257082	25990	4.89	37.04	29914	2446	0.92	10.37	13888	1356	2.93	10.91
1987	273424	29742	4.93	35.50	29958	3937	1.12	10.98	15839	204	2.82	9.73
1988	292110	31749	4.02	35.90	33029	2915	1.10	10.34	17350	1028	2.69	7.67
1989	315659	34627	5.14	35.81	37877	3630	1.13	15.34	16900	890	3.29	7.67
1990	339361	36181	5.33	36.71	39130	3430	1.13	12.92	17966	1980	3.00	12.53
1991	361186	38639	5.29	36.56	41512	4125	1.11	13.49	35805	1869	2.83	12.39
1992	372676	40343	5.01	33.96	41622	3794	1.00	20.97	35548	2204	3.01	15.93
1993	386150	40748	4.97	34.06	44455	2573	1.07	20.97	35548	2279	3.43	16.67
1994	399083	41636	5.69	33.92	44359	3307	0.98	10.35	37119	2553	3.01	15.91
1995	412609	41509	5.69	33.92	46309	3435	0.82	12.53	38364	3688	2.90	14.15
1996	424396	41029	6.17	34.76	47714	3245	1.10	13.27	41347	3934	3.11	17.89

Table 12: US PSFs - entry & exit patterns.