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Growth patterns of professional services firms.

Edouard Ribes^{*†}

April 10, 2018

Abstract

Based on data collected by the US Census, this paper proposes an investigation of the firm level dynamics inherent to the professional services industry. I show that PSFs do not follow Gibrat's law but that their growth speed is increasing and convex in size. This translates into asymptotically linear transformation costs with size.

But size is not the only key feature that has to be considered to explain PSFs growth. I show that PSFs are subject to a maturation process with age that is crystallized by their distribution across the marketplace as well as their margin structure. This aspect of PSFs is then compared to other industrial sectors and a possible interpretation for this behavior in terms of employee sorting is proposed.

Keywords and phrases. Firm growth Professional Services Firms Learning Firm performance Employee Selection Transformation Costs Mean Field Games

Economic Classification [JEL]. E17; L16; L25; L84; D23.

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1 Introduction

This work investigates the structure of the professional services firms sector and the performance of its constituents. This question is addressed both theoretically and empirically using a publicly available and representative sample of the 2007 US firms prepared by the US Census bureau. The analysis of the PSFs sector shows that firm age and size yield a positive relationship to both revenue and costs that can be approached by a set of multivariate power laws. The data shows that the PSFs sector exhibits different performance patterns compared to other more capital intensive sector such as the manufacturing or mining industry. I show that linking firms performance and sector structure with a mean field game approach yields interesting results in terms of PSFs growth patterns and transformation costs. This application shows that PSFs growth speed is increasing, convex and sustained by approximately linear costs with respect to the PSFs size. I finally show that a possible micro level interpretation for this evolutionary pattern could be found in terms of employee sorting.

This work contributes to two main strands of the related literature. First this work builds upon the studies that stress the effects of size and age on firms growth. Since the pioneering work of Gibrat [15] that stated that the relationship between firm growth and its initial size can be described by a constant growth rate and a random element that is normally distributed, it has been empirically shown that firm growth is industry specific and exhibits power law patterns ([21], [25]). However size is not the only relevant variable to describe firms growth dynamics. Starting with [13], authors have started to complement the classical firms growth dependency in size by observing a dependency in age. The relationship between firm development and its age has spurred an increased interest over the last years [6]. This has motivated numerous applications on firms entry and exit rates [1],[26], firms internationalization strategies [16] and resilience to extreme random events [9]. With respect to this literature, the present paper brings a standard size - age approximation of firm growth speed that is shown to be easily calibrated to macro economic sectorial data, such as the ones publicly collected by most countries. Additionally this paper's mechanistic considerations showcase the professional services industry (see [12] for a definition), that has not been explored, to my knowledge, in the above mentioned literature.

Second this study goes beyond the descriptive evolutionary mechanisms related to firm growth to explore its motivations. The initial explanations in terms productivity and technology shocks of [18] gave birth to numerous discussion between product and firm lifecycle [7], [3]. This has caught up with the evolutionary literature focusing on age as an important determinant by providing some evidence of the learning behavior [2] of firms towards their customer preferences [17]. Growing beyond considerations on demand, the field has had a special interest in financial considerations associated to growth and transformation. The main areas of interest revolved around convex entry costs [10] and capital management [8], [5]. This paper provides a novel connection between evolutionary mechanism and financial consideration through the notion of transformation costs. If the notion of entry costs appear relatively weak in the context of PSFs as it is not a capital intensive industry, transformation comes with indirect costs that have, to my knowledge, not been measured. I show that the observed patterns of firm growth, exhibiting a form of viscosity, impose conditions on what these indirect costs are and demonstrate how to measure them.

From a technical standpoint, this work builds upon the mean field game literature [19] (referred to as MFG thereafter). This type of structure indeed mixes population dynamics through Kolmogorov forward equations (see [14] for a review in economy and [20] for an example on firm growth) with profit optimization expressed as Bellman equations (see [23] for an example). If this serves well mechanistic description, it also leads to interesting results associated to costs optimization (see [11] for

an example). If the MFG related literature is growing, there has been numerous discussion about its inherent general complexity. However when paired with economic concepts that are technically related to power laws [24], the MFG frameworks can be simplified to yield closed formulas that can easily be interpreted. This is applied in this paper to demonstrate how sectorial firm considerations can be used to estimate firm transformation costs. Note that one of the key features of the MFG apparatus lies in its capacity to indirectly tackle competition questions that appear to have yet eluded the evolutionary field in economic with respect to firm growth. MFGs indeed address actors behavior in a given environment. The MFG application developed in this paper therefore opens an interesting doorway for further research.

The remaining of this paper is organized as follows. Section 2 provides an analysis of the size-age relationships for PSFs sectorial distribution as well as entry and exit rates. Section 3 leverages a mean field game approach to extrapolate PSFs growth patterns and transformation costs. Section 4 concludes.

Important Legal Remarks. The findings and opinions expressed in this paper are those of the authors and do not reflect any positions from any company or institution.

2 Empirical Description of PSFs in the US

This section analyzes the importance of size and age for professional services firms. After a brief data description, it highlights three main facts. First, young PSFs, that are typically small, exhibit different dynamics than their older counterparts. Second, at equal size, PSFs profit is increasing with age and becomes more predictable. Third, when compared with other industrial sectors, PSFs profit and dynamics differ from capital intensive industries such as mining or manufacturing but appear close to fast paced sectors such as the wholesale one.

2.1 Data Description.

The underlying data in this study is based on a public use micro sample (PUMS) of the 2007 survey of business owners (SBO) sponsored by the US census bureau. The SBO is conducted on a company or firm basis. It includes information such as the number of employer and nonemployer firms, sales and receipts, annual payroll, and employment. It covers 20 industrial sectors and firms are mapped according to the 2012 North American Industry Classification System (NAICS).

The dataset contains 294882 records of professional services firms, indexed under the NAICS code 54. The dataset encompasses about 38% of the total 771611 firms in the US professional services sector. Among this dataset, the firms that have not been operating during the entire year and that had no recorded permanent employee were filtered out. This reduced the sample to 96713 PSFs (e.g. 12,5% of the total sector). The dataset also contained 755 mining firms (NAICS code 22), 57508 wholesale firms (NAICS code 42) and 32110 financial firms (NAICS code 52). This accounts respectively for 13%, 17.2% and 6.4% of the total US sectors.

As per the PSFs sector, the dataset accounts for firms that employ between 1 and 13000 persons with an average of 39.4 and a standard deviation of 101.12. The average firm revenue is of 5.07M\$ per year with a standard deviation of 13.2M\$, while its average payroll is of 1.98M\$ per year (standard deviation 5.19 M\$). Across all firms, the average pay per employee is of 55.2k\$, while the average revenue generated per employee is of 163.8k\$. Note that those numbers aligns with the overall US

PSFs sectors that generated 1,251 B\$ in 2007 in revenue for a total 502 B\$ payroll dispatched over 7.87M employees.

Aside of revenue, payroll and employment information, the dataset also presented records of the year in which the business was established. This was used in this study as the age of a firm, while the number of employees was used to define the size of a firm. Note that choosing the number of employees as a proxy for the firm size appears reasonable in the case of the PSFs as it is not a capital but workforce heavy industry. Firms age records (expressed in years) belong to one of the 9 following possible groups $\{0; 1; 2; 3; 4; [5 - 7]; [8 - 17]; [18 - 27]; 27+\}$. Note that in the following study, the average of each age group was used to carry over the regression analysis, with the exception of the last group (firm age superior to 27 years) for which an average of 32 years was inferred.

2.2 PSFs Dynamics.

In this section, I show that young PSFs, that are typically small, exhibit different dynamics than their older counterparts. To do so, I run linear regressions respective to firms size s and age a on three macro-economic indicators: the number of PSFs $m(s, a)$ as well as the number of firms exiting $j(s, a)$ (resp. entering $h(s, a)$) the sector.

Regressions are first run on size for the entire dataset in section (2.2.1). This shows that the three main macro economic indicators can be approximated by power laws respective to firm size which complement the classical laws of [15]. Regressions are then rerun for each age categories (see (2.1)) in the dataset in section (2.2.2) where changes in the regression paramaters according to age are modeled and discussed.

2.2.1 PSFs Dynamics according to size.

To measure the distribution of existing (resp. entering,exiting) PSFs with respect to size (e.g. $m(s)$ (resp. $h(s), j(s)$)), I restricted the study to the firms below 5000 employees and split the overall size continuum in segments of length 100. I therefore ended up with 50 size segments: $\{[0; 100], \dots, [4900; 5000]\}$. For each of those segments, the number of operating firms was used as a proxy for m , the number of operating firms of age 0 was used as a proxy for h and the number of non operating firms was used as proxy for j . Note that the middle of the segment was used as a reference to plot the functions in the 2D space.

The proposed distributions (m, j, h) can be well approximated by one dimensional power laws with respect to size, which is a standard in the field (see [14]). An efficient way to represent that is to plot the log of the firms distribution with respect to the log of the firms size. This way, power laws can be calibrated with a simple regression (see eq.(1)). Such a representation is displayed in figure (1). Regression results are then presented in table (1). Note that there may be link between PSFs exit and entry because of mergers and acquisitions, however this is not something that is tackled in this paper due to a lack of data but could consist in another avenue of future research.

Noting ϵ_m (resp. ϵ_j and ϵ_h), the gaussian error term associated to the regression on all PSFs (resp. the exiting firms and new entrants), regression can be represented by:

$$\log(m(s)) = \alpha.\log(s) + \log(m_0) + \epsilon_m \leftrightarrow m(s) = m_0.s^\alpha \quad (1)$$

$$\log(j(s)) = \gamma.\log(s) + \log(j_0) + \epsilon_j \leftrightarrow j(s) = j_0.s^\gamma \quad (2)$$

$$\log(h(s)) = \beta.\log(s) + \log(h_0) + \epsilon_h \leftrightarrow h(s) = h_0.s^\beta \quad (3)$$

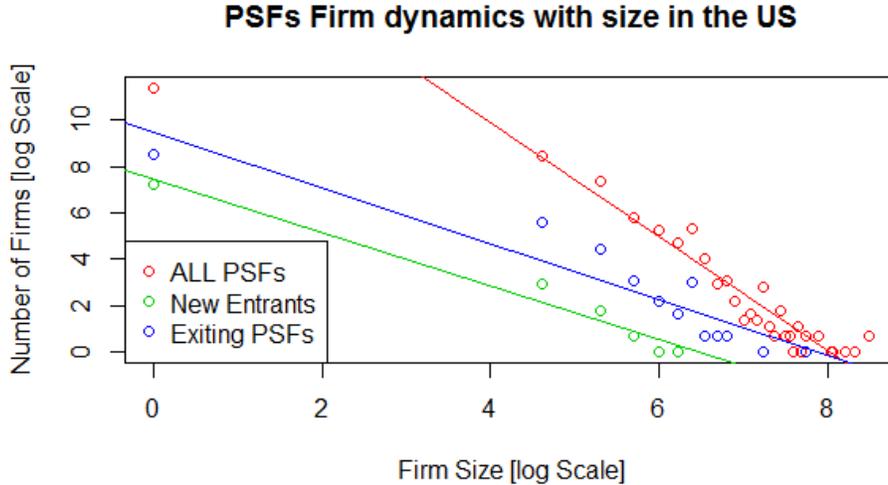


Figure 1: US PSFs size distributions.

Parameter	Estimate (std)	Parameter	Estimate (std)	Parameter	Estimate (std)
α	-2.45 (0.15)	γ	-1.19 (0.15)	β	-0.95 (0.12)
$\log(m_0)$	19.71 (1.06)	$\log(j_0)$	9.46 (1.19)	$\log(h_0)$	7.09 (0.71)

Table 1: PSFs distributions Calibration

The results displayed in table (1) show that the bigger the firm size category, the bigger the proportion of new entrants over existing firms (e.g. $\beta < -\alpha$). Additionally it appears that the bigger the PSFs the bigger the chance of ceasing operations (e.g. $\gamma < -\alpha$). This means that the professional services sector appears less stable for bigger firms. This doesn't mean that bigger firms have an higher chance of running out of business. It can however be interpreted in terms of mergers, acquisitions and divestitures (e.g. the larger PSFs landscape is subject to a high number of recompositions.)

Note that if the proposed regression appears reasonable (e.g. $R^2 > 90\%$) the distributions associated to the mid to large sized firms doesn't seem to be similar to the small - mid segment. When looking at figure (1), one can indeed see that the slopes of the regressions are higher on the large size firms than the small ones. This is not yet something that will be covered in this paper.

2.2.2 PSFs Dynamics modulation with age.

To assess the age dependency of the dynamics (m, j) described in the previous section in equation (1), I reformed the associated regressions on each of the age segments in the dataset (e.g. regression conditional on age)(see (2.1)). Note that because of the definition of the entry distribution h (e.g. PSFs of age 0), there is no need to investigate its age dependency.

The obtained results are described in the figures (2) and (3). This showed that the previously described parameters $(\alpha; \gamma; m_0; j_0)$ for the existing and exiting PSFs distribution are age dependent.

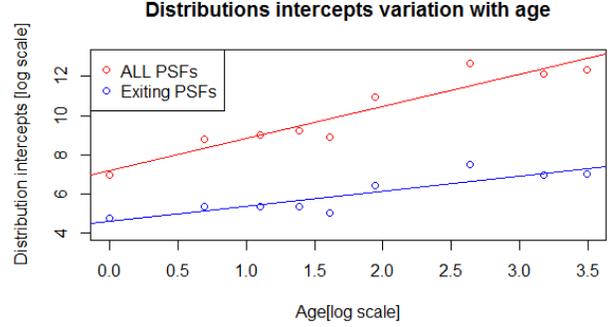
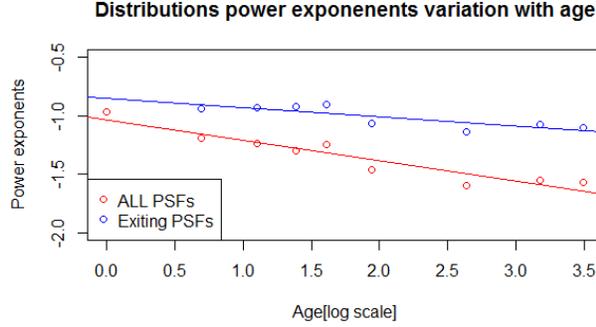


Figure 2: US PSFs dynamics Power exponent Age modulation . Figure 3: US PSFs dynamics intercepts age modulation.

This naturally suggests the following regressions:

$$\alpha(a) = \alpha_0 + \alpha_a \cdot \log(a) + \epsilon_\alpha \quad (4)$$

$$\gamma(a) = \gamma_0 + \gamma_a \cdot \log(a) + \epsilon_\gamma \quad (5)$$

$$\log(m_0)(a) = \log(M) + m_a \cdot \log(a) + \epsilon_M \leftrightarrow m_0(a) = M \cdot a^{m_a} \quad (6)$$

$$\log(j_0)(a) = \log(J) + j_a \cdot \log(a) + \epsilon_J \leftrightarrow j_0(a) = J \cdot a^{j_a} \quad (7)$$

The calibration results are shown in tables (2) and (3).

Parameter	Estimate	Parameter	Estimate
m_a	1.63 (0.21)	j_a	0.77 (0.15)
$\log(M)$	7.19 (0.43)	$\log(J)$	4.58 (0.32)

Table 2: PSFs distributions intercepts age calibration

Parameter	Estimate	Parameter	Estimate
α_a	-0.17 (0.02)	γ_a	-0.078 (0.02)
α_0	-1.04 (0.05)	γ_0	-0.85 (0.05)

Table 3: PSFs distributions power exponent age calibration

The results displayed on table (2) and (3) show that the probability that a PSF exit the market lowers with age at constant size as $j_a < m_a + (\alpha_a - \gamma_a) \log(s)$. It also shows that a constant age, PSFs exit probability still grows with respect to size but in a more balanced fashion than previously anticipated in section (2.2.1) as $\gamma - \alpha > \gamma_0 - \alpha_0 + (\gamma_a - \alpha_a) \cdot \log(a + 1)$.

Note that this aligns with the large body of literature regarding the age effect on firms. It also shows that exit rates can be perceived as having separate variables as $\gamma_a \ll \gamma_0$, which means that whatever the firm size, the same exit patterns with age hold. However this doesn't hold from a overall firm

distribution perspective. This then translates into the first heuristically justified approximation for m and j :

$$m(s, a) = M.(a + 1)^{m_a} .s^{\alpha_0 + \alpha_a .\log(a+1)} \quad (8)$$

$$j(s, a) = J.(a + 1)^{j_a} .s^\gamma \quad (9)$$

2.3 PSFs' empirical profit structure.

In this section, I show that, at equal size, PSFs profit, defined as the difference between firms revenues r and direct costs (approximated as the firm's payroll p), not only increases with age but also becomes more predictable. To do so, I run regressions respective to firms size s and age a on the dataset revenue and payroll informations.

Regressions are first run on size only for the entire dataset in section (2.3.1) to show that size is a good preliminary proxy to assess PSFs profit. Regressions are then rerun for each age categories (see (2.1)) in the dataset in section (2.3.2) where changes in the regression parameters according to age are modeled and discussed. Finally the regressions error rates evolution is analyzed in section (2.3.3) to illustrate the predictability results previously highlighted.

2.3.1 Links between PSFs size and their profit structure.

PSFs revenues r and costs p appear to be increasing and to exhibit decreasing returns with respect to their size s . They can be mathematically approximated by power laws which can easily be calibrated using simple linear regressions when rescaling the data using a log transformation (see figures (4) and (5)):

$$\log(r(s)) = \log(r_0) + \nu.\log(s) + \epsilon_r \leftrightarrow r(s) = r_0.s^\nu \quad (10)$$

$$\log(p(s)) = \log(p_0) + \phi.\log(s) + \epsilon_p \leftrightarrow p(s) = p_0.s^\phi \quad (11)$$

Where ϵ_r (resp. ϵ_p) represents the revenue (resp. payroll) regression error term.

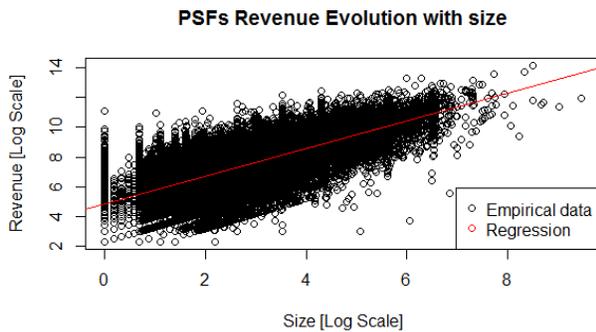


Figure 4: US PSFs Revenue structure.

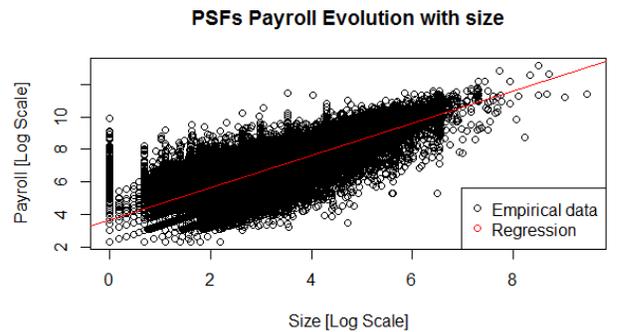


Figure 5: US PSFs Direct costs structure.

The calibration results are shown in table (4). Without surprises, revenue concavity is statistically significant, while direct costs (e.g. payroll) can actually be considered as linear in size.

Note that the proposed regression have a $R^2 \approx 60\%$ on both the revenue and cost data. Size is indeed not the only explanatory variable in a firm financial activity. To firm explore the nature of PSFs profit structure heterogeneity, it would be tempting to get a quality indicator as per services delivered but this is not something, to my knowledge, that is publicly available.

Parameter	Estimate	Parameter	Estimate
ν	0.93 (0.003)	ϕ	0.99 (0.002)
$\log(r_0)$	4.82 (0.008)	$\log(p_0)$	3.65 (0.008)

Table 4: PSFs revenue and cost functions calibration

2.3.2 Differences in PSFs profit structure with age.

To assess the age dependency of PSFs revenues r and costs p , I have reperformed the regressions associated to equations (10) in a similar fashion to section (2.2.2). The results displayed in figures (7) and (6) show that both the revenue (and costs) intercept and power exponent are changing with respect to age. This led to considering the following equations:

$$\log(r_0)(a) = \log(R) + r_a \cdot \log(a) + \epsilon_R \leftrightarrow r_0(a) = R \cdot a^{r_a} \quad (12)$$

$$\log(p_0)(a) = \log(P) + p_a \cdot \log(a) + \epsilon_P \leftrightarrow p_0(a) = P \cdot a^{p_a} \quad (13)$$

$$\nu(a) = \nu_0 + \nu_a \cdot \log(a) + \epsilon_\nu \quad (14)$$

$$\phi(a) = \phi_0 + \phi_a \cdot \log(a) + \epsilon_\phi \quad (15)$$

Where $\epsilon_\nu, \epsilon_\phi, \epsilon_R, \epsilon_P$ represent the regression error terms.

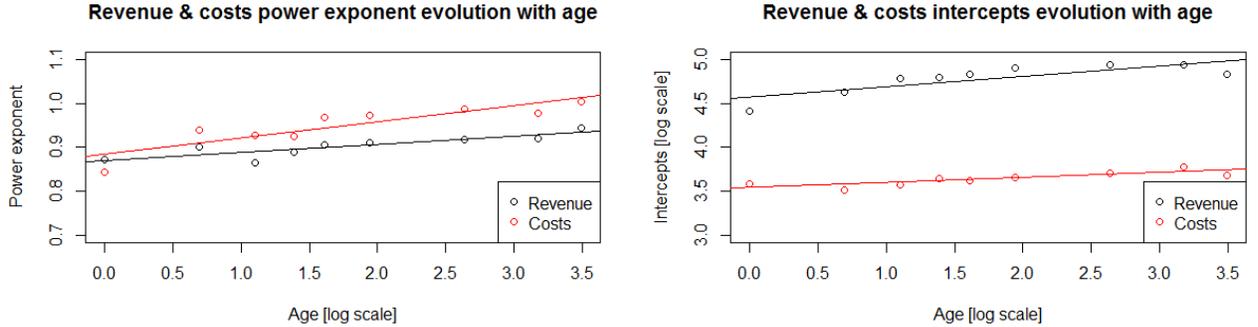


Figure 6: US PSFs elasticities modulation with age. Figure 7: US PSFs intercepts modulation with age.

Calibration results are displayed in tables (5) and (6). They show that the power exponent modulation associated to age ν_a is weak and can be neglected, such that the revenue function can be approximated by a function with separate variables. On the costs side, the intercept modulation with age p_a is also weak and will be neglected.

Parameter	Estimate	Parameter	Estimate
r_a	0.12 (0.032)	p_a	0.057 (0.01)
$\log(R)$	4.57 (0.07)	$\log(P)$	3.54 (0.03)

Table 5: PSFs revenue and cost age calibration

Parameter	Estimate	Parameter	Estimate
ν_a	0.018 (0.01)	ϕ_a	0.037 (0.01)
ν_0	0.87 (0.01)	ϕ_0	0.88 (0.02)

Table 6: PSFs revenue and cost age calibration

This then can be approximated into a second heuristical result related to r and p :

$$r(s, a) = R.(a + 1)^{r_a} .s^\nu \quad (16)$$

$$p(s, a) = p_0 .s^{\phi_0 + \phi_a . \log(a+1)} \quad (17)$$

The form of the revenue is interesting as it could suggest that prices approximated by $R(a + 1)^{r_a}$ are increasing with age, which raises the question of the establishment of a brand for PSFs that enables them to charge higher rates. Note that this effect is independent of size considerations. On the other hand, the payroll structure shows that age and size effects are mixed, so that older firm indeed have a quasi linear cost structure in size, but younger firms exhibit a diminishing return payroll structure with size. This means that at equal size PSF profit (defined as a % of their revenue (e.g. $1 - p/r$)) is increasing with age for small firms ($-r_a + \phi_a . \log(s) > 0$) and decreasing with age for large ones. This could suggest that young firms may be decreasing the share of their profit they transfer to their employees to provision for unexpected event as they know that they have a higher probability to exit the PSFs sector than their older counterparts. However in the absence of additional data, it is difficult to generate a robust interpretation.

2.3.3 PSFs profit becomes more predictable as they age.

The age patterns on PSFs profits described in section (2.3.2) do not yet fully account for firm profit heterogeneity. However, it appears that the variance of regressions error terms (ϵ_R and ϵ_P) on the difference age bands is decreasing with age (see figures (8) and (9) below). This suggests that the older the firms, the more predictable their profit structure is.

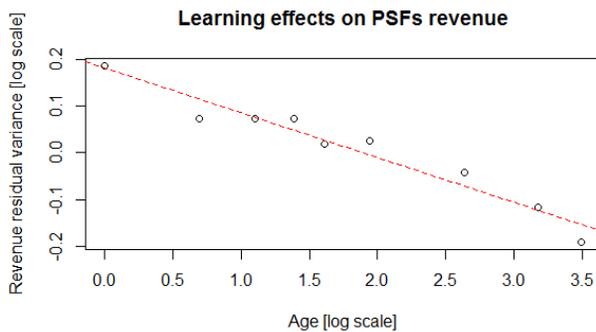


Figure 8: PSFs revenue uncertainty evolution

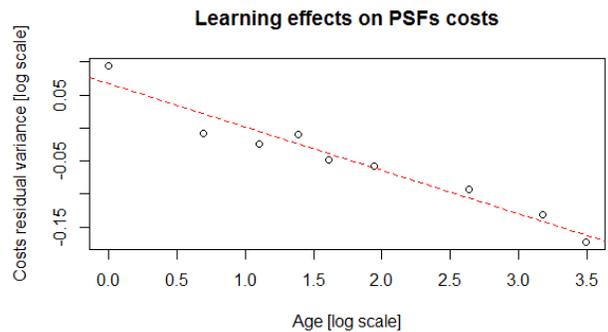


Figure 9: PSFs cost uncertainty evolution.

When rescaled with a log transformation, the variances evolution with age exhibit a linearly decreasing behavior which suggest that they can be reasonably modeled by a power law, so that the revenue r

and costs p become:

$$r(s, a) = R.(a + 1)^{r_a}.s^{\nu+\sigma_R.(a)^{\Theta_R}.W_R} \quad (18)$$

$$p(s, a) = p_0.s^{\phi_0+\phi_a.\log(a+1)+\sigma_P.(a)^{\Theta_P}.W_P} \quad (19)$$

Where W_R and W_P represents two gaussian random variable that are centered and of variance 1. The calibration results associated to the equations (18) are given in table (7).

Parameter	Estimate	Parameter	Estimate
Θ_R	-0.095 (0.009)	Θ_P	-0.066 (0.005)
$\log(\sigma_R)$	0.18 (0.018)	$\log(\sigma_P)$	0.067 (0.012)

Table 7: PSFs revenue and cost uncertainty calibration

These results show that revenue is less predictable than costs ($\sigma_R > \sigma_p$) and that the increase in revenue predictability with age is higher on the revenue than on the costs (e.g. $\Theta_R < \Theta_P < 0$). Additionally revenue & costs uncertainties (e.g W_R and W_P) are strongly correlated (e.g. correlation $> 70\%$) as displayed on figure (10). This correlation appears to slightly drop by about 10 points in the firms first years of operations (firm's age < 5 years) to stabilize around 75%. Noting $\rho_W(a)$ the

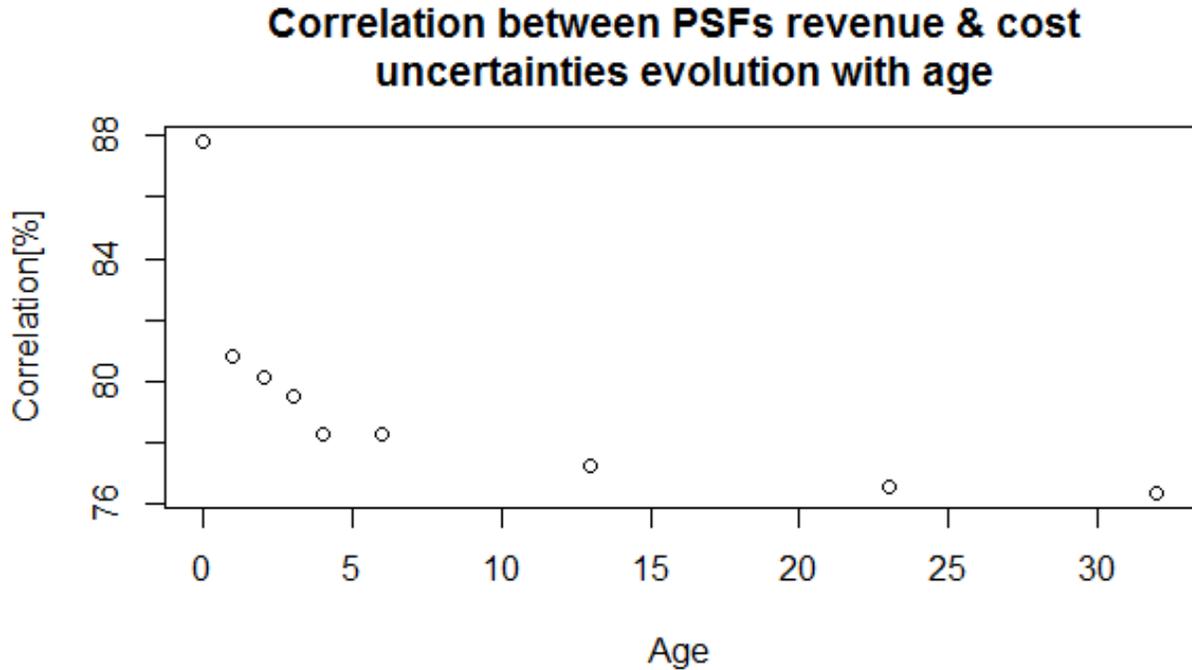


Figure 10: PSFs revenue and costs uncertainty correlations

correlation between PSFs revenue and costs, a reasonable approximation could be to fine tune the

profit structure model as:

$$r(s, a) = R.(a + 1)^{r_a}.s^{\nu+\sigma_R.(a)^{\Theta_R}.\hat{W}_R} \quad (20)$$

$$p(s, a) = p_0.s^{\phi_0+\phi_a.log(a+1)+\sigma_P.(a)^{\Theta_P}.\sqrt{(\rho_W(a)).\hat{W}_R}+\sqrt{(1-\rho_W(a))\hat{W}_P}} \quad (21)$$

Where \hat{W}_P and \hat{W}_R are independent. The magnitude of the correlation between costs and revenue suggest that the question of firm heterogeneity in profit is actually a question of firm heterogeneity in revenue. To improve the current model and understand the source of the heterogeneity between firms, it would be interesting to explore competition effects as well as potential quality discrepancies between firms.

2.4 Comparison to other industrial sectors.

To discuss the specificities of PSFs growth pattern compared to other industries, I re-ran the analysis of sections (2.2) and (2.3) for different industrial sectors. I first compared PSFs dynamics (2.4.1) and then PSFs profit structures (2.4.2) to the one of the US mining industry (NAICS code 21) as well as to the US finance and insurance industry (NAICS code 52) and the US Wholesale industry (NAICS code 42). This section shows that PSFs appear close to fast paced sector such as the wholesale one.

2.4.1 Differences in dynamics.

From a firm dynamics standpoint, the calibration results reported in table (8) show that the PSFs sector distribution m appears to have the same aging dependencies as the wholesale sector (α_a and m_a). This may be explained by the fact that both sectors share a similar business to business model. Out of the 4 selected sectors, the PSFs one appears to be the easiest to enter. The coefficient α_0 is indeed the lowest for PSFs, which means that the proportion of small news entrants is more important for PSFs than for the other sectors. With respect to the entry process, PSFs appears close to both financial and wholesale firms.

Parameter	PSFs	Finance	Mining	Wholesale
m_a	1.63 (0.20)	1.31 (0.14)	1.07 (0.14)	1.97 (0.19)
$\log(M)$	7.19 (0.43)	6.01 (0.30)	1.68 (0.29)	5.40 (0.39)
α_a	-0.17 (0.02)	-0.11 (0.019)	-0.16 (0.03)	-0.18 (0.02)
α_0	-1.04 (0.04)	-0.96 (0.04)	-0.37 (0.07)	-0.89 (0.04)
j_a	0.77 (0.15)	0.28 (0.14)	-	0.75 (0.14)
$\log(J)$	4.58 (0.32)	4.43 (0.29)	-	3.48 (0.29)
γ_a	-0.078 (0.02)	-0.059 (0.042)	-	-0.01 (0.033)
γ_0	-0.85 (0.047)	-0.78 (0.10)	-	-0.89 (0.069)

Table 8: Overall and exiting firms distribution parameters across selected industries

As per entry dynamics, differences arise among sectors as seen on figure (11). The PSF sector exhibits a far greater number of small new entrants compared to the finance and wholesale sector (which is reflected in α_0). Power exponents associated to the power law models similar to equation (1) are

indeed leading to a β that is close to -1.19 for PSFs, -0.82 for the financial industry, -0.97 for the wholesale sector and -0.53 for mining. The mining industry model appears to be substantially different, and similar exercises do not lead to good fits as there are only small number of new entrants due to capitalisation issues that are creating entry costs. But the other industries exhibit a smaller number of small size new entrants, which may indicate that competition is sharp and doesn't leave room for small players that are not that efficient from a margin standpoint.

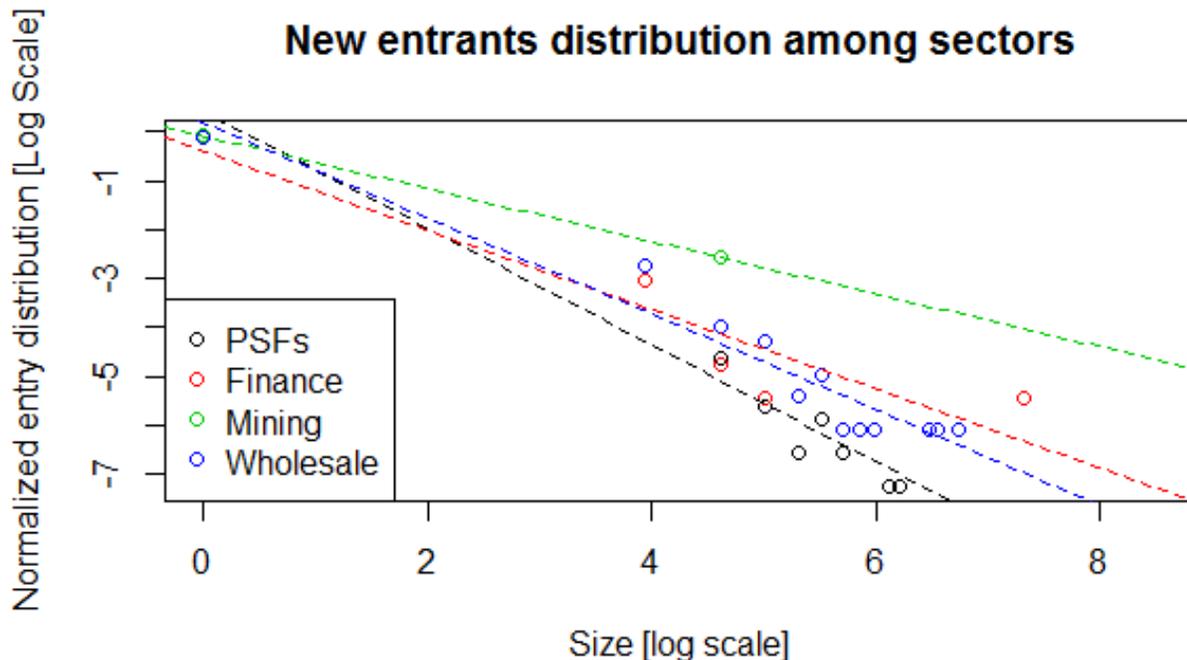


Figure 11: Selected industries normalized entry distributions.

From a exiting dynamics point of view, the PSFs sector behavior is closer to the wholesale sector than from the financial one. At constant size, there is indeed an concave increase j_a in the exiting flow amongst firm that is shared between wholesale & PSFs that is not appearing in the financial sector. This could suggest a need to adapt the firms to make them last. Note that there is not enough exits in the mining industry sample to draw any conclusions.

2.4.2 Differences in profit structure.

PSFs exhibit strong similarities with wholesale firms from both a dynamic standpoint and a profit structure perspective (see table (9) and figure (12)).

First, from a revenue perspective, the PSFs and wholesale sector share a similar "price" evolution process with age r_a . This could suggest that both sectors share a similar brand establishment mechanism towards their client. Also note that both sectors do not exhibit strong aging effect with respect to their "production" elasticities ($\nu_a \ll \nu_0$). Interestingly, if the financial sectors shares some of the "production" features of the PSFs sector, it has a different behavior when it comes to "price". As per the mining industry, production and price features are completely different to the point that age is

not even statistically significant when it comes to the profit structure.

Parameter	PSFs	Finance	Mining	Wholesale
r_a	0.12 (0.032)	0.077 (0.03)	0.029 (0.207)	0.14 (0.03)
$\log(R)$	4.57 (0.067)	4.74 (0.072)	5.06 (0.43)	5.95 (0.06)
ν_a	0.018 (0.004)	0.034 (0.005)	-0.016 (0.062)	0.014 (0.010)
ν_0	0.87 (0.008)	0.85 (0.012)	1.06 (0.131)	0.82 (0.021)
p_a	0.057 (0.01)	0.058 (0.03)	-0.059 (0.13)	0.048 (0.05)
$\log(P)$	3.54 (0.02)	3.54 (0.06)	3.51 (0.29)	3.60 (0.10)
ϕ_a	0.037 (0.007)	0.036 (0.014)	0.017 (0.044)	0.036 (0.015)
ϕ_0	0.88 (0.015)	0.88 (0.029)	1.00 (0.09)	0.86 (0.033)

Table 9: Revenue and cost functions calibration across selected industries

From a cost structure perspective (e.g. payroll), the PSFs, financial and wholesale industries present exactly the same structure (p_a, ϕ_a, ϕ_0). They share the same age effect that doesn't appear on the mining industry: at age constant, the average payroll per employee may be decreasing with firm size ($\phi_0 + \phi_a \cdot \log(a + 1) < 1$).

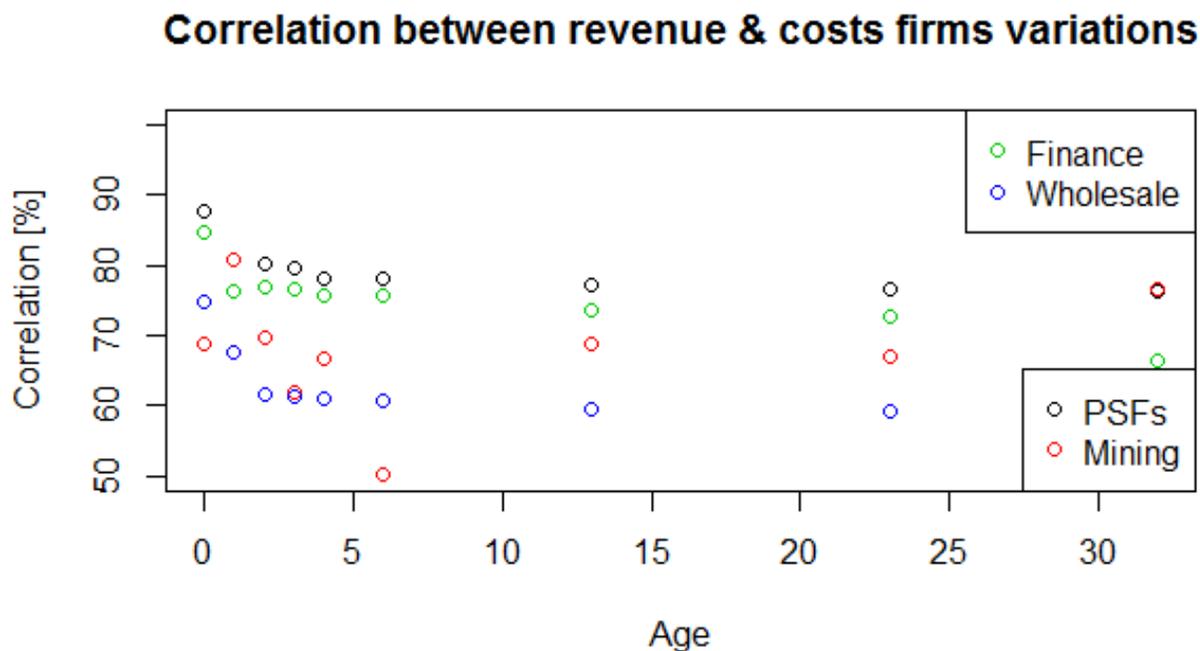


Figure 12: Selected industries revenue vs cost variations correlations.

Finally from a profit predictability standpoint, with the expectations of the mining industry, all the selected sectors exhibit an increased predictability with age, while there is a decreasing correlation between their firm level revenue and payroll structure variations with age. This could mean that old

firms do not share the revenue fluctuations risks with their employee the same way young firms do.

In summary, in section (2) three main facts were highlighted. First, young PSFs, that are typically small, exhibit different dynamics than their older counterparts. Second, at equal size, PSFs profit is increasing with age and becomes more predictable. Third, when compared with other industrial sectors, PSFs firms profit and dynamics differ from capital intensive industries such as mining or manufacturing but appear close to fast paced sectors such as the wholesale one. I now propose to investigate how PSFs profit maximization behavior can yield the observed macro dynamics via the notion of firm growth (section (3)).

3 A model of PSFs growth.

In the previous section (2), it was shown that PSFs macro economic structure (e.g. distribution, entry/exit rate) could be modeled by multivariate power laws with respect to size and age. The same type of mathematical models was shown to be a suitable approximation for PSFs revenues and costs. I now demonstrate that micro economic profit foundations can be used in conjunction with the main macro economic indicators to propose an optimal growth path for PSFs that ends up being convex in size (e.g. the bigger the firm, the faster its development), which means that PSFs growth doesn't follow gibrat's law. I also show that this growth comes at a cost, that can be considered linear in size (e.g. there is no return to scale on transformation).

To do so, I start by investigating the effect of size on growth speed and transformation costs with respect to size in section (3.1). I then expand the demonstration to take into account the previously mentioned age dependencies of PSFs in section (3.2). Finally I show that the age effects related to PSFs growth could be linked to their ability to sort high performing workers and suggest several future avenues of research in the field of personal economics.

3.1 Relationships between US PSFs growth & size.

In this section, the PSFs growth and transformation costs are investigated in a power law setting with respect to firm size. First, leveraging Kolmogorov equations in a fashion similar to [14], I show that US PSFs evolution can be approximated by a power law in size in (3.1.1). I then demonstrate in (3.1.2) that profit maximizing PSFs must be subject to linear transformation costs to follow this growth pattern. This second part is based on the underlying theory behind Hamilton Jacobi Bellman (HJB) equations. This section hence relies heavily on an application of the mean field game theory developed by [19].

3.1.1 US PSFs growth speed.

To investigate the PSFs growth pattern, I assume that PSFs of size S_t are growing at a speed $\mu(S_t)$ such that: $dS_t = \mu(s)dt$. Leveraging the notation of section (2), where $m(t, s)$ is the number of PSFs of size s at time t , $h(s)$ (resp. $j(s)$) the number of new firms (resp. exiting firms) in the country, it comes that:

$$\partial_t m + \partial_s(\mu \cdot m) = h(s) - j(s) \tag{22}$$

Leveraging the work done in the previous section (2.2), I assume a steady state with $m(s) = m_0 s^\alpha$, $h(s) = h_0 s^\beta$ and $j(s) = j_0 s^\gamma$. Note that this hypothesis of steady state with respect to size appears reasonable in light of the data presented in the statistic of the US businesses programs covering the last 20 years

(e.g. 1998 to 2018). This indeed shows that PSFs distribution, entry and exit rates haven't drastically changed.

Under the steady state assumption, firms' growth speed obeys the following relationship:

$$d_s \mu = -\alpha \cdot \mu_1 s^{-\alpha-1} + \frac{h_0}{m_0(\beta+1)}((\beta-\alpha+1) \cdot s^{\beta-\alpha} + \alpha \cdot s^{-\alpha-1}) - \frac{j_0}{m_0(\gamma+1)}((\gamma-\alpha+1) \cdot s^{\gamma-\alpha} + \alpha \cdot s^{-\alpha-1})$$

Which leads to:

$$\mu(s) = \mu_1 s^{-\alpha} + \frac{h_0}{m_0(\beta+1)} \cdot (s^{\beta-\alpha+1} - s^{-\alpha}) - \frac{j_0}{m_0(\gamma+1)} \cdot (s^{\gamma-\alpha+1} - s^{-\alpha}) \quad (23)$$

Using the data described in (2.2), the growth speed of PSFs appears convex (see figure (13)). Note that this may generate the concentration phenomena previously observed in figure (1). In any case, the main fact associated to this model is that PSFs evolution does not follow Gibrat's law.

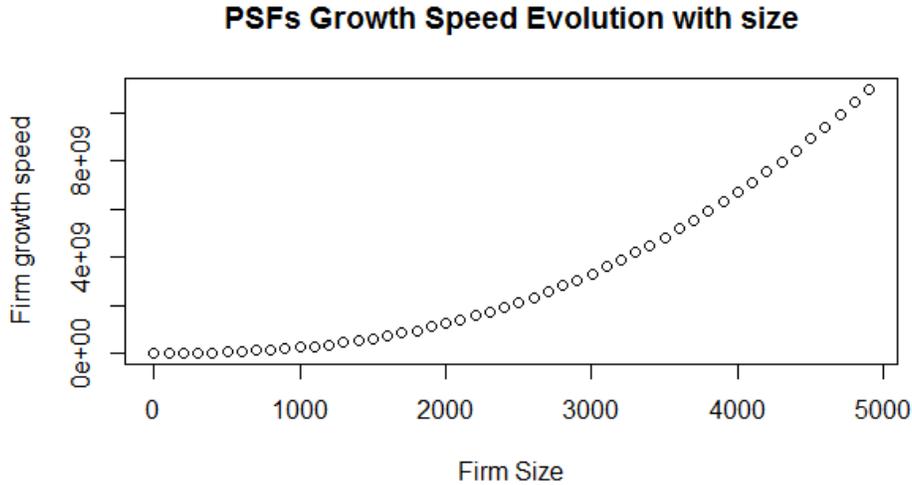


Figure 13: US PSFs Growth.

3.1.2 PSFs Costs of transformation.

As stressed in section (3.1.1), PSFs growth is not instantaneous. This intuitively means that there is a form of friction that prevent firms to jump from one state to the other. The idea developed in this section is that this friction can be modeled and estimated by a transformation cost term c . This cost can for example account for advertisement fees necessary to connect to new clients and to source new employees and from a firm age perspective, this cost could be linked to a reputation building phenomenon. In a one dimensional framework, cost c would depend in the growth speed $\mu(\cdot)$ and the firm size s . However as shown ealier, $\mu(\cdot)$ can be approximated by a power law of s , making c only dependent in μ or s .

In one dimension, PSFs revenue (resp. payroll) structure follows a concave relationship $r(s) = r_0 s^\nu$ (resp. $p(s) = p_0 s^\phi$). Given their initial state s , PSFs are interested in growing at speed $\mu(s)$ to increase their margin. As the profit optimization Π problem over a given time horizon T is the same as defining a growth program, it is better to look at the transformation costs as a function of $\mu(\cdot)$. PSFs margin $\pi(s)$ can therefore be expressed as $\pi(s, \mu) = r(s) - p(s) - c(\mu)$. Adding an interest rate ρ , the PSFs optimization program is given by:

$$\Pi(t, s) = \max_{\mu(\cdot)} \mathbb{E} \left(\int_t^{t+T} e^{-\rho \cdot (x-t)} \cdot (r(S(x)) - p(S(x)) - c(\mu(x))) dx \right)$$

The PSF growth therefore obeys a classical Hamilton Jacobi Bellman type of problem that can be rewritten as:

$$\partial_t \Pi + \max_{\mu(\cdot)} (\mu \cdot \partial_s \Pi - \rho \cdot \Pi + r(s) - p(s) - c(\mu)) = 0 \quad (24)$$

This leads to $\partial_s \Pi = d_\mu c(\mu)$. Because μ depends in s , it is possible to look for a functional form for the transformation costs that is s dependent: $c(\mu(s)) = \hat{c}(s)$. Further differentiating the equation (24) toward s leads to:

$$\partial_{ss}(\hat{c}) - \frac{\rho}{\mu} \partial_s \hat{c} = \frac{\partial_s(p - r)}{\mu \cdot \partial_\mu(s)}$$

This can be fully integrated using a simple solver to estimate the transformation cost of a PSF. But when simplifying the problem, the following proposition holds.

Proposition 1 *Assuming the PSFs growth speed and the PSFs margin structure follows a power law (e.g. $\mu = \mu_1 \cdot s^{-\alpha}$, $r(s) - p(s) = \omega_0 \cdot s^\psi$) yields linear transformation costs in size:*

$$\hat{c}(s) = \hat{c}_1 + \left(\partial_s \hat{c}_1 - \frac{\alpha \cdot \omega_0 \cdot \psi}{(1 - \psi)} \right) (s - 1) + \frac{\alpha \cdot \omega_0}{(\psi - 1)} \cdot (s^\psi - 1) \approx \left(\partial_s \hat{c}_1 - \frac{\alpha \cdot \omega_0 \cdot \psi}{(1 - \psi)} \right) (s - 1)$$

Proof* If PSF growth speed (see eq.(23)) follows a power law $\mu = \mu_1 \cdot s^{-\alpha} \leftrightarrow s = \left(\frac{\mu}{\mu_1} \right)^{-\frac{1}{\alpha}}$, it comes that:

$$\partial_{ss} \hat{c}(s) - \frac{\rho \cdot s^\alpha}{\mu_1} \cdot \partial_s \hat{c}(s) = \alpha \cdot (r_0 \cdot \nu \cdot s^{\nu-2} - p_0 \cdot \phi \cdot s^{\phi-2})$$

Hence:

$$\partial_s \hat{c}(s) = \partial_s \hat{c}_1 \cdot e^{\frac{\rho}{\mu_1 \cdot (\alpha+1)} \cdot s^{\alpha+1}} + \alpha \cdot \int_1^s (r_0 \cdot \nu \cdot s^{\nu-2} - p_0 \cdot \phi \cdot s^{\phi-2}) \cdot e^{\frac{\rho}{\mu_1 \cdot (\alpha+1)} \cdot s^{\alpha+1}} dx$$

Let's assume for the sake of simplicity that $\rho = 0$ (e.g. no interest rate) and that the profit structure can be simplified to $r(s) - p(s) = \omega_0 \cdot s^\psi$ with $\omega_0 > 0$ and $\psi \in [0; 1]$. In this case:

$$\partial_s \hat{c} = \partial_s \hat{c}_1 + \frac{\omega_0 \cdot \psi \cdot \alpha}{(\psi - 1)} \cdot (s^{\psi-1} - 1)$$

Assuming that $\partial_s \hat{c}_1 > \frac{\omega_0 \cdot \psi \cdot \alpha}{(\psi-1)}$, this means that transformation costs increase (otherwise decrease) linearly for large PSFs as:

$$\hat{c}(s) = \hat{c}_1 + \left(\partial_s \hat{c}_1 - \frac{\alpha \cdot \omega_0 \cdot \psi}{(1 - \psi)} \right) (s - 1) + \frac{\alpha \cdot \omega_0}{(\psi - 1)} \cdot (s^\psi - 1) \approx \left(\partial_s \hat{c}_1 - \frac{\alpha \cdot \omega_0 \cdot \psi}{(1 - \psi)} \right) (s - 1)$$

In light of the results of this paper, the proposition (1) holds asymptotically. This linearity of the transformation costs c is interesting because, assuming that those costs are associated to marketing activities, it shows there is a fixed fee per head in branding without any economies of scale. As the profit exhibit decreasing returns, this linearity means that PSFs growth is actually capped as there is point where the revenue increase doesn't mitigate the fixed branding fees (e.g. $\partial_s \pi < 0$).

3.2 Impact of age on PSFs evolution.

In the previous section (3.1), it was shown that with respect to size, PSFs dynamics and profit maximization behavior leads to a convex growth speed and a size cap because of linear transformation costs. If these size considerations are meaningful, they have be nuanced by taking into account the effect of age on PSFs evolution as shown in section (2). This will be done here by first looking at the growth speed modulation with age (3.2.1) and then by discussing the effects of age on transformation costs (3.2.2).

3.2.1 Age & growth speed.

To account for age A_t impact on PSFs dynamics, a couple of updates to the framework described in (3.1.1) are required. First of all, PSFs size S_t is growing s.t: $dS_t = \mu(S_t, A_t)dt$. And the equation (22) on the PSFs repartition $m(t, s, a)$ becomes:

$$\partial_t m + \partial_s(\mu.m) + \partial_a(m) = -j(s, a) \quad (25)$$

Subject to the boundary conditions $m(t, s, a = 0) = h(s)$ (e.g. firms that enter the market have an age equal to 0). Leveraging the work done in section (2.2) (e.g. steady states of distribution and 2D formulations), the following can be written:

$$\partial_s(\mu.s^{\alpha_0+\alpha_a.\log(a+1)}) + (m_a + \frac{\alpha_a}{s}).(a+1)^{-1}.s^{\alpha_0+\alpha_a.\log(a+1)} = -\frac{J}{M}.(a+1)^{j_a-m_a}.s^\gamma \quad (26)$$

Therefore:

$$\begin{aligned} \mu(s, a) = & \mu(1, a).s^{-\alpha_0-\alpha_a.\log(a+1)} - \\ & \frac{J}{M.(\gamma+1)}.(a+1)^{j_a-m_a}.(s^{\gamma-\alpha_0-\alpha_a.\log(a+1)} - s^{-\alpha_0-\alpha_a.\log(a+1)}) \\ & - \frac{m_a}{\alpha_0 + \alpha_a.\log(a+1) + 1}.(a+1)^{-1}.(s - s^{-\alpha_0-\alpha_a.\log(a+1)}) \\ & - \frac{\alpha_a}{\alpha_0 + \alpha_a.\log(a+1)}.(a+1)^{-1}.(1 - s^{-\alpha_0-\alpha_a.\log(a+1)}) \end{aligned}$$

In light of the quantitative structure described in (3.1.1), the dominating term in μ is the one in $s^{-\alpha_0-\alpha_a.\log(a+1)}$, which means that the power law structure seen in figure (13) is preserved:

$$\mu(s, a) \approx \hat{\mu}(a).s^{-\alpha_0-\alpha_a.\log(a+1)} \quad (27)$$

Note that here additional longitudinal data would be required to further calibrate the term in $\hat{\mu}(a)$. Yet it is easy to see that older firms are proportionally changing faster than younger ones (e.g. at constant size $\frac{\partial_s \mu}{\mu}$ is higher for older firms).

3.2.2 Age & transformation costs.

Now that is has been shown that including the age considerations on growth preserves the growth speed power law structure and that with respect to size PSFs evolution is boosted by age, an update can be performed to estimate the impact of age on the transformation costs. The HJB equation (24) must be updated towards:

$$\partial_t \Pi + \max_{\mu(\cdot)}(\mu.\partial_s \Pi + \partial_a \Pi - \rho.\Pi + r(s, a) - p(s, a) - c(\mu, \cdot)) = 0 \quad (28)$$

This leaves the condition $\partial_s \Pi = \partial_\mu c$ unchanged. Further differentiating in s leads to:

$$\partial_s(\mu \cdot \partial_\mu c) + \partial_a(\partial_\mu c) - \rho \cdot \partial_\mu c - \partial_s(r - p - c) = 0$$

Because the power law approximation for PSFs growth speed still holds, transformation costs c are actually a function of s and a . Simplifying the problem by putting $\rho = 0$ and looking for asymptotic solutions that are linear in size $c(s, a) \approx \hat{c}(a) \cdot s$ to be consistent with the previously developed results, it comes that:

$$\hat{c}'(a) + \hat{c}(a)(\partial_\mu(s) + \partial_s(\mu \cdot \partial_\mu(s))) \approx 0$$

As for large s the terms $\partial_\mu(s)$ can be neglected, it comes that:

$$\hat{c}'(a) - \hat{c}(a) \cdot \left(\frac{1}{\alpha_0 + \alpha_a \cdot \log(a+1)} \right) \approx 0 \quad (29)$$

The equation (29) therefore shows that asymptotically age reduces transformation costs. Thinking again in terms of marketing activity, this can be associated to notions of brand. The older the firm, the more established its brand. Therefore the need for marketing budget to source employee and clients is getting lower. Interestingly, this also means that if there is a cap in terms of PSFs size above which it is not interesting to grow (see section (3.1.2)), this cap is age dependent. This could mean that older firms, because of their brand can grow above the level of their younger peers. Alternatively, this could mean that older firm do not have the same financial constraints than their younger peers, because for instance younger firms need to provision for unexpected events, which ties back to some of the alternative explanations offered in the previous sections.

3.3 Learning and employee sorting.

It was previously shown that PSFs growth speed could be approximated by a convex power law in size and that older firm were changing faster than young PSFs. Additionally, it was shown that to evolve, PSFs had to incur a cost that could be associated to branding activities that was asymptotically linear in size. The cost per head of branding was also shown to decrease with age at constant size. However if this description was aligned with the patterns observed on the US empirical data, it did not account for interfirm profit heterogeneity. While I do not propose here a full-blown extension of the MFG approach to account for this heterogeneity, I sketch here a direction that appears relevant regarding the workers differences in terms of quality.

Firm profit heterogeneity can be assumed as linked to its employee pool. For the sake of simplicity, the following demonstration is based in one dimension. Imagine that employees can be either high performer H or low performers L , so that a firm size $s = s_H + s_L$. From a profit perspective, assume that the two types of workers yield different results:

$$\omega(s_H, s_L) = \omega_H \cdot s_H^{\psi_H} + \omega_L \cdot s_L^{\psi_L}$$

Where $(\psi_H, \psi_L) \in [0; 1]^2$ and $\psi_H > \psi_L$. Let's note $dS_H = \mu_H \cdot dt$ and $dS_L = \mu_L \cdot dt$. In the spirit of sections (3.1.2), assume linear transformation costs: $c(\mu_H, \mu_L) = C_H \cdot \mu_H^\ominus + C_L \cdot \mu_L^\ominus$. Under this structure it is possible to separate the objectives associated to the high performers from the low ones, such that assuming $\rho = 0$, the HJB equation becomes:

$$\partial_t \Pi_H + \max_{\mu_H(\cdot)} (\mu_H \cdot \partial_{s_H} \Pi_H + \omega_H \cdot s_H^{\psi_H} - C_H \cdot \mu_H^\ominus) = 0$$

Therefore $\partial_{s_H} \Pi = C_H \cdot \Theta \cdot \mu_H^{\Theta-1}$ and:

$$\partial_{s_H} (C_H \cdot (\Theta - 1) \cdot \mu_H^\Theta + \omega_H \cdot s_H^{\psi_H}) = 0$$

This leads to different growth speeds for each population with the firm:

$$\mu_H = \left(\frac{-C_H + \omega_H \cdot s_H^{\psi_H}}{C_H \cdot (\Theta - 1)} \right)^{\frac{1}{\Theta}} \approx \hat{C}_H \cdot s_H^{\frac{\psi_H}{\Theta}}$$

Similarly I take:

$$\mu_L \approx \hat{C}_L \cdot s_L^{\frac{\psi_L}{\Theta}}$$

This means that:

$$\forall k \in \{H; L\}; dS_k = \hat{C}_k \cdot S_k^{\frac{\psi_k}{\Theta}} dt \leftrightarrow S_k(t) = (S_k(0))^{-\frac{\psi_k}{\Theta} + 1} + \frac{\hat{C}_k \cdot \Theta}{-\psi_k + \Theta} \cdot t)^{\frac{\Theta}{-\psi_k + \Theta}}$$

As firms grow and age, because $\psi_H > \psi_L$ the proportion of low performer H is asymptotically going to:

$$\frac{S_H}{S_H + S_L} \rightarrow 1$$

This means that as firms evolve, they are accumulating high performers. This obviously raises a question of competition but from a profit standpoint, this means this sorting explains the observed age effect in section (3.2.1):

$$\omega(a, s_H) \approx \omega_H \cdot (s_H)^{\psi_L} \left(1 - \frac{\omega_L}{\omega_H} \cdot \frac{(s_L)^{\psi_L}}{(s_H)^{\psi_H}} \right) \approx \omega_H \cdot (s_H)^{\psi_L} \left(1 - \tilde{C}_a \frac{(\psi_L - \psi_H) \cdot \Theta^2}{(\Theta - \psi_L)(\Theta - \psi_H)} \right)$$

As $\psi_L < \psi_H$ this leads to the same age effect as in the observation. The profit is indeed not only impacted by size but as firms mature, their sorting power enables them to increase their margin per employee. Even though not possible with the current dataset, measuring endogenous ψ_H and ψ_L could be prove interesting, as the higher the fraction at a given point in time of H , the higher the relative increase in s_H . Additionally, further discussion on the tools used to sort employees could be of interest. For recent considerations on employee sorting, see [4]. Note that this sktech is yet limited as this doesn't account for any resource constraint (e.g. the supply of high performers H is infinite) nor does it take into considerations competition questions between the firms that may be linked again to questions of brand (see [22] for a preliminary discussion in the legal sector which constitutes a sub-sector of the PSFs industry).

4 Conclusion & next steps.

In this paper, I have shown three main facts relative to US PSFs based on US census data. First the young US PSFs, that are typically small, exhibit different dynamics than their older counterparts. Second, at equal size, PSFs profits are increasing with age and becomes more predictable. Third, when compared with other industrial sectors, PSFs firms profit and dynamics differ from capital intensive industries such as mining or manufacturing but appear close to fast paced sectors such as the wholesale one.

Additionally, I have shown that US PSFs dynamics and profit structure can be used to yield results on

PSFS growth speed and transformation costs associated to PSFs branding activities. Under the current observations, US PSFs growth can be approximated by a convex power law in size and is increasing with age, so that older PSFs change more quickly than their younger counterparts. Moreover the current US observations lead to an approximation of PSFs transformation costs that are linear in size and decreasing with age. Assuming that transformation costs consist of branding activities, this can be understood as an improvement in PSFs brands with age that require less effort to get access to customer and resources/employees.

I do believe that the final remarks developed in section (3) in terms of PSFs growth pave the way for future research in the field of personal economics and industrial organization applied to the professional service industry. From a personal economics standpoint, it would indeed be interesting to clarify the notion of PSF brand and to link its development to the construction of a firm workforce. From an industrial organization point of view, the questions of customer and resources access that have been approached through the notion of transformation costs could benefit from further micro economic investigations.

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