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Logarithmic foliations (*)

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Abstract. — The purpose of this paper is to study singular holomorphic foliations of arbitrary codimension defined by logarithmic forms on projective spaces.

Résumé. — Nous étudions dans cet article les feuilletages holomorphes singuliers de codimension arbitraire définis par des formes logarithmiques sur les espaces projectifs.

1. Basic definitions and results

Recall that a logarithmic form on a complex manifold M is a meromorphic q-form η on M such that the pole divisors of η and $\mathrm{d}\eta$ are reduced. It is known that a holomorphic form on a compact Kähler manifold is closed. This statement were generalized by Deligne in the context of logarithmic forms as follows:

Theorem 1.1. — *Let η be a logarithmic q-form on a compact Kähler manifold M. Assume that the pole divisor $(\eta)_\infty$ of η is an hypersurface with normal crossing singularities. Then η is closed.*

In the case of germs of closed meromorphic 1-forms there are "normal forms" describing them in terms of the poles and residues (cf. [7]). These normal forms can be translated to the projective spaces and in the logarithmic case they are of the type

$$\eta = \sum_j \lambda_j \frac{\mathrm{d}f_j}{f_j}, \quad \lambda_j \in \mathbb{C}^*, \quad f_j \text{ holomorphic.}$$

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One of our purposes is to generalize the above normal form for p-forms, $p \geqslant 2$, in a special case. We need a definition.

DEFINITION 1.2. — *Let $X \subset (\mathbb{C}^n, 0)$ be a germ at $0 \in \mathbb{C}^n$ of holomorphic hypersurface and $f \in \mathcal{O}_n$ be a reduced germ $f = f_1 \dots f_r$, defining X: $X = (f = 0)$. We say that X has* strictly ordinary singularities outside 0 *if $0 \in \mathbb{C}^n$ is an isolated singularity of f_i (i.e. $(f_i = 0) \setminus \{0\}$ is smooth), $\forall\ 1 \leqslant i \leqslant r$, and X is normal crossing outside the origin.*

It is important to note that Definition 1.2 is different from the usual definition of a germ of hypersurface having normal crossing outside 0. For example, is f is irreducible then $(f = 0)$ has strictly ordinary singularities outside 0 if, and only if, f has an isolated singularity at 0.

Our first main result (see Theorem 2.1) is a generalization of Cerveau–Mattei theorem on normal forms for germs of closed meromorphic 1-forms [7]. For instance, in the case of 2-forms it says that a germ of a closed 2-form η with poles at the strictly ordinary singularities hypersurface $X = (f_1 \dots f_r = 0)$ can be written as

$$\eta = \sum_{i<j} \lambda_{ij} \frac{\mathrm{d}f_i}{f_i} \wedge \frac{\mathrm{d}f_j}{f_j} + \sum_j \mathrm{d}g_j \wedge \frac{\mathrm{d}f_j}{f_j} + \mathrm{d}\alpha,$$

where $\lambda_{ij} \in \mathbb{C}$, $1 \leqslant i < j \leqslant r$, $g_1, \dots, g_r \in \mathcal{O}_n$ and $\alpha \in \Omega^1(\mathbb{C}^n, 0)$. The numbers λ_{ij} are the residues and can be calculated by integral formulas (see Section 2). In Section 2 we will see a precise statement of Theorem 2.1 for germs of closed logarithmic p-forms on $(\mathbb{C}^n, 0)$.

As a consequence of Theorem 2.1 in the general case we get normal forms in the case of logarithmic p-forms on $\mathbb{P}^n$:

COROLLARY 1.3. — *Let η be a logarithmic p-form on $\mathbb{P}^n$, $p \leqslant n-1$. Assume that the divisor of poles $(\eta)_\infty$ is given in homogeneous coordinates by $f_1 \dots f_r$, where the $f_{i's}$ are irreducible homogeneous polynomials on $\mathbb{C}^{n+1}$. Furthermore suppose that the hypersurface $X = (f_1 \dots f_r = 0)$ has strictly ordinary singularities outside $0 \in \mathbb{C}^{n+1}$. Then $r \geqslant p+1$ and there are numbers λ_I, $I \in \mathcal{S}_r^p$, such that in homogeneous coordinates we have*

$$\eta = \sum_{\substack{I \in \mathcal{S}_r^p \\ I=(i_1<\dots<i_p)}} \lambda_I \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \dots \wedge \frac{\mathrm{d}f_{i_p}}{f_{i_p}}, \tag{1.1}$$

where $i_R\eta = 0$.

Notation. — Let us fix homogeneous polynomials $f_1, \dots, f_r \in \mathbb{C}[z_0, \dots, z_n]$. The projectivization of the vector space of p-forms η that can be written as in (1.1) (not satisfying $i_R\eta = 0$ necessarily) will be denoted by

$\mathcal{L}^p(f_1,\dots,f_r)$. The subspace of forms $\eta \in \mathcal{L}^p(f_1,\dots,f_r)$ such that $i_R\eta = 0$ will be denoted by $\mathcal{L}^p_R(f_1,\dots,f_r)$. Note that $\mathcal{L}^p_R(f_1,\dots,f_r) \subsetneq \mathcal{L}^p(f_1,\dots,f_r)$.

We now turn our attention to p-forms defining codimension p foliations. A holomorphic p-form ω, on an open subset $U \subset \mathbb{C}^n$, defines a codimension p distribution, outside its singular set $\mathrm{Sing}(\omega) = \{z \in U \,|\, \omega(z) = 0\}$, if it is *locally totally decomposable* on $U \setminus \mathrm{Sing}(\omega)$. This means that for any $z \in U \setminus \mathrm{Sing}(\eta)$ there are holomorphic 1-forms $\omega_1,\dots,\omega_p$, defined in some neighborhood V of z, such that $\omega|_V = \omega_1 \wedge \cdots \wedge \omega_p$. The distribution $\mathcal{D}$ is then defined on $U \setminus \mathrm{Sing}(\omega)$ by the codimension p planes

$$\mathcal{D}_z = \ker(\omega(z)) := \{v \in T_zU \,|\, i_v\omega(z) = 0\} = \bigcap_{1\leqslant j\leqslant p} \ker(\omega_j(z)).$$

Definition 1.4. — *A holomorphic p-form ω will be said integrable if it is locally totally decomposable outside its singular set and satisfies Frobenius integrability condition. In this context it means that, if $\omega|_V = \omega_1 \wedge \cdots \wedge \omega_p$ as above then $\mathrm{d}\omega_j \wedge \omega = 0$ for all $j = 1,\dots,p$.*

Remark that if ω is closed and locally totally decomposable then the Frobenius condition is automatic:

$$\omega_j \wedge \omega = 0,\ \forall\ j \implies \mathrm{d}\omega_j \wedge \omega = \mathrm{d}(\omega_j \wedge \omega) = 0,\ \forall\ j.$$

In particular, if ω is a closed logarithmic p-form then it is integrable if, and only if, it is locally totally decomposable outside $(\omega)_\infty \cup \mathrm{Sing}(\omega)$.

Example 1.5. — Let $f_1,\dots,f_r$ be irreducible homogeneous polynomials on $\mathbb{C}^{n+1}$. Then any 1-form $\theta \in \mathcal{L}^1_R(f_1,\dots,f_r)$ defines a logarithmic codimension one foliation on $\mathbb{P}^n$, denoted by $\mathcal{F}_\theta$. Let $\theta_1,\dots,\theta_p \in \mathcal{L}^1_R(f_1,\dots,f_r)$ and $\eta := \theta_1 \wedge \cdots \wedge \theta_p$. If $\eta \not\equiv 0$ then $\eta \in \mathcal{L}^p_R(f_1,\dots,f_r)$ and defines a singular codimension p foliation on $\mathbb{P}^n$, denoted by $\mathcal{F}_\eta$. The leaves of $\mathcal{F}_\eta$, outside the pole divisor $f_1 \dots f_r = 0$, are contained in the intersection of the leaves of $\mathcal{F}_{\theta_1},\dots,\mathcal{F}_{\theta_p}$. By this reason, $\mathcal{F}_\eta$ is called the intersection of the foliations $\mathcal{F}_{\theta_1},\dots,\mathcal{F}_{\theta_p}$.

Notation 1.6. — We will use the notation

$$\mathcal{L}^p_{\mathcal{F}}(f_1,\dots,f_r) = \{\eta \in \mathcal{L}^p_R(f_1,\dots,f_r) \,|\, \eta \text{ is integrable}\}.$$

Remark 1.7. — We would like to observe that $\mathcal{L}^p_{\mathcal{F}}(f_1,\dots,f_r)$ is an algebraic subset of $\mathcal{L}^p_R(f_1,\dots,f_r)$. The proof is left as an exercise to the reader.

Definition 1.8. — *We say that $\eta \in \mathcal{L}^p(f_1,\dots,f_r)$ is* totally decomposable into logarithmic forms *if $\eta = \theta_1 \wedge \cdots \wedge \theta_p$, where $\theta_1,\dots,\theta_p \in \mathcal{L}^1(f_1,\dots,f_r)$. We will use the notation*

$$\mathcal{L}^p_{td}(f_1,\dots,f_r) = \left\{\eta \in \mathcal{L}^p_R(f_1,\dots,f_r) \,\middle|\, \begin{array}{c}\eta \text{ is totally decomposable} \\ \text{into logarithmic forms}\end{array}\right\}.$$

Observe that $\mathcal{L}^p_{td}(f_1,\dots,f_r)$ is an irreducible algebraic subset of $\mathcal{L}^p_{\mathcal{F}}(f_1,\dots,f_r)$.

PROBLEM 1.9. — *When $\mathcal{L}^p_{td}(f_1,\dots,f_r)=\mathcal{L}^p_{\mathcal{F}}(f_1,\dots,f_r)$? In other words, does a foliation on $\mathbb{P}^n$ defined by a logarithmic p-form, $2\leqslant p<n$, is an intersection of p codimension one logarithmic foliations?*

A partial answer to Problem 1.9 is given by Theorem 1.10 (see Section 3):

THEOREM 1.10. — *Let $f_1,\dots,f_r$ be homogeneous polynomials on $\mathbb{C}^{n+1}$ and assume that $(f_1\dots f_r=0)$ has strictly ordinary singularities outside $0\in\mathbb{C}^{n+1}$. Then:*

(a) *If $p=2$, or $r\in\{p+1,p+2\}$ then $\mathcal{L}^p_{td}(f_1,\dots,f_r)=\mathcal{L}^p_{\mathcal{F}}(f_1,\dots,f_r)$.*
(b) *If $2<p\leqslant n-2$ and $r>p+2$ then $\mathcal{L}^p_{td}(f_1,\dots,f_r)$ is an irreducible component of $\mathcal{L}^p_{\mathcal{F}}(f_1,\dots,f_r)$. In particular, if $\mathcal{L}^p_{\mathcal{F}}(f_1,\dots,f_r)$ is irreducible then $\mathcal{L}^p_{td}(f_1,\dots,f_r)=\mathcal{L}^p_{\mathcal{F}}(f_1,\dots,f_r)$.*

An interesting consequence of Theorem 1.10 is the following:

COROLLARY 1.11. — *In the hypothesis of Theorem 1.10 if $r=p+1$ and $\eta\in\mathcal{L}^p_{\mathcal{F}}(f_1,\dots,f_{p+1})$ then the foliation $\mathcal{F}_\eta$ in $\mathbb{P}^n$ is a rational fibration of codimension p on $\mathbb{P}^n$. In other words, $\mathcal{F}_\eta$ has a rational first integral $F\colon\mathbb{P}^n\dashrightarrow\mathbb{P}^p$ that in homogeneous coordinates can be written as*

$$F=\left(f_1^{k_1},\dots,f_{p+1}^{k_{p+1}}\right),$$

where $k_1.\deg(f_1)=\cdots=k_{p+1}.\deg(f_{p+1})$ and $\gcd(k_1,\dots,k_{p+1})=1$.

Remark 1.12. — We would like to observe that the statement of Theorem 1.10 cannot be true in the case of $p=n-1$.

In fact, if $p=n-1$ then $\mathcal{L}^{n-1}_{\mathcal{F}}(f_1,\dots,f_r)=\mathcal{L}^{n-1}_{R}(f_1,\dots,f_r)$, because any $(n-1)$-form on $\mathbb{C}^n$ is locally decomposable outside its singular set. Moreover, if $r\geqslant p+2$ then $\mathcal{L}^{n-1}_{td}(f_1,\dots,f_r)$ is a proper algebraic subset of $\mathcal{L}^{n-1}_{\mathcal{F}}(f_1,\dots,f_r)$. The reason is that if η is decomposable, $\eta=\theta_1\wedge\cdots\wedge\theta_p$, where $\theta_1,\dots,\theta_p$ are logarithmic 1-forms as in Theorem 1.10, then η cannot have isolated singularities outside its pole divisor. A specific example on $\mathbb{P}^3$ is given in homogeneous coordinates by the logarithmic 2-form

$$\eta=\sum_{1\leqslant i<j\leqslant 6}\lambda_{ij}\frac{\mathrm{d}\ell_i}{\ell_i}\wedge\frac{\mathrm{d}\ell_j}{\ell_j},$$

where $\lambda_{ij}\in\mathbb{C}$, $1\leqslant i<j\leqslant 6$, and $\ell_j\in\mathbb{C}[z_0,\dots,z_3]$ is homogeneous of degree one, $1\leqslant j\leqslant 6$. If we choose the $\ell_{j's}$ and $\lambda_{ij's}$ generic then the foliation $\mathcal{F}_\eta$ defined by η has degree three and $40=3^3+3^2+3+1$ isolated singularities. Each plane ℓ_j is $\mathcal{F}_\eta$-invariant and the restriction $\mathcal{F}_\eta|_{\ell_j}$ also

defines a degree three foliation and so contains $13 = 3^2 + 3 + 1$ singularities, $1 \leqslant j \leqslant 6$, each line $\ell_i \cap \ell_j$ contains $4 = 3 + 1$ singularities, $1 \leqslant i < j \leqslant 6$, and each point $\ell_i \cap \ell_j \cap \ell_k$ one singularity, $1 \leqslant i < j < k \leqslant 6$. In particular, there are $13 \times 6 - 4 \times \#(\ell_i \cap \ell_j) + \#(\ell_i \cap \ell_j \cap \ell_k) = 38$ singularities contained in $\bigcup_{j=1}^{6} \ell_j$ and so $2 = 40 - 38$ singularities not contained in the pole divisor. If η was decomposable as in Theorem 1.10 then these two singularities could not be isolated.

As a consequence of Theorem 1.10 we can assert that if $\mathcal{G}$ is a codimension two logarithmic foliation on $\mathbb{P}^4 \supset \mathbb{P}^3$ such that $\mathcal{G}|_{\mathbb{P}^3} = \mathcal{F}_\eta$ then $\mathcal{G}$ cannot be tangent to $\mathbb{P}^3$ outside the pole divisor $\bigcup_j \ell_j$. As a consequence $\mathcal{G}$ will be a pull-back $\Pi^*(\mathcal{F}_\eta)$, where Π is induced by a linear map $\widetilde{\Pi}\colon \mathbb{C}^5 \to \mathbb{C}^4$.

In fact, the example of Remark 1.12 has motivated Theorems 5.1 and 5.2 that will be proved in Section 5. These results give necessary conditions for a codimension p foliation $\mathcal{F}$ to be a local or global product in terms of the codimension of the singular set of its intersection with a $(p+1)$-plane: if there is a $(p+1)$-plane Σ such that $\operatorname{cod}(\operatorname{Sing}(\mathcal{F}|_\Sigma)) \geqslant 3$ then $\mathcal{F} = F^*(\mathcal{F}|_\Sigma)$, where $F\colon \mathbb{P}^n - \to \mathbb{P}^{p+1}$ is induced by a linear map of maximal rank $f\colon \mathbb{C}^{n+1} \to \mathbb{C}^{p+2}$ (Theorem 5.1). Theorem 5.2 is a local version of Theorem 5.1.

Another kind of result that we will prove concerns the "stability" of logarithmic foliations on $\mathbb{P}^n$, $n \geqslant 3$. In order to precise this phrase we recall the definition of the degree of a foliation on $\mathbb{P}^n$.

Definition 1.13. — *Let $\mathcal{F}$ be a holomorphic foliation of codimension p on $\mathbb{P}^n$, $1 \leqslant p < n$. The degree of $\mathcal{F}$, $\deg(\mathcal{F})$, is defined as the degree of the divisor of tangencies of $\mathcal{F}$ with a generic plane of complex dimension p of $\mathbb{P}^n$.*

Remark 1.14. — In the particular case of codimension one foliations the degree is the number of tangencies of the foliation with a generic line $\mathbb{P}^1 \subset \mathbb{P}^n$. More generally, a codimension p foliation $\mathcal{F}$ on $\mathbb{P}^n$ can be defined by a meromorphic integrable p-form on $\mathbb{P}^n$, say η, with $\operatorname{cod}_{\mathbb{C}}(\operatorname{Sing}(\eta)) \geqslant 2$. If we consider a generic p-plane $\Sigma \simeq \mathbb{P}^p \subset \mathbb{P}^n$ then the degree of $\mathcal{F}$ is the degree of the divisor of zeroes of $\eta|_\Sigma$.

Note that, if $\Pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is the canonical projection, then the foliation $\Pi^*(\mathcal{F})$ can be extended to a foliation $\mathcal{F}^*$ on $\mathbb{C}^{n+1}$. This foliation is represented by a holomorphic p-form η whose coefficients are homogeneous polynomials of degree $\deg(\mathcal{F})+1$ and such that $i_R\eta = 0$, where R is the radial vector field on $\mathbb{C}^{n+1}$. We say that the form η represents $\mathcal{F}$ in homogeneous coordinates.

A consequence of the definition, is that if $T\colon \mathbb{P}^m - \to \mathbb{P}^n$ is a linear map of maximal rank, where $m > p$, then $\deg(T^*(\mathcal{F})) = \deg(\mathcal{F})$. In particular, if $\mathbb{P}^m \subset \mathbb{P}^n$ is a generic m-plane, where $m > p$, then the degree of $\mathcal{F}|_{\mathbb{P}^m}$ is equal to the degree of $\mathcal{F}$.

The space of dimension k (codimension $p = n - k$) foliations on $\mathbb{P}^n$ of degree d will be denoted by $\mathbb{F}\mathrm{ol}(d; k, n)$. Note that $\mathbb{F}\mathrm{ol}(d; k, n)$ can be identified with the subset of the projectivisation of the space of $(n-k)$-forms η on $\mathbb{C}^{n+1}$ such that: η is integrable, η has homogeneous coefficients of degree $d+1$, $\mathrm{cod}_{\mathbb{C}}(\mathrm{Sing}(\eta)) \geqslant 2$ and $i_R\eta = 0$.

When $k = 1$ the integrability condition is automatic and $\mathbb{F}\mathrm{ol}(d; 1, n)$ is a Zariski open and dense subset of some projective space $\mathbb{P}^N$. However, if $k \geqslant 2$ then the integrability condition is non-trivial and $\mathbb{F}\mathrm{ol}(d; k, n)$ is an algebraic subset of some Zariski open and dense subset of a projective space.

Example 1.15. — Let $\mathcal{F}$ be the logarithmic foliation on $\mathbb{P}^n$ defined in homogeneous coordinates by an integrable p-form η on $\mathbb{C}^{n+1}$ as below:

$$\eta = \sum_{\substack{I \in \mathcal{S}_r^p \\ I=(i_1<\cdots<i_p)}} \lambda_I \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_p}}{f_{i_p}}, \tag{1.2}$$

where $f_1, \ldots, f_r$ are homogeneous polynomials on $\mathbb{C}^{n+1}$ with $\deg(f_j) = d_j$, $1 \leqslant j \leqslant r$. We assume also that $f_1, \ldots, f_r$ are normal crossing outside the origin and $\lambda_I \neq 0$, $\forall\ I \in \mathcal{S}_r^p$. With these conditions then the holomorphic form $\widetilde{\eta} := f_1 \ldots f_r \eta$ has singular set of codimension $\geqslant 2$ and so defines $\mathcal{F}$ in homogeneous coordinates. Since the degree of the coefficients of $\widetilde{\eta}$ is $\sum_{j=1}^r d_j - p$ we obtain

$$\deg(\mathcal{F}) = \sum_{j=1}^r d_j - p - 1 := D(d_1, \ldots, d_r, p)$$
$$\Longrightarrow\ \mathcal{F} \in \mathbb{F}\mathrm{ol}(D(d_1, \ldots, d_r, p); n-p, n)$$

Notation 1.16. — The space of dimension $k = n - p$ logarithmic foliations of $\mathbb{P}^n$ defined by a closed p-form as in (1.2) will be denoted by $\mathcal{L}_{\mathcal{F}}(d_1, \ldots, d_r; k, n)$. Note that

$$\mathcal{L}_{\mathcal{F}}(d_1, \ldots, d_r; k, n) \subset \mathbb{F}\mathrm{ol}(D(d_1, \ldots, d_r, p); k, n).$$

The sub-space of $\mathcal{L}_{\mathcal{F}}(d_1, \ldots, d_r; k, n)$ of foliations that are defined by totally decomposable into logarithmic forms p-forms will be denoted by $\mathcal{L}_{td}(d_1, \ldots, d_r; k, n)$.

Our main result generalizes a theorem by Calvo-Andrade [3]:

THEOREM 1.17. — *If $k \geqslant 2$ and $r \geqslant p+2 = n-k+2$ then $\overline{\mathcal{L}_{td}(d_1,\ldots,d_r;k,n)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}(D(d_1,\ldots,d_r,p);k,n)$ for all $r > p$ and $d_1,\ldots,d_r \geqslant 1$.*

Remark 1.18. — The above result is also true in the case $r = p+1$. In fact, in [8] it is proven the stability of foliations induced by rational maps. On the other hand, by Corollary 1.11 the set $\mathcal{L}_{\mathcal{F}}(d_1,\ldots,d_{p+1};n-p,n)$ coincides with the set of foliations induced by a rational map

$$F = (f_1^{k_1},\ldots,f_{p+1}^{k_{p+1}})\colon \mathbb{C}^{n+1} \to \mathbb{C}^{p+1},$$

where $\deg(f_j) = d_j$ and $k_1.d_1 = \cdots = k_{p+1}.d_{p+1}$.

Theorem 1.17 and Problem 1.9 motivate the following question:

PROBLEM 1.19. — *When $\mathcal{L}_{\mathcal{F}}(d_1,\ldots,d_r;k,n) = \mathcal{L}_{td}(d_1,\ldots,d_r;k,n)$?*

Finally, in the appendix we give a proof of Theorem 2.10. This result, which is used in the proof of Theorem 2.1, gives sufficient conditions for the extension of forms defined in a hypersurface X with an strictly ordinary singularities to the ambient space.

Remark 1.20. — Just before finishing this paper we have found a work by Javier Gargiulo Acea [11] in which he studies some of the problems that we have treated in our paper. For instance, he obtains the same results (decomposability and stability) of our Theorems 1.10 and 1.17 in the case $p = 2$ (2-forms). He also proves the normal form for logarithmic p-forms on $\mathbb{P}^n$ if the pole divisor is normal crossing and $p \leqslant n-1$ (our Corollary 1.3). The local case and the logarithmic foliations of codimension $\geqslant 3$ are not treated by him. We would like to observe that his proof of the stability of logarithmic 2-forms is purely algebraic: he computes the Zariski tangent space at a generic point.

2. Normal forms

The aim of this section is to prove Theorem 2.1 and its corollary (see Corollary 1.3).

THEOREM 2.1. — *Let η be a germ at $0 \in \mathbb{C}^n$ of closed logarithmic p-form with poles along a hypersurface $X = (f_1 \ldots f_r = 0)$ with strictly ordinary singularities outside $0 \in \mathbb{C}^{n+1}$. Assume that $n \geqslant p+2$. Then:*

(a) *If $r < p$ then η is exact; $\eta = \mathrm{d}\Theta$, where Θ is logarithmic non-closed and has the same pole divisor as η.*

(b) *If $r \geqslant p$ then there are numbers $\lambda_I \in \mathbb{C}$, $I \in \mathcal{S}_p^r$, such that*

$$\eta = \sum_{\substack{I \in \mathcal{S}_r^p \\ I=(i_1<\cdots<i_p)}} \lambda_I \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_p}}{f_{i_p}} + \mathrm{d}\Theta, \tag{2.1}$$

where, either $\Theta = 0$, or Θ is logarithmic non-closed and has pole divisor contained in X.

Remark 2.2. — In the above statement, if $r = 0$ then $X = \emptyset$ and η is holomorphic and closed. In this case it can be written as $\eta = \mathrm{d}\Theta$, where Θ is a holomorphic $(p-1)$-form, by Poincaré lemma. On the other hand, if $p = 1$ and $r \geqslant 1$ then η can be written as

$$\eta = \sum_j \lambda_j \frac{\mathrm{d}f_j}{f_j} + \mathrm{d}g, \quad g \in \mathcal{O}_n,$$

whereas when $p = 2$ and $r \geqslant 2$ then Theorem 2.1 implies that

$$\eta = \sum_{i<j} \lambda_{ij} \frac{\mathrm{d}f_i}{f_i} \wedge \frac{\mathrm{d}f_j}{f_j} + \sum_j \mathrm{d}g_j \wedge \frac{\mathrm{d}f_j}{f_j} + \mathrm{d}\alpha,$$

where $g_1, \ldots, g_r \in \mathcal{O}_n$ and $\alpha \in \Omega^1(\mathbb{C}^n, 0)$.

Remark 2.3. — The numbers λ_I in (2.1), $I \in \mathcal{S}_p^r$, are called the numerical residues of η (see Section 2.1.1). Given $I = (i_1 < \cdots < i_p)$ then λ_I can be calculted by integrating η as follows: since $1 \leqslant p < n$ the germ of analytic set $X_I := (f_{i_1} = \cdots = f_{i_p} = 0)$ has dimension $n - p \geqslant 1$. Moreover, by the normal crossing condition the set $\widetilde{X}_I := X_I \setminus \bigcup_{j \notin I}(f_j = 0)$ is not empty. If we fix $m \in \widetilde{X}_I$ then there are local coordinates $z = (z_1, \ldots, z_n)$ such that $z(m) = 0$ and $f_{i_j} = z_j$ for all $j = 1, \ldots, p$. Given $\epsilon > 0$ small, consider the real p-dimensional torus

$$T_\epsilon^p = \{z \,|\, |z_j| = \epsilon \text{ if } 1 \leqslant j \leqslant p, \text{ and } z_j = 0 \text{ if } j > p\}.$$

It follows from (2.1) that

$$\lambda_I = \frac{1}{(2\pi i)^p} \int_{T_\epsilon^p} \eta.$$

As a consequence, a logarithmic p-form η on $\mathbb{P}^n$, $p \leqslant n-1$, with pole divisor given in homogeneous coordinates by $f_1 \ldots f_r = 0$, where the $f_{i's}$ are irreducible and the hypersurface $X = (f_1 \ldots f_r = 0)$ has strictly ordinary singularities outside $0 \in \mathbb{C}^{n+1}$ can be written as in Corollary 1.3:

$$\eta = \sum_{\substack{I \in \mathcal{S}_r^p \\ I=(i_1<\cdots<i_p)}} \lambda_I \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_p}}{f_{i_p}} \qquad \text{with } \; i_R\eta = 0.$$

Another observation is that Theorem 2.1 is false if $p = n-1$ and $n \geqslant 3$ as shows the following example in $\mathbb{C}^n$, $n \geqslant 3$:

Example 2.4. — Let P be an irreducible homogeneous polynomial of degree n on $\mathbb{C}^n$ and set

$$\eta = \frac{i_R(\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n)}{P(z_1, \ldots, z_n)} = \frac{\sum_{j=1}^{n}(-1)^{j-1} z_j \mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_j} \wedge \cdots \wedge \mathrm{d}z_n}{P(z_1, \ldots, z_n)},$$

where in the symbol $\widehat{\mathrm{d}z_j}$ means omission of $\mathrm{d}z_j$ in the product.

We would like to observe that the same example shows that Corollary 1.3 is false in $\mathbb{P}^m = \mathbb{P}^{n-1}$ if $p = m$: the form η represents in homogeneous coordinates a closed logarithmic m-form on $\mathbb{P}^m$ which is not like in the statement of the corollary.

2.1. Preliminaries

Let η be a germ at $0 \in \mathbb{C}^n$ of meromorphic p-form with reduced pole divisor $X = (f_1 \ldots f_r = 0)$, $r \geqslant 1$. At the begining we will not assume that η is closed.

It follows from the definition that η is logarithmic if, and only if, $f_1 \ldots f_r.\mathrm{d}\eta$ is holomorphic. Since $(\eta)_\infty = (f_1 \ldots f_r)$ we can write $\eta = \frac{1}{f_1 \ldots f_r}\omega$ where $\omega \in \Omega^p_n$ is a germ of holomorphic p-form. We would like to observe that the following assertions are equivalent:

(a) $\eta = \frac{1}{f_1 \ldots f_r}\omega$ is logarithmic.
(b) f_j divides $\mathrm{d}f_j \wedge \omega$, for all $1 \leqslant j \leqslant r$.

In particular, we have:

(c) $\frac{1}{f_1 \ldots f_s}\omega$ is logarithmic, for all $s \leqslant r$.

In fact:

$$\begin{aligned}
\text{(a)} &\iff f_1 \ldots f_r.\mathrm{d}\eta = \mathrm{d}\omega - \frac{\mathrm{d}(f_1 \ldots f_r)}{f_1 \ldots f_r} \wedge \omega = \mu \text{ is holomorphic} \\
&\iff f_1 \ldots f_r . \sum_j \frac{\mathrm{d}f_j}{f_j} \wedge \omega = f_1 \ldots f_r(\mathrm{d}\omega - \mu) \\
&\iff f_j \text{ divides } \mathrm{d}f_j \wedge \omega, 1 \leqslant j \leqslant r.
\end{aligned}$$

The proof of Theorem 2.1 will be based in the following:

LEMMA 2.5. — *Let $\eta = \frac{1}{f_1 \dots f_r}\omega$ be a germ of at $0 \in \mathbb{C}^n$ of logarithmic p-form, where $1 \leqslant p \leqslant n-2$. Assume that the pole divisor of η is $X = (f_1 \dots f_r = 0)$, $r \geqslant 1$, has strictly ordinary singularities outside 0. Then η can be written as*

$$\eta = \alpha_0 + \sum_{s=1}^{p-1} \left(\sum_{I \in \mathcal{S}_r^s} \alpha_I \wedge \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_s}}{f_{i_s}} \right) + \sum_{J \in \mathcal{S}_r^p} g_J . \frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{j_p}}{f_{j_p}}, \quad (2.2)$$

where $\alpha_0 \in \Omega_n^p$, $\alpha_I \in \Omega_n^{p-s}$ if $I \in \mathcal{S}_r^s$, $s < p$, and $g_J \in \mathcal{O}_n$ if $J \in \mathcal{S}_r^p$.

The proof of Lemma 2.5 relies in the concept of *residue* of a logarithmic form along an irreducible pole (cf. [10]).

2.1.1. Residues of a logarithmic form

Let $\eta = \frac{1}{f_1 \dots f_r}\omega$ be a germ at $0 \in \mathbb{C}^n$ of logarithmic p-form with pole divisor $X = (f_1 \dots f_r = 0)$. Let us define the residue along $Y_k := (f_k = 0)$, $1 \leqslant k \leqslant r$.

Fix representatives of $f_1, \dots, f_r$ and η, denoted by the same symbols, on some polydisc Q. We will assume that the $f_{j's}$ are irreducible in Q, and that the divisor $f_1 \dots f_r$ has strictly ordinary singularities on $Q \setminus \{0\}$. In particular, the $f_{j's}$ have isolated singularity at $0 \in Q$. We have seen that f_k divides $\mathrm{d}f_k \wedge \omega$. In particular, we can write $\mathrm{d}f_k \wedge \omega = f_k.\theta$ where $\theta \in \Omega_n^{p+1}$. This implies that $\mathrm{d}f_k \wedge \theta = 0$. Since $\mathrm{d}f_k$ has an isolated singularity at $0 \in Q$ and $p+1 \leqslant n-1$, it follows from de Rham's division theorem [21] that $\theta = \mathrm{d}f_k \wedge \beta_k$, where $\beta_k \in \Omega^p(Q)$. Therefore, we can write $\mathrm{d}f_k \wedge (\omega - f_k.\beta_k) = 0$ which implies, via the division theorem [21], that there exists $\alpha_k \in \Omega^{p-1}(Q)$ such that $\omega = \alpha_k \wedge \mathrm{d}f_k + f_k.\beta_k$. The residue of $\frac{1}{f_k}\omega$ along Y_k is the $(p-1)$-form along Y_k defined as $\mathrm{Res}\left(\frac{1}{f_k}\omega, Y_k\right) := \alpha_k|_{Y_k}$. Finaly, the residue of $\eta = \frac{1}{f_1 \dots f_r}\omega$ along Y_k is defined as $\mathrm{Res}(\eta, Y_k) := \frac{1}{f_1 \dots \widehat{f_k} \dots f_r}\alpha_k|_{Y_k}$, where $\widehat{f_k}$ means omission of the factor f_k in the product.

Remark 2.6. — Let η and Y_k be as above. It is well known that $\mathrm{Res}\,(\eta, Y_k)$ does not depend on the particular decomposition $\omega = \alpha_k \wedge \mathrm{d}f_k + f_k\beta_k$ and on the particular equation of Y_k (cf. [10]).

The above remark allow us to define the residue of a logarithmic form η on a arbitrary complex manifold M along any codimension one smooth irreducible submanifold Y contained in the pole divisor of η. In particular, we can define the iterated residue. Given $I = (i_1 < \cdots < i_k) \in \mathcal{S}_r^k$, set

$X_I = (f_{i_1} = \cdots = f_{i_k} = 0)$ and $X_I^* = X_I \setminus \{0\}$. We define $\mathrm{Res}(\eta, X_I)$ inductively. If $k = 1$ then $\mathrm{Res}(\eta, X_I) = \mathrm{Res}(\eta, Y_{i_1})$ and for $k \geqslant 2$, $\mathrm{Res}(\eta, X_I) = \mathrm{Res}\big(\mathrm{Res}(\eta, Y_{i_k}), X_{I\setminus\{i_k\}}\big)$. This definition depends only of the ordering of the $f_{j's}$, that we will assume fixed.

Example 2.7. — If $\eta = \alpha \wedge \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_k}}{f_{i_k}}$, where α is holomorphic, then $\mathrm{Res}(\eta, X_I) = \alpha|_{X_I}$, $I = (i_1 < \cdots < i_k)$. We leave the proof to the reader.

Remark 2.8. — Let $\eta = \frac{1}{f_1 \cdots f_k}\omega$ be logarithmic as above, $Y_k = (f_k = 0)$, where we assume $n \geqslant p + 2$. We would like to observe the following facts:

(a) If $\mathrm{Res}(\eta, Y_k) = 0$ then f_k divides ω, or equivalently f_k is not contained in the pole divisor of η.
(b) If $p = 1$ then $\mathrm{Res}(\eta, Y_k)$ is a holomorphic function on Y_k.
(c) If $p \geqslant 2$ then $\mathrm{Res}(\eta, Y_k)$ is logarithmic on Y_k. Moreover, the pole divisor of $\mathrm{Res}(\eta, Y_k)$ is $f_1 \ldots \widehat{f_k} \ldots f_r|_{Y_k}$.
(d) $\mathrm{d}\eta$ is logarithmic and $\mathrm{Res}(\mathrm{d}\eta, Y_k) = \mathrm{d}\,\mathrm{Res}(\eta, Y_k)$. In particular, if η is closed then $\mathrm{Res}(\eta, Y_k)$ is closed.

We will prove (b) and (c). The proofs of (a) and (d) will be left to the reader. Write $\omega = \alpha_k \wedge \mathrm{d}f_k + f_k\beta_k$ (resp. $\omega = g_k\mathrm{d}f_k + f_k\beta_k$ if $p = 1$) as before. It is sufficient to prove that if $\ell \neq k$ then $f_\ell|_{Y_k}$ divides $\alpha_k \wedge \mathrm{d}f_\ell|_{Y_k}$ (resp. $f_\ell|_{Y_k}$ divides $g_k|_{Y_k}$ if $p = 1$). Note that $\dim(Y_k \cap Y_\ell) \geqslant 1$, because $n \geqslant p + 2 \geqslant 3$. Therefore we can fix a point $m \in Y_k \cap Y_\ell$ where $\mathrm{d}f_k(m) \wedge \mathrm{d}f_\ell(m) \neq 0$. Let $(U, z = (z_1, \ldots, z_n))$ be a coordinate system around m such that $f_k|_U = z_1$ and $f_\ell|_U = z_2$. Write $\omega = \omega_1 \wedge \mathrm{d}z_1 + \omega_2 \wedge \mathrm{d}z_2 + \omega_{12} \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2 + \theta$, where ω_1 does not contain terms with $\mathrm{d}z_2$, ω_2 does not contain terms with $\mathrm{d}z_1$ and θ does not contain terms in $\mathrm{d}z_1$ or $\mathrm{d}z_2$ (resp. $\omega = \sum_j h_j\mathrm{d}z_j$ if $p = 1$).

Let us consider the case $p > 1$. In this situation, $\omega \wedge \mathrm{d}z_1 = \omega_2 \wedge \mathrm{d}z_2 \wedge \mathrm{d}z_1 + \theta \wedge \mathrm{d}z_1$ and $\theta \wedge \mathrm{d}z_1$ does not contain terms with $\mathrm{d}z_1 \wedge \mathrm{d}z_2$, so that z_1 divides ω_2 and θ. Similarly, z_2 divides ω_1 and θ. Therefore, we can write $\omega = z_2\widetilde{\omega}_1 \wedge \mathrm{d}z_1 + z_1\widetilde{\omega}_2 + \omega_{12} \wedge \mathrm{d}z_1 \wedge \mathrm{d}z_2 + z_1z_2\widetilde{\theta}$, which implies

$$\begin{aligned}\omega = (z_2\widetilde{\omega}_1 \wedge -\omega_{12} \wedge \mathrm{d}z_2) \wedge \mathrm{d}z_1 + z_1\big(\widetilde{\omega}_2 + z_2\widetilde{\theta}\big)\\ \Longrightarrow\ \alpha_k|_{Y_k\cap U} = (z_2\widetilde{\omega}_1 \wedge -\omega_{12} \wedge \mathrm{d}z_2)\,|_{Y_k\cap U}\\ \Longrightarrow\ \alpha_k \wedge \mathrm{d}z_2|_{Y_k\cap U} = z_2\widetilde{\omega}_2 \wedge \mathrm{d}z_2|_{Y_k\cap U},\end{aligned}$$

which implies (c).

In the case $p = 1$, since z_j divides $\omega \wedge \mathrm{d}z_j$, $\forall\ j$, then z_1 divides h_j if $j > 1$ and z_2 divides h_1, so that $\omega|_U = z_2\widetilde{h}_1\mathrm{d}z_1 + z_1\sum_{j>1}\widetilde{h}_j\mathrm{d}z_j$. Hence, $g_k|_{Y_k\cap U} = z_2\widetilde{h}_1|_{Y_k\cap U}$, which implies (b).

2.1.2. Proof of Lemma 2.5 in the case $p = 1$

Since the proof of Lemma 2.5 is rather technical in the general case, we give first the proof in the case $p = 1$ which contains essentially the idea of the general case.

As before, write $\eta = \frac{1}{f_1 \cdots f_r}\omega$. The proof will be by induction on the number r of components of the pole divisor.

Formula (2.2) *of Lemma 2.5 is true if* $r = 1$ *and* $p \geqslant 1$. — When $\eta = \frac{1}{f_1}\omega$ is logarithmic we have seen in Section 2.1.1 that $\omega = \alpha_1 \wedge \mathrm{d}f_1 + f_1.\beta_1$ (resp. $\omega = g_1 \mathrm{d}f_1 + f_1\beta_1$ if $p = 1$). Hence $\eta = \alpha_1 \wedge \frac{\mathrm{d}f_1}{f_1} + \beta_1$ (resp. $\eta = g_1 \frac{\mathrm{d}f_1}{f_1} + \beta_1$ if $p = 1$), as we wished.

If $p = 1$ *and* (2.2) *is true for* $r - 1 \geqslant 1$ *then it is true for* r. — Let $\eta = \frac{1}{f_1 \cdots f_r}\omega$ and $Q \subset \mathbb{C}^n$ be a polydisc where $f_1, \ldots, f_r$ and ω have representatives as before. As before we set $Y_j = (f_j = 0) \subset Q \subset \mathbb{C}^n$. We will use the following well known result in the case $n \geqslant 3$:

Lemma 2.9. — *Any holomorphic function* $h \in \mathcal{O}(Y_j \setminus \{0\})$ *has an extension* $g \in \mathcal{O}(Q)$.

In fact, Lemma 2.9 is a particular case of Theorem 2.10 stated below and that will proved in Section 6 (see Remark 2.11).

Let $h = \mathrm{Res}(\eta, Y_r) \in \mathcal{O}(Y_r \setminus \{0\})$. By Lemma 2.9, h has an extension $g_r \in \mathcal{O}(Q)$. The form $g_r \frac{\mathrm{d}f_r}{f_r}$ is logarithmic and $\mathrm{Res}\big(g_r \frac{\mathrm{d}f_r}{f_r}, Y_r\big) = h$. Therefore, the form $\widetilde{\eta} = \eta - g_r \frac{\mathrm{d}f_r}{f_r}$ is also logarithmic and $\mathrm{Res}\,(\widetilde{\eta}, Y_r) = 0$. In particular, f_r is not a pole of $\widetilde{\eta}$ by Remark 2.8. Since the pole divisor of $\widetilde{\eta}$ has $r - 1$ irreducible components, by the induction hypothesis we can write

$$\eta - g_r \frac{\mathrm{d}f_r}{f_r} = \widetilde{\eta} = \alpha_0 + \sum_{j=1}^{r-1} g_j \frac{\mathrm{d}f_j}{f_j} \implies \text{Lemma 2.5 in the case } p = 1.$$

2.1.3. Proof of Lemma 2.5 when $p \geqslant 2$

The case $p \geqslant 2$ is more involved, but the idea of the proof is the same as in the case $p = 1$. Before given the details let us sketch the proof.

Given $s \in \{0, 1, \ldots, r\}$ set $Y_s = (f_s = 0)$ if $s \geqslant 1$, $X_0 = Q$ and $X_s = Y_1 \cap \cdots \cap Y_s$ if $s \geqslant 1$. Set also $X_s^* = X_s \setminus \{0\}$, $0 \leqslant s \leqslant r$. Note that $X_s = \{0\}$ if $s \geqslant n$. On the other hand, if $1 \leqslant s \leqslant n - 1$ then X_s is an analytic reduced germ of codimension s and X_s^* is a complex smooth manifold of dimension

$n-s$. The proof will involve two induction arguments. In order to state properly these arguments we need a definition.

Given $1 \leqslant s \leqslant p \leqslant n-2$ and $q \geqslant 1$, we will say that X_s^* satisfies the q *decomposition property* if any logarithmic q-form θ on X_s^* with pole divisor on the zeroes of $f_{s+1}\dots f_r|_{X_s^*} := \widetilde{f}_{s+1}\dots\widetilde{f}_r$ can be decomposed as in formula (2.2):

$$\theta = \alpha_0 + \sum_{\ell=1}^{q-1}\left(\sum_{I\in\mathcal{S}^\ell_{r-s}} \alpha_I \wedge \frac{\mathrm{d}\widetilde{f}_{i_1}}{\widetilde{f}_{i_1}} \wedge\cdots\wedge \frac{\mathrm{d}\widetilde{f}_{i_\ell}}{\widetilde{f}_{i_\ell}}\right) + \sum_{J\in\mathcal{S}^q_{r-s}} g_J.\frac{\mathrm{d}\widetilde{f}_{j_1}}{\widetilde{f}_{j_1}} \wedge\cdots\wedge \frac{\mathrm{d}\widetilde{f}_{j_q}}{\widetilde{f}_{j_q}},$$

where α_0 is a holomorphic q-form, the $\alpha_{I's}$ are holomorphic $(q-\ell)$-forms on X_s^* and the $g_{J's}$ are holomorphic functions on X_s^*. We resume below the main steps in the arguments.

1st step. — If $0 \leqslant s \leqslant p-1$ then X_s^* satisfies the 1 decomposition property.

2nd step. — If $2 \leqslant q \leqslant p-s$, where $s \geqslant 0$, and X_{s+1}^* satisfies the $q-1$ decomposition property then X_s^* satisfies the q decomposition property.

The 1st and 2nd steps above will be proved by induction on the number of $r \geqslant 1$ of factors in the pole divisor $f_1\dots f_r$. In the proof we will use the following result:

THEOREM 2.10. — *Let X_s and X_s^* be as above, where $s = n-k$ and $2 \leqslant k \leqslant n-1$ ($\dim_{\mathbb{C}}(X_s) = k$). Then there are representatives of X_s and X_s^* in a polydisc $Q \subset \mathbb{C}^n$, denoted by the same letters, such that:*

(a) *If $0 \leqslant q \leqslant k-2$ then any form $\widetilde{\alpha} \in \Omega^q(X_s^*)$ can be extended to a form $\alpha \in \Omega^q(Q)$.*
(b) *If $q \geqslant 1$, $\ell \geqslant 0$ and $1 \leqslant q+\ell \leqslant k-2$ then $H^q(X_s^*, \Omega^\ell) = 0$.*

Remark 2.11. — Note that Lemma 2.9 is a particular case of Theorem 2.10 (a).

Theorem 2.10 implies that, if X_s^* is as before and $0 \leqslant q \leqslant n-s-2$, then any holomorphic q-form on X_s^* can be extended to a holomorphic q-form on Q. The proof of Theorem 2.10 will be done in Section 6. Let us finish the proof of Lemma 2.5 assuming Theorem 2.10.

Proof of the 1st step. — It is similar to the case $p = 1$ done above with Lemma 2.9 (which corresponds to the case $q = 0$ in Theorem 2.10). Therefore, we will assume $1 \leqslant s \leqslant p-1$. Note that the 1st step is trivially true if $r = s$, because in this case the pole divisor is empty and the 1-form is holomorphic.

Assume that the assertion is true for any logarithmic 1-form on X_s^* with pole divisor containing $\ell-1\geqslant 0$ functions in the set $\{\widetilde{f}_j=f_j|_{X_s^*}\,|\,s+1\leqslant j\leqslant r\}$. Let θ be a logarithmic 1-form on X_s^* with pole divisor $\widetilde{f}_{s+1}\dots\widetilde{f}_{s+\ell}$. By Remark 2.8, $\widetilde{g}_{s+1}:=\mathrm{Res}(\theta,X_{s+1}^*)\in\mathcal{O}(X_{s+1}^*)$. By Theorem 2.10, $\widetilde{g}_{s+1}$ admits an extension $g_{s+1}\in\mathcal{O}(Q)$. In particular, $\widehat{g}_{s+1}:=g_{s+1}|_{X_s^*}$ is a holomorphic extension of $\widetilde{g}_{s+1}$ on X_s^*. Let $\widetilde{\theta}:=\widehat{g}_{s+1}\frac{\mathrm{d}\widetilde{f}_{s+1}}{\widetilde{f}_{s+1}}$. Then $\widetilde{\theta}$ is logarithmic and $\mathrm{Res}(\widetilde{\theta},X_{s+1}^*)=\mathrm{Res}(\theta,X_{s+1}^*)$. In particular, $\widetilde{f}_{s+1}$ is not contained in the pole divisor of $\theta-\widetilde{\theta}$, by Remark 2.8(a). By the induction hypothesis, $\theta-\widetilde{\theta}$ can be decomposed as in (2.2):

$$\theta-\widetilde{\theta}=\alpha_0+\sum_{j=2}^{\ell}\widehat{g}_{s+j}\frac{\mathrm{d}\widetilde{f}_{s+j}}{\widetilde{f}_{s+j}}\implies 1^{\mathrm{st}}\text{ step.}$$

Proof of the 2^{nd} step. — The proof is again by induction on the number $r-s$ of factors of the pole divisor. The assertion is trivially true if $r=s$. Assume that the assertion is true for any logarithmic q-form, $2\leqslant q\leqslant p-s$, on X_s^* with pole divisor containing $\ell-1\geqslant 0$ functions in the set $\{\widetilde{f}_j=f_j|_{X_s^*}\,|\,s+1\leqslant j\leqslant r\}$. Let θ be a logarithmic q-form on X_s^* with pole divisor $\widetilde{f}_{s+1}\dots\widetilde{f}_{s+\ell}$. By Remark 2.8 the $(q-1)$-form $\mu:=\mathrm{Res}(\theta,X_{s+1}^*)$ is logarithmic on X_{s+1}^* with pole divisor $\widehat{f}_{s+2}\dots\widehat{f}_{s+\ell}:=\widetilde{f}_{s+2}\dots\widetilde{f}_{s+\ell}|_{X_{s+1}^*}$ (or holomorphic if $\ell=1$). Since X_{s+1}^* satisfies the $q-1$ decompostion property, we can write

$$\mu=\alpha_0+\sum_{t=1}^{q-2}\sum_{I\in\mathcal{S}_r^t}\alpha_I\wedge\frac{\mathrm{d}\widehat{f}_{s+i_1-1}}{\widehat{f}_{s+i_1-1}}\wedge\dots\wedge\frac{\mathrm{d}\widehat{f}_{s+i_t-1}}{\widehat{f}_{s+i_t-1}}$$
$$+\sum_{J\in\mathcal{S}_r^{q-1}}g_J.\frac{\mathrm{d}\widehat{f}_{s+j_1-1}}{\widehat{f}_{s+j_1-1}}\wedge\dots\wedge\frac{\mathrm{d}\widehat{f}_{s+j_{(q-1)}-1}}{\widehat{f}_{s+j_{(q-1)}-1}}$$

where α_0 and the $\alpha_{I's}$ are holomorphic forms on X_{s+1}^* and the $g_{J's}$ are holomorphic functions on X_{s+1}^*. By Theorem 2.10 each α_I (resp. each g_J) has a holomorphic extension $\widetilde{\alpha}_I$ (resp. $\widetilde{g}_J$) on X_s^*. Therefore, μ has a logarithmic extension $\widetilde{\mu}$ on X_s^*,

$$\widetilde{\mu}=\widetilde{\alpha}_0+\sum_{t=1}^{q-2}\sum_{I\in\mathcal{S}_r^t}\widetilde{\alpha}_I\wedge\frac{\mathrm{d}\widetilde{f}_{s+i_1-1}}{\widetilde{f}_{s+i_1-1}}\wedge\dots\wedge\frac{\mathrm{d}\widetilde{f}_{s+i_t-1}}{\widetilde{f}_{s+i_t-1}}$$
$$+\sum_{J\in\mathcal{S}_r^{q-1}}\widetilde{g}_J.\frac{\mathrm{d}\widetilde{f}_{s+j_1-1}}{\widetilde{f}_{s+j_1-1}}\wedge\dots\wedge\frac{\mathrm{d}\widetilde{f}_{s+j_{(q-1)}-1}}{\widetilde{f}_{s+j_{(q-1)}-1}}$$

Therefore, $\theta_1 := \widetilde{\mu} \wedge \frac{\mathrm{d}\tilde{f}_{s+1}}{\tilde{f}_{s+1}}$ is logarithmic on X_s^* and

$$\mathrm{Res}(\theta_1, X_{s+1}^*) = \mathrm{Res}(\theta, X_{s+1}^*) \implies \mathrm{Res}(\theta - \theta_1, X_{s+1}^*) = 0.$$

Hence, $\widetilde{f}_{s+1}$ is not contained in the pole divisor of $\theta - \theta_1$. By the induction hypothesis, $\theta - \theta_1 := \theta_2$ admits a decomposition as in (2.2), and so $\theta = \theta_1 + \theta_2$ admits a decomposition as in (2.2). This finishes the proof of Lemma 2.5. □

2.2. Proof of Theorem 2.1

In the proof of Theorem 2.1 we will use Theorem 2.10 and Hamm's generalization of Milnor's theorem (cf. [13], [14], [20] and [23]):

THEOREM 2.12. — *Let $X = (f_1 = \cdots = f_\ell = 0)$ be a germ at $0 \in \mathbb{C}^m$ of a complete intersection with an isolated singularity at 0, so that $\dim_{\mathbb{C}}(X) = m - \ell := n$. Then there exist representatives of $f_1, \ldots, f_\ell$ and X defined in a ball $B_\epsilon = B(0, \epsilon)$, denoted by the same letters, such that:*

(a) *$X^* = X \setminus \{0\}$ is rectratible to the link $K := X \cap \mathbb{S}_\epsilon^{2m-1}$, $\mathbb{S}_\epsilon^{2m-1} = \partial B_\epsilon$.*
(b) *If $n \geqslant 3$ then K is $(n-2)$-connected. In particular, X^* is connected and $H_{DR}^k(X^*) = \{0\}$ if $1 \leqslant k \leqslant n-2$.*
(c) *If $n = 2$ then X^* is connected.*

When $n = 1$, X^* is not necessarily connected, as shows the example $X = (x^2 + y^2 + z^2 = z = 0) \subset \mathbb{C}^3$.

Let η be a germ at $0 \in \mathbb{C}^n$ of a closed logarithmic p-form, $1 \leqslant p \leqslant n-2$, with pole divisor $f_1 \ldots f_r$ with a strictly ordinary singularity outside 0. According to Lemma 2.5 we can write η as a sum of a holomorphic p-form α_0, and "monomial" p-forms of the type $\alpha_I \wedge \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_s}}{f_{i_s}}$, or $g_J \frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{j_p}}{f_{i_p}}$, where $I \in \mathcal{S}_r^s$ and $J \in \mathcal{S}_r^p$.

Given a monomial $\mu = \alpha_I \wedge \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_s}}{f_{i_s}}$ we define the *pseudo depth* of μ as $\widetilde{\mathrm{dep}}(\mu) = s$. Given $\eta = \sum_{j=1}^m \mu_j$, where the μ_j are monomials as above, we set $\widetilde{\mathrm{dep}}(\eta) = \max\{\widetilde{\mathrm{dep}}(\mu_j) \,|\, 1 \leqslant j \leqslant m\}$.

Observe that $\widetilde{\mathrm{dep}}$, as defined above, is not well defined. For instance, if $g \in \langle f_1, \ldots, f_p \rangle$, the ideal generated by $f_1, \ldots, f_p$, $g = \sum_{j=1}^p h_j . f_j$, then

$$g \frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \frac{\mathrm{d}f_p}{f_p} = \sum_j \alpha_j \wedge \frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \widehat{\frac{\mathrm{d}f_j}{f_j}} \wedge \frac{\mathrm{d}f_p}{f_p},$$

where $\alpha_j = \pm h_j \mathrm{d}f_j$, $1 \leqslant j \leqslant p$. Therefore, if η is a logarithmic form as above, then we define its depth as

$$\mathrm{depth}(\eta) = \min\left\{ \widetilde{\mathrm{dep}}\left(\sum_j \mu_j\right) \,\middle|\, \eta = \sum_j \mu_j, \text{where the } \mu_{j's} \text{ are monomials} \right\}.$$

When η is holomorphic we define $\mathrm{depth}(\eta) = 0$.

Claim 2.13. — *Let η be a germ at $0 \in \mathbb{C}^n$ of logarithmic closed p-form, $1 \leqslant p \leqslant n-2$. If $\mathrm{depth}(\eta) = p$ there exists a collection $(\lambda_J)_{J \in \mathcal{S}_r^p}$, $\lambda_J \in \mathbb{C}$, such that*

$$\mathrm{depth}\left(\eta - \sum_{J \in \mathcal{S}_r^p} \lambda_J \frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{j_p}}{f_{j_p}}\right) \leqslant p-1.$$

Proof. — If $\mathrm{depth}(\eta) = p$ then the decomposition of η as in (2.2) contains at least one monomial of the form $\mu_J = g_J \frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{j_p}}{f_{j_p}}$, where $g_J \notin \langle f_{j_1}, \ldots, f_{j_p} \rangle$. As before, set $X_J := (f_{j_1} = \cdots = f_{j_p} = 0)$ and $X_J^* = X_J \setminus \{0\}$. We assert that $g_J|_{X_J^*}$ is a constant $\lambda_J \in \mathbb{C}^*$.

In fact, since $\dim_{\mathbb{C}} X_J = n - p \geqslant 2$, X_J^* is connected, by Theorem 2.12. Note that $\mathrm{Res}(\mu_J, X_J) = \mathrm{Res}(\eta, X_J) = g_J|_{X_J^*}$ (see Example 2.7). Since η is closed, we have $\mathrm{Res}(\mathrm{d}\eta, X_J) = dg_J|_{X_J^*} = 0$. Hence, $g_J|_{X_J^*} = \lambda_J \in \mathbb{C}$. On the other hand, if $\lambda_J = 0$ then $g_J|_{X_J} = 0$ and since X_J is a complete intersection we get $g_J \in \langle f_{j_1}, \ldots, f_{j_p} \rangle$, a contradiction.

Let $\mu_J := \lambda_J \frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{j_p}}{f_{j_p}}$. Note that $\eta - \mu_J$ is still logarithmic, closed and does not contain terms multiples of $\frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{j_p}}{f_{j_p}}$. By repeating this procedure finitely many times we can find the collection $(\lambda_J)_{J \in \mathcal{S}_r^p}$ as in the statement of the claim. □

Claim 2.14. — *Let η be logarithmic closed p-form with pole divisor $f_1 \ldots f_r = 0$, with a strictly ordinary singularities at $0 \in \mathbb{C}^n$. If $\mathrm{depth}(\eta) < p$ then η is exact: $\eta = \mathrm{d}\Theta$, where Θ is either zero, or is logarithmic with pole divisor contained in $f_1 \ldots f_r = 0$.*

Proof. — The proof will be by induction on the depth of η. If $\mathrm{depth}(\eta) = 0$ then η is holomorphic and so it is exact by Poincaré lemma. □

Assume that any closed logarithmic p-form ω with $\mathrm{depth}(\omega) \leqslant q-1 < p-1$ is exact: $\omega = \mathrm{d}\theta$ with θ logarithmic as above. Let η be a logarithmic p-form with pole divisor $f_1 \ldots f_r = 0$ with $\mathrm{depth}(\eta) = q < p$. By the definition of depth, when we write η as in (2.2) then we get

$$\eta = \sum_{I \in \mathcal{S}_r^q} \alpha_I \wedge \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_q}}{f_{i_q}} + \beta,$$

where β is logarithmic and $\mathrm{depth}(\beta) < q$. Recall that, if $I = (i_1, \ldots, i_q) \in \mathcal{S}_r^q$ then $X_I = (f_{i_1} = \cdots = f_{i_q} = 0)$ and $X_I^* = X_I \setminus \{0\}$. As the reader can check, we have

$$\mathrm{Res}(\eta, X_I) = \alpha_I|_{X_I} := \widetilde{\alpha}_I \in \Omega^{p-q}(X_I^*),\ \forall\, I \in \mathcal{S}_r^q,$$

where $\widetilde{\alpha}_I$ is closed, by Remark 2.8. Now, we use Theorem 2.12 and Theorem 2.10 (b): since $\dim(X_I^*) = n-q$ we get $H^k_{DR}(X^*) = 0$ if $1 \leqslant k \leqslant n-q-2$. But, $p \leqslant n-2$ and so $p-q \leqslant n-q-2 = \dim_{\mathbb{C}}(X^*) - 2$ which implies that $\widetilde{\alpha}_I \in \Omega^{p-q}(X_I^*)$ is exact: $\widetilde{\alpha}_I = \mathrm{d}\widetilde{\beta}_I$, where $\widetilde{\beta}_I$ is in principle a $C^\infty(p-q-1)$-form. However, the fact that $H^{sr}_{\overline{\partial}}(X_I^*) \simeq H^r(X_I^*, \Omega^s) = 0$ if $r+s = p-q-1$ implies that we can assume $\widetilde{\beta}_I \in \Omega^{p-q-1}(X_I^*)$ (cf. [12]).

Therefore, there are $(p-q-1)$-forms $\widetilde{\beta}_I \in \Omega^{p-q-1}(X_I^*)$ such that $\widetilde{\alpha}_I = \mathrm{d}\widetilde{\beta}_I$, $\forall\, I \in \mathcal{S}_r^q$. By Theorem 2.10 (a) each form $\widetilde{\beta}_I$ admits an extension $\beta_I \in \Omega^{p-q-1}(Q)$, where Q is some polydisc of $\mathbb{C}^n$ where X_I has a representative. Define a logarithmic form μ by

$$\mu = \sum_{I \in \mathcal{S}_r^q} \beta_I \wedge \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_q}}{f_{i_q}}$$

so that

$$\mathrm{d}\mu = \sum_{I \in \mathcal{S}_r^q} \mathrm{d}\beta_I \wedge \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_q}}{f_{i_q}} \implies \mathrm{Res}(\mathrm{d}\mu, X_I) = \mathrm{d}\beta_I|_{X_I} = \mathrm{Res}(\eta, X_I),$$

for all $I \in \mathcal{S}_r^q$. In particular, $\mathrm{Res}(\eta - \mathrm{d}\mu, X_I) = 0$ for all $I \in \mathcal{S}_r^q$, and this implies that $\mathrm{depth}(\eta - \mathrm{d}\mu) < q$. Finally, since $\eta - \mathrm{d}\mu$ is closed the induction hypothesis implies that $\eta - \mathrm{d}\mu = \mathrm{d}\theta$, where either $\theta = 0$, or θ is logarithmic with pole divisor contained in $f_1 \ldots f_r = 0$. This finishes the proof of Claim 2.14 and of Theorem 2.1. □

2.3. Proof of Corollary 1.3

Let η be a logarithmic p-form on $\mathbb{P}^n$, where $p \leqslant n-1$, with pole divisor in homogeneous coordinates $(f_1 \ldots f_r = 0)$ with strictly ordinary singularities outside $0 \in \mathbb{C}^{n+1}$. Let $\Pi\colon \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ be the canonical projection and $\widetilde{\eta} = \Pi^*(\eta)$. We want to prove that $\widetilde{\eta}$ can be written as

$$\widetilde{\eta} = \sum_{I \in \mathcal{S}_r^p} \lambda_I \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \ldots \frac{\mathrm{d}f_{i_p}}{f_{i_p}}$$

where $\lambda_I \in \mathbb{C}\ \forall\, I \in \mathcal{S}_r^p$. We know that η is closed (Deligne's theorem 1.1).

The pull-back $\widetilde{\eta} = \Pi^*(\eta)$ can be extended to a closed logarithmic p-form on $\mathbb{C}^{n+1}$ which is called the expression of η in homogeneous coordinates. The

pole divisor of $\widetilde{\eta}$ is of course the pull-back of the pole divisor of η, and by assumption it is $f_1 \dots f_r$, where f_j is a homogeneous polynomial of degree d_j, $1 \leqslant j \leqslant r$. In particular, we can write

$$\widetilde{\eta} = \frac{1}{f_1 \dots f_r} \sum g_J \mathrm{d}z^J$$

where $\mathrm{d}z^J = \mathrm{d}z_{j_1} \wedge \dots \wedge \mathrm{d}z_{j_p}$ and g_J is a homogeneous polynomial. Using that $\widetilde{\eta}$ is invariant by any homothety $H_t(z) = t.z$, $\forall\ t \in \mathbb{C}^*$, and with a straighforward computation we see that g_J is homogeneous of degree $\deg(g_J) = \deg(f_1 \dots f_r) - p$. This implies that the coefficients $g_J/f_1 \dots f_r$ of $\widetilde{\eta}$ are meromorphic homogenous of degree $-p$.

Now, the hypothesis on the pole divisor of η implies that the pole divisor of $\widetilde{\eta}$, $f_1 \dots f_r$, has strictly ordinary singularities outside $0 \in \mathbb{C}^{n+1}$. Therefore, by Theorem 2.1 we have

$$\widetilde{\eta} = \sum_{I \in \mathcal{S}_r^p} \lambda_I \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \dots \frac{\mathrm{d}f_{i_p}}{f_{i_p}} + \mathrm{d}\Theta,$$

where Θ is logarithmic with pole divisor contained in $(\eta)_\infty$. It is enough to prove that $\mathrm{d}\Theta = 0$.

The proof of Theorem 2.1 implies that the monomials of Θ have depth $< p$ and are, either of the form $\alpha \wedge \frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \dots \wedge \frac{\mathrm{d}f_{j_q}}{f_{j_q}}$, where α is a $(p-q-1)$-form, or of the form $g.\frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \dots \wedge \frac{\mathrm{d}f_{j_{p_1}}}{f_{j_{p_1}}}$, where g is a holomorphic function. In particular, the monomials of $\mathrm{d}\Theta$ are, either of the form $\mathrm{d}\alpha \wedge \frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \dots \wedge \frac{\mathrm{d}f_{j_q}}{f_{j_q}}$, or of the form $\mathrm{d}g \wedge \frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \dots \wedge \frac{\mathrm{d}f_{j_{p_1}}}{f_{j_{p_1}}}$. In both cases, the meromorphic degree of the coefficients of the monomial is $> -p$ and this implies that $\mathrm{d}\Theta = 0$. The proof that $i_R\widetilde{\eta} = 0$ follows from the fact that $D\Pi(z).R(z) = 0$ for all $z \in \mathbb{C}^{n+1} \setminus \{0\}$. Finally, $i_R\widetilde{\eta} = 0$ implies that $r \geqslant p+1$, as the reader can check. □

3. Decomposition of logarithmic foliations

The purpose of this section is to study the question posed in Problem 1.9: is an integrable logarithmic p-form on $\mathbb{P}^n$ totally decomposable into logarithmic 1-forms?

The main theorem to be proved here gives a partial answer to the above problem:

THEOREM 1.10. — *Let $f_1, \ldots, f_r$ be homogeneous polynomials on $\mathbb{C}^{n+1}$ and assume that the pole divisor $f_1 \ldots f_r = 0$ has strictly ordinary singularities outside $0 \in \mathbb{C}^{n+1}$. Then:*

(a) *If $p = 2$, or $r \in \{p+1, p+2\}$ then $\mathcal{L}^p_{td}(f_1, \ldots, f_r) = \mathcal{L}^p_{\mathcal{F}}(f_1, \ldots, f_r)$.*
(b) *If $2 < p \leqslant n-2$ and $r > p+2$ then $\mathcal{L}^p_{td}(f_1, \ldots, f_r)$ is an irreducible component of $\mathcal{L}^p_{\mathcal{F}}(f_1, \ldots, f_r)$. In particular, if $\mathcal{L}^p_{\mathcal{F}}(f_1, \ldots, f_r)$ is irreducible then $\mathcal{L}^p_{td}(f_1, \ldots, f_r) = \mathcal{L}^p_{\mathcal{F}}(f_1, \ldots, f_r)$.*

An interesting consequence of Theorem 1.10 is Corollary 1.11: when $r = p+1$ in Theorem 1.10 then $\mathcal{F}_\eta$ is a rational fibration (see Section 1). Since the case $r = p+1$ is the easier one, we will do it, together with the proof of Corollary 1.11, in Section 3.1.

In Section 3.2 we prove the theorem in the case $p = 2$. In Section 3.3 we will see that the proof of (b) can be reduced to the case of 2-dimensional foliations (in which $p = n-2$). The proof of the case $r = p+2$ will be done in Section 3.4. We note that item (b) is an easy consequence of Theorem 1.17 and so it will be not done in this section.

3.1. Proof of the case $r = p+1$ and of Corollary 1.11

The proof will be based in the remark that a p-vector Ω in a vector space V of dimension $p+1$ is always decomposable. In fact, if $\{v_1, \ldots, v_{p+1}\}$ is a basis of V, then we can write

$$\Omega = \sum_{j=1}^{p+1} a_j v_1 \wedge \cdots \wedge \widehat{v_j} \wedge \cdots \wedge v_{p+1}, \quad a_j \in K, \quad 1 \leqslant j \leqslant p+1.$$

Since $\Omega \neq 0$, we can assume that $a_1 \neq 0$. Dividing Ω by a_1 if necessary, we can assume that $a_1 = 1$.

Let $\{g_1, \ldots, g_{p+1}\}$ be dual basis of the basis $\{v_1, \ldots, v_{p+1}\}$; $g_j(v_i) = \delta_{ij}$. If $X = g_1 + \sum_{j=2}^{p+1} (-1)^{j-1} a_j g_j$ then $\Omega = i_X v_1 \wedge \cdots \wedge v_{p+1}$. Now, if we set $\theta_j := v_j + (-1)^j a_j v_1$, $2 \leqslant j \leqslant p+1$, then $i_X \theta_j = 0$ and the reader can verify that $\Omega = \theta_2 \wedge \cdots \wedge \theta_{p+1}$.

Let $\widetilde{\eta}$ be the extension of $\Pi^*(\eta)$ to $\mathbb{C}^{n+1}$, as in Corollary 1.3. Let $f_1 \ldots f_{p+1}$ be the pole divisor $(\widetilde{\eta})_\infty$, so that

$$\widetilde{\eta} = \sum_{j=1}^{p+1} \lambda_j \frac{\mathrm{d}f_1}{f_1} \wedge \cdots \wedge \widehat{\frac{\mathrm{d}f_j}{f_j}} \wedge \cdots \wedge \frac{\mathrm{d}f_{p+1}}{f_{p+1}}. \tag{3.1}$$

By the above remark $\widetilde{\eta}$ is decomposable: if we assume $\lambda_1 \neq 0$ then there exist $\mu_2, \ldots, \mu_{p+1} \in \mathbb{C}$ such that, if we set $\theta_j = \frac{\mathrm{d}f_j}{f_j} - \mu_j \frac{\mathrm{d}f_1}{f_1}$ then

$$\widetilde{\eta} = \lambda_1 \theta_2 \wedge \cdots \wedge \theta_{p+1}.$$

We assert that $\mu_j \in \mathbb{Q}_+$, $2 \leqslant j \leqslant p+1$. In fact, from $i_R\widetilde{\eta} = 0$ we get

$$\begin{aligned} i_R(\theta_1 \wedge \cdots \wedge \theta_{p+1}) = 0 &\implies \sum_{j=2}^{p+1} (-1)^j i_R(\theta_j).\theta_2 \wedge \cdots \wedge \widehat{\theta_j} \wedge \cdots \wedge \theta_{p+1} = 0 \\ &\implies i_R(\theta_j) = 0, 2 \leqslant j \leqslant p+1 \\ &\implies \mu_j = \frac{\deg(f_j)}{\deg(f_1)} := \frac{d_j}{d_1} \in \mathbb{Q}_+. \end{aligned}$$

In particular, the rational function $f_j^{d_1}/f_1^{d_j}$ is a first integal of θ_j;$\mathrm{d}\big(f_j^{d_1}/f_1^{d_j}\big) \wedge \theta_j = 0$, $2 \leqslant j \leqslant p+1$. This of course implies that $F = \big(f_1^{k_1}, \ldots, f_{p+1}^{k_{p+1}}\big)$ is a first integral of $\widetilde{\eta}$ if $k_j := d_1 \ldots d_{p+1}/d_j$. □

3.2. Proof of Theorem 1.10 in the case $p = 2$: foliations of codimension two

Let $\mathcal{F}$ be a logarithmic foliation of codimension two on $\mathbb{P}^n$ defined by a logarithmic 2-form $\widetilde{\eta} \in \mathcal{L}^2_{\mathcal{F}}(f_1, \ldots, f_r)$. Note that the hypothesis $p = 2 \leqslant n-2$ implies that $n \geqslant 4$.

Remark 3.1. — The condition of local decomposability of $\widetilde{\eta}$ outside the singular set is equivalent to $\widetilde{\eta} \wedge \widetilde{\eta} = 0$. This is a consequence of the fact that a two vector θ on a complex vector space is decomposable if, and only if, $\theta \wedge \theta = 0$.

In particular, we have

$$\mathcal{L}^2_{\mathcal{F}}(f_1, \ldots, f_r) = \{\omega \in \mathcal{L}^2_R(f_1, \ldots, f_r) \,|\, \omega \wedge \omega = 0\}.$$

As we have seen, a form $\omega \in \mathcal{L}^2_R(f_1, \ldots, f_r)$ can be written as

$$\omega = \sum_{1 \leqslant i < j \leqslant r} \mu_{ij} \frac{\mathrm{d}f_i}{f_i} \wedge \frac{\mathrm{d}f_j}{f_j}. \tag{3.2}$$

As the reader can check,

$$\omega \wedge \omega = \sum_{1 \leqslant i<j<k<\ell \leqslant r} 2\Psi(\mu_{ij}, \mu_{k\ell}, \mu_{ik}, \mu_{j\ell}, \mu_{i\ell}, \mu_{jk}) \frac{\mathrm{d}f_i}{f_i} \wedge \frac{\mathrm{d}f_j}{f_j} \wedge \frac{\mathrm{d}f_k}{f_k} \wedge \frac{\mathrm{d}f_\ell}{f_\ell},$$

where $\Psi(a,b,c,d,e,f) = ab - cd + ef$. If $\omega \wedge \omega = 0$ their numerical residues must vanish (see Remark 2.3). This implies that $\mathcal{L}^2_{\mathcal{F}}(f_1,\ldots,f_r)$ is isomorphic to the algebraic subset $\mathcal{A}$ of $\mathbb{C}^{r(r-1)/2}$ defined by

$$\mathcal{A} = \big\{(\lambda_{ij})_{1\leqslant i<j\leqslant r} \,\big|\, \Psi(\lambda_{ij},\lambda_{k\ell},\lambda_{ik},\lambda_{j\ell},\lambda_{i\ell},\lambda_{jk}) = 0,\ \forall\, 1 \leqslant i<j<k<\ell\leqslant r\big\},$$

where the isomorphism is given by

$$(\lambda_{ij})_{1\leqslant i<j\leqslant r} \in \mathcal{A} \longmapsto \sum_{1\leqslant i<j\leqslant r} \lambda_{ij} \frac{\mathrm{d}f_i}{f_i} \wedge \frac{\mathrm{d}f_j}{f_j}$$

On the other hand, if we fix a base $\{e_1,\ldots,e_r\}$ of $\mathbb{C}^r$, a 2-vector θ on $\mathbb{C}^r$ can be written as

$$\theta = \sum_{1\leqslant i<j\leqslant r} a_{ij} e_i \wedge e_j.$$

Since

$$\theta\wedge\theta = \sum_{1\leqslant i<j<k<\ell\leqslant r} 2\Psi(a_{ij},a_{k\ell},a_{ik},a_{j\ell},a_{i\ell},a_{jk}) e_i\wedge e_j\wedge e_k\wedge e_\ell,$$

we obtain $\theta\wedge\theta = 0$ if, and only if, $(a_{ij})_{1\leqslant i<j\leqslant r} \in \mathcal{A}$. Now, if $\theta\wedge\theta = 0$ then θ is decomposable: $\theta = \alpha\wedge\beta$, where $\alpha,\beta\in\mathbb{C}^r$. In fact, if $\theta\neq 0$ let u,v be in the dual of $\mathbb{C}^r$ and such that $\theta(u,v)\neq 0$. Then

$$0 = i_u(\theta\wedge\theta) = 2i_u(\theta)\wedge\theta \implies \theta = c.i_u(\theta)\wedge i_v(\theta),\ c = 1/\theta(u,v).$$

Finally, if ω is as in (3.2) and satisfies $\omega\wedge\omega = 0$ then the 2-vector $\theta = \sum_{i<j}\mu_{ij}e_i\wedge e_j$ is decomposable: $\theta = \alpha\wedge\beta$, $\alpha = \sum_i a_i e_i$ and $\beta = \sum_j b_j e_j$, so that $\omega = \omega_1\wedge\omega_2$, $\omega_1 = \sum_i a_i \frac{\mathrm{d}f_i}{f_i}$ and $\omega_2 = \sum_j b_j \frac{\mathrm{d}f_j}{f_j}$. Moreover, if $i_R\omega = 0$ then $i_R\omega_1 = i_R\omega_2 = 0$ because

$$0 = i_R(\omega_1\wedge\omega_2) = i_R\omega_1.\omega_2 - i_R\omega_2.\omega_1 \implies i_R\omega_1 = i_R\omega_2 = 0 \qquad \square$$

3.3. Some remarks

From now on, we fix homogeneous polynomials $f_1,\ldots,f_r\in\mathbb{C}[z_0,\ldots,z_n]$, where $r > p+1$, the divisor $f_1,\ldots,f_r$ has strictly ordinary singularities outside $0\in\mathbb{C}^{n+1}$ and $\deg(f_j) = d_j$, $1\leqslant j\leqslant r$. Recall that $\mathcal{L}^p(f_1,\ldots,f_r)$ denotes the set of logarithmic p-forms that can be written as below:

$$\widetilde{\eta} = \sum_{I\in\mathcal{S}^p_r} \lambda_I \frac{\mathrm{d}f_{i_1}}{f_{i_1}}\wedge\cdots\wedge\frac{\mathrm{d}f_{i_p}}{f_{i_p}}, \quad \lambda_I\in\mathbb{C}, \quad \forall\, I\in\mathcal{S}^p_r. \tag{3.3}$$

Given a base $\{\mathrm{d}u_1,\ldots,\mathrm{d}u_r\}$ of $\mathbb{C}^{r*}$ there exists an unique linear map $\Phi^p\colon \bigwedge^p(\mathbb{C}^{r*}) \to \mathcal{L}^p(f_1,\ldots,f_r)$ such that

$$\Phi^p(\mathrm{d}u_{i_1}\wedge\cdots\wedge\mathrm{d}u_{i_p}) = \frac{\mathrm{d}f_{i_1}}{f_{i_1}}\wedge\cdots\wedge\frac{\mathrm{d}f_{i_p}}{f_{i_p}}.$$

LEMMA 3.2. — *Φ^p is an isomorphism for all $p \geqslant 1$. Moreover, if $\alpha \in \bigwedge^p(\mathbb{C}^{r*})$ and $\beta \in \bigwedge^q(\mathbb{C}^{r*})$ then*

$$\Phi^{p+q}(\alpha \wedge \beta) = \Phi^p(\alpha) \wedge \Phi^q(\beta). \tag{3.4}$$

Proof. — On one hand, it is clear that Φ^p is surjective. On the other hand, if $\widetilde{\eta} = \sum_{I \in \mathcal{S}_r^p} \lambda_I \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_p}}{f_{i_p}}$ then each numerical residue λ_I, $I \in \mathcal{S}_r^p$, can be calculated by an integral as in Remark 2.3:

$$\lambda_I = \frac{1}{(2\pi i)^p} \int_{T_\epsilon^p} \eta.$$

It follows that

$$\sum_{I \in \mathcal{S}_r^p} \lambda_I \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_p}}{f_{i_p}} \equiv 0 \iff \lambda_I = 0,\ \forall\, I \in \mathcal{S}_r^p$$

and so Φ^p is injective.

Finally, formula (3.4) is consequence of

$$\begin{aligned} \Phi^{p+q}\left((\mathrm{d}u_{i_1} \wedge \cdots \wedge \mathrm{d}u_{i_p}) \wedge (\mathrm{d}u_{j_1} \wedge \cdots \wedge \mathrm{d}u_{j_q})\right) & \\ = \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_p}}{f_{i_p}} \wedge \frac{\mathrm{d}f_{j_1}}{f_{j_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{j_q}}{f_{j_q}} & \\ = \Phi^p(\mathrm{d}u_{i_1} \wedge \cdots \wedge \mathrm{d}u_{i_p}) \wedge \Phi^q(\mathrm{d}u_{j_1} \wedge \cdots \wedge \mathrm{d}u_{j_q}) & \quad \square \end{aligned}$$

Remark 3.3. — Given a p-form $\alpha \in \bigwedge^p(\mathbb{C}^{r*})$ its kernel is defined as

$$\ker(\alpha) = \{v \in \mathbb{C}^r \mid i_v\alpha = 0\}.$$

We say that $\alpha \in \bigwedge^p(\mathbb{C}^{r*})$ is totally decomposable if there are p 1-forms $\alpha_1, \ldots, \alpha_p$ such that $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p$. It is well known that:

(a) $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p$ is totally decomposable if, and only if, $\dim(\ker(\alpha)) = r - p$.
(b) If $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p$ then $\ker(\alpha) = \bigcap_{j=1}^p \ker(\alpha_j)$.
(c) The projectivization of the set of totally decomposable p-forms of $\bigwedge^p(\mathbb{C}^{r*})$ is isomorphic to the grassmanian of p planes through the origin in $\mathbb{C}^r$. In particular, it is an algebraic subset of $\mathbb{P}\left(\bigwedge^p(\mathbb{C}^{r*})\right)$.

Recall that $\widetilde{\eta} \in \mathcal{L}_{td}^p(f_1, \ldots, f_r)$ if it is totally decomposable into logarithmic forms (totally decomposable into logarithmic forms). An easy consequence of Lemma 3.2 and of Remark 3.3(c) is the following:

COROLLARY 3.4. — *Let $p \geqslant 2$. A p-form $\widetilde{\eta} \in \mathcal{L}^p(f_1, \ldots, f_r)$ is* totally decomposable into logarithmic forms *if, and only if, there are 1-forms $\alpha_1, \ldots, \alpha_p \in \mathbb{C}^{r*}$ such that*

$$\widetilde{\eta} = \Phi^p(\alpha_1 \wedge \cdots \wedge \alpha_p).$$

In particular, $\mathcal{L}^p_{td}(f_1,\dots,f_r)$ is an algebraic irreducible subset of $\mathcal{L}^p_R(f_1,\dots,f_r)$.

Another consequence of Lemma 3.2 is that Theorem 1.10(b) can be reduced to the case of 2-dimensional foliations. Let $\Sigma \simeq \mathbb{P}^q$ be a q-plane linearly embedded in $\mathbb{P}^n$. We say that Σ is in general position with respect to the divisor $f_1 \dots f_r$ if for all $J = \{j_1, \dots, j_k\} \subset \{1, \dots, r\}$ then Σ is transverse to $\bigcap_{j\in J} \Pi(f_j = 0)$. By transversality theory, the set of q-planes of $\mathbb{P}^n$ in general position with respect to $f_1 \dots f_r$ is a Zariski open and dense subset of the grassmanian of q-planes on $\mathbb{P}^n$.

Remark 3.5. — Let $\widetilde{\eta} \in \mathcal{L}^p_R(f_1, \dots, f_r)$. Let Σ be $(p+k)$-plane of $\mathbb{P}^n$ in general position with respect to $f_1 \dots f_r$, $p < p+k < n$, and $\widetilde{\Sigma}$ be a $p+k+1$ plane through $0 \in \mathbb{C}^{n+1}$ such that $\Pi(\widetilde{\Sigma}) = \Sigma$. Then $\widetilde{\eta}|_{\widetilde{\Sigma}}$ is a logarithmic p-form on $\widetilde{\Sigma}$. An easy consequence of Lemma 3.2 and Corollary 3.4 is the following:

Corollary 3.6. — *Let $\widetilde{\eta}$, Σ and $\widetilde{\Sigma}$ be as in Remark 3.5. Then $\widetilde{\eta}$ is* totally decomposable into logarithmic forms *if, and only if, $\widetilde{\eta}|_{\widetilde{\Sigma}}$ is* totally decomposable into logarithmic forms.

Taking $k = 2$ in the above statement, we reduce the proofs of the case $r = p + 2$ and of Theorem 1.10(b) to the case of 2-dimensional foliations. From now on, we will assume that $\widetilde{\eta} = \Pi^*(\eta) \in \mathcal{L}^p_{\mathcal{F}}(f_1, \dots, f_r)$ and that $n = p + 2$. By Section 3.1 we will assume also that $r \geqslant p + 2$. As we have seen, we can write

$$\widetilde{\eta} = \sum_{I\in\mathcal{S}^p_r} \lambda_I \frac{\mathrm{d}f_{i_1}}{f_{i_1}} \wedge \cdots \wedge \frac{\mathrm{d}f_{i_{n-2}}}{f_{i_{n-2}}}. \tag{3.5}$$

The foliation $\mathcal{F}_\eta$ is defined in homogeneous coordinates by the (n-2)-form $\omega = f_1 \dots f_r \widetilde{\eta}$. As a consequence, the part of $\mathrm{Sing}(\mathcal{F}_{\tilde{\eta}})$ contained in the pole divisor contains an union of curves: given $J = (j_1, \dots, j_{n-1}) \in \mathcal{S}^{n-1}_r$ let $S_J = \Pi(f_{j_1} = \cdots = f_{j_{n-1}} = 0)$. By the assumption on the pole divisor $f_1 \dots f_r$, S_J is a smooth complex curve and

$$\mathrm{Sing}(\mathcal{F}_\eta) \cap \Pi(f_1 \dots f_j \dots f_r = 0) \supset \bigcup_{J\in\mathcal{S}^{n-1}_r} S_J.$$

A point $q = \Pi(p) \in S_J$, for a fixed $J \in \mathcal{S}^{n-1}_r$, will be said *generic* if for all $i \notin J$ then $f_i(p) \neq 0$. Otherwise, q will be said *non-generic*. By the assumption on the pole divisor, if $q = \Pi(p)$ is non-generic and $f_i(p) = 0$ then $f_\ell(p) \neq 0$ for all $\ell \notin J \cup \{i\}$.

Let us fix $J = (j_1 < \cdots < j_{n-1}) \in \mathcal{S}^{n-1}_r$ and a point $q = \Pi(q) \in S_J$. After an automorphism of $\mathbb{P}^n$ we can assume that $q = (0, \dots, 0)$ in the affine

chart $(x_0 = 1) \simeq \mathbb{C}^n$. In this chart, the pole divisor of η is $g_1 \dots g_r$, where $g_j(x) = g_j(x_1, \dots, x_n) = f_j(1, x_1, \dots, x_n)$. Since the equation of the curve S_J is $(g_{j_1} = \cdots = g_{j_{n-1}} = 0)$, there exists a holomorphic coordinate system $(U, z = (z_1, \dots, z_n))$ around q such that $g_{j_i}|_U = z_i$, $1 \leqslant i \leqslant n-1$.

Remark 3.7. — Let $q \in S_J$ and (U, z) be as above. We would like to observe that:

(a) If q is a generic point of S_J then we can write

$$\eta|_U = \sum_{j=1}^{n-1} \mu_j \frac{\mathrm{d}z_1}{z_1} \wedge \cdots \wedge \widehat{\frac{\mathrm{d}z_j}{z_j}} \wedge \cdots \wedge \frac{\mathrm{d}z_{n-1}}{z_{n-1}} + \mathrm{d}\Theta, \tag{3.6}$$

where, either $\Theta = 0$, or Θ is a non-closed logarithmic (n-3)-form with pole divisor contained in $x = z_1 \dots z_{n-1}$, and $\mu_i = \lambda_{I_i}$, $I_i = J \setminus \{j_i\}$.

(b) If $q \in S_J$ is a non-generic point then there exists $j \notin J$ such that $g_j(q) = 0$ and $g_i(q) \neq 0$ if $i \notin J \cup \{j\}$. In this case, we can assume that $g_j|_U = z_n$. Moreover, we can write

$$\eta|_U = \sum_{1 \leqslant k < \ell \leqslant n} \mu_{k\ell} \frac{\mathrm{d}z_1}{z_1} \wedge \cdots \wedge \widehat{\frac{\mathrm{d}z_k}{z_k}} \wedge \cdots \wedge \widehat{\frac{\mathrm{d}z_\ell}{z_\ell}} \wedge \cdots \wedge \frac{\mathrm{d}z_n}{z_n} + \mathrm{d}\Theta, \tag{3.7}$$

where Θ is as in (a) and $\mu_{k\ell} = \lambda_{I_{k\ell}}$, $I_{k\ell} = J \cup \{j\} \setminus \{j_k, j_\ell\}$ if $\ell < n$, $\mu_{kn} = J \setminus \{j_k\}$.

The proof can be done directly by using (3.5) or Theorem 2.1.

3.4. Proof of the case $r = p + 2$

In this case $r = p + 2 = n$ and the non generic points of $\mathrm{Sing}(\mathcal{F}_\eta) \cap \Pi(f_1 \dots f_n = 0)$ are in the finite set $\Pi(f_1 = \cdots = f_n = 0)$. In particular, if we fix a non-generic point $q \in \Pi(f_1 = \cdots = f_n = 0)$ there exists a local coordinate system $(U, z = (z_1, \dots, z_n))$ around q such that $g_j|_U = z_j$, $1 \leqslant j \leqslant n$. In particular, by (3.5) we have

$$\eta|_U = \sum_{1 \leqslant k < \ell \leqslant n} \mu_{k\ell} \frac{\mathrm{d}z_1}{z_1} \wedge \cdots \wedge \widehat{\frac{\mathrm{d}z_k}{z_k}} \wedge \cdots \wedge \widehat{\frac{\mathrm{d}z_\ell}{z_\ell}} \wedge \cdots \wedge \frac{\mathrm{d}z_n}{z_n}.$$

Since $\eta \in \mathcal{L}^{n-2}_{\mathcal{F}}(f_1, \dots, f_n)$ then $\eta|_U$ is locally decomposable outside the polar set $z_1 \dots z_n = 0$. The foliation $\mathcal{F}_\eta$ is defined in U by the holomorphic form

$$\omega := z_1 \dots z_n \eta|_U = \sum_{1 \leqslant k < \ell \leqslant n} \mu_{k\ell} z_k z_\ell \mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_k} \wedge \cdots \wedge \widehat{\mathrm{d}z_\ell} \wedge \cdots \wedge \mathrm{d}z_n. \quad (*)$$

Remark 3.8. — Let α be a holomorphic (n-2)-form on an open subset $V \subset \mathbb{C}^n$. Given $1 \leqslant j \leqslant n$ and $p \in V$ such that $\alpha|_{(z_j=z_j(p))} \not\equiv 0$ we can define a vector field X^j_p, tangent to the hyperplane $(z_j = z_j(p))$, by

$$\alpha|_{(z_j=z_j(p))} = i_{X^j_p}\nu_j, \nu_j = \mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_j} \wedge \cdots \wedge \mathrm{d}z_n.$$

This procces defines a holomorphic vector field X^j on V, tangent to the fibration $(z_j = cte)$, by

$$X^j(p) = X^j_p(p), p \in V.$$

Altough $i_{X^j_p}\alpha|_{(z_j=z_j(p))} = 0$, in general $i_{X^j}\alpha \not\equiv 0$. However, if the form α is locally decomposable outside its singular set then $i_{X^j}\alpha \equiv 0$, so that X^j is tangent to the distribution defined by α. The proof is straightforward and is left to the reader.

If we apply Remark 3.8 to the (n-2)-form ω in $(*)$ we obtain $X^j = z_j.Y^j$, where

$$Y^j = \sum_{k\neq j} \rho^j_k z_k \frac{\partial}{\partial z_k},$$

and $\rho^j_k = (-1)^{k-1}\mu_{kj}$, with the convention $\mu_{rs} = -\mu_{sr}$, $\forall\, r, s$. Since $\mathcal{F}_\eta$ has dimension two, at least two of the linear vector fields above, that we can suppose to be Y^1 and Y^2, are not identically zero and generically linearly independent. In this case, the form $\widetilde{\omega} = i_{Y^1}i_{Y^2}\nu$, $\nu = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$, defines the same distribution as ω. The reader can check that

$$\widetilde{\omega} = \sum_{1\leqslant k<\ell\leqslant n} (-1)^{k+\ell}(\rho^1_k\rho^2_\ell - \rho^1_\ell\rho^2_k) z_k z_\ell \mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_k} \wedge \cdots \wedge \widehat{\mathrm{d}z_\ell} \wedge \cdots \wedge \mathrm{d}z_n.$$

Since the coefficients of ω and $\widetilde{\omega}$ are homogeneous of degree two it follows that $\widetilde{\omega} = c.\omega$, where $c \in \mathbb{C}^*$. From $\rho^j_k = (-1)^{k-1}\mu_{kj}$ and the above expression, we get

$$\mu_{k1}\mu_{\ell 2} - \mu_{\ell 1}\mu_{k2} = c.\mu_{k\ell} \underset{k=2}{\Longrightarrow} c = \mu_{12}.$$

Now, consider the n-2 closed logarithmic 1-forms $\widetilde{\theta}_3, \ldots, \widetilde{\theta}_n$ defined by

$$\widetilde{\theta}_j = \rho^2_j \frac{\mathrm{d}z_1}{z_1} - \rho^1_j \frac{\mathrm{d}z_2}{z_2} + \rho^1_2 \frac{\mathrm{d}z_j}{z_j}.$$

Using that $\rho^2_1 = -\rho^1_2$ we get $i_{Y^1}\widetilde{\theta}_j = i_{Y^2}\widetilde{\theta}_j = 0$, $3 \leqslant j \leqslant n$, and this implies that $\widetilde{\theta}_3 \wedge \cdots \wedge \widetilde{\theta}_n = k.\eta|_U$, $k \in \mathbb{C}^*$. Comparing the coefficients of $\frac{\mathrm{d}z_3}{z_3} \wedge \cdots \wedge \frac{\mathrm{d}z_n}{z_n}$ of the two members of the relation we get $k = (\rho^1_2)^{n-3} = \mu^{n-3}_{12}$. Finally, if we define $\theta_j = \rho^2_j \frac{\mathrm{d}f_1}{f_1} - \rho^1_j \frac{\mathrm{d}f_2}{f_2} + \rho^1_2 \frac{\mathrm{d}f_j}{f_j}$ then $\theta_3 \wedge \cdots \wedge \theta_n$ then

$$\theta_3 \wedge \cdots \wedge \theta_n = \mu^{n-3}_{12}\widetilde{\eta},$$

which proves that $\mathcal{L}^{n-2}_{\mathcal{F}}(f_1, \ldots, f_n) = \mathcal{L}^{n-2}_{td}(f_1, \ldots, f_n)$.

4. Proof of Theorem 1.17

The purpose of this section is to prove the following:

THEOREM 1.17. — *If $k \geqslant 2$ and $r \geqslant p+2 = n-k+2$ then $\overline{\mathcal{L}_{td}(d_1,\ldots,d_r;k,n)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}(D(d_1,\ldots,d_r,p);k,n)$ for all $r > p$ and $d_1,\ldots,d_r \geqslant 1$.*

The proof of Theorem 1.17 will be done first in the case of foliations of dimension two. The general case will be reduced to this one by using the following result:

THEOREM 4.1. — *Let $\mathcal{F}$ be a codimension p holomorphic foliation on $\mathbb{P}^n$, $n > p+1$. Assume that there is an algebraic smooth submanifold $M \subset \mathbb{P}^n$, $\dim_{\mathbb{C}}(M) = m$, where $p+1 \leqslant m < n$, such that:*

- *The set of tangencies of $\mathcal{F}$ with M has codimension $\geqslant 2$ on M.*
- *$\mathcal{F}|_M$ can be defined by a closed meromorphic p-form on M, say η.*

Then η can be extended to a closed meromorphic p-form $\widetilde{\eta}$ on $\mathbb{P}^n$ defining $\mathcal{F}$. Moreover, if η is logarithmic so is $\widetilde{\eta}$.

In fact, Theorem 4.1 is a generalization of a result in [4] (see also [16]).

The proof of Theorem 1.17 in the two dimensional case will be reduced to a result that we state next (Theorem 4.3). In order to state it properly let us recall the definition of a Kupka or generalized Kupka singularity for two dimensional foliations (see also [17]).

Let ω be a germ at $p \in \mathbb{C}^n$ of integrable $(n-2)$-form with $\omega(p) = 0$. Recall that the rotational of ω is the vector field $X = \mathrm{rot}(\omega)$ (cf. [17]) defined by

$$\mathrm{d}\omega = i_X\nu, \nu = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n.$$

The singularity p of ω is of Kupka type if $X(p) \neq 0$ and it is of *generalized Kupka type* (briefly g.K) if $X(p) = 0$ and p is an isolated singularity of X. When $X(p)=0$ and the linear part of X at p is non singular ($\det(DX(p)) \neq 0$) we say that p is non degenerated g.K.

The *Kupka set* of $\mathcal{F}_\omega$ is the set of Kupka singularities of $\mathcal{F}_\omega$.

If p is of Kupka or g.K type then the division theorem [21] implies that there exists another germ of holomorphic vector field, say Y, such that $\omega = i_Y i_X \nu$.

Remark 4.2. — If p is of Kupka type then there exists a local coordinate system $z = (z_1,\ldots,z_n)$ around p such that $z(p) = 0$, $X = \frac{\partial}{\partial z_n}$ and $Y =$

$\sum_{j=1}^{n-1} Y_j(z_1,\ldots,z_{n-1})\frac{\partial}{\partial z_j}$, $Y(0)=0$. In particular, the foliation $\mathcal{F}_\omega$ has the structure of a local product, the germ of curve $\gamma=(z_1=\cdots=z_{n-1}=0)$ is contained in the Kupka set of $\mathcal{F}_\omega$ and the vector field Y defines the *normal type* of $\mathcal{F}_\omega$ along γ [17].

In the next result we will consider the following situation: let $\mathcal{F}$ be a two-dimensional foliation on $\mathbb{P}^n$, $n\geqslant 4$. Assume that $\mathrm{Sing}(\mathcal{F})$ contains a smooth irreducible curve, say S, with the following properties:

(I) There is a finite subset $F=\{p_1,\ldots,p_k\}\subset S$ such that $S\setminus F\subset K(\mathcal{F})$, the Kupka set of $\mathcal{F}$. Since $S\setminus F$ is connected, the normal type of $\mathcal{F}$ is the same at all points of $S\setminus F$. We will denote by Y a germ at $0\in\mathbb{C}^{n-1}$ of holomorphic vector representing this normal type.

(II) The eigenvalues of the linear part of Y, say $\rho_1,\ldots,\rho_{n-1}$, are in the Poincaré domain and satisfy the following non-resonant conditions
   ($\star$) $\rho_j\neq\sum_{i\neq j}m_i\rho_i$ for all $m=(m_1,\ldots,\widehat{m_j},\ldots,m_{n-1})\in\mathbb{Z}_{\geqslant 0}^{n-2}$ with $\sum_i m_i\geqslant 1$.
   In particular, we have $\rho_i\neq\rho_j$ if $i\neq j$. Recall that $\rho_1,\ldots,\rho_{n-1}$ are in the Poincaré domain if there exists $a\neq 0$ such that $\mathrm{Re}(a.\rho_j)>0$, $1\leqslant j\leqslant n-1$. With these conditions the germ of vector field Y is linearizable and semi-simple (cf. [1] and [19]).

(III) Given $p\in F$ let ω be a germ of (n-2)-form defining the germ of $\mathcal{F}$ at p. We will assume that there is a local coordinate system $(U,z=(z_1,\ldots,z_n))$ around p with the following properties:
   (i) $z(p)=0$ and $S\cap U=(z_1=\cdots=z_{n-1}=0)$.
   (ii) Set $X=\mathrm{rot}(\omega)$, so that $\mathrm{d}\omega=i_X\nu$, $\nu=\mathrm{d}z_1\wedge\cdots\wedge\mathrm{d}z_n$. Let $\lambda_1,\ldots,\lambda_n$ be the eigenvalues of the linear part $DX(p)$. We will assume that there exists $a\neq 0$ such that $\mathrm{Re}(a.\lambda_n)<0$ and $\mathrm{Re}(a.\lambda_j)>0$, $\forall\ 1\leqslant j\leqslant n-1$. Moreover, we will assume that the eingenspace of $DX(p)$ associated to the eigenvalue λ_n is the tangent space T_pS.
   (iii) Setting $\rho_n=0$, we will assume that $\lambda_i\rho_j-\lambda_j\rho_i\neq 0$, $\forall\ 1\leqslant i<j\leqslant n$.

Theorem 4.3. — *If $\mathcal{F}$ satisfies conditions* (I), (II) *and* (III) *above then there exist homogeneous polynomials $g_1,\ldots,g_r$ such that $\mathcal{F}\in\mathcal{L}_{td}^{n-2}(g_1,\ldots,g_r)$.*

A crucial fact that will be used in the proof of Theorem 4.3 is that, if $r\geqslant n$ then there are foliations $\mathcal{F}\in\mathcal{L}_{td}(d_1,\ldots,d_r;2,n)$ with a curve $S\subset\mathrm{Sing}(\mathcal{F})$ that satisfies conditions (I), (II) and (III) above:

LEMMA 4.4. — *If $r \geqslant n$ then there are $\theta_2, \ldots, \theta_{n-1} \in \mathcal{L}^1_{\mathcal{F}}(d_1, \ldots, d_r; n-1, n)$ such that, if $\eta_o := \theta_2 \wedge \cdots \wedge \theta_{n-1}$, then $\mathcal{F}_{\eta_0}$ satisfies* (I), (II) *and* (III) *along the curve $S = \Pi(f_1 = \cdots = f_{n-1} = 0)$.*

Since the proof of Lemma 4.4 is not difficult we begin by it.

4.1. Proof of Lemma 4.4

Let $f_1, \ldots, f_r$ be homogeneous polynomials on $\mathbb{C}^{n+1}$ with $\deg(f_j) = d_j$, $1 \leqslant j \leqslant r$, and such that $(f_1 \ldots f_r = 0)$ has strictly ordinary singularities outside $0 \in \mathbb{C}^{n+1}$.

When $r = n-1$ and $S = \Pi(f_1 = \cdots = f_{n-1} = 0)$ we have seen in the proof of Corollary 1.3 in Section 2.3 that the foliation $\mathcal{F}_0 \in \mathcal{L}_{\mathcal{F}}(f_1, \ldots, f_{n-1})$ is unique and defined by $\theta_0^2 \wedge \cdots \wedge \theta_0^{n-1}$, where

$$\theta_0^j = \frac{\mathrm{d}f_j}{f_j} - A_j \frac{\mathrm{d}f_1}{f_1}, \quad A_j = \frac{d_j}{d_1}, \quad 2 \leqslant j \leqslant n-1, \tag{4.1}$$

with $d_j = \deg(f_j)$. In a neighborhood U of any point $p \in S$ we can find local coordinates $z = (z_1, z_2, \ldots, z_n)$ such that $f_j|_U = z_j$, $1 \leqslant j \leqslant n-1$, $\Longrightarrow \theta_0^j = \frac{\mathrm{d}z_j}{z_j} - A_j \frac{\mathrm{d}z_1}{z_1}$ and $\mathcal{F}_{\eta_0}$ is defined by $\omega = z_1 \ldots z_{n-1} \theta_0^2 \wedge \cdots \wedge \theta_0^{n-1}|_U$. Since the $\theta^0_{j's}$ are closed we get

$$\mathrm{d}\omega = z_1 \ldots z_{n-1} \left(\frac{\mathrm{d}z_1}{z_1} + \cdots + \frac{\mathrm{d}z_{n-1}}{z_{n-1}} \right) \wedge \theta_0^2 \wedge \cdots \wedge \theta_0^{n-1}|_U = \rho \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_{n-1},$$

where $\rho = \frac{1}{d_1} \sum_j d_j \neq 0$, as the reader can check. Therefore, all points in S are of Kupka type and S is a Kupka component of $\mathcal{F}_{\eta_0}$. The normal type of $\mathcal{F}_{\eta_0}$ at p can be defined by the linear vector field $Y = \sum_{j=1}^{n-1} d_j z_j \frac{\partial}{\partial z_j}$ because it satisfies $i_Y \theta_0^j = 0$, $\forall\, 2 \leqslant j \leqslant n-1$. In particular, the eigenvalues $\rho_j = d_j$, $1 \leqslant j \leqslant n-1$, are in the Poincaré domain: $\mathrm{Re}(\rho_j) > 0$, $\forall\, 1 \leqslant j \leqslant n-1$.

When $r \geqslant n$ we will consider small deformations of the forms θ_0^j above. For instance, if $r = n$ then the non-generic points of S are the points of the set $F = S \cap \Pi(f_n = 0)$.

Let us consider the case $r = n$. Given $\tau = (t_2, \ldots, t_{n-1}) \in \mathbb{C}^{n-2}$ consider the family of 1-forms

$$\theta_\tau^j = \frac{\mathrm{d}f_j}{f_j} - A_j(\tau) \frac{\mathrm{d}f_1}{f_1} - B_j(\tau) \frac{\mathrm{d}f_n}{f_n}, 2 \leqslant j \leqslant n-1,$$

where $A_j(\tau) = \frac{d_j}{d_1} - t_j$ and $B_j(\tau) = \frac{t_j d_1}{d_n}$. Note that $i_R \theta_\tau^j = 0$, $\forall\, 2 \leqslant j \leqslant n-1$, so that $\theta_\tau^j \in \mathcal{L}^1_{\mathcal{F}}(f_1, \ldots, f_n)$, $\forall\, \tau \in \mathbb{C}^{n-2}$, $\forall\, 2 \leqslant j \leqslant n-1$. Let $\mathcal{F}_\tau$ be the foliation defined by $\eta_\tau = \theta_\tau^2 \wedge \cdots \wedge \theta_\tau^{n-1}$.

If $p \in S$ is a generic point then $f_n(p) \neq 0$ and there are local coordinates at p, $(U, z = (z_1, \dots, z_{n-1}, z_n))$, such that $f_j|_U = z_j$, $1 \leqslant j \leqslant n-1$. In these coordinates we have $S \cap U = (z_1 = \cdots = z_{n-1} = 0)$ and the normal type can be calculated by considering the restriction of $\mathcal{F}_\tau$ to a normal section, for instance $\Sigma := (f_n = f_n(p)) \cap U$. The foliation $\mathcal{F}_\tau|_\Sigma$ is defined by the (n-2)-form

$$z_1 \dots z_{n-1}\eta_\tau|_\Sigma = z_1 \dots z_{n-1}\left(\frac{\mathrm{d}z_2}{z_2} - A_2(\tau)\frac{\mathrm{d}z_1}{z_1}\right) \wedge \cdots \wedge \left(\frac{\mathrm{d}z_{n-1}}{z_{n-1}} - A_{n-1}(\tau)\frac{\mathrm{d}z_1}{z_1}\right).$$

In particular, the normal type can be defined by the vector field

$$Y_\tau = \sum_{j=1}^{n-1} \rho_j(\tau) z_j \frac{\partial}{\partial z_j},$$

where $\rho_1(\tau) = d_1$ and $\rho_j(\tau) = d_j - t_j d_1$, because $i_{Y_\tau}\theta^i_\tau = 0$, $\forall\ 2 \leqslant i \leqslant n-1$.

If $|\tau|$ is small enougth then the genereric points of S are of Kupka type and the eigenvalues of the normal type are in the Poincaré domain (these are open conditions). Moreover, the parameter τ can be chosen in such a way that the eigenvalues $\rho_1(\tau), \dots, \rho_{n-1}(\tau)$ satisfy the non-resonance conditions $(\star)$ of (II). This is a consequence of the fact that the set $\{(\rho_2, \dots, \rho_{n-1}) \in \mathbb{C}^{n-2} \,|\, d_1 = \rho_1, \rho_2, \dots, \rho_{n-1}$ satisfy conditions $(\star)\}$ is dense in $\mathbb{C}^{n-2}$.

At a point $p \in F$ we can find local coordinates $(U, z = (z_1, \dots, z_n))$ such that $z(p) = 0$ and the foliation is defined by the form $\omega_\tau = z_1 \dots z_n \widetilde{\theta}^2_\tau \wedge \cdots \wedge \widetilde{\theta}^{n-1}_\tau$, where

$$\widetilde{\theta}^j_\tau = \frac{\mathrm{d}z_j}{z_j} - A_j(\tau)\frac{\mathrm{d}z_1}{z_1} - B_j(\tau)\frac{\mathrm{d}z_n}{z_n}.$$

Since the forms $\widetilde{\theta}^j_\tau$ are closed, we get

$$\mathrm{d}\omega_\tau = \left(\frac{\mathrm{d}z_1}{z_1} + \cdots + \frac{\mathrm{d}z_n}{z_n}\right) \wedge \omega_\tau.$$

The rotational X_τ of ω_τ is defined by $\mathrm{d}\omega_\tau = i_{X_t}\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$ and so $X_\tau = \sum_{j=1}^n \lambda_j(\tau) z_j \frac{\partial}{\partial z_j}$ is linear and must satisfy $i_{X_\tau}\left(\frac{\mathrm{d}z_1}{z_1} + \cdots + \frac{\mathrm{d}z_n}{z_n}\right) = 0$ and $i_{X_\tau}\widetilde{\theta}^j_\tau = 0$, $\forall\ 2 \leqslant j \leqslant n-1$. It follows that the eigenvalues $\lambda_1(\tau), \dots, \lambda_n(\tau)$ must satisfy the homogeneous system

$$\begin{cases} x_1 + \cdots + x_n = 0 \\ x_j - A_j(\tau)x_1 - B_j(\tau)x_n = 0, \quad 1 \leqslant j \leqslant n-2 \end{cases} \tag{4.2}$$

When $\tau = 0$ we are in the situation of the case $r = n-1$ and a solution of (4.2) is $x_j = d_j > 0$, if $1 \leqslant j \leqslant n-1$, and $x_n = -(d_1 + \cdots + d_{n-1}) < 0$. Therefore, $\lambda_1(0), \dots, \lambda_n(0)$ satisfy condition $\mathrm{Re}(a.\lambda_n(0)) < 0$

and $\mathrm{Re}(a.\lambda_j(0)) > 0$, $1 \leqslant j \leqslant n-1$, for some $a \neq 0$. Of course, this implies that for small $|\tau|$ the eigenvalues of X_τ has eigenvalues that satisfy $\mathrm{Re}(a_\tau.\lambda_n(\tau)) < 0$ and $\mathrm{Re}(a_\tau.\lambda_j(\tau)) > 0$, $1 \leqslant j \leqslant n-1$, for some $a_\tau \neq 0$. It remains to verify that $\mathcal{F}_\tau$ satisfies condition (iii) of (III) near p.

First of all, recall that X_τ and $Z_\tau = Y_\tau = \sum_{j=1}^{n-1} \rho_j(\tau) z_j \frac{\partial}{\partial z_j}$ are tangent to $\mathcal{F}_\tau$. Moreover, since $X_\tau \wedge Z_\tau \not\equiv 0$ these vector fields generate the foliation in a neighborhood of $p = 0$. In particular, we must have $\omega_\tau = b.i_{X_\tau} i_{Z_\tau} \nu$ for some $b \neq 0$. If we set $\rho_n(\tau) = 0$ then the coefficient of $\frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ in $X_\tau \wedge Z_\tau$ is $\lambda_i(\tau)\rho_j(\tau) - \lambda_j(\tau)\rho_i(\tau) \neq 0$ if $i < j$. Therefore (iii) is equivalent to prove that all coefficients of ω_τ are not zero.

Set $\alpha = z_1 \dots z_{n-1} \widetilde{\theta}_0^2 \wedge \dots \wedge \widetilde{\theta}_0^{n-1}$. By the case $r = n-1$ we have $\mathrm{Sing}(\alpha) = S \cap U = (z_1 = \dots = z_{n-1} = 0)$. Since $\omega_0 = z_n.\alpha$ we have

$$\mathrm{Sing}(\omega_0) = (z_n = 0) \cup (z_1 = \dots = z_{n-1} = 0).$$

On the other hand, if $\tau \neq 0$ then the form ω_τ can be written as

$$\omega_\tau = z_n.\alpha_\tau + \mathrm{d}z_n \wedge \beta_\tau$$

where $\alpha_0 = \alpha$, α_τ has linear coefficients and

$$\beta_\tau = \sum_{1 \leqslant i < j \leqslant n-1} A_{ij}(\tau) z_i z_j \mathrm{d}z_1 \wedge \dots \wedge \widehat{\mathrm{d}z_i} \wedge \dots \wedge \widehat{\mathrm{d}z_j} \wedge \dots \wedge \mathrm{d}z_{n-1},$$

where $A_{ij}(\tau) = \pm(A_i(\tau)B_j(\tau) - A_j(\tau)B_i(\tau))$, if $i, j > 1$ and $A_{1j}(\tau) = \pm B_j(\tau)$, if $2 \leqslant j \leqslant n-1$. We leave this computation for the reader. If $t_j \neq 0$ then $A_{1j}(\tau) = \pm B_j(\tau) = \pm t_j.d_1/d_n \neq 0$. If $i, j > 1$ then

$$A_{ij}(\tau) = \pm \frac{d_1}{d_n}(t_j d_i - t_i d_j).$$

Hence, we can choose τ small so that $t_j d_i - t_i d_j \neq 0$, $\forall\, 1 < i < j \leqslant n-1$.

In the case $r > n$ we consider the parameter $\tau = (t_{ji})_{n \leqslant i \leqslant r}^{2 \leqslant j \leqslant n-1}$ and

$$\theta_\tau^{rj} = \frac{\mathrm{d}f_j}{f_j} - A_j(\tau)\frac{\mathrm{d}f_1}{f_1} - \sum_{i=n}^{r} B_{ji}(\tau)\frac{\mathrm{d}f_i}{f_i},$$

where $A_j(\tau) = \frac{d_j}{d_1} - \sum_{i=n}^{r} t_{ji}$ and $B_{ji}(\tau) = \frac{d_1}{d_i} t_{ji}$, $2 \leqslant j \leqslant n-1$. It can be checked directly that $i_R \theta_\tau^{rj} = 0$, $\forall\, j$. The proof of the lemma in this case can be done by induction on $r \geqslant n$. We leave the details for the reader. □

4.2. Proof of Theorem 1.17 in the case of two dimensional foliations

In the proof we will use Lemma 4.4 and Theorem 4.3. Theorem 4.3 will be proved in the next section.

We want to prove that for all $r \geqslant n$ and $d_1, \ldots, d_r \geqslant 1$ then $\overline{\mathcal{L}_{td}(d_1, \ldots, d_r; 2, n)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}(D; 2, n)$, where $D = \sum_j d_j - n + 1$. To avoid confusion we assume $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_r$. Recall that the definition implies

$$\mathcal{L}_{td}(d_1, \ldots, d_r; 2, n) = \bigcup_{\substack{dg(f_j)=d_j \\ 1 \leqslant j \leqslant r}} \mathcal{L}_{td}^{n-2}(f_1, \ldots, f_r).$$

Since $\overline{\mathcal{L}_{td}^{n-2}(f_1, \ldots, f_r)}$ is irreducible for all polynomials $f_1, \ldots, f_r$ with $\deg(f_j) = d_j$, $1 \leqslant j \leqslant r$, it is clear that $\overline{\mathcal{L}_{td}(d_1, \ldots, d_r; 2, n)}$ is an irreducible algebraic subset of $\mathbb{F}\mathrm{ol}(D; 2, n)$. The idea is to exhibit a foliation $\mathcal{F}_0 \in \mathcal{L}_{td}(d_1, \ldots, d_r; 2, n)$ such that for any germ of holomorphic deformation $t \in (\mathbb{C}, 0) \mapsto \mathcal{F}_t \in \mathbb{F}\mathrm{ol}(D; 2.n)$, with $\mathcal{F}_t|_{t=0} = \mathcal{F}_0$, then $\mathcal{F}_t \in \mathcal{L}_{td}(d_1, \ldots, d_n; 2, n)$ $\forall\, t \in (\mathbb{C}, 0)$.

In order to do that, first of all let us fix homogeneous polynomials $f_1, \ldots, f_r$ in $\mathbb{C}^{n+1}$ with $\deg(f_j) = d_j$, $1 \leqslant j \leqslant r$, such that the hypersurface $(f_1 \ldots f_r = 0) \subset \mathbb{C}^{n+1}$ has a strictly ordinary singularity outside $0 \in \mathbb{C}^{n+1}$. In particular, for any $J = (1 \leqslant j_1 < \cdots < j_{n-1} \leqslant r)$ then the curve $S_J = \Pi(f_{j_1} = \cdots = f_{j_{n-1}} = 0) \subset \mathbb{P}^n$ is a smooth complete intersection.

From now on we fix $J = (1, 2, \ldots, n-1)$ and set $S_J = S$. By Lemma 4.4 there exists $\eta_\tau = \theta_\tau^2 \wedge \cdots \wedge \theta_\tau^{n-1} \in \mathcal{L}_{td}^{n-2}(f_1, \ldots, f_r)$ such that the foliation $\mathcal{F}_{\eta_\tau}$ defined by η_τ satisfies (I), (II) and (III) along the curve S. The finite set of (I) is $F = S \cap \bigcup_{j \geqslant n} \Pi(f_j = 0)$.

Remark 4.5. — The parameter $\tau = (t_{ji})_{n \leqslant i \leqslant r}^{2 \leqslant j \leqslant n-1}$ in Lemma 4.4 can be chosen in such a way that if $\mathcal{F}_{\eta_\tau} \in \mathcal{L}_{td}(d'_1, \ldots, d'_s; 2, n)$, where $d'_1 \leqslant d'_2 \leqslant \cdots \leqslant d'_r$ then $d'_i = d_i$, $\forall\, i$. Recalling the definition of the $\theta_\tau^{j's}$, an example in which $\mathcal{F}_{\eta_\tau}$ belongs to two different $\mathcal{L}_{td's}$ is when $B_{jn}(\tau) = B_{jn+1}(\tau)$ for all $2 \leqslant j \leqslant n-1$. In this case, in the sum that defines θ_τ^j there are terms as below

$$B_{jn}(\tau)\frac{\mathrm{d}f_n}{f_n} + B_{jn+1}(\tau)\frac{\mathrm{d}f_{n+1}}{f_{n+1}} = B_{jn}(\tau)\frac{\mathrm{d}(f_n f_{n+1})}{f_n f_{n+1}}.$$

In this case

$$\mathcal{F}_{\eta_\tau} \in \mathcal{L}_{td}(d_1, \ldots, d_r : 2, D) \cap \mathcal{L}_{td}(d_1, \ldots, d_{n-1}, d_n + d_{n+1}, \ldots, d_r; 2, D).$$

On the other hand, if we choose the parameters t_{ji} linearly independent over $\mathbb{Z}$ then the required property is true. From now on, we will assume this propety.

Since $\mathcal{F}_{\eta_o}$ satisfies property (III) along the curve S, all points of the finite set F are non degenerated g.K singularities of $\mathcal{F}_{\eta_o}$. Fix any holomorphic germ of deformation $t \in (\mathbb{C}, 0) \mapsto \mathcal{F}_t \in \mathbb{F}\mathrm{ol}(D; 2, n)$. The main fact that we will

use is that the curve S admits a C^∞ deformation $t \in (\mathbb{C},0) \mapsto S(t)$ such that $S(t) \subset \mathrm{Sing}(\mathcal{F}_t)$ and the foliation $\mathcal{F}_t$ satisfies properties (I), (II), (III) along $S(t)$.

LEMMA 4.6. — *There exists a germ of C^∞ isotopy $\Phi\colon (\mathbb{C},0)\times S \mapsto \mathbb{P}^n$, such that, if we denote $S(t) := \phi(t,S)$, then:*

(a) *$S(0) = S$ and $S(t) \subset \mathrm{Sing}(\mathcal{F}_t)$ is smooth $\forall\ s \in (\mathbb{C},0)$. In particular, $S(t)$ is an algebraic complete intersection, $\forall\ t \in (\mathbb{C},0)$.*
(b) *If $r = n-1$ then all poins of $S(t)$ are of Kupka type.*
(c) *If $r > n-1$ then any point $p \in F = S \cap \bigcup_{k\geqslant n} \Pi(f_k = 0)$, $n-1 < k \leqslant r$, has a holomorphic deformation $t \in (\mathbb{C},0) \mapsto P_p(t)$ such that $P_p(t) \in S(t)$ is a non degenerated g.K singularity of $\mathcal{F}_t$. Set $F(t) := \{P_p(t) \,|\, p \in F\}$.*
(d) *The points of $S(t) \setminus F(t)$ are in the Kupka set of $\mathcal{F}(t)$. Moreover, if we denote by Y_t the normal type of $\mathcal{F}_t$ along $S(t) \setminus F(t)$ then the correspondence $t \in (\mathbb{C},0) \mapsto Y_t$ is holomorphic.*

Proof. — The argument for the proof of (c) uses the stability under deformations of the non degenerated g.K points [17, Theorem 6]. The argument for the existence of the isotopy Φ is similar to [16, Lemma 2.3.3, p. 83] and uses essentially the local stability under deformations of the Kupka set [9] and of the non degenerated g.K singular points [17]. The fact that the deformed curve $S(t)$ satisfies (I), (II) and (III) for the foliation $\mathcal{F}(t)$ is a consequence of the fact that these conditions are open. We leave the details for the reader. □

Let us finish the proof. We will assume that $\mathcal{F}_{\eta_0}$ satisfies Remark 4.5. Lemma 4.6 implies that the foliation $\mathcal{F}(t)$ has a curve $S(t)$ in the singular set that satisfies (I), (II) and (III). In particular, there are homogeneous polynomials $g_1(t),\dots,g_{s(t)}(t)$ such that $\mathcal{F}_t \in \mathcal{L}^{n-2}_{td}(g_1(t),\dots,g_{s(t)}(t))$. Set $\deg(g_j(t)) = d_j(t)$. We assert that $s(t) = r$ and that we can assume $d_j(t) = d_j$, $1 \leqslant j \leqslant r$.

In fact, since $D = \sum_{j=1}^{s(t)} d_j(t) - n + 1$ we have $s(t) \leqslant D + n - 1$ and the number of possilities for the degrees $d_j(t)$, $1 \leqslant j \leqslant s(t)$ is finite. In particular, there is a germ of non-contable set $A \subset (\mathbb{C},0)$ such that the functions $t \in A \mapsto s(t)$ and $t \in A \mapsto d_j(t)$, $1 \leqslant j \leqslant s(t)$, are all constants, say $s|_A = r'$ and $d_j|_A = d'_j$. In particular, $\mathcal{F}(t) \in \mathcal{L}_{td}(d'_1,\dots,d'_{r'};2;D)$ for all $t \in A$. Since 0 is in the adherence of A we get $\mathcal{F}_{\eta_0} \in \mathcal{L}_{td}(d'_1,\dots,d'_{r'};2,D)$. Hence, $r' = r$ and $d'_j = d_j$, $1 \leqslant j \leqslant r$, and $\mathcal{F}(t) \in \mathcal{L}_{td}(d_1,\dots,d_r;2,D)$. □

4.3. Proof of Theorem 4.3

Let $\mathcal{F}$ be a two dimensional foliation on $\mathbb{P}^n$, $n \geqslant 4$, having a curve S in the singular set and that satisfies (I), (II) and (III). The idea is to construct closed logarithmic 1-forms $\theta_2, \ldots, \theta_{n-1}$, defined in a neighborhood U of the curve S, such that $\theta_2 \wedge \cdots \wedge \theta_{n-1}$ defines the foliation $\mathcal{F}|_U$. By using an extension theorem of meromorphic functions (cf. [2] and [22]), each form θ_j can be extended to a global closed meromorphic 1-form on $\mathbb{P}^n$, denoted again by θ_j, $2 \leqslant j \leqslant n-1$. The fact that $\theta_j|_U$ is logarithmic implies that θ_j is also logarithmic: there are homogeneous polynomials in $\mathbb{C}^{n+1}$, say $g_1, \ldots, g_r$, such that $\theta_j \in \mathcal{L}^1_R(g_1, \ldots, g_r)$, $\forall\ 1 \leqslant j \leqslant n-2$, $\implies \mathcal{F} \in \mathcal{L}^{n-2}_{td}(g_1, \ldots, g_r)$. The following result will be usefull:

THEOREM 4.7 (Parametric linearization). — *Let $(W_\tau)_{\tau \in (\mathbb{C}^k,0)}$ be a germ at $0 \in \mathbb{C}^k$ of a holomorphic family of germs of holomorphic vector fields at $0 \in \mathbb{C}^m$. Assume that:*

(a) *The linear part $L_\tau = DW_\tau(0)$ is diagonal of the form $L_\tau = \sum_{j=1}^m \rho_j(\tau) z_j \frac{\partial}{\partial z_j}$ in some local coordinate system $z = (z_1, \ldots, z_m)$ around $0 \in \mathbb{C}^m$.*
(b) *$\rho_1(0), \ldots, \rho_m(0)$ are in the Poincaré domain and satisfy the non-resonance condition $(\star)$ in (II).*

Then there exists a holomorphic family of germs of biholomorphisms $(\Psi_\tau)_{\tau \in (\mathbb{C}^k,0)}$ such that $D\Psi_\tau(0) = I$ and

$$\Psi_\tau^*(W_\tau) = L_\tau = \sum_{j=1}^m \rho_j(\tau) w_j \frac{\partial}{\partial w_j}.$$

Theorem 4.7 is a parametric version of Poincaré's linearization theorem. Its proof can be found in [1] or [19].

Let us continue the proof of Theorem 4.3. First of all, we will prove that there are n-2 closed logarithmic 1-forms $\theta_2, \ldots, \theta_{n-1}$, defined in some neighborhood W of $S \setminus F$, such that $\eta = \theta_2 \wedge \cdots \wedge \theta_{n-1}$ defines $\mathcal{F}|_W$.

Fix $p \in S \setminus F$. Since $p \in K(\mathcal{F})$ there are local coordinates $(V, z = (z_1, \ldots, z_n))$, with $p \in V$, such that

(i) $z(p) = 0$ and $S \cap V = (z_1 = \cdots = z_{n-1} = 0)$.
(ii) $\mathcal{F}|_V$ is defined by a holomorphic (n-2)-form ω that can be written as $\omega = i_Y i_X \nu$, where $X = \frac{\partial}{\partial z_n}$, $Y = \sum_{j=1}^{n-1} Y_j(z_1, \ldots, z_{n-1}) \frac{\partial}{\partial z_j}$ is the normal type and $\nu = \mathrm{d}z_n \wedge \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_{n-1}$.

Since the eigenvalues $\rho_1, \ldots, \rho_{n-1}$ of $DY(0)$ satisfy the non-resonance conditions $(\star)$, by Theorem 4.7 (without parameters) we can assume that Y is linear

$$Y = \sum_{j=1}^{n-1} \rho_j z_j \frac{\partial}{\partial z_j},$$

which implies

$$\omega = \sum_{j=1}^{n-1} (-1)^{j-1} \rho_j z_j \mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_j} \wedge \cdots \wedge \mathrm{d}z_{n-1}.$$

In particular, the form $\eta_V := \frac{1}{z_1 \ldots z_{n-1}} \omega$ is logarithmic

$$\eta_V = \sum_{j=1}^{n-1} (-1)^{j-1} \rho_j \frac{\mathrm{d}z_1}{z_1} \wedge \cdots \wedge \widehat{\frac{\mathrm{d}z_j}{z_j}} \wedge \cdots \wedge \frac{\mathrm{d}z_{n-1}}{z_{n-1}}.$$

Note that η_V can be decomposed as $\eta_V = \rho_1 \theta_V^2 \wedge \cdots \wedge \theta_V^{n-1}$, where

$$\theta_V^j = \frac{\mathrm{d}z_j}{z_j} - \frac{\rho_j}{\rho_1} \frac{\mathrm{d}z_1}{z_1}, \tag{4.3}$$

because $i_Y \theta_V^j = i_X \theta_V^j = 0$, $\forall\, 2 \leqslant j \leqslant n-1$.

The above argument implies that there exists a covering $\mathcal{V}$ of $S \setminus F$, by open sets, such that

(iii) For each $V \in \mathcal{V}$ there exists a coordinate system $z_V = (z_1, \ldots, z_n) : V \to \mathbb{C}^n$ with $V = \{z \,||\, z_j| < 1, 1 \leqslant j \leqslant n\}$ and $S \cap V = (z_1 = \cdots = z_{n-1} = 0)$.

(iv) If θ_V^j is as (4.3), $2 \leqslant j \leqslant n-1$, then the logarithmic form $\theta_V^2 \wedge \cdots \wedge \theta_V^{n-1}$ defines $\mathcal{F}|_V$.

(v) The vector fields $X_V = \frac{\partial}{\partial z_n}$ and $Y_V = \sum_{j=1}^{n-1} \rho_j z_j \frac{\partial}{\partial z_j}$ generate $\mathcal{F}|_V$.

We assert that if $V, \widetilde{V} \in \mathcal{V}$ are such that $V \cap \widetilde{V} \neq \emptyset$ then $\theta_V^j \equiv \theta_{\widetilde{V}}^j$ on $V \cap \widetilde{V}$.

In fact, first of all let us remark that

(vi) For all $j \in \{1, \ldots, n-1\}$ the hypersurface $\Sigma_V^j := (z_j = 0) \subset V$ is invariant by $\mathcal{F}|_V$. Moreover, if $\widehat{\Sigma}^j \subset V$ is another smooth hypersurface which is $\mathcal{F}|_V$ invariant and is tangent to Σ_V^j along S then $\widehat{\Sigma}^j \subset \Sigma_V^j$.

Note that (vi) above is equivalent to the fact that the hyperplane $(z_j = 0)$ is Y-invariant, $1 \leqslant j \leqslant n-1$. Moreover, it is the unique smooth hypersurface which is Y-invariant and tangent to $(z_j = 0)$. This is well-known and is a consequence of the fact that $\rho_1, \ldots, \rho_{n-1}$ satisfy $(\star)$ (see [1]).

Let $z_V = (z_1, \ldots, z_n)$ and $z_{\widehat{V}} = (\widehat{z}_1, \ldots, \widehat{z}_n)$ be the coordinate systems of V and $\widehat{V}$ on which (iv), (v) and (vi) are true. We assert that $\widehat{z}_j = u_j.z_j$ on $V \cap \widehat{V}$, where $u_j(z) \neq 0 \; \forall \; z \in V \cap \widehat{V}$, $\forall \; 1 \leqslant j \leqslant n-1$.

In fact, if we fix $1 \leqslant j \leqslant n-1$, by (vi) we must have $\widehat{z}_j = u.z_i$, where $u(z) \neq 0 \; \forall \; z \in V \cap \widehat{V}$, for some $1 \leqslant i \leqslant n-1$. However, the fact that $\rho_\ell \neq \rho_j$ if $\ell \neq j$ implies that $i = j$. It follows that

$$\theta^j_{\widehat{V}} = \frac{\mathrm{d}\widehat{z}_j}{\widehat{z}_j} - \frac{\rho_j}{\rho_1}\frac{\mathrm{d}\widehat{z}_1}{\widehat{z}_1} = \frac{\mathrm{d}z_j}{z_j} - \frac{\rho_j}{\rho_1}\frac{\mathrm{d}z_1}{z_1} + \mathrm{d}v = \theta^j_V + \mathrm{d}v, 2 \leqslant j \leqslant n-1,$$

where $v = \log(u_j) - \frac{\rho_j}{\rho_1}\log(u_1)$ is holomorphic. Now, (iv) and (v) imply that

$$\begin{aligned} i_{X_V}\theta^j_{\widehat{V}} = i_{X_V}\theta^j_V = i_{Y_V}\theta^j_{\widehat{V}} = i_{Y_V}\theta_V = 0 \\ \implies X_V(v) = Y_V(v) = 0 \\ \implies v \text{ is a first integral of both vector fields } X_V \text{ and } Y_V \end{aligned}$$

Since $X_V = \frac{\partial}{\partial z_n}$ we get $v = v(z_1, \ldots, z_{n-1})$. Since the eigenvalues of Y_V are in the Poincaré domain $v(z_1, \ldots, z_{n-1})$ must be a constant and $\mathrm{d}v = 0$. Hence, $\theta^j_{\widehat{V}} = \theta^j_V$ on $V \cap \widehat{V}$, as asserted. Therefore there are closed logarithmic 1-forms $\theta_2, \ldots, \theta_{n-1}$, defined on $W = \bigcup_{\mathcal{V}} V$, such that $\mathcal{F}|_W$ is defined by $\theta_2 \wedge \cdots \wedge \theta_{n-1}$, as asserted. Let us prove that the forms θ_j extend to a neighborhood of any point in F.

Given $p \in F$ let ω be a germ of (n-2)-form defining the germ of $\mathcal{F}$ at p. Let $(U, z = (x = z_1, \ldots, z_n))$ be a coordinate system around p as in (III), so that $z(p) = 0$ and $S \cap U = (z_1 = \cdots = z_{n-1} = 0)$. The rotational X of ω has eigenvalues $\lambda_1, \ldots, \lambda_n$ and there exists $a \neq 0$ such that $\mathrm{Re}(a.\lambda_n) < 0$ and $\mathrm{Re}(a.\lambda_j) > 0$, $\forall \; 1 \leqslant j \leqslant n-1$. Since $p = 0$ is an isolated singularity of X there exists another germ of vector field Z such that $\omega = i_Z i_X \nu$, $\nu = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n$. The vector fields X and Z generate the germ of $\mathcal{F}$ at 0.

Lemma 4.8. — *There are germs at p of vector fields $\widetilde{X}$ and $\widetilde{Z}$ that generate the germ of $\mathcal{F}$ at p and a holomorphic coordinate system $(U_1, w = (w_1, \ldots, w_n))$ around p, with the following properties:*

(a) *$w(p) = 0$ and $S \cap U_1 = (w_1 = \cdots = w_{n-1} = 0)$.*
(b) *$\widetilde{Z}(w) = \sum_{j=1}^{n-1} \rho_j w_j \frac{\partial}{\partial w_j}$. In particular, $\widetilde{Z}$ is the normal type of $\mathcal{F}$ along $S \setminus F$.*
(c) *$\widetilde{X} = \sum_{j=1}^{n} \lambda_j w_j (1 + \phi_j(w_n)) \frac{\partial}{\partial w_j}$, where $\phi_j(0) = 0$, $\forall \; 1 \leqslant j \leqslant n-1$.*

Proof. — Let W_u be the hyperplane of $T_p\mathbb{P}^n$ generated by the eigenspaces of $DX(p)$ associated to the eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ and W_s be the eigenspace associated to λ_n. Recall that we have assumed $W_s = T_pS$, which implies that W_u is transverse to S at p. The conditions $\mathrm{Re}(a.\lambda_n) < 0$ and $\mathrm{Re}(a.\lambda_j) > 0$

implies that the vector field $a.X$ has an unique invariant smooth hypersurface Σ_u tangent to W_u, which meets S transversely at p. This is a consequence of the existence of invariant manifolds for hyperbolic singularities of vector fields (see [15]). The hypersurface Σ_u is the unstable manifold of the vector field $a.X$. We assert that Σ_u is also Z-invariant. For simplicity, we will assume $a = 1$.

In the proof we will use the relation:

$$[Z, X] = hX \tag{4.4}$$

where $h \in \mathcal{O}_n$ and $h(0) = 0$. Let us assume (4.4) and prove that Σ is Z-invariant.

Take representatives of Z, X and h defined in some small ball B around 0. Let Z_t and X_t be the local flows of Z and X, respectively. Since Σ_u is the unstable manifold of X the real flow X_t of X satisfies $\lim_{t\to-\infty} X_t(z) = 0\ \forall\ z \in \Sigma \cap B$. Integrating (4.4) we get

$$Z_t^*(X) = \phi_t.X, \quad \text{where } \phi_t(z) = \exp\left(\int_0^t h(Z_s(z))\mathrm{d}s\right).$$

The above relation implies that Z_t sends orbits of X on orbits of X. Given $z \in \Sigma \cap B$ denote $O(z) = \{X_t(z) \,|\, t \leqslant 0\}$. Since $\lim_{t\to-\infty} X_t(z) = 0$ we get $\overline{O(z)} = O(z) \cup \{0\}$. Let $\widetilde{O}(z)$ be the germ of $\overline{O(z)}$ at 0. Note that $\widetilde{O}(z) \subset \Sigma$ and that $Z_t(\widetilde{O}(z))$ is a germ of curve through 0 such that $Z_t(\widetilde{O}(z)) \setminus \{0\}$ is an orbit of X. This of course implies that $Z_t(\widetilde{O}(z)) \subset \Sigma$. Hence, Σ is Z-invariant.

Proof of (4.4). — Since $\mathrm{d}\omega = i_X\nu$ and $\omega = i_Z i_X \nu$ we have

$$\begin{aligned} L_Z\omega = i_Z(\mathrm{d}\omega) + d(i_Z\omega) = \omega \\ \Longrightarrow\ L_Z(\mathrm{d}\omega) = \mathrm{d}\omega \\ \Longrightarrow\ i_X\nu = L_Z(i_X\nu) = i_{[Z,X]}\nu + i_X L_Z\nu = i_{[Z,X]}\nu + \nabla(Z) i_X\nu, \end{aligned}$$

where $\nabla(Z) = \sum_j \frac{\partial Z_j}{\partial z_j}$ is the divergence of Z. From this relation we get $[Z, X] = hX$, where $h = 1 - \nabla(Z)$. We assert that $h(0) = 0$.

In fact, let $X_1 = DX(0)$ and $Z_1 = DZ(0)$. Relation (4.4) implies that

$$[Z_1, X_1] = h(0).X_1.$$

The above relation implies that if $h(0) \neq 0$ then X_1 is nilpotent, so that $\lambda_1 = \cdots = \lambda_n = 0$, a contradiction (see [17]). In particular, we have proved that X_1 and Z_1 commute. □

Let us continue the proof of Lemma 4.8. After a holomorphic change of variables, we can assume that $\Sigma_u \subset (z_n = 0)$. Since $(z_n = 0)$ is invariant for both vector fields, in the new coordinate system we can write the n^{th}

component of X and Z as $\lambda_n z_n(1+h_1(z))$ and $z_n f(z)$, respectively, where $h_1(0)=0$. If we set $\Psi := -\frac{f(z)}{\lambda_n(1+h_1(z))}$ then the n^{th} component of $\widetilde{Z} := Z+\Psi X$ vanishes. Moreover, $\omega = i_{\widetilde{Z}} i_X \nu = i_Z i_X \nu$ and $[\widetilde{Z}, X] = gX$, where $g = h - X(\Psi)$ and $g(0)=0$. We assert that there are coordinates $(W, w = (w_1, \ldots, w_{n-1}, w_n))$ around p such that

(i) $w(p)=0$, $\Sigma_u \cap W = (w_n = 0)$ and $S \cap W = (w_1 = \cdots = w_{n-1} = 0)$.
(ii) $\widetilde{Z} = \phi(w_n).\sum_{j=1}^{n-1} \rho_j w_j \frac{\partial}{\partial w_j}$, where $\phi(0) \neq 0$.

In fact, since the n^{th} component of $\widetilde{Z}$ vanishes, the hyperplanes $\Sigma_c := (z_n = c)$ are $\widetilde{Z}$-invariant. On the other hand, if $c \neq 0$ then Σ_c is transverse to $S = (z_1 = \cdots = z_{n-1} = 0)$ and so $\widetilde{Z}|_{\Sigma_c}$ represents the normal type of $\mathcal{F}$ in the section Σ_c. Therefore, the eigenvalues of $D\widetilde{Z}(0,c)$ are proportional to $\rho_1, \ldots, \rho_{n-1}$. In other words there exists a $\phi \in \mathcal{O}_1$ such that the eigenvalues of $D\widetilde{Z}(0,c)|_{\Sigma_c}$ are $\phi(c).\rho_1, \ldots, \phi(c).\rho_{n-1}$. Considering $\widetilde{Z}$ as a 1-parameter family of germs of vector fields at $0 \in \mathbb{C}^{n-1}$ and applying Theorem 4.7 to this family we get (ii) of the assertion. Now, we assert that there exists $\Phi \in \mathcal{O}_n$ such that if we set $\widetilde{X} := e^{\Phi}.X$ then

$$[\widetilde{Z}, \widetilde{X}] = 0. \tag{4.5}$$

In fact, if $\Phi \in \mathcal{O}_n$ then

$$[\widetilde{Z}, \widetilde{X}] = [\widetilde{Z}, e^{\Phi}.X] = e^{\Phi}.\widetilde{Z}(\Phi).X + e^{\Phi}.[\widetilde{Z}, X] = e^{\Phi}(\widetilde{Z}(\Phi) + g)X.$$

Therefore, we have to prove that $\widetilde{Z}(\Phi) = -g$ has a solution $\Phi \in \mathcal{O}_n$. Recall that $\widetilde{Z} = \phi(w_n).L$, where $L = \sum_{j=1}^{n-1} \rho_j w_j \frac{\partial}{\partial w_j}$. Set $w = (x, w_n)$, $x = (w_1, \ldots, w_{n-1})$. We can write

$$-g(x, w_n) = \sum_{\sigma} b_\sigma(w_n).x^\sigma$$

where $\sigma = (\sigma_1, \ldots, \sigma_{n-1}) \in \mathbb{Z}_{\geqslant 0}^{n-1}$, $b_\sigma \in \mathcal{O}_1$ and $x^\sigma = w_1^{\sigma_1} \ldots w_{n-1}^{\sigma_{n-1}}$.

Let $\sigma_0 = (0, \ldots 0)$. We will prove below that $b_{\sigma_0} \equiv 0$. Assuming this fact, the equation $\widetilde{Z}(\Phi) = -g$ has a formal solution $\Phi = \sum_\sigma c_\sigma(w_n) x^\sigma$ where

$$c_\sigma(w_n) = \frac{b_\sigma(w_n)}{\phi(w_n)\,\langle \rho, \sigma \rangle}, \langle \rho, \sigma \rangle = \sum_{j=1}^{n-1} \rho_j \sigma_j.$$

Since $\rho_1, \ldots, \rho_{n-1}$ are in the Poincaré domain we have

$$\inf\{\, |\langle \rho, \sigma \rangle| \;|\; \sigma \in \mathbb{Z}_{\geqslant 0}^{n-1}, \sigma \neq (0, \ldots, 0)\} \geqslant C,$$

where $C > 0$. This implies that the formal series converges; $\Phi \in \mathcal{O}_n$.

Proof that $b_{\sigma_0}(w_n) \equiv 0$, or equivalently $g(0, w_n) \equiv 0$. — First of all, the n^{th} component of $[\widetilde{Z}, X]$ is $\widetilde{Z}(X_n)$, where $X_n = \lambda_n w_n(1 + h_1(x, w_n))$ is the n^{th} component of X. Hence, $[\widetilde{Z}, X] = gX$ implies that

$$g.\lambda_n w_n(1+h_1) = \widetilde{Z}\left(\lambda_n w_n(1+h_1)\right) = \lambda_n w_n \widetilde{Z}(h_1) = \lambda_n w_n \phi(w_n) L(h_1)$$
$$\implies g(x, w_n) = \frac{\phi(w_n) \sum_{j=1}^{n-1} \rho_j w_j \frac{\partial h_1}{\partial w_j}}{1+h_1}$$
$$\implies g(0, w_n) \equiv 0 \qquad \square$$

From $\widetilde{Z} = \phi(w_n).L$ we get that (4.5) implies:

(1) $D\widetilde{X}(0)$ commutes with L. In particular, $D\widetilde{X}(0)$ is diagonal in the coordinate system w and we can write $\widetilde{X} = \sum_{j=1}^{n} X^j(w) \frac{\partial}{\partial w_j}$, where $X^j(w) = \lambda_j w_j + h.o.t$, $1 \leqslant j \leqslant n$.
(2) $L(X^n) = 0$. Since the first integrals of L are functions of w_n we get $X^n = \lambda_n w_n(1 + \psi_n(w_n))$, $\psi_n \in \mathcal{O}_1$.
(3) $\phi(w_n).L(X^j) = \widetilde{X}(\phi(w_n)\rho_j w_j)$, $1 \leqslant j \leqslant n-1 \iff$

$$L\left(X^j\right) - \rho_j X^j = \frac{\phi'(w_n)}{\phi(w_n)}(1 + \psi_n(w_n))\rho_n \rho_j w_j w_n, 1 \leqslant j \leqslant n. \qquad (4.6)$$

Note that (4.6) implies that $w_j | X^j$ if $j < n$ and so $X^j = \lambda_j w_j u$, where $u(0) = 1$. But, in this case we get that $\frac{1}{w_j}(L(X^j) - \rho_j X_j) \in \mathcal{O}_1^*$ which can happens only if $\phi' \equiv 0$ and $L(X^j) = \rho_j X_j$. We can assume that $\phi \equiv 1$. Finally, the solutions of $L(X^j) = \rho_j X^j$ with linear part $\lambda_j w_j$ are of the form $X^j(w) = \lambda_j w_j(1 + \psi_j(w_n))$, $\psi_j(0) = 0$. This finishes the proof of Lemma 4.8. $\square$

Let us finish the proof that the forms θ_j, $2 \leqslant j \leqslant n-1$, extend to a neighborhood of $p \in F$. Define closed logarithmic 1-forms $\widetilde{\theta}_j$, $2 \leqslant j \leqslant n-1$, by

$$\widetilde{\theta}_j = \frac{\mathrm{d}w_j}{w_j} - \frac{\rho_j}{\rho_1}\frac{\mathrm{d}w_1}{w_1} - \zeta_j(w_n)\frac{\mathrm{d}w_n}{w_n},$$

where

$$\zeta_j(w_n) = \frac{\lambda_j(1+\phi_j(w_n)) - \frac{\rho_j}{\rho_1}\lambda_1(1+\phi_1(w_n))}{\lambda_n(1+\phi_n(w_n))}.$$

Note that $\zeta_j(0) \neq 0$, because $\rho_1\lambda_j - \rho_j\lambda_1 \neq 0$. In particular, the pole divisor of $\widetilde{\theta}_j$ contains w_n with multiplicity one.

The reader can check directly that $i_{\widetilde{Z}}\widetilde{\theta}_j = i_{\widetilde{X}}\widetilde{\theta}_j = 0$, $\forall\ 2 \leqslant j \leqslant n-1$, so that $\widetilde{\eta} := \widetilde{\theta}_2 \wedge \cdots \wedge \widetilde{\theta}_{n-1}$ defines the germ of $\mathcal{F}$ at p. Taking representatives,

we can assume that the $\widetilde{\theta}_{j's}$ are defined in some polydisc $\widetilde{W}$ containing p and with $F \cap \widetilde{W} = \{p\}$. We assert that $\widetilde{\theta}_j = \theta_j$ on $\widetilde{W} \cap W$, $2 \leqslant j \leqslant n-1$.

In fact, fix a point $q \in S \cap \widetilde{W} \cap W$. We have seen that there are coordinates $(V, z = (z_1, \ldots, z_{n-1}, z_n))$ around q such that $z(q) = 0$, $S \cap V = (z_1 = \cdots = z_{n-1} = 0)$, $\mathcal{F}|_V$ is generated by the vector fields $X = \frac{\partial}{\partial z_n}$ and $Y = \sum_{j=1}^{n-1} \rho_j z_j \frac{\partial}{\partial z_j}$ and $\theta_j|_V = \frac{\mathrm{d}z_j}{z_j} - \frac{\rho_j}{\rho_1}\frac{\mathrm{d}z_1}{z_1}$, $2 \leqslant j \leqslant n-1$. Note that $w_n|_V \in \mathcal{O}^*(V)$ and that $w_j|_V = v_j.z_j$, where $v_j \in \mathcal{O}^*(V)$, $1 \leqslant j \leqslant n-1$. This implies that $\widetilde{\theta}_j|_V = \theta_j|_V + \mathrm{d}f_j$, where f_j is a primitive of the closed holomorphic form

$$\frac{\mathrm{d}v_j}{v_j} - \frac{\rho_j}{\rho_1}\frac{\mathrm{d}v_1}{v_1} - \zeta_j(w_n)\frac{\mathrm{d}w_n}{w_n}|_V.$$

Finally, the fact that $i_X\theta_j = i_Y\theta_j = i_X\widetilde{\theta}_j = i_Y\widetilde{\theta}_j = 0$ implies that $X(f_j) = Y(f_j) = 0$ and so f_j is a constant and $\widetilde{\theta}_j|_V = \theta_j|_V$, $2 \leqslant j \leqslant n-1$.

We have proved that there are closed logarithmic 1-forms $\theta_2, \ldots, \theta_{n-1}$ defined in a neighborhood U of S such that $\eta := \theta_2 \wedge \cdots \wedge \theta_{n-1}$ defines $\mathcal{F}|_U$. By the extension theorem in [22] the form θ_j can be extended to closed meromorphic 1-forms on $\mathbb{P}^n$, $2 \leqslant j \leqslant n-1$, denoted by the same symbol. The pole divisor of θ_j must be reduced because the pole divisor of the restriction $\theta_j|_U$ is reduced. Therefore θ_j is logarithmic, $2 \leqslant j \leqslant n-1$, and η is totally decomposable into logarithmic forms. In particular, there exist $g_1, \ldots, g_r$ such that $\eta \in \mathcal{L}_{td}^{n-1}(g_1, \ldots, g_r)$. This finishes the proof of Theorem 4.3. □

4.4. Proof of Theorem 4.1

Let $M \subset \mathbb{P}^n$ be a m-dimensional smooth algebraic submanifold, where $m < n$, and $\mathcal{F}$ be a codimension p holomorphic foliation on $\mathbb{P}^n$, where $p+1 \leqslant m$. Assume that:

(a) The set of tangencies of $\mathcal{F}$ with M has codimension $\geqslant 2$ on M.
(b) $\mathcal{F}|_M$ can be defined by a meromorphic closed p-form η.

We want to prove that η admits a closed meromorphic extension $\widetilde{\eta}$ defining $\mathcal{F}$ on $\mathbb{P}^n$. In fact, this proof is similar to the proof of the extension theorem of [4] (see also [16, Proposition 3.1.1]). The idea is to prove that η admits a closed extension $\widehat{\eta}$, defined in a neighborhood U of M, such that $\mathcal{F}|_U$ is represented by $\widehat{\eta}$. After that, by [2] and [22], the form $\widehat{\eta}$ admits a meromorphic extension $\widetilde{\eta}$ to $\mathbb{P}^n$. Since U is an open non-empty subset of $\mathbb{P}^n$, it is clear that $\widetilde{\eta}$ is closed and defines $\mathcal{F}$ on $\mathbb{P}^n$.

Let $X = \mathrm{Sing}(\mathcal{F}|_M)$. Note that $X = \mathrm{Tang}(\mathcal{F}, M) \cup (\mathrm{Sing}(\mathcal{F}) \cap M)$, where $\mathrm{Tang}(\mathcal{F}, M)$ denotes the set of tangencies of $\mathcal{F}$ and M. By (a) we have $\mathrm{cod}_M(X) \geqslant 2$. We begin by extending η to a neighborhood of $M \setminus X$.

Extension to a neighborhood of $M \setminus X$. — By definition, the foliation $\mathcal{F}$ is transverse to M at the points of $M \setminus X$. In particular, given $q \in M \setminus X$ there exists a local coordinate system around q, $z = (z_1, \ldots, z_n) \colon W \to \mathbb{C}^n$, with $z(W)$ a polydisc of $\mathbb{C}^n$, $z(q) = 0 \in \mathbb{C}^n$, and such that

(i) $M \cap W = (z_{m+1} = \cdots = z_n = 0)$.
(ii) The leaves of $\mathcal{F}|_W$ are the levels $z_1 = ct_1, \ldots, z_p = ct_p$.

In particular, $\mathcal{F}|_W$ is defined by the form $\Omega_W = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p$. Since $\mathcal{F}|_{W\cap M}$ is also defined by $\eta|_{W\cap M}$ we must have $\eta|_{W\cap M} = f.\Omega_W|_{W\cap M}$, where $f = f(z_1, \ldots, z_m)$ is meromorphic on $W \cap M$. Since η and Ω_W are closed we get $\mathrm{d}f \wedge \Omega_W = 0$, which is equivalent to

$$\frac{\partial f}{\partial z_j} = 0, \ \ \forall\, p+1 \leqslant j \leqslant m$$
$$\Longrightarrow \ f(z_1, \ldots, z_m) = f(z_1, \ldots, z_p) : f \text{ depends only of } z_1, \ldots, z_p.$$

In particular, $\eta|_{W\cap M}$ admits an unique closed meromorphic extension to W defining $\mathcal{F}|_W \colon \widehat{\eta}_W = f(z_1, \ldots, z_p)\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p$. This proves that $\eta|_{M\setminus X}$ admits an unique closed meromorphic extension $\widehat{\eta}$ to a neighborhood V of $M \setminus X$ representing $\mathcal{F}|_V$.

Extension of $\widehat{\eta}$ to a neighborhood of M. — Since $\mathrm{cod}_M(X) \geqslant 2$, given $q \in X$ we can find a Hartog's domain $H \subset V$ such that $q \in \widehat{H}$, the holomorphic closure of H (for the details see [16]). Therefore, $\widehat{\eta}$ admits a meromorphic extension to a neighborhood of q, by Levi's extension theorem [24]. In particular, η can be extended to a closed meromorphic p-form $\widetilde{\eta}$ defining $\mathcal{F}$ on $\mathbb{P}^n$ by [2] and [22].

Let us assume now that η is logarithmic and let $(\widetilde{\eta})_\infty = \widetilde{S}_1^{k_1} \ldots \widetilde{S}_r^{k_r}$ be the decomposition of the pole divisor of $\widetilde{\eta}$ into irreducible components. The pole divisor of η will be then $(\eta)_\infty = (\widetilde{\eta})_\infty \cap M$, which is reduced because η is logarithmic. Hence, $k_1 = \cdots = k_r = 1$ and $\widetilde{\eta}$ is logarithmic. □

4.5. End of the proof of Theorem 1.17

Recall that we want to prove that, if $k \geqslant 3$, $n \geqslant 5$ and $r \geqslant n - k + 2 := p + 2$ then $\overline{\mathcal{L}_{td}(d_1, \ldots, d_r; k, n)}$ is an irreducible component of $\mathbb{F}\mathrm{ol}(D, k, n)$, where $D = \sum_j d_j - n + k - 1$. Fix $f_1, \ldots, f_r$ homogeneous polynomials on $\mathbb{C}^{n+1}$ with the following properties:

(i) $\deg(f_j) = d_j$, $1 \leqslant j \leqslant r$.
(ii) the hypersurface $(f_1 \dots f_r = 0)$ has strictly ordinary singularities outside $0 \in \mathbb{C}^{n+1}$.

Set $m = n - k + 2$ and let $\mathbb{P}^m \simeq \Sigma \subset \mathbb{P}^n$ be a m-plane such that:

(iii) If $\mathbb{C}^{m+1} \simeq E = \Pi^{-1}(\Sigma) \cup \{0\} \subset \mathbb{C}^{n+1}$ and $\widetilde{f}_j = f_j|_E$, $1 \leqslant j \leqslant r$, then $(\widetilde{f}_1 \dots \widetilde{f}_r = 0)$ has strictly ordinary singularities outside $0 \in E$.

Such m-plane E exists by transversality theory. In fact, it is sufficient to choose E in such a way that for any sequence $I = (i_1 < \dots < i_s) \in \mathcal{S}^r_s$, where $1 \leqslant s \leqslant n-1$, then the algebraic smooth set $\Pi(f_{i_1} = \dots = f_{i_s} = 0) \subset \mathbb{P}^n$ meets transversely $\Sigma = \Pi(E)$ (see Definition 1.2). We leave the details for the reader.

Since $m - 2 = n - k = p$, then for any $\mathcal{F} \in \mathcal{L}^p_{td}(f_1, \dots, f_r)$ we have $\mathcal{F}|_\Sigma \in \mathcal{L}^{m-2}_{td}(\widetilde{f}_1, \dots, \widetilde{f}_r)$, so that $\mathcal{F}|_\Sigma$ is a two dimensional foliation. Given a 1-form $\theta = \sum_j \lambda_j \frac{\mathrm{d}f_j}{f_j} \in \mathcal{L}^1_{\mathcal{F}}(f_1, \dots, f_r)$, we set $\widetilde{\theta} = \sum_{j=1}^r \lambda_j \frac{\mathrm{d}\widetilde{f}_j}{\widetilde{f}_j}$.

Choose $\mathcal{F}_o \in \mathcal{L}^p_{td}(f_1, \dots, f_r)$ defined by a logarithmic form $\eta_o = \theta_2 \wedge \dots \wedge \theta_{p+1}$, where $\theta_2, \dots \theta_{p+1}$ are as in Lemma 4.4. We assume also $\mathcal{F}_o|_\Sigma$ satisfies Remark 4.5: if $\mathcal{F}_o|_\Sigma \in \mathcal{L}_{td}(d'_1, \dots, d'_s; 2, m)$ then $s = r$ and $d'_j = d_j$, $1 \leqslant j \leqslant r$.

Let $(\mathcal{F}_t)_{t \in (\mathbb{C},0)}$ be a germ of holomorphic 1-parameter family of foliations in $\mathbb{F}\mathrm{ol}(D, k, n)$ such that $\mathcal{F}_t|_{(t=0)} = \mathcal{F}_o$. Consider the germ of 1-parameter family of two dimensional foliations $\widetilde{\mathcal{F}}_t := \mathcal{F}_t|_\Sigma$, $t \in (\mathbb{C}, 0)$. By the proof in Section 4.2 we get $\widetilde{\mathcal{F}}_t \in \mathcal{L}_{td}(d_1, \dots, d_r; 2, m)$, $\forall\, t \in (\mathbb{C}, 0)$, so that $\widetilde{\mathcal{F}}_t$ can be defined in homogeneous coordinates by a $m - 2 = n - k$ logarithmic form $\widetilde{\eta}_t \in \mathcal{L}^{m-2}(\widetilde{f}_{1t}, \dots, \widetilde{f}_{rt})$, where $\widetilde{f}_{jt}|_{t=0} = \widetilde{f}_j$, $1 \leqslant j \leqslant r$. By Theorem 4.1 the foliation $\mathcal{F}_t \in \mathbb{F}\mathrm{ol}(D, k, n)$ is logarithmic, $\forall\ t \in (\mathbb{C}, 0)$, so that $\mathcal{F}_t \in \mathcal{L}(d_1(t), \dots, d_{s_t}(t); k, n)$. We assert that $s_t = r$ and $d_j(t) = d_j$, $1 \leqslant j \leqslant r$.

In fact, since $\mathcal{F}_t \in \mathcal{L}(d_1(t), \dots, d_{s_t}(t); k, n)$ we get $\widetilde{\mathcal{F}}_t \in \mathcal{L}(d_1(t), \dots, d_{s_t}(t); 2, m)$. Therefore, as in the proof of the two dimensional case, we have $s_t = r$ and $d_j(t) = d_j$, $1 \leqslant j \leqslant r$, $\forall\ t \in (\mathbb{C}, 0)$. Finally, by Corollary 3.6 we get $\mathcal{F}_t \in \mathcal{L}_{td}(d_1, \dots, d_r; k, n)$, $\forall\ t \in (\mathbb{C}, 0)$. This finishes the proof of Theorem 1.17. □

5. Linear pull-back foliations

The purpose of this section is to prove a result of "trivialisation" of holomorphic foliations on projective spaces:

THEOREM 5.1. — *Let $\mathcal{G}$ be a codimension p foliation on $\mathbb{P}^n$, where $n \geqslant p+2$. Assume that there is a $p+1$ plane $\Sigma \simeq \mathbb{P}^{p+1}$ such that the foliation $\mathcal{F} := \mathcal{G}|_\Sigma$ has singular set $\mathrm{Sing}(\mathcal{F})$ of codimension $\geqslant 3$. Then $\mathcal{G}$ is the pull-back of $\mathcal{F}$ by some linear projection $T\colon \mathbb{P}^n - \to \Sigma$. In particular, there exists an affine coordinate system $(z,w) \in \mathbb{C}^{p+1} \times \mathbb{C}^{n-p-1} = \mathbb{C}^n \subset \mathbb{P}^n$ such that $\mathcal{G}$ is represented in these coordinates by a p-form depending only of z and $\mathrm{d}z$:*

$$\eta = \sum_{j=1}^{p+1} P_j(z)\mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_j} \wedge \cdots \wedge \mathrm{d}z_{p+1} = i_X \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n,$$

where $X = \sum_{j=1}^{p+1}(-1)^{j-1} P_j(z)\frac{\partial}{\partial z_j}$.

Theorem 5.2, that will be used in the proof of Theorem 5.1, is a local version of it. Let $Z_o \not\equiv 0$ be a germ at $(\mathbb{C}^{p+1}, 0)$ of holomorphic vector field, where $p+1 \geqslant 3$. The germ of foliation defined by Z_o is also defined by the germ of p-form $\eta_o = i_{Z_o}\nu$, where $\nu = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_{p+1}$. If $Z_o = \sum_{j=1}^{p+1} f_j(z)\frac{\partial}{\partial z_j}$ then

$$\eta_o = \sum_{j=1}^{p+1}(-1)^{j-1} f_j(z)\mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_j} \wedge \cdots \wedge \mathrm{d}z_{p+1}.$$

We will assume that there is a germ of integrable holomorphic p-form η at $0 \in \mathbb{C}^n$, where $\mathbb{C}^n = \mathbb{C}^{p+1} \times \mathbb{C}^{n-p-1}$, $n > p+1$, such that $\eta_o = i^*\eta$, where i is the inclusion $\mathbb{C}^{p+1} \mapsto \mathbb{C}^{p+1} \times \mathbb{C}^{n-p-1}$.

THEOREM 5.2. — *In the above situation, assume that* $\mathrm{cod}\,(\mathrm{Sing}(Z_o)) \geqslant 3$. *Then there exists a local coordinate system $(z,w) \in (\mathbb{C}^{p+1} \times \mathbb{C}^{n-p-1}, (0,0))$ and an unity $\phi \in \mathcal{O}_n^*$ such that*

$$\eta = \phi \sum_{j=1}^{p+1}(-1)^{j-1} f_j(z)\mathrm{d}z_1 \wedge \cdots \wedge \widehat{\mathrm{d}z_j} \wedge \cdots \wedge \mathrm{d}z_{p+1} = \phi i_{Z_o}\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_{p+1}.$$

In particular, the foliation generated by η is equivalent to the product of the singular one dimensional foliation generated by Z_o by the non-singular foliation of dimension $n-p-1$ with leaves $z = constant$.

We begin with the proof of Theorem 5.2.

5.1. Proof of Theorem 5.2

Let $\eta_o = i_{Z_o}\nu$ be the germ of p-form on $(\mathbb{C}^{p+1}, 0)$ which can be extended to a germ of integrable p-form η on $(\mathbb{C}^n = \mathbb{C}^{p+1} \times \mathbb{C}^{n-p-1}, (0,0))$. As in the hypothesis of Theorem 5.2 we will assume that $\mathrm{cod}(\mathrm{Sing}(Z_o)) \geqslant 3$.

The points in $\mathbb{C}^n = \mathbb{C}^{p+1} \times \mathbb{C}^{n-p-1}$ will be denoted (z, y), where $z \in \mathbb{C}^{p+1}$ and $y \in \mathbb{C}^{n-p-1}$. We will consider representatives of Z_o, η_o and η, denoted by the same letters, the first two defined in a neighborhood $V \subset \mathbb{C}^{p+1}$ of a closed polydisc $\overline{U}$ and the last defined in a neighborhood of $\overline{U} \times \{0\}$ in $\mathbb{C}^n$, so that

$$\eta_o = \eta|_{V \times \{0\}} = i_{Z_o} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_{p+1}.$$

We will assume $\mathrm{cod}_V(\mathrm{Sing}(Z_o)) \geqslant 3$. Define a holomorphic vector field Z in a neighborhood of $\overline{U} \times \{0\}$ by $Z(z, y_o) = Z_{y_o}(z) = \sum_{j=1}^{p+1} g_j(z, y_o) \frac{\partial}{\partial z_j}$, where

$$\eta|_{(y=y_o)} = i_{Z_{y_o}} \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_{p+1}.$$

Since η is locally totally decomposable. outside its singular set, we have $i_Z \eta = 0$.

Note that $Z(z, 0) = Z_o$. Therefore, the hypothesis implies that there is a neighborhood W of $\overline{U} \times \{0\}$ in $\mathbb{C}^n$ such that $\mathrm{cod}(\mathrm{Sing}(Z)) \geqslant 3$ and $W \cap (y = 0) = V \times \{0\}$.

We assert that there are holomorphic vector fields $X_1, \ldots, X_{n-p-1}$ defined in a smaller neighborhood of $\overline{U} \times \{0\}$, such that $i_{X_j} \eta = 0$ and

$$X_j(x, y) = \frac{\partial}{\partial y_j} + \sum_{i=1}^{p+1} h_{ji}(z, y) \frac{\partial}{\partial z_i}, \quad \forall\, 1 \leqslant j \leqslant n - p - 1. \tag{5.1}$$

First of all, we note that the above assertion is true in a neighborhood of any point $(z_o, 0) \in (V \times \{0\}) \setminus \mathrm{Sing}(Z_o)$. This is true because for (z, y) in a neighborhood U_α of $(z_o, 0)$ some component of $Z(z, y)$ does not vanishes, say $g_{p+1}(z, y) \neq 0$, so that

$$\frac{(-1)^p}{g_{p+1}} \eta|_{U_\alpha} = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_p + \widetilde{\Theta},$$

where $\widetilde{\Theta} \wedge \mathrm{d}z_{p+1} \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_{n-p-1} \equiv 0$. As the reader can check this implies the existence of holomorphic vector fields $X_{j\alpha}$ on U_α as in (5.1), $1 \leqslant j \leqslant n - p - 1$. It follows that there are:

- a polydisc $Q = Q_1 \times Q_2 \subset \mathbb{C}^{p+1} \times \mathbb{C}^{n-p-1}$, with $Q_1 \supset \overline{U}$ and $0 \in Q_2$.
- a covering $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ of $Q \setminus \mathrm{Sing}(\eta)$ by polydiscs,
- n-p-1 collections of holomorphic vector fields $(X_{j\alpha})_{\alpha \in A}$, $1 \leqslant j \leqslant n - p - 1$, $X_{j\alpha} \in \mathcal{X}(U_\alpha)$,

such that

(i) $\mathrm{cod}_Q(\mathrm{Sing}(Z|_Q)) \geqslant 3$.
(ii) $X_{j\alpha} = \frac{\partial}{\partial y_j} + \sum_{i=1}^{p+1} g_{i\alpha}(z, y) \frac{\partial}{\partial z_i}$.
(iii) $i_{X_{j\alpha}} \eta = 0$, $\forall\, 1 \leqslant j \leqslant n - p - 1$, $\forall\, \alpha \in A$.

(iv) for all $q = (z, y) \in U_\alpha$ then $\ker(\eta(q)) = \langle Z(q), X_{1\alpha}(q), \ldots, X_{n-p-1\alpha}(q)\rangle_{\mathbb{C}}$.

If $1 \leqslant j \leqslant n-p-1$ and $U_\alpha \cap U_\beta \neq \emptyset$ then

$$X_{j\alpha} - X_{j\beta} = \sum_{i=1}^{p+1}(g_{j\alpha} - g_{j\beta})\frac{\partial}{\partial z_i} = h^j_{\alpha\beta}.Z,$$

where $h^j_{\alpha\beta} \in \mathcal{O}(U_\alpha \cap U_\beta)$. The collection $(h^j_{\alpha\beta})_{U_{\alpha\beta}\neq\emptyset}$ is an additive cocycle. Since $\operatorname{cod}(\operatorname{Sing}(Z)) \geqslant 3$ by Cartan's theorem (cf. [5] and [12]) the cocycle is trivial; $h^j_{\alpha\beta} = h^j_\alpha - h^j_\beta$, $h^j_\alpha \in \mathcal{O}(U_\alpha)$. Hence, there exists a holomorphic vector field X_j on $Q \setminus \operatorname{Sing}(Z)$ as in (5.1) such that $i_{X_j}\eta = 0$; $X_j|_{U_\alpha} = X_{j\alpha} - h^j_\alpha Z$. By Hartog's theorem X_j can be extended to a holomorphic vector field on Q, denoted by the same letter. In particular, we have

$$\ker(\eta)|_Q = \langle Z, X_1, \ldots, X_{n-p-1}\rangle_{\mathcal{O}(Q)}. \tag{5.2}$$

Finally, (5.2) and [6, Theorem 11] imply the theorem:

(i) There exists a smaller polydisc $\overline{U} \times \{0\} \subset Q' \subset Q$ and holomorphic vector fields $Z', Y_1, \ldots, Y_{n-p-1} \in \langle Z, X_1, \ldots, X_{n-p-1}\rangle_{\mathcal{O}(Q')}$ such that $[Y_i, Y_j] = 0$, $[Z', Y_j] = 0$, $\forall\, 1 \leqslant j \leqslant n-p-1$, and

$$\langle Z', Y_1, \ldots, Y_{n-p-1}\rangle_{\mathcal{O}(Q')} = \langle Z|_{Q'}, X_1|_{Q'}, \ldots, X_{n-p-1}|_{Q'}\rangle_{\mathcal{O}(Q')}.$$

(ii) There are coordinates $(z, w) = (z, w_1, \ldots, w_{n-p-1})$ in Q' such that $Y_j = \frac{\partial}{\partial w_j}$, $\forall\, 1 \leqslant j \leqslant n-p-1$.

This finishes the proof of Theorem 5.2. □

A simple consequence of Theorem 5.2 is the following:

Corollary 5.3. — $\operatorname{Sing}(\eta)$ *is biholomorphic to* $\operatorname{Sing}(Z_o) \times (\mathbb{C}^{n-p-1}, 0)$.

5.2. Proof of Theorem 5.1

In this section we consider a holomorphic codimension p foliation $\mathcal{G}$ on $\mathbb{P}^n$, $2 \leqslant p \leqslant n-2$. We assume that there is a $p+1$ plane $\mathbb{P}^{p+1} = \Sigma_o \subset \mathbb{P}^n$ such that $\operatorname{cod}_\Sigma(\mathcal{G}|_{\Sigma_o}) \geqslant 3$. We want to prove that there is a linear projection $T\colon \mathbb{P}^n - \to \Sigma_o$ such that $\mathcal{G} = T^*(\mathcal{G}|_{\Sigma_o})$. We will prove Theorem 5.1 in the case $n = p+2$, or equivalently, when the foliation is two-dimensional. The general case will be reduced to this case using [18, Section 3.4].

The foliation $\mathcal{G}|_{\Sigma_o}$ is one dimensional and so it can be defined by a finite covering $(Q_\alpha)_{\alpha\in A}$ of Σ_o by polydiscs of Σ_o, a collection $(X_\alpha)_{\alpha\in A}$ of holomorphic vector fields $X_\alpha \in \mathcal{X}(Q_\alpha)$, and a multiplicative cocycle $(g_{\alpha\beta})_{Q_\alpha\cap Q_\beta\neq\emptyset}$

such that $X_\alpha = g_{\alpha\beta}.X_\beta$ on $Q_\alpha \cap Q_\beta \neq \emptyset$. A consequence of Theorem 5.2 is the following:

COROLLARY 5.4. — *There is a finite covering of Σ_o by polydiscs of $\mathbb{P}^{p+2}$, say $(U_\alpha)_\alpha$, and two collections of holomorphic vector fields $(Z_\alpha)_{\alpha\in A}$ and $(Y_\alpha)_{\alpha\in A}$, $Z_\alpha, Y_\alpha \in \mathcal{X}(U_\alpha)$, with the following properties:*

(a) *$U_\alpha \cap \Sigma_o = Q_\alpha$ and Z_α is an extension of X_α to U_α. In particular, Z_α is tangent to Σ_o along Q_α.*
(b) *$\mathrm{Sing}(Y_\alpha) = \emptyset$ and Y_α is transverse to Σ_o along Q_α.*
(c) *If $z \notin \mathrm{Sing}(\mathcal{G}) \cap U_\alpha$ then $T_z\mathcal{G} = \langle Z_\alpha(z), Y_\alpha(z)\rangle_{\mathbb{C}}$.*
(d) *If $z \in \mathrm{Sing}(\mathcal{G}) \cap U_\alpha$ then $Z_\alpha(z) = 0$. Moreover, the orbit of Y_α through z is contained in $\mathrm{Sing}(\mathcal{G})$.*

The proof is a straightforward consequence of Theorem 5.2 and is left to the reader.

Our goal now is to prove the following:

LEMMA 5.5. — *Under the hypothesis of Theorem 5.1 assume that $n = p+2$. Then there is a one-dimensional foliation $\mathcal{H}$ of degree zero transverse to Σ_o whoose leaves are $\mathcal{G}$-invariant.*

Proof. — The foliation $\mathcal{H}$ will be constructed in homogeneous coordinates. Let $\Pi\colon \mathbb{C}^{p+3}\setminus\{0\} \to \mathbb{P}^{p+2}$ be the canonical projection and $\widetilde{\mathcal{G}} = \Pi^*(\mathcal{G})$. Consider homogeneous coordinates $z = (z_0, \ldots, z_{p+2}) \in \mathbb{C}^{p+3}$ such that $\Pi^{-1}(\Sigma_o) \cup \{0\} = (z_o = 0) := \widetilde{\Sigma}_o$. In these homogeneous coordinates the foliation $\widetilde{\mathcal{G}}$ is defined by an integrable homogeneous p-form η such that $i_R\eta = 0$, where R denotes the radial vector field on $\mathbb{C}^{p+3}$. The foliation $\mathcal{H}$ will be defined in homogeneous coordinates by R and a constant vector field v such that $i_v\eta = 0$.

The idea is to construct a formal series of vector fields of the form $V = \frac{\partial}{\partial z_0} + \sum_{j\geqslant 0} z_0^j V_j$, where $V_j = \sum_{i=1}^{p+2} f_{ji}(z_1, \ldots, z_{p+2})\frac{\partial}{\partial z_i}$, the $f_{ji's}$ are holomorphic in some polydisc Q of $\mathbb{C}^{p+2}$ containing the origin of $\mathbb{C}^{p+2}$ and such that $i_V\eta = 0$. If $v := V(0) = \frac{\partial}{\partial z_0} + V_0(0) \neq 0$ then $i_v\eta = 0$ because the coefficients of η are homogeneous of the same degree. The constant vector field v and R induce a foliation $\mathcal{H}$ of degree zero on $\mathbb{P}^{p+2}$ tangent to $\mathcal{G}$.

Let us construct the series V. The covering $(U_\alpha)_{\alpha\in A}$, given by Corollary 5.4, induces the covering $\big(\widetilde{U}_\alpha = \Pi^{-1}(U_\alpha)\big)_{\alpha\in A}$ of $\widetilde{\Sigma}_o \setminus \{0\}$. Without lost of generality, we can suppose that for any $\alpha \in A$ then U_α is contained in some affine chart $(z_{j(\alpha)} \neq 0)$, where $j(\alpha) \neq 0$.

Claim 5.6. — *There are collections of holomorphic vector fields $(\widetilde{Z}_\alpha)_{\alpha\in A}$ and $(\widetilde{Y}_\alpha)_{\alpha\in A}$, with $\widetilde{Z}_\alpha, \widetilde{Y}_\alpha \in \mathcal{X}(\widetilde{U}_\alpha)\ \forall\ \alpha\in A$, with the following properties:*

(i) $D\Pi(z).\widetilde{Z}_\alpha(z) = Z_\alpha\circ\Pi(z)$ *and* $D\Pi(z).\widetilde{Y}_\alpha(z) = Y_\alpha\circ\Pi(z)$, $\forall\ z\in \widetilde{U}_\alpha$. *In particular,* $\widetilde{Y}_\alpha$ *and* $\widetilde{Z}_\alpha$ *are tangent to* $\widetilde{\mathcal{G}}|_{\widetilde{U}_\alpha}$*:* $i_{\widetilde{Y}_\alpha}\eta = 0$ *and* $i_{\widetilde{Z}_\alpha}\eta = 0$, $\forall\ \alpha$.

(ii) $\widetilde{Y}_\alpha$, $\widetilde{Z}_\alpha$ *and* R *generate* $\widetilde{\mathcal{G}}$ *in the sense that:*
- *if* $z\in\widetilde{U}_\alpha\setminus \mathrm{Sing}(\widetilde{\mathcal{G}})$ *then* $T_z\widetilde{\mathcal{G}} = \langle \widetilde{Y}_\alpha(z), \widetilde{Z}_\alpha(z), R(z)\rangle_{\mathbb{C}}$.
- $z\in\widetilde{U}_\alpha\cap\mathrm{Sing}(\widetilde{\mathcal{G}}) \iff \widetilde{Y}_\alpha(z)\wedge\widetilde{Z}_\alpha(z)\wedge R(z) = 0$.

(iii) $\widetilde{Z_\alpha}$ *is tangent to* $\widetilde{\Sigma}_o$ *along* $\widetilde{\Sigma}_o\cap\widetilde{U}_\alpha$, $\forall\ \alpha\in A$. *This means that*

$$\widetilde{Z}_\alpha(0, z_1, \ldots, z_{p+2}) \in \left\langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{p+2}} \right\rangle_{\mathcal{O}}.$$

(iv) $\widetilde{Y}_\alpha = g_\alpha(z)\frac{\partial}{\partial z_0} + V_\alpha$, *where* $V_\alpha \in \left\langle \frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_{p+2}}\right\rangle_{\mathcal{O}}$ *and* $g_\alpha \in \mathcal{O}^*(\widetilde{U}_\alpha)$.

In particular, $\mathrm{Sing}(\widetilde{Y}_\alpha) = \emptyset$ *and* $\widetilde{Y}_\alpha$ *is transverse to* $\widetilde{\Sigma}_o$ *along* $\widetilde{\Sigma}_o\cap\widetilde{U}_\alpha$, $\forall\ \alpha\in A$.

Proof. — Let us construct $\widetilde{Y}_\alpha$ and $\widetilde{Z}_\alpha$, $\alpha\in A$. Let $j\neq 0$ be such that $U_\alpha\subset(z_j\neq 0)$. Let us assume that $U_\alpha\subset(z_n = 1)$, for instance, and that Y_α and Z_α are vector fields tangent to the affine plane $(z_n = 1)$: $Y_\alpha = \sum_{i<n} g_i^\alpha(z_0,\ldots,z_{n-1})\frac{\partial}{\partial z_i}$ and $Z_\alpha = \sum_{i<n} h_i^\alpha(z_0,\ldots,z_{n-1})\frac{\partial}{\partial z_i}$, where $g_i^\alpha, h_i^\alpha\in\mathcal{O}(U_\alpha)$, $\forall\ \alpha$. Since Y_α is transverse to Σ_o we have $g_0^\alpha\in\mathcal{O}^*(U_\alpha)$, $\forall\ \alpha$. The vector fields $\widetilde{Y}_\alpha$ and $\widetilde{Z}_\alpha$ are then constructed by extending Y_α and Z_α "radially": we set $\widetilde{Y}_\alpha := \sum_{i<n}\widetilde{g}_i^\alpha(z)\frac{\partial}{\partial z_i}$ and $\widetilde{Z}_\alpha := \sum_{i<n}\widetilde{h}_i^\alpha(z)\frac{\partial}{\partial z_i}$, where $\widetilde{g}_i^\alpha(z) = z_0.g_i^\alpha(z_0/z_n,\ldots,z_{n-1}/z_n)$ and $\widetilde{h}_i^\alpha(z) = z_0.h_i^\alpha(z_0/z_n,\ldots,z_{n-1}/z_n)$. We leave the proof of (i), (ii), (iii) and (iv) for the reader. □

We now define a multiplicative cocycle of 3×3 matrices $(A_{\alpha\beta})_{\widetilde{U}_\alpha\cap\widetilde{U}_\beta\neq\emptyset}$. Since $\mathrm{cod}(\mathrm{Sing}(\mathcal{G}|_{U_\alpha}))\geqslant 3$, we get $\mathrm{cod}(\mathrm{Sing}(\widetilde{\mathcal{G}}|_{\widetilde{U}_\alpha}))\geqslant 3$, which implies

$$\mathrm{cod}\left(\{z\in\widetilde{U}_\alpha | \widetilde{Y}_\alpha(z)\wedge\widetilde{Z}_\alpha(z)\wedge R(z) = 0\}\right)\geqslant 3.$$

From this and (ii) we get that, if $\widetilde{U}_\alpha\cap\widetilde{U}_\beta\neq\emptyset$ then we can write

$$\begin{cases}\widetilde{Y}_\alpha(z) = a_{\alpha\beta}(z)\widetilde{Y}_\beta(z) + b_{\alpha\beta}(z)\widetilde{Z}_\beta(z) + c_{\alpha\beta}(z)R(z)\\ \widetilde{Z}_\alpha(z) = d_{\alpha\beta}(z)\widetilde{Y}_{\alpha\beta}(z) + e_{\alpha\beta}(z)\widetilde{Z}_\beta(z) + f_{\alpha\beta}(z)R(z),\end{cases}\quad \forall\ z\in\widetilde{U}_\alpha\cap\widetilde{U}_\beta,$$

where $a_{\alpha\beta},\ldots,f_{\alpha\beta}\in\mathcal{O}(\widetilde{U}_\alpha\cap\widetilde{U}_\beta)$. The matrix

$$A_{\alpha\beta} := \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} & c_{\alpha\beta}\\ d_{\alpha\beta} & e_{\alpha\beta} & f_{\alpha\beta}\\ 0 & 0 & 1\end{pmatrix}$$

defines the transition

$$\begin{pmatrix} \widetilde{Y}_\alpha \\ \widetilde{Z}_\alpha \\ R \end{pmatrix} = A_{\alpha\beta}. \begin{pmatrix} \widetilde{Y}_\beta \\ \widetilde{Z}_\beta \\ R \end{pmatrix}. \tag{5.3}$$

Of course, $A_{\alpha\beta} = A_{\beta\alpha}^{-1}$ and if $\widetilde{U}_\alpha \cap \widetilde{U}_\beta \cap \widetilde{U}_\gamma \neq \emptyset$ then $A_{\alpha\beta}A_{\beta\gamma}A_{\gamma\alpha} = I$ on $\widetilde{U}_\alpha \cap \widetilde{U}_\beta \cap \widetilde{U}_\gamma$. Since $\widetilde{U}_\alpha \cap \widetilde{U}_\beta$ is a neighborhood of $Q_\alpha \subset (z_0 = 0)$ in $\mathbb{C}^{p+3}$ we can write $A_{\alpha\beta}$ as a power series in z_0:

$$A_{\alpha\beta} = \sum_{j \geqslant 0} z_0^j A_{\alpha\beta}^j,$$

where $A_{\alpha\beta}^j$ is a matrix with coefficients in $\mathcal{O}(Q_\alpha \cap Q_\beta)$, $Q_\alpha = \widetilde{U}_\alpha \cap (z_0 = 0)$. Now, the proof of Lemma 5.5 can be reduced to the following:

Lemma 5.7. — *The matrix cocycle $(A_{\alpha\beta})_{\widetilde{U}_\alpha \cap \widetilde{U}_\beta \neq \emptyset}$ is formally trivial: there exist a collection $(A_\alpha)_{\alpha \in A}$ of formal power series*

$$A_\alpha = \sum_{j \geqslant 0} z_0^j A_\alpha^j,$$

where

(a) *A_α^j is a matrix with coefficients in $\mathcal{O}(Q_\alpha)$, $Q_\alpha = \widetilde{U}_\alpha \cap (z_0 = 0)$, $\forall\, \alpha$, $\forall\, j \geqslant 0$.*
(b) *A_α is invertible as a matrix formal series and its third line is $(0, 0, 1)$, $\forall\, \alpha$.*
(c) *if $Q_\alpha \cap Q_\beta \neq \emptyset$ then $A_{\alpha\beta} = A_\alpha^{-1}.A_\beta$.*
(d) *A_α^0 is triangular superior $\forall\, \alpha \in A$.*

The proof of Lemma 5.7 will be done at the end of the section. Let us see how it implies Lemma 5.5. From (5.3) we have

$$\begin{pmatrix} \widetilde{Y}_\alpha \\ \widetilde{Z}_\alpha \\ R \end{pmatrix} = A_\alpha^{-1}.A_\beta. \begin{pmatrix} \widetilde{Y}_\beta \\ \widetilde{Z}_\beta \\ R \end{pmatrix} \implies A_\alpha. \begin{pmatrix} \widetilde{Y}_\alpha \\ \widetilde{Z}_\alpha \\ R \end{pmatrix} = A_\beta. \begin{pmatrix} \widetilde{Y}_\beta \\ \widetilde{Z}_\beta \\ R \end{pmatrix}.$$

Since the third line of A_α and A_β is $(0, 0, 1)$, it follows that there are formal series of vector fields $Y = \sum_{j \geqslant 0} z_0^j Y_j$ and $Z = \sum_{j \geqslant 0} z_0^j Z_j$ such that

$$\begin{pmatrix} Y \\ Z \\ R \end{pmatrix}\Bigg|_{Q_\alpha \times (\mathbb{C},0)} = A_\alpha. \begin{pmatrix} \widetilde{Y}_\alpha \\ \widetilde{Z}_\alpha \\ R \end{pmatrix}, \ \forall\, \alpha.$$

Note that $i_Y \eta = 0$. Since the coefficients of η are homogeneous of the same degree, we obtain $i_v \eta = 0$, where $v = Y(0) = Y_0(0)$. Therefore, it

is sufficient to see that $Y(0) = \sum_{j=0}^{n} a_j \frac{\partial}{\partial z_j}$, where $a_0 \neq 0$. This is a consequence of Lemma 5.7(d) and the fact that the $\frac{\partial}{\partial z_0}$ component of $\widetilde{Y}_\alpha$ does not vanishes at $(z_0 = 0)$, as the reader can check. This finishes the proof of Theorem 5.1. □

5.2.1. Proof of Lemma 5.7

The restriction of $A_{\alpha\beta}$ to $\widetilde{\Sigma}_o \cap \widetilde{U}_\alpha \cap \widetilde{U}_\beta$ is triangular:

$$A_{\alpha\beta}|_{\widetilde{\Sigma}_o \cap \widetilde{U}_\alpha \cap \widetilde{U}_\beta} = \begin{pmatrix} a_{\alpha\beta} & b_{\alpha\beta} & c_{\alpha\beta} \\ 0 & e_{\alpha\beta} & f_{\alpha\beta} \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.4}$$

The cocycle defined by (5.4) is trivial, when restricted to a domain of $\widetilde{\Sigma}_o$ where we can apply Cartan's theorem [5]. Fix two polydiscs $Q_1, \overline{Q}_2 \subset \widetilde{\Sigma}_o$, where $Q_1 = \{(z_1, \ldots, z_{p+2}) \,|\, |z_i| < 1\}$ and $\overline{Q}_2 = \{(z_1, \ldots, z_{p+2}) \,|\, |z_i| \leqslant 1/2\}$, for instance. The open set $H := Q_1 \setminus \overline{Q}_2$ is a Hartog's domain in $\widetilde{\Sigma}_o$, so that any $f \in \mathcal{O}(H)$ extends to a holomorphic function $\widetilde{f} \in \mathcal{O}(Q_1)$ (cf. [24]). By Cartan's theorem [5] we have $H^1(H, \mathcal{O}) = 0$, because $n \geqslant 3$. Since $H^2(H, \mathbb{Z}) = 0$ we have also $H^1(H, \mathcal{O}^*) = 0$. Consider the Leray covering $(W_\alpha)_{\alpha \in A}$ of H given by $W_\alpha = \widetilde{U}_\alpha \cap \widetilde{\Sigma}_o$. The restriction $A_{\alpha\beta}|_{W_\alpha \cap W_\beta}$ in (5.4) will be denoted by $B_{\alpha\beta}$. Since $B_{\alpha\beta}$ is triangular, the entries $a_{\alpha\beta}$ and $e_{\alpha\beta}$ define multiplicative cocycles $(a_{\alpha\beta})_{W_\alpha \cap W_\beta \neq \emptyset}$ and $(e_{\alpha\beta})_{W_\alpha \cap W_\beta \neq \emptyset}$, which are trivial: there are collections $(a_\alpha)_{\alpha \in A}$ and $(e_\alpha)_{\alpha \in A}$, $a_\alpha, e_\alpha \in \mathcal{O}^*(W_\alpha)$ such that $a_{\alpha\beta} = a_\alpha^{-1}.a_\beta$ and $e_{\alpha\beta} = e_\alpha^{-1}.e_\beta$ on $W_\alpha \cap W_\beta \neq \emptyset$. Hence, the cocycle $(B_{\alpha\beta})_{W_\alpha \cap W_\beta \neq \emptyset}$ is equivalent to a cocycle $(C_{\alpha\beta})_{W_\alpha \cap W_\beta \neq \emptyset}$, where

$$C_{\alpha\beta} = \begin{pmatrix} 1 & g_{\alpha\beta} & h_{\alpha\beta} \\ 0 & 1 & k_{\alpha\beta} \\ 0 & 0 & 1 \end{pmatrix}.$$

By writing explicitly that $(C_{\alpha\beta})_{W_\alpha \cap W_\beta \neq \emptyset}$ is a multiplicative cocycle, we get that $(g_{\alpha\beta})_{W_\alpha \cap W_\beta \neq \emptyset}$ and $(k_{\alpha\beta})_{W_\alpha \cap W_\beta \neq \emptyset}$ are aditive cocycles. In particular, there are collections $(g_\alpha)_\alpha$ and $(k_\alpha)_\alpha$ with $g_\alpha, k_\alpha \mathcal{O}(W_\alpha)$ such that $g_{\alpha\beta} = g_\beta - g_\alpha$ and $k_{\alpha\beta} = k_\beta - k_\alpha$ on $W_\alpha \cap W_\beta \neq \emptyset$. If we set

$$M_\alpha = \begin{pmatrix} 1 & -g_\alpha & 0 \\ 0 & 1 & -k_\alpha \\ 0 & 0 & 1 \end{pmatrix}$$

then

$$D_{\alpha\beta} := M_\alpha^{-1} C_{\alpha\beta} M_\beta = \begin{pmatrix} 1 & 0 & \ell_{\alpha\beta} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Using that $(D_{\alpha\beta})_{V_\alpha\cap V_\beta\neq\emptyset}$ is a multiplicative cocycle we obtain that $(\ell_{\alpha\beta})_{W_\alpha\cap W_\beta\neq\emptyset}$ is an aditive cocycle and $\ell_{\alpha\beta}=\ell_\beta-\ell_\alpha$ on $W_\alpha\cap W_\beta\neq\emptyset$. Finally, $L_\alpha^{-1}D_{\alpha\beta}L_\beta=I$, where

$$L_\alpha=\begin{pmatrix}1&0&-\ell_\alpha\\0&1&0\\0&0&1\end{pmatrix}$$

as the reader can check. From this, we obtain that there is a collection of invertible triangular superior matrices $(B_\alpha)_\alpha$ such that $B_{\alpha\beta}=B_\alpha^{-1}B_\beta$ on $W_\alpha\cap W_\beta\neq\emptyset$. Let

$$B_\alpha=\begin{pmatrix}r_\alpha&s_a&t_\alpha\\0&u_\alpha&v_\alpha\\0&0&1\end{pmatrix}.$$

If $W_\alpha\cap W_\beta\neq\emptyset$ then $r_\alpha\widetilde{Y}_\alpha+s_\alpha\widetilde{Z}_\alpha+t_\alpha R=r_\beta\widetilde{Y}_\beta+s_\beta\widetilde{Z}_\beta+t_\beta R$ and $u_\alpha Z_\alpha+v_\alpha R=u_\beta Z_\beta+v_\beta R$ on $W_\alpha\cap W_\beta$. This defines two holomorphic vector fields V_0 and Z_0 on H by

$$V_0|_{W_\alpha}=r_\alpha\widetilde{Y}_\alpha+s_\alpha\widetilde{Z}_\alpha+t_\alpha R\quad\text{and}\quad Z_0|_{W_\alpha}=u_\alpha\widetilde{Z}_\alpha+v_\alpha R.$$

Since H is a Hartog's domain with holomorphic closure is the polydisc Q_1, V_0 and Z_0 can be extended to Q_1. We denote these extensions by the same symbols. Moreover, we have $i_{V_0}\eta=i_{Z_0}\eta=0$. We assert that $V_0(0)\neq 0$.

In fact, write $V_0(z)=\sum_{j=0}^{p+2}g_j(z)\frac{\partial}{\partial z_j}$, $z\in Q_1$. If $V_0(0)=0$ then $g_0(0)=0$ and so the analytic set $C:=\{z\in Q_1|g_0(z)=0\}$ must intersect the boundary ∂Q_1 of Q_1. If $z_0\in C\cap\partial Q_1$ then there is $\alpha\in A$ such that $z_0\in W_\alpha$. However, since $\widetilde{Z_\alpha}$ and R are tangent to $\widetilde{\Sigma}_o$, we get $g_0(z_0)=r_\alpha(z_0).g_\alpha(z_0)\neq 0$ (see (iv)), because $g_\alpha\in\mathcal{O}^*(\widetilde{U}_\alpha)$ and the matrix B_α is invertible.

Now, let us prove that there is a formal vector field $V=V_0+\sum_{j\geqslant 1}z_0^jV_j$ such that $i_V\eta=0$. To do that we recall that $A_{\alpha\beta}|_{W_\alpha\cap W_\beta}=B_{\alpha\beta}$ and $B_{\alpha\beta}=B_\alpha^{-1}B_\beta$. Consider a collection of invertible matrices $(\widetilde{B}_\alpha)_{\alpha\in A}$, where $\widetilde{B}_\alpha$ is an extension of B_α to $\widetilde{U}_\alpha$. Consider also the cocycle of matrices $\widetilde{A}_{\alpha\beta}:=\widetilde{B}_\alpha.A_{\alpha\beta}.\widetilde{B}_\beta^{-1}$. This cocycle is equivalent to $A_{\alpha\beta}$ and $\widetilde{A}_{\alpha\beta}|_{W_\alpha\cap W_\beta}=I$, $\forall$ $W_\alpha\cap W_\beta\neq\emptyset$. Since $W_\alpha\cap W_\beta=(z_0=0)\cap\widetilde{U}_\alpha\cap\widetilde{U}_\beta$ we can write

$$\widetilde{A}_{\alpha\beta}=I+\sum_{j\geqslant 1}z_0^jA_{\alpha\beta}^j,$$

where the entries of $A_{\alpha\beta}^j$ are holomorphic in $W_\alpha\cap W_\beta$. We claim that there are collections of power series of matrices of the form

$$A_\alpha=I+\sum_{j\geqslant 1}z_0^jA_\alpha^j,\alpha\in A,\tag{5.5}$$

such that the entries of A^j_α are holomorphic in W_α and $\widetilde{A}_{\alpha\beta} = A_\alpha^{-1}A_\beta$. Given a power series in z_0, say $\mathcal{S} = \sum_{j\geqslant 0} z_0^j \mathcal{S}_j$, we will use the notation $J^k(\mathcal{S})$ for the truncation $\sum_{0\leqslant j\leqslant k} z_0^j\mathcal{S}_j$. The matrices A^j_α will be constructed by induction on $j \geqslant 0$ in such a way that

$$J^k\left(\left(I + \sum_{1\leqslant j\leqslant k} z_0^j A^j_\alpha\right)^{-1} .\widetilde{A}_{\alpha\beta}. \left(I + \sum_{1\leqslant j\leqslant k} z_0^j A^j_\beta\right)\right) = I. \qquad (I_k)$$

Note that (I_0) is true and assume that we can construct collections $(A^j_\alpha)_{0\leqslant j\leqslant \ell-1}$, $\alpha \in A$, such that (I_k) is true for all $0 \leqslant k \leqslant \ell - 1$. Set $\widetilde{A}^\ell_{\alpha\beta} = I + \sum_{j=1}^{\ell} z_0^j A^j_{\alpha\beta}$ and $C^{\ell-1}_\alpha = I + \sum_{j=1}^{\ell-1} z_0^j A^j_\alpha$. Since $(I_{\ell-1})$ is true, we get

$$J^\ell\left[(C^{\ell-1}_\alpha)^{-1}.\widetilde{A}^\ell_{\alpha\beta}.C^{\ell-1}_\beta\right] = I + z_0^\ell A^\ell_{\alpha\beta}.$$

Writing explicitly that the above expression is a multiplicative cocycle of matrices we get that $(A^\ell_{\alpha\beta})_{W_\alpha\cap W_\beta\neq\emptyset}$ is an aditive cocycle. Since $H^1(H,\mathcal{O}) = 0$ we get collections $(A^\ell_\alpha)_{\alpha\in A}$ such that if $C^\ell_\alpha = I + \sum_{j=1}^{\ell} z_0^j A^j_\alpha$ then (I_ℓ) is true. In particular, the collection of formal series $C_\alpha = I + \sum_{j\geqslant 1} z_0^j A^j_\alpha$, $\alpha \in A$, satisfies $C_\alpha^{-1}\widetilde{A}_{\alpha\beta}C_\beta = I$, so that

$$\widetilde{A}_{\alpha\beta} = C_\alpha.C_\beta^{-1} \implies A_{\alpha\beta} = \widetilde{B}_\alpha^{-1}.C_\alpha.C_\beta^{-1}.\widetilde{B}_\beta = (\widetilde{B}_\alpha^{-1}.C_\alpha).(\widetilde{B}_\beta^{-1}.C_\beta)^{-1}.$$

This proves that the cocycle $(A_{\alpha\beta})_{a\beta}$ is formally trivial and finishes the proof of the existence of the constant vector field v such that $i_v\eta = 0$.

6. Appendix. Proof of Theorem 2.10 (by Alcides Lins Neto)

Let $(f_1 \dots f_s = 0)$ be a germ at $0 \in \mathbb{C}^n$ of hypersurface with a strictly ordinary singularity at 0, where $s = n - k$ and $2 \leqslant k \leqslant n - 1$. Set $X_s = (f_1 = \cdots = f_s = 0)$ and $X_s^* = X_s \setminus \{0\}$. Note that $\dim_{\mathbb{C}}(X_s) = k$. Our aim is to prove the following:

THEOREM 2.10. — *In the above situation there are representatives of X_s and X_s^* in a polydisc $Q \subset \mathbb{C}^n$, denoted by the same letters, such that:*

(a) *If $0 \leqslant q \leqslant k - 2$ then any form $\widetilde{\alpha} \in \Omega^q(X_s^*)$ can be extended to a form $\alpha \in \Omega^q(Q)$.*
(b) *If $q \geqslant 1$, $\ell \geqslant 0$ and $1 \leqslant q + \ell \leqslant k - 2$ then $H^q(X_s^*, \Omega^\ell) = 0$.*

Example 6.1. — We would like to observe that the statement of Theorem 2.10 is not true for k and $k-1$ forms. For instance, let $f \in \mathbb{C}[z_0, z_1, \dots, z_n]$, $n \geqslant 3$, be a homogeneous polynomial of degree $\geqslant n + 1$ and $X = (f = 0) \subset$

$\mathbb{C}^{n+1}$, so that $k-1=n-1$. Assume that $Z:=\Pi(X^*)\subset\mathbb{P}^n$ is smooth. It is known that there exists a non-vanishing holomorphic (n-1)-form on Z, say α. The (n-1)-form $\Pi^*(\alpha)$ is holomorphic on X^* and has no holomorphic extension to any neighborhood of $0\in\mathbb{C}^{n+1}$.

In the situation of the hypothesis of Theorem 2.10, if we fix representatives of $f_1,\ldots,f_s$ in a polydisc Q (denoted by the same letters), $0\in Q\subset\mathbb{C}^n$, we use the notation $X_0=Q$ and $X_t=\{z\in Q\,|\,f_1(z)=\cdots=f_t(z)=0\}$, $1\leqslant t\leqslant s$. We set also $X_t^*=X_t\setminus\{0\}$.

The "strictly ordinary singularity" assumption implies that for any $1\leqslant t\leqslant s$ then $X_t^*:=(f_1=\cdots=f_t=0)\setminus\{0\}$ has an isolated singularity at 0:

$$\mathrm{d}f_1(z)\wedge\cdots\wedge\mathrm{d}f_t(z)\neq 0,\forall\, z\in X_t^*. \tag{6.1}$$

From the above remark we get the following:

Lemma 6.2. — *There are representatives of $f_1,\ldots,f_s$ in a polydisc Q such that:*

(a) *0 is the unique singularity of X_t in Q, $\forall\ 1\leqslant t\leqslant s$. In particular, X_t^* is smooth of codimension t, $\forall\ 1\leqslant t\leqslant s$.*
(b) *For all $0\leqslant t\leqslant s-1$ the function $f_{t+1}|_{X_t^*}$ is a submersion. In particular, $\mathrm{d}f_{t+1}(z)\neq 0$ for all $z\in X_t^*$.*

With Lemma 6.2 the proof of Theorem 2.10 is reduced to the following:

Claim 6.3. — *Let $Q\subset\mathbb{C}^n$ be a polydisc with $0\in Q$. Let $X\subset Q$ be a connected complete intersection with a singularity $0\in X$, defined by $X=(f_1=\cdots=f_{n-k}=0)$. Assume $2\leqslant k\leqslant n-1$ and:*

(1) *f_j has an isolated singularity at $0\in Q$, $\forall\ 1\leqslant j\leqslant n-k$.*
(2) *$\forall\ I=(i_1,\ldots,i_t)$, where $i_j\neq i_k$ if $j\neq k$, and $\forall\ z\in(f_{i_1}=\cdots=f_{i_t}=0)\setminus\{0\}$ then $\mathrm{d}f_{i_1}(z)\wedge\cdots\wedge\mathrm{d}f_{i_t}(z)\neq 0$. In particular:*
   (a) *$(f_{i_1}=\cdots=f_{i_t}=0)\setminus\{0\}$ is smooth of codimension t.*
   (b) *$\dim_{\mathbb{C}}(X)=k$.*

If $0\leqslant\ell\leqslant k-2$ then any ℓ-form $\omega_\ell\in\Omega^\ell(X\setminus\{0\})$ admits an extension $\widetilde{\omega}_\ell\in\Omega^\ell(Q)$.

In order to prove Claim 6.3 we will consider the situation below:

Let Y be a connected complex manifold of dimension $n\geqslant 3$ and $Z\subset Y$ be a codimension one complex codimension one submanifold defined by $f=0$, where $f\in\mathcal{O}(Y)$ and 0 is a regular value of f. In particular, Z is a smooth submanifold of Y. For simplicity, we will use the notations Ω^ℓ for the sheaf of holomorphic ℓ-forms on Y and Z. Of course $\Omega^0=\mathcal{O}$.

LEMMA 6.4. — *In the above situation assume that* $H^k(Y,\Omega^\ell)=0$ *for all* k *and* ℓ *such that* $k\geqslant 1$ *and* $1\leqslant k+\ell\leqslant r+1$. *Then:*

(a) *If* $k\geqslant 1$, $\ell\geqslant 0$ *and* $r\geqslant 1$ *are such that* $1\leqslant k+\ell\leqslant r$ *then* $H^k(Z,\Omega^\ell)=0$.
(b) *If* $r\geqslant 0$ *and* $0\leqslant\ell\leqslant r$ *then any* ℓ*-form* $\omega\in\Omega^\ell(Z)$ *can be extended to a* ℓ*-form* $\widetilde{\omega}\in\Omega^\ell(Y)$.

Proof. — We will use Leray's theorem (cf. [12, p. 43]). Let us consider Leray coverings $\mathcal{U}=(U_\alpha)_{\alpha\in A}$ and $\widetilde{\mathcal{U}}=(\widetilde{U}_{\tilde{\alpha}})_{\tilde{\alpha}\in\tilde{A}}$ of Z and Y by open sets, respectively, such that: $A\subset\widetilde{A}$ and if $\alpha\in A$ then:

(i) $\widetilde{U}_\alpha$ is the domain of a holomorphic chart $z^\alpha=(z_1,\ldots,z_n)\colon\widetilde{U}_\alpha\to\mathbb{C}^n$, such that $\widetilde{U}_\alpha=\{z^\alpha\,|\,|z_j|<1,j=1,\ldots,n\}$ and $f|_{\widetilde{U}_\alpha}=z_1$
(ii) $U_\alpha=\widetilde{U}_\alpha\cap Z$. In particular, $U_\alpha=\{z^\alpha\in\widetilde{U}_\alpha\,|\,z_1=0\}$.

Note that $\widetilde{U}_\alpha$ is biholomorphic to a polydisc of $\mathbb{C}^n$ and U_α to a polydisc of $\mathbb{C}^{n-1}$. We assume also that:

(iii) If $\alpha\in\widetilde{A}\setminus A$ then $\widetilde{U}_\alpha\cap Z=\emptyset$. This implies that $A=\{\alpha\in\widetilde{A}\,|\,\widetilde{U}_\alpha\cap Z\neq\emptyset\}$.

Given $J=(j_0,\ldots,j_k)\in\widetilde{A}^{k+1}$ (resp. $J\in A^{k+1}$) we set $\widetilde{U}_J=\widetilde{U}_{j_0}\cap\cdots\cap\widetilde{U}_{j_k}$ (resp. $U_J=U_{j_0}\cap\cdots\cap U_{j_k}$). Note that by construction, if $J\in A^{k+1}$ is such that $U_J\neq\emptyset$ then $U_J=\widetilde{U}_J\cap Z$. Moreover, if $z^{\alpha_0}=(z_1,\ldots,z_n)$ is a chart as in (i) then $U_J\subset\{z_1=0\}$.

CLAIM 6.5. — *Given* $0\leqslant\ell\leqslant n-1$ *and* $J=(j_0,\ldots,j_k)\in A^{k+1}$ *such that* $U_J\neq\emptyset$ *then any* ℓ*-form* ω *on* U_J *can be extended to a* ℓ*-form on* $\widetilde{U}_J$. *Moreover, if* $\widetilde{\omega}_1$ *and* $\widetilde{\omega}_2$ *are two extensions of* ω *to* $\widetilde{U}_J$ *then:*

(a) $\widetilde{\omega}_2-\widetilde{\omega}_1=g.f$, $g\in\mathcal{O}(\widetilde{U}_J)$ *if* $\ell=0$.
(b) $\widetilde{\omega}_2-\widetilde{\omega}_1=\alpha\wedge\mathrm{d}f+f.\beta$, *where* $\alpha\in\Omega^{\ell-1}(\widetilde{U}_J)$ *and* $\beta\in\Omega^\ell(\widetilde{U}_J)$, *if* $\ell\geqslant 1$.

Proof of the Claim. — Since $\widetilde{U}_\alpha$ is biholomorphic to a polydisc, for any $\alpha\in\widetilde{A}$ it follows that $\widetilde{U}_J$ is a Stein open subset of Y. Since $U_J=f^{-1}(0)\cap\widetilde{U}_J$ it follows that any holomorphic function $h\in\mathcal{O}(U_J)$ admits an extension $\widetilde{h}\in\mathcal{O}(\widetilde{U}_J)$ (cf. [12]). This proves the case $\ell=0$. When $\ell\geqslant 1$, we consider the chart $z^{\alpha_0}=(z_1,\ldots,z_n)\colon\widetilde{U}_{\alpha_0}\to\mathbb{C}^n$ where $f|_{U_{\alpha_0}}=z_1$ and $U_{\alpha_0}=\{z^\alpha\in\widetilde{U}_{\alpha_0}\,|\,z_1=0\}$, so that $U_J=\{z^{\alpha_0}\in\widetilde{U}_J\,|\,z_1=0\}$. In particular, any ℓ-form $\omega\in\Omega^\ell(U_J)$ can be written as

$$\omega=\sum_{I=(2\leqslant i_1<\cdots<i_\ell\leqslant n)}h_I.\mathrm{d}z_{i_1}\wedge\cdots\wedge\mathrm{d}z_{i_\ell},\ \text{ where } h_I=h_I(z_2,\ldots,z_n)\in\mathcal{O}(U_J).$$

By the case $\ell = 0$ any function h_I admits an extension $\widetilde{h}_I \in \mathcal{O}(\widetilde{U}_J)$. Therefore, ω admits the extension

$$\widetilde{\omega} = \sum_{I=(2\leqslant i_1<\cdots<i_\ell\leqslant n)} \widetilde{h}_I.\mathrm{d}z_{i_1}\wedge\cdots\wedge\mathrm{d}z_{i_\ell} \in \Omega^\ell(\widetilde{U}_J).$$

If $\widetilde{\omega}_2$ and $\widetilde{\omega}_1$ are two extensions of ω to $\widetilde{U}_J$ then $(\widetilde{\omega}_2-\widetilde{\omega}_1)|_{z_1=0} = 0$. Therefore, if $\ell = 0$ then $\widetilde{\omega}_2 - \widetilde{\omega}_1 = g.z_1 = g.f$ as in (a), whereas if $\ell \geqslant 1$ then $\widetilde{\omega}_2 - \widetilde{\omega}_1 = \alpha\wedge\mathrm{d}z_1 + z_1.\beta = \alpha\wedge\mathrm{d}f + f.\beta$ as in (b). □

Since Ω^ℓ is a holomorphic sheaf, by Leray's theorem we have $H^k(Z,\Omega^\ell) = H^k(\mathcal{U},\Omega^\ell)$ and $H^k(Y,\Omega^\ell) = H^k(\widetilde{\mathcal{U}},\Omega^\ell)$ for all $k \geqslant 1$ and $\ell \geqslant 0$. Of course, $H^0(Z,\Omega^\ell) = \Omega^\ell(Z)$ and $H^0(Y,\Omega^\ell) = \Omega^\ell(Y)$. Let us fix some notations (cf. [12]):

(1) $C^k(\mathcal{U},\Omega^\ell)$ (resp. $C^k(\widetilde{\mathcal{U}},\Omega^\ell)$) the $\mathcal{O}$-module of k-cochains of ℓ-forms with respect to $\mathcal{U}$ (resp. with respect to $\widetilde{\mathcal{U}}$).
(2) $\delta = \delta_k\colon C^k(*,\Omega^\ell) \to C^{k+1}(*,\Omega^\ell)$ the coboundary operator, where $* = \mathcal{U}$ or $\widetilde{\mathcal{U}}$. In this way, we have:

$$H^k(*,\Omega^\ell) = \ker(\delta_k)/\operatorname{Im}(\delta_{k-1}), k \geqslant 0.$$

Recall that

$$C^k(\mathcal{U},\Omega^\ell) = \prod_{J\in A^{k+1}} \Omega^\ell(U_J),$$

where $J = (\alpha_0,\ldots,\alpha_k) \in A^{k+1}$ and $U_J = U_{\alpha_0}\cap\cdots\cap U_{\alpha_k}$. In particular, a cochain in $\omega_\ell^k \in C^k(\mathcal{U},\Omega^\ell)$ is of the form

$$\omega_\ell^k = (\omega_J)_{J\in A^{k+1}}, \omega_J \in \Omega^\ell(U_J).$$

When $U_J = \emptyset$ by convenction we set $\omega_J = 0$. Anagolously, a cochain $\widetilde{\omega}_\ell^k \in C^k(\widetilde{\mathcal{U}},\Omega^\ell)$ is of the form

$$\widetilde{\omega}_\ell^k = (\widetilde{\omega}_J)_{J\in \tilde{A}^{k+1}}, \omega_J \in \Omega^\ell(\widetilde{U}_J).$$

Restriction of cochains. — Given a cochain $\widetilde{\omega}_\ell^k \in C^k(\widetilde{\mathcal{U}},\Omega^\ell)$, where $\widetilde{\omega}_\ell^k = (\widetilde{\omega}_J)_{J\in\tilde{A}^{k+1}}$, its restriction to Z is defined as

$$\widetilde{\omega}_\ell^k|_Z := (\widetilde{\omega}_J|_{U_J})_{J\in A^{k+1}} \in C^k(\mathcal{U},\Omega^\ell).$$

Recall that if $J \in A^{k+1}$ then $U_J = \widetilde{U}_J \cap Z$.

Remark 6.6. — Let $\widetilde{\omega}_\ell^k, \widetilde{\eta}_\ell^k \in C^k(\widetilde{\mathcal{U}},\Omega^\ell)$ be two cochains with the same restriction to Z: $(\widetilde{\eta}_\ell^k - \widetilde{\omega}_\ell^k)|_Z = 0$. It follows from Claim 6.5 that:

(a) If $\ell = 0$ then there exists a cochain $g_0^k = (g_J)_{J\in\tilde{A}^{k+1}} \in C^k(\widetilde{\mathcal{U}},\mathcal{O})$ such that $\widetilde{\eta}_J - \widetilde{\omega}_J = g_J.f$, for all $J \in \widetilde{A}^{k+1}$. In this case we will write $\widetilde{\eta}_\ell^k - \widetilde{\omega}_\ell^k = f.g_0^k$.

(b) If $\ell \geqslant 1$ then there are cochains $\widetilde{\alpha}^k_{\ell-1} = (\widetilde{\alpha}_J)_{J\in\tilde{A}^{k+1}} \in C^k(\widetilde{\mathcal{U}}, \Omega^{\ell-1})$ and $\beta^k_\ell = (\widetilde{\beta}_J)_{J\in\tilde{A}^{k+1}} \in C^k(\widetilde{\mathcal{U}}, \Omega^\ell)$ such that $\widetilde{\eta}_J - \widetilde{\omega}_J = \widetilde{\alpha}_J \wedge \mathrm{d}f + f.\widetilde{\beta}_J$, for all $J \in \widetilde{A}^{k+1}$. In this case, we will write $\widetilde{\eta}^k_\ell - \widetilde{\omega}^k_\ell = \widetilde{\alpha}^k_{\ell-1} \wedge \mathrm{d}f + f.\widetilde{\beta}^k_\ell$.

We leave the details to the reader.

Extension of cochains. — Claim 6.5 implies that given a cochain $\omega^k_\ell \in C^k(\mathcal{U}, \Omega^\ell)$ then there exists a cochain $\widetilde{\omega}^k_\ell \in C^k(\widetilde{\mathcal{U}}, \Omega^\ell)$ whoose restriction to Z coincides with ω^k_ℓ. We leave the details to the reader. The cochain $\widetilde{\omega}^k_\ell$ will be called an *extension* of the cochain ω^k_ℓ.

Division of cochains. — Given a cochain $\beta^k_\ell = (\beta_J)_{J\in\tilde{A}^{k+1}} \in C^k(\widetilde{\mathcal{U}}, \Omega^\ell)$ we define the cochain $\beta^k_\ell \wedge \mathrm{d}f := (\beta_J \wedge \mathrm{d}f)_{J\in\tilde{A}^{k+1}} \in C^k(\widetilde{\mathcal{U}}, \Omega^{\ell+1})$. We would like to observe that, if $\ell \geqslant 1$ and $\mathrm{d}f \wedge \beta^k_\ell = 0$ then there exists a cochain $\beta^k_{\ell-1} \in C^k(\widetilde{\mathcal{U}}, \Omega^{\ell-1})$ such that $\beta^k_\ell = \beta^k_{\ell-1} \wedge \mathrm{d}f$. The proof is easy and is left to the reader.

Let us assume the hypothesis of Lemma 6.4: $H^k(Y, \Omega^\ell) = 0$ if $k \geqslant 1$ and $1 \leqslant k + \ell \leqslant r + 1$.

Claim 6.7. — *In the above situation, if $k \geqslant 0$ and $\ell \geqslant 0$ are such that $k + \ell \leqslant r$ then any cocycle $\omega^k_\ell \in C^k(\mathcal{U}, \Omega^\ell)$ such that $\delta\omega^k_\ell = 0$ admits an extension $\widetilde{\omega}^k_\ell \in C^k(\widetilde{\mathcal{U}}, \Omega^\ell)$ such that $\delta\widetilde{\omega}^k_\ell = 0$.*

Proof. — Let $\omega^k_\ell \in C^k(\mathcal{U}, \Omega^\ell)$ be such that $\delta\omega^k_\ell = 0$. As we have seen before, ω^k_ℓ admits an extension $\widehat{\omega}^k_\ell \in C^k(\widetilde{\mathcal{U}}, \Omega^\ell)$. Then $\delta\widehat{\omega}^k_\ell \in C^{k+1}(\widetilde{\mathcal{U}}, \Omega^\ell)$ and so $\delta\widehat{\omega}^k_\ell|_Z = \delta\omega^k_\ell = 0$.

Let us assume first that $\ell = 0$, so that $k + \ell = k \leqslant r$. In this case, from Remark 6.6 we obtain $\delta\widehat{\omega}^k_0 = f.g^{k+1}_0$, where $g^{k+1}_0 \in C^{k+1}(\widetilde{\mathcal{U}}, \mathcal{O})$. Now, since $\delta^2 = 0$, we have $f.\delta g^{k+1}_0 = 0$, and so $\delta g^{k+1}_0 = 0$. Since $k + 1 \leqslant r + 1$ the hypothesis implies that $H^{k+1}(\widetilde{\mathcal{U}}, \mathcal{O}) = 0$ and so there exists a cochain $h^k_0 \in C^k(\widetilde{\mathcal{U}}, \mathcal{O})$ with $g^{k+1}_0 = \delta h^k_0$. Therefore,

$$\delta\widehat{\omega}^k_0 = f.\delta h^k_0 \implies \delta(\widehat{\omega}^k_0 - f.h^k_0) = 0.$$

If we set $\widetilde{\omega}^k_0 = \widehat{\omega}^k_0 - f.h^k_0$ then $\widetilde{\omega}^k_0|_Z = \omega^k_0$ and $\delta\widetilde{\omega}^k_0 = 0$, which proves in the case $\ell = 0$.

Let us assume now that $\ell \geqslant 1$. In this case, Remark 6.6 implies that

$$\delta\widehat{\omega}^k_\ell = \widehat{\alpha}^{k+1}_{\ell-1} \wedge \mathrm{d}f + f.\widehat{\beta}^{k+1}_\ell, \tag{6.2}$$

where $\widehat{\alpha}^{k+1}_{\ell-1} \in C^{k+1}(\widetilde{\mathcal{U}}, \Omega^{\ell-1})$ and $\widehat{\beta}^{k+1}_\ell \in C^{k+1}(\widetilde{\mathcal{U}}, \Omega^\ell)$. We assert that we can choose $\widehat{\alpha}^{k+1}_{\ell-1} \in C^{k+1}(\widetilde{\mathcal{U}}, \Omega^{\ell-1})$ and $\widehat{\beta}^{k+1}_\ell \in C^{k+1}(\widetilde{\mathcal{U}}, \Omega^\ell)$ such that (6.2) is true and $\delta\widehat{\beta}^{k+1}_\ell = 0$. Let us prove this assertion.

First we construct by induction a sequence of cochains

$$\beta_{\ell-j}^{k+j+1} \in C^{k+j+1}(\widetilde{\mathcal{U}}, \Omega^{\ell-j}), j = 0, \ldots, \ell$$

such that $\beta_\ell^{k+1} = \widehat{\beta}_\ell^{k+1}$ and:

(i) $\delta\beta_{\ell-j}^{k+j+1} \wedge \mathrm{d}f = 0$, $\forall\, j = 0, \ldots, \ell$.
(ii) $\delta\beta_{\ell-j}^{k+j+1} = \beta_{\ell-j-1}^{k+j+2} \wedge \mathrm{d}f$, $\forall\, j = 0, \ldots, \ell - 1$.

The construction is based in the division property. Since $\delta^2 = 0$, relation (6.2) implies that

$$\delta\widehat{\alpha}_{\ell-1}^{k+1} \wedge \mathrm{d}f + f.\delta\widehat{\beta}_\ell^{k+1} = 0 \implies \delta\widehat{\beta}_\ell^{k+1} \wedge \mathrm{d}f = 0 \implies \delta\widehat{\beta}_\ell^{k+1} = \beta_{\ell-1}^{k+2} \wedge \mathrm{d}f$$
$$\delta\beta_{\ell-1}^{k+2} \wedge \mathrm{d}f = 0 \implies \delta\beta_{\ell-1}^{k+2} = \beta_{\ell-2}^{k+3} \wedge \mathrm{d}f \implies \cdots \implies \delta\beta_{\ell-j}^{k+j+1} \wedge \mathrm{d}f = 0$$
$$\delta\beta_{\ell-j}^{k+j+1} = \beta_{\ell-j-1}^{k+j+2} \wedge \mathrm{d}f \implies \cdots \implies \delta\beta_1^{k+\ell} \wedge \mathrm{d}f = 0 \implies \delta\beta_1^{k+\ell} = \beta_0^{k+\ell+1}.\mathrm{d}f.$$

Next, we will see that the sequence can be constructed in such a way that $\delta\beta_j^{k+\ell-j+1} = 0$, $\forall\, j = 0, \ldots, \ell$. This involves another induction argument.

1st step: $j = 0$. — From $\delta\beta_1^{k+\ell} = \beta_0^{k+\ell+1}.\mathrm{d}f$ we get $\delta\beta_0^{k+\ell+1} = 0$. Hence $\beta_0^{k+\ell+1} \in \ker(\delta)$.

2nd step. — Assume that we have constructed the sequence satisfying (i), (ii) with $\delta\beta_i^{k+\ell-i+1} = 0$ for $i = 0, \ldots, j-1 \leqslant \ell - 1$ and let us prove that can assume that $\delta\beta_j^{k+\ell-j+1} = 0$.

From (ii) we have $\delta\beta_j^{k+\ell-j+1} = \beta_{j-1}^{k+\ell-j+2} \wedge \mathrm{d}f$, where $\delta\beta_{j-1}^{k+\ell-j+2} = 0$ by the induction hypothesis. Since $(k + \ell - j + 2) + (j - 1) = k + \ell + 1 \leqslant r + 1$ we have $H^{k+\ell-j+2}(\widetilde{\mathcal{U}}, \Omega^{j-1}) = 0$ and so there exists a cochain $\gamma_{j-1}^{k+\ell-j+1} \in C^{k+\ell-j+1}(\widetilde{\mathcal{U}}, \Omega^{j-1})$ such that $\beta_{j-1}^{k+\ell-j+2} = \delta\gamma_{j-1}^{k+\ell-j+1}$. Therefore, if we set $\widetilde{\beta}_j^{k+\ell-j+1} = \beta_j^{k+\ell-j+1} - \gamma_{j-1}^{k+\ell-j+1} \wedge \mathrm{d}f$ then

$$\delta\widetilde{\beta}_j^{k+\ell-j+1} = \delta\Big(\beta_j^{k+\ell-j+1} - \gamma_{j-1}^{k+\ell-j+1} \wedge \mathrm{d}f\Big) = 0.$$

Moreover,

$$\widetilde{\beta}_j^{k+\ell-j+1} \wedge \mathrm{d}f = \beta_j^{k+\ell-j+1} \wedge \mathrm{d}f = \beta_{j+1}^{k+\ell-j}.$$

Hence, if we replace $\beta_j^{k+\ell-j+1}$ by $\widetilde{\beta}_j^{k+\ell-j+1}$ in the sequence, then the new sequence still satisfies (i) and (ii).

The induction process implies that there exists a cochain $\gamma_{\ell-1}^{k+1} \in C^{k+1}(\widetilde{\mathcal{U}}, \Omega^{\ell-1})$ such that $\delta(\widehat{\beta}_\ell^{k+1} - \gamma_{\ell-1}^{k+1} \wedge \mathrm{d}f) = 0$. Hence, if we set $\widetilde{\beta}_\ell^{k+1} = \widehat{\beta}_\ell^{k+1} - \gamma_{\ell-1}^{k+1} \wedge \mathrm{d}f$ and $\widetilde{\alpha}_{\ell-1}^{k+1} = \widehat{\alpha}_{\ell-1}^{k+1} + f.h_{\ell-1}^{k+1}$ then (6.2) can be written as

$$\delta\widehat{\omega}_\ell^k = \widetilde{\alpha}_{\ell-1}^{k+1} \wedge \mathrm{d}f + f.\widetilde{\beta}_\ell^{k+1}, \quad \text{where } \delta\widetilde{\beta}_\ell^{k+1} = 0.$$

Since $H^{k+1}(\widetilde{\mathcal{U}},\Omega^\ell)=0$, there exists a cochain $\gamma_\ell^k\in C^k(\widetilde{\mathcal{U}},\Omega^\ell)$ such that $\widetilde{\beta}_\ell^{k+1}=\delta\gamma_\ell^k$. In particular, if we set $\overline{\omega}_\ell^k=\widehat{\omega}_\ell^k-f.\gamma_\ell^k$ then $\overline{\omega}_\ell^k|_Z=\widehat{\omega}_\ell^k|_Z=\omega_\ell^k$ and

$$\delta\overline{\omega}_\ell^k=\widetilde{\alpha}_{\ell-1}^{k+1}\wedge \mathrm{d}f. \tag{6.3}$$

If $\ell=1$ then $\widetilde{\alpha}_0^{k+1}\in H^{k+1}(\widetilde{\mathcal{U}},\mathcal{O})$ and (6.3) implies that $\delta\alpha_{\ell-1}^{k+1}=0$ and there exists a cochain $g_0^k\in C^k(\widetilde{\mathcal{U}},\mathcal{O})$ such that $\widetilde{\alpha}_0^{k+1}=\delta g_0^k$. In particular, the cochain $\widetilde{\omega}_1^k=\overline{\omega}_1^k-g_0^k.\mathrm{d}f$ satisfies $\delta\widetilde{\omega}_1^k=0$ and $\widetilde{\omega}_1^k|_Z=\omega_1^k$, proving Claim 6.7 in this case.

Finally, when $\ell\geqslant 2$ using (6.3) and an induction argument similar to that used in the case of $\widehat{\beta}_\ell^{k+1}$ it is possible to obtain a cochain $\gamma_{\ell-2}^{k+1}\in C^{k+1}(\widetilde{\mathcal{U}},\Omega^{\ell-2})$ such that $\delta(\widetilde{\alpha}_{\ell-1}^{k+1}-\gamma_{\ell-2}^{k+1}\wedge\mathrm{d}f)=0$. Since $(\ell-1)+k+1=\ell+k\leqslant r+1$ we have $H^{k+1}(\widetilde{\mathcal{U}},\Omega^{\ell-1})=0$, so that $\widetilde{\alpha}_{\ell-1}^{k+1}-\gamma_{\ell-2}^{k+1}\wedge\mathrm{d}f=\delta\eta_{\ell-1}^k$, where $\eta_{\ell-1}^k\in C^k(\widetilde{\mathcal{U}},\Omega^{\ell-1})$. From (6.3) we get

$$\delta\overline{\omega}_\ell^{k+1}=\widetilde{\alpha}_{\ell-1}^{k+1}\wedge\mathrm{d}f=\delta\eta_{\ell-1}^k\wedge\mathrm{d}f\implies\delta(\overline{\omega}_\ell^k-\eta_{\ell-1}^k\wedge\mathrm{d}f)=0.$$

Hence, if we set $\widetilde{\omega}_\ell^k=\overline{\omega}_\ell^k-\eta_{\ell-1}^k\wedge\mathrm{d}f$ then $\delta\widetilde{\omega}_k^\ell=0$ and $\widetilde{\omega}_\ell^k|_Z=\omega_\ell^k$, which proves Claim 6.7.

Let us finish the proof of Lemma 6.4.

Proof of (a). — By Leray's theorem it is suficient to prove that $H^k(\mathcal{U},\Omega^\ell)=0$, if $k\geqslant 1$ and $k+\ell\leqslant r$. If $\omega_\ell^k\in C^k(\mathcal{U},\Omega^\ell)$ is such that $\delta\omega_\ell^k=0$ then by Claim 6.7, ω_ℓ^k admits an extension $\widetilde{\omega}_\ell^k$ such that $\delta\widetilde{\omega}_\ell^k=0$. Since $k+\ell\leqslant r<r+1$ then $H^k(\widetilde{\mathcal{U}},\Omega^\ell)=0$, so that $\widetilde{\omega}_\ell^k=\delta\widetilde{\eta}_\ell^{k-1}$ for some cochain $\widetilde{\eta}_\ell^{k-1}\in C^{k-1}(\widetilde{\mathcal{U}},\Omega^\ell)$. As the reader can check, this implies that $\omega_\ell^k=\delta\left(\widetilde{\eta}_\ell^{k-1}|_Z\right)$, which proves the assertion.

Proof of (b). — Let $\omega_\ell\in\Omega^\ell(Z)$, where $\ell\leqslant r$. We can associate to ω_ℓ a 0-cochain $\omega_\ell^0=(\omega_\ell|_{U_\alpha})_{\alpha\in A}$ with $\delta\omega_\ell^0=0$. By Claim 6.7, ω_ℓ^0 admits an extension $\widetilde{\omega}_\ell^0\in C^0(\widetilde{\mathcal{U}},\Omega^\ell)$ such that $\delta\widetilde{\omega}_\ell^0=0$. This is equivalent to say that there exists a section $\widetilde{\omega}_\ell\in\Omega^\ell(Y)$ such that $\widetilde{\omega}_\ell^0=(\widetilde{\omega}_\ell|_{\tilde{U}_\alpha})_{\alpha\in\tilde{A}}$. Hence, $\widetilde{\omega}_\ell$ extends ω_ℓ proving Lemma 6.4. □

We are now in position to prove the statement of Theorem 2.10. Let $0\in Q\subset\mathbb{C}^n$, Q a polydisc, and $X=(f_1=\cdots=f_{n-k}=0)$ be as in the statement of Lemma 2.3. Define a sequence of analytic complete intersections $X_0\supset X_1\supset\cdots\supset X_{n-k}$, where $X_0=Q$ and $X_q=(f_1=\cdots=f_q=0)$ if $1\leqslant q\leqslant n-k$, and set $X_q^*:=X_q\setminus\{0\}$, $0\leqslant q\leqslant n-k$. The hypothesis implies the following:

(i) $\dim_{\mathbb{C}}(X_q)=k(q):=n-q$ and X_q^* is smooth, $\forall\,0\leqslant q\leqslant k$.

(ii) $X_q = f_q^{-1}(0) \cap X_{q-1}$, $\forall\, 1 \leqslant q \leqslant n-k$. Moreover, 0 is a regular value of $f_q|_{X^*_{q-1}}$.

Recall that $k \geqslant 2$, so that $k(q) \geqslant 3$ if $q \leqslant n-3$.

Claim 6.8. — *Let $p \geqslant 1$, $\ell \geqslant 0$ and $0 \leqslant q \leqslant n-k-1$ be such that $1 \leqslant p+\ell \leqslant k(q)-2$. Then $H^p(X_q^*, \Omega^\ell) = 0$.*

Proof. — The proof is by induction on $q = 0, \ldots, n-3$. The case $q = 0$ is consequence of a generalization of Cartan's theorem (cf. [5]): since $X_0 = Q$ is Stein then ([12, p. 133]):

$$H^p(X_0^*, \Omega^\ell) = 0, \forall\, p = 1, \ldots, n-2, \forall\, \ell \geqslant 0.$$

In particular, $H^p(X_0^*, \Omega^\ell) = 0$ if $p \geqslant 1$, $\ell \geqslant 0$ and $1 \leqslant p+\ell \leqslant n-2 = k(0)-2$. The induction step is consequence of Lemma 6.4(a): let us assume that Claim 6.8 is true for q, where $1 \leqslant q \leqslant n-k-2$. Set $Y = X_q^*$, $Z = X_{q+1}^*$ and $f = f_{q+1}|_{X^*_{q+1}}$ in Lemma 6.4. The induction hypothesis implies that, if $p \geqslant 1$ and $\ell \geqslant 0$ are such that $1 \leqslant p+\ell \leqslant k(q)-2$ then $H^p(X_q^*, \Omega^\ell) = 0$. In particular, Lemma 6.4(a) implies that $H^p(X_{q+1}^*, \Omega^\ell) = 0$, $\forall\, p \geqslant 1$, $\ell \geqslant 0$ such that $1 \leqslant p+\ell \leqslant k(q)-3 = k(q+1)-2$.

The extension property is consequence of Lemma 6.4(b). The idea is to use Claim 6.8 and Lemma 6.4(b) inductively. Let $\omega_\ell \in \Omega^\ell(X \setminus \{0\})$, where $\ell \leqslant k-2$. In the first step we set $Z = X^*_{n-k} = X \setminus \{0\}$, $Y = X^*_{n-k-1}$ and $f = f_{n-k}|_{X^*_{n-k-1}}$. From Claim 6.8 we have $H^p(X^*_{n-k-1}, \Omega^\ell) = 0$ if $p \geqslant 1$ and $\ell \geqslant 0$ are such that $1 \leqslant p+\ell \leqslant k(n-k-1)-2 = k-1$. Hence, Lemma 6.4(b) implies that if $0 \leqslant \ell \leqslant k-2$ then any form $\omega_\ell \in \Omega(X^*_{n-k})$ has an extension $\omega_\ell^1 \in \Omega^\ell(X^*_{n-k-1})$. The induction step is similar: assume that ω_ℓ has an extension $\omega_\ell^j \in \Omega^\ell(X^*_{n-k-j})$, where $1 \leqslant j \leqslant n-k-1$. Since $\ell \leqslant k-2 < k-2+j = k(n-k-j)-2$, Lemma 6.4(b) implies that ω_ℓ^j has an extension $\omega_\ell^{j+1} \in \Omega^\ell(X^*_{n-k-j-1})$. Finally, ω_ℓ has an extension $\omega_\ell^{n-k} \in \Omega^\ell(X_0^*)$, which by Hartog's theorem has an extension $\widetilde{\omega}_\ell \in \Omega^\ell(Q)$. □

Bibliography

[1] V. I. Arnol'd, *Chapitres supplémentaires de la théorie des équations différentielles ordinaires*, Mir, 1980.

[2] W. Barth, "Fortsetzung meromorpher Funktionen in Tori und komplex-projektiven Räumen", *Invent. Math.* **5** (1968), no. 1, p. 42-62.

[3] O. Calvo-Andrade, "Irreducible components of the space of holomorphic foliations", *Math. Ann.* **299** (1994), no. 1, p. 751-767.

[4] C. Camacho, A. Lins Neto & P. Sad, "Foliations with algebraic limit sets", *Ann. Math.* **136** (1992), no. 2, p. 429-446.

[5] H. Cartan, "Sur le premier problème de Cousin", *C. R. Math. Acad. Sci. Paris* **207** (1938), p. 558-560.
[6] D. Cerveau, "Distributions involutives singulières", *Ann. Inst. Fourier* **29** (1979), no. 3, p. 261-294.
[7] D. Cerveau & J.-F. Mattei, *Formes intégrables holomorphes singulières*, Astérisque, vol. 97, Société Mathématique de France, 1982.
[8] F. Cukierman, J. V. Pereira & I. Vainsencher, "Stability of foliations induced by rational maps", *Ann. Fac. Sci. Toulouse, Math.* **18** (2009), no. 4, p. 685-715.
[9] A. S. De Medeiros, "Singular foliations and differential p-forms", *Ann. Fac. Sci. Toulouse, Math.* **9** (2000), no. 3, p. 451-466.
[10] P. Deligne, "Théorie de Hodge II", *Publ. Math., Inst. Hautes Étud. Sci.* **40** (1971), no. 1, p. 5-57.
[11] J. Gargiulo Acea, "Logarithmic forms and singular projective foliations", *Ann. Inst. Fourier* **70** (2020), no. 1, p. 171-203.
[12] H. Grauert & R. Remmert, *Theory of Stein spaces*, Springer, 1979.
[13] H. A. Hamm, "Lokale topologische Eigenschaften komplexer Räume", *Math. Ann.* **191** (1971), no. 3, p. 235-252.
[14] ———, "Lefschetz theorems for singular varieties", in *Singularities, Part 1 (Arcata, Calif., 1981)*, Proceedings of Symposia in Pure Mathematics, vol. 40, American Mathematical Society, 1983, p. 547-557.
[15] M. W. Hirsch, C. C. Pugh & M. Shub, *Invariant manifolds*, Lecture Notes in Mathematics, vol. 583, Springer, 1977.
[16] A. Lins Neto, "Componentes Irredutíveis dos Espacos de Folheacões", 2007, 26^o Colóquio Brasileiro de Matemática, Instituto de Matemática Pura e Aplicada (IMPA), Rio de Janeiro. Available at: `http://www.impa.br/opencms/pt/biblioteca/cbm/26CBM/26CBM02.pdf`.
[17] ———, "Germs of complex two dimensional foliations", *Bull. Braz. Math. Soc. (N.S.)* **46** (2015), no. 4, p. 645-680.
[18] A. Lins Neto & W. Costa e Silva, "Pull-back components of the space of foliations of codimension$\geqslant 2$", *Trans. Am. Math. Soc.* **371** (2019), no. 2, p. 949-969.
[19] J. Martinet, "Normalisation des champs de vecteurs holomorphes", in *Séminaire Bourbaki vol. 1980/81*, Lecture Notes in Mathematics, vol. 901, Springer, 1981, p. 55-70.
[20] J. Milnor, *Singular points of complex hypersurfaces*, no. 61, Princeton University Press, 1968.
[21] G. de Rham, "Sur la division de formes et de courants par une forme linéaire", *Comment. Math. Helv.* **28** (1954), no. 1, p. 346-352.
[22] H. Rossi, "Continuation of subvarieties of projective varieties", *Am. J. Math.* **91** (1969), no. 2, p. 565-575.
[23] J. Seade, *On the topology of isolated singularities in analytic spaces*, Progress in Mathematics, vol. 241, Birkhäuser, 2006.
[24] Y.-T. Siu, *Techniques of extension of analytic objects*, Lecture Notes in Pure and Applied Mathematics, vol. 8, Marcel Dekker, 1974.