



HAL
open science

Integral inequalities of Hermite–Hadamard type for (α, s) -convex and (α, s, m) -convex functions

Bo-Yan Xi, Dan-Dan Gao, Feng Qi

► **To cite this version:**

Bo-Yan Xi, Dan-Dan Gao, Feng Qi. Integral inequalities of Hermite–Hadamard type for (α, s) -convex and (α, s, m) -convex functions. *Italian Journal of Pure and Applied Mathematics*, 2020, 44, pp.499–510. hal-01761678v2

HAL Id: hal-01761678

<https://hal.science/hal-01761678v2>

Submitted on 14 Nov 2020

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Integral inequalities of Hermite–Hadamard type for (α, s) -convex and (α, s, m) -convex functions

Bo-Yan Xi

*College of Mathematics and Physics, Inner Mongolia University for Nationalities, Tongliao 028043, Inner Mongolia, China
baoyintu78@qq.com*

Dan-Dan Gao

*College of Mathematics and Physics, Inner Mongolia University for Nationalities, Tongliao 028043, Inner Mongolia, China
Tongliao Huaxin Aishangxue Education and Training Limited Company, No. 02872, Yongqing Branch Office, Horqin District, Tongliao 028043, Inner Mongolia, China
dan-dangao@hotmail.com*

Feng Qi*

*Institute of Mathematics, Henan Polytechnic University, Jiaozuo 454010, Henan, China
School of Mathematical Sciences, Tianjin Polytechnic University, Tianjin 300387, China
qifeng618@gmail.com*

Abstract. In the paper, the authors introduce the concepts “ (α, s) -convex function” and “ (α, s, m) -convex function” and establish some new integral inequalities of the Hermite–Hadamard type for (α, s) -convex and (α, s, m) -convex functions in terms of the classical Euler beta, gamma, and polygamma functions.

Keywords: integral inequality; (α, s) -convex function; (α, s, m) -convex function; Hermite–Hadamard type; beta function; gamma function; polygamma function

1. Introduction

We first recite some definitions of various convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2 ([24]). For $f : [0, b] \rightarrow \mathbb{R}_0 = [0, \infty)$ with $b > 0$ and some $m \in (0, 1]$, if

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)$$

for $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m -convex function on $[0, b]$.

*. Corresponding author

Definition 1.3 ([12]). For $f : [0, b] \rightarrow \mathbb{R}_0$ with $b > 0$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an (α, m) -convex function on $[0, b]$.

Definition 1.4 ([4, 9]). Let $s \in (0, 1]$ be a real number. A function $f : \mathbb{R} \rightarrow \mathbb{R}_0$ is said to be s -convex (in the second sense) if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.5 ([28]). For some $s \in [-1, 1]$, a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be extended s -convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$.

Definition 1.6 ([33]). For some $(s, m) \in [-1, 1] \times (0, 1]$, a function $f : [0, b] \rightarrow \mathbb{R}_0$ is said to be extended (s, m) -convex if

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$.

The famous Hermite–Hadamard integral inequality for convex functions and some of its diverse generalizations can be reformulated as follows.

Theorem 1.1. *If $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

Theorem 1.2 ([5]). *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.$$

Theorem 1.3 ([10, Theorems 2.3 and 2.4]). *Let $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^p$ is s -convex on $[a, b]$ for some $s \in (0, 1]$ and $p > 1$, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} (|f'(a)| + |f'(b)|)$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{1/p} \left\{ [|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)}]^{1-1/p} + [3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}]^{1-1/p} \right\}.$$

For more information on integral inequalities of the Hermite–Hadamard type for various kinds of convex functions, please refer to the monograph [6, 8, 13, 14], to recently published papers [2, 3, 5, 7, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, 31, 32], and to the closely related references therein.

In this paper, we introduce two new concepts “ (α, s) -convex function” and “ (α, s, m) -convex function” and present inequalities of Hermite–Hadamard type for functions whose twice differentiation are of (α, s, m) -convexity.

2. Two definitions and two lemmas

We introduce the notions of “ (α, s) -convex function” and “ (α, s, m) -convex function”.

Definition 2.1. For some $s \in [-1, 1]$ and $\alpha \in (0, 1]$, a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be (α, s) -convex if

$$f(tx + (1-t)y) \leq t^{\alpha s} f(x) + (1-t^\alpha)^s f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$.

Definition 2.2. For some $s \in [-1, 1]$ and $(\alpha, m) \in (0, 1]^2$, a function $f : [0, b] \rightarrow \mathbb{R}$ is said to be (α, s, m) -convex if

$$f(tx + m(1-t)y) \leq t^{\alpha s} f(x) + m(1-t^\alpha)^s f(y)$$

for all $x, y \in [0, b]$ and $t \in (0, 1)$.

Remark 2.1. By Definition 2.2, we see that

1. if $s = 1$, then $f(x)$ is an (α, m) -convex function on $(0, b]$;
2. if $\alpha = 1$, then $f(x)$ is an extended (s, m) -convex function on $(0, b]$;
3. if $\alpha = m = 1$, then $f(x)$ is an extended s -convex function on $(0, b]$;
4. if $\alpha = s = m = 1$, then $f(x)$ is a convex function on $(0, b]$.

This means that Definitions (2.1) and (2.2) are significant.

For establishing new integral inequalities of the Hermite–Hadamard type for (α, s) -convex and (α, s, m) -convex functions, we need the following lemmas.

Lemma 2.1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I$ with $a < b$. If $f'' \in L_1([a, b])$, then

$$\begin{aligned} & \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx = \frac{(b-a)^2}{16} \\ & \quad \times \int_0^1 t(1-t) \left[f''\left(at + (1-t)\frac{a+b}{2}\right) + f''\left(t\frac{a+b}{2} + (1-t)b\right) \right] dt. \end{aligned}$$

Proof. Integrating by parts and changing variable of integral give

$$\begin{aligned} & \int_0^1 t(1-t) f''\left(at + (1-t)\frac{a+b}{2}\right) dt \\ &= \frac{2}{b-a} \int_0^1 (1-2t) f'\left(at + (1-t)\frac{a+b}{2}\right) dt \\ &= \frac{4}{(b-a)^2} \left[f(a) + f\left(\frac{a+b}{2}\right) \right] - \frac{8}{(b-a)^2} \int_0^1 f\left(at + (1-t)\frac{a+b}{2}\right) dt \\ &= \frac{4}{(b-a)^2} \left[f(a) + f\left(\frac{a+b}{2}\right) \right] - \frac{16}{(b-a)^3} \int_a^{(a+b)/2} f(x) dx \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 t(1-t) f''\left(t\frac{a+b}{2} + (1-t)b\right) dt \\ &= \frac{4}{(b-a)^2} \left[f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{16}{(b-a)^3} \int_{(a+b)/2}^b f(x) dx. \end{aligned}$$

The proof of Lemma 2.1 is complete. \square

Lemma 2.2. Let $\alpha \in (0, 1]$ and $s \in [-1, 1]$. Then

1. when $s \in (-1, 1]$, we have

$$\begin{aligned} (2.1) \quad M(\alpha, s) & \triangleq \int_0^1 t(1-t)(1-t^\alpha)^s dt \\ &= \frac{1}{\alpha} \left[B\left(s+1, \frac{2}{\alpha}\right) - B\left(s+1, \frac{3}{\alpha}\right) \right]; \end{aligned}$$

2. when $s = -1$, we have

$$\int_0^1 \frac{t(1-t)}{1-t^\alpha} dt = \frac{1}{\alpha} \left[\psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right) \right],$$

where $\Gamma(x)$, $B(x, y)$, and $\psi(x)$ are the classical Euler gamma, beta, and psi functions defined respectively by

$$(2.2) \quad \Gamma(x) = \int_0^1 t^{x-1} e^{-t} dt, \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

and

$$(2.3) \quad \psi(x) = \frac{d \ln \Gamma(x)}{dx} = \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right] dt$$

for $\Re(x), \Re(y) > 0$.

Proof. Let $u = t^\alpha$ for $t \in [0, 1]$. If $s \in (-1, 1]$, we have

$$\begin{aligned} M(\alpha, s) &= \int_0^1 t(1-t)(1-t^\alpha)^s dt \\ &= \frac{1}{\alpha} \int_0^1 (u^{2/\alpha-1} - u^{3/\alpha-1})(1-u)^s du = \frac{1}{\alpha} \left[B\left(s+1, \frac{2}{\alpha}\right) - B\left(s+1, \frac{3}{\alpha}\right) \right]. \end{aligned}$$

When $s = -1$, from the formulas

$$\psi(z) + \gamma = \int_0^1 \frac{1 - tz^{-1}}{1 - t} dt, \quad \text{and} \quad \gamma = \int_0^\infty \left(\frac{1}{1+t} - e^{-t} \right) \frac{dt}{t}$$

in [1, p. 259, 6.3.22], it is easily to deduce

$$\int_0^1 \frac{t(1-t)}{1-t^\alpha} dt = \frac{1}{\alpha} \int_0^1 \frac{u^{2/\alpha-1} - u^{3/\alpha-1}}{1-u} du = \frac{1}{\alpha} \left[\psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right) \right].$$

The proof of Lemma 2.2 is complete. □

3. Integral inequalities of Hermite–Hadamard type

Now we start out to establish some new integral inequalities of the Hermite–Hadamard type for (α, s, m) -convex functions.

Theorem 3.1. For $(\alpha, m) \in (0, 1]^2$ and $s \in (-1, 1]$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $q \geq 1$, then

$$(3.1) \quad \begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \\ & \times \left\{ \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + mM(\alpha, s) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \left. + \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + mM(\alpha, s) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $M(\alpha, s)$ is defined by (2.1).

Proof. By Lemma 2.1 and Hölder’s integral inequality, we obtain

$$\begin{aligned}
 (3.2) \quad & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{16} \left[\int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right| dt \right. \\
 & \quad \left. + \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \leq \frac{(b-a)^2}{16} \\
 & \times \left\{ \left[\int_0^1 t(1-t) dt \right]^{1-1/q} \left[\int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \right]^{1/q} \right. \\
 & \quad \left. + \left(\int_0^1 t(1-t) dt \right)^{1-1/q} \left[\int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \right]^{1/q} \right\}.
 \end{aligned}$$

From the (α, s, m) -convexity of $|f''|^q$ and Lemma 2.2, we arrive at

$$\begin{aligned}
 & \int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \\
 & \leq \int_0^1 t(1-t) \left(t^{\alpha s} |f''(a)|^q + m(1-t)^{\alpha s} \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right) dt \\
 & = \frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + mM(\alpha, s) \left| f''\left(\frac{a+b}{2m}\right) \right|^q
 \end{aligned}$$

and

$$\begin{aligned}
 (3.3) \quad & \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \\
 & \leq \frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + mM(\alpha, s) \left| f''\left(\frac{b}{m}\right) \right|^q.
 \end{aligned}$$

By the inequalities between (3.2) and (3.3), we conclude the inequality (3.1). The proof of Theorem 3.1 is complete. \square

Corollary 3.1. For $\alpha \in (0, 1]$ and $s \in (-1, 1]$, let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an (α, s) -convex function on I for $q \geq 1$, then

$$\begin{aligned}
 & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + M(\alpha, s) \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\
 & \quad \left. + \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + M(\alpha, s) |f''(b)|^q \right]^{1/q} \right\},
 \end{aligned}$$

where $M(\alpha, s)$ is defined by (2.1).

Proof. This is a special case of Theorem 3.1 for $m = 1$. \square

Theorem 3.2. For $(\alpha, m) \in (0, 1]^2$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is $(\alpha, -1, m)$ -convex function on $(0, \frac{b^*}{m}]$ for $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{|f''(a)|^q}{(2-\alpha)(3-\alpha)} + \frac{m[\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})]}{\alpha} \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{(2-\alpha)(3-\alpha)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{m[\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})]}{\alpha} \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $\psi(x)$ is defined by (2.3).

Proof. This follows from similar argument to Theorem 3.1. □

Corollary 3.2. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an $(\alpha, -1)$ -convex function on I for $\alpha \in (0, 1]$ and $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{1}{(2-\alpha)(3-\alpha)} |f''(a)|^q + \frac{\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})}{\alpha} \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{(2-\alpha)(3-\alpha)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})}{\alpha} |f''(b)|^q \right]^{1/q} \right\}, \end{aligned}$$

where $\psi(x)$ is defined by (2.3).

Proof. This is a special case of Theorem 3.2 for $m = 1$. □

Theorem 3.3. For $(\alpha, m) \in (0, 1]^2$, $s \in (-1, 1]$, and $\ell \geq 0$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $q > 1$ and $q \geq \ell$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left[B\left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \\ & \quad \times \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{\alpha s + \ell + 1} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. By Lemma 2.1, Hölder’s integral inequality, and the (α, s, m) -convexity of $|f''|^q$, one has

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\int_0^1 t^{(q-\ell)/(q-1)} (1-t)^{q/(q-1)} dt \right)^{1-1/q} \left\{ \left[\int_0^1 t^\ell \left| f'' \left(at + (1-t)\frac{a+b}{2} \right) \right|^q dt \right]^{1/q} + \left[\int_0^1 t^\ell \left| f'' \left(t\frac{a+b}{2} + (1-t)b \right) \right|^q dt \right]^{1/q} \right\} \\ & \leq \frac{(b-a)^2}{16} \left(\int_0^1 t^{(q-\ell)/(q-1)} (1-t)^{q/(q-1)} dt \right)^{1-1/q} \\ & \quad \times \left\{ \left[\int_0^1 t^\ell \left(t^{\alpha s} |f''(a)|^q + m(1-t^\alpha)^s \left| f'' \left(\frac{a+b}{2m} \right) \right|^q \right) dt \right]^{1/q} + \left[\int_0^1 t^\ell \left(t^{\alpha s} \left| f'' \left(\frac{a+b}{2} \right) \right|^q + m(1-t^\alpha)^s \left| f'' \left(\frac{b}{m} \right) \right|^q \right) dt \right]^{1/q} \right\} \\ & = \frac{(b-a)^2}{16} \left[B \left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q} \\ & \quad \times \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B \left(s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left(\frac{a+b}{2m} \right) \right|^q \right]^{1/q} + \left[\frac{1}{\alpha s + \ell + 1} \left| f'' \left(\frac{a+b}{2} \right) \right|^q + \frac{m}{\alpha} B \left(s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \right\}. \end{aligned}$$

The proof of Theorem 3.3 is thus complete. □

Corollary 3.3. Under assumptions of Theorem 3.3, if $\ell = 0$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{16} \\ & \quad \times \left[B \left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q} \left\{ \left[\frac{1}{\alpha s + 1} |f''(a)|^q + \frac{m}{\alpha} B \left(s+1, \frac{1}{\alpha} \right) \right. \right. \\ & \quad \left. \left. \times \left| f'' \left(\frac{a+b}{2m} \right) \right|^q \right]^{1/q} + \left[\frac{|f''(a+b/2)|^q}{\alpha s + 1} + \frac{m}{\alpha} B \left(s+1, \frac{1}{\alpha} \right) \left| f'' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Corollary 3.4. For $\alpha \in (0, 1]$, $s \in (-1, 1]$, and $\ell \geq 0$, let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an (α, s) -convex function on I for $q \geq 1$ and $q \geq \ell$, then

$$\left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{16} \left[B \left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q} \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B \left(s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left(\frac{a+b}{2m} \right) \right|^q \right]^{1/q} + \left[\frac{1}{\alpha s + \ell + 1} \left| f'' \left(\frac{a+b}{2} \right) \right|^q + \frac{m}{\alpha} B \left(s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \right\}.$$

$$\frac{2q-1}{q-1} \Big)^{1-1/q} \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{1}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[\frac{1}{\alpha s + \ell + 1} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{1}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) |f''(b)|^q \right]^{1/q} \right\},$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. This is a special case of Theorem 3.3 for $m = 1$. □

Theorem 3.4. For $(\alpha, m) \in (0, 1]^2$ and $s \in (-1, 1]$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $b^* > 0$. If $f \in L_1([a, b])$ for $a, b \in (0, b^*]$ and $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + m(2^\alpha - 1)^s f\left(\frac{x}{m}\right)}{2^{\alpha s}} dx$$

and

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \max \left\{ \frac{1}{\alpha s + 1} f(a) + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{b}{m}\right), \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{a}{m}\right) + \frac{1}{\alpha s + 1} f(b) \right\},$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. By the (α, s, m) -convexity of f , we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{\alpha s}} \int_0^1 \left[f(ta + (1-t)b) + m(2^\alpha - 1)^s f\left(\frac{(1-t)a + tb}{m}\right) \right] dt.$$

Letting $x = ta + (1-t)b$ or $x = (1-t)a + tb$ for all $t \in [0, 1]$ leads to

$$\int_0^1 f(c[ta + (1-t)b]) dt = \int_0^1 f(c[(1-t)a + tb]) dt = \frac{1}{b-a} \int_a^b f(cx) dx$$

for $c \in \mathbb{R}$.

On the other hand, when putting $x = ta + (1-t)b$ for all $t \in [0, 1]$, by the (α, s, m) -convexity of f , we can gain

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) dx &\leq \int_0^1 \left[t^{\alpha s} f(a) + m(1-t^\alpha)^s f\left(\frac{b}{m}\right) \right] dt \\ &= \frac{1}{\alpha s + 1} f(a) + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{b}{m}\right). \end{aligned}$$

The proof of Theorem 3.4 is complete. □

Corollary 3.5. *Under conditions of Theorem 3.4, if $m = 1$, then*

$$\begin{aligned} \frac{2^{\alpha s}}{(2^\alpha - 1)^s + 1} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \max \left\{ \frac{f(a)}{\alpha s + 1} + \frac{1}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f(b), \frac{1}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f(a) + \frac{f(b)}{\alpha s + 1} \right\}. \end{aligned}$$

Remark 3.1. This paper is a corrected and revised version of the preprint [27].

Acknowledgements

This work was partially supported by the National Natural Science Foundation of China (Grant No. 11361038).

The authors are thankful to Jamshed Nasir in Pakistan for his careful corrections to and valuable comments on the original version of this paper.

References

- [1] M. Abramowitz and I. A. Stegun (Eds), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, National Bureau of Standards, Applied Mathematics Series **55**, 10th printing, Dover Publications, New York and Washington, 1972.
- [2] M. Alomari, M. Darus, and U. S. Kirmaci, *Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means*, *Comput. Math. Appl.* **59** (2010), no. 1, 225–232; available online at <https://doi.org/10.1016/j.camwa.2009.08.002>.
- [3] R.-F. Bai, F. Qi, and B.-Y. Xi, *Hermite–Hadamard type inequalities for the m - and (α, m) -logarithmically convex functions*, *Filomat* **27** (2013), no. 1, 1–7; available online at <https://doi.org/10.2298/FIL1301001B>.
- [4] W. W. Breckner, *Stetigkeitsaussagen für eine Klasse verallgemeinerter konvexer Funktionen in topologischen linearen Räumen*, *Publ. Inst. Math. (Beograd) (N.S.)* **23(37)** (1978), 13–20.
- [5] S. S. Dragomir and R. P. Agarwal, *Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula*, *Appl. Math. Lett.* **11** (1998), no. 5, 91–95; available online at [https://doi.org/10.1016/S0893-9659\(98\)00086-X](https://doi.org/10.1016/S0893-9659(98)00086-X).
- [6] S. S. Dragomir and C. E. M. Pearce, *Selected Topics on Hermite-Hadamard Type Inequalities and Applications*, RGMIA Monographs, Victoria University, 2000; available online at http://rgmia.org/monographs/hermite_hadamard.html.

- [7] S. Erden and M. Z. Sarikaya, *New Hermite Hadamard type inequalities for twice differentiable convex mappings via Green function and applications*, Moroccan J. Pure and Appl. Anal. **2** (2016), no. 2, 107–117.
- [8] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1934.
- [9] H. Hudzik and L. Maligranda, *Some remarks on s -convex functions*, Aequationes Math. **48** (1994), no. 1, 100–111; available online at <https://doi.org/10.1007/BF01837981>.
- [10] U. S. Kirmaci, *Inequalities for differentiable mappings and applications to special means of real numbers to midpoint formula*, Appl. Math. Comp. **147** (2004), no. 1, 137–146; available online at [https://doi.org/10.1016/S0096-3003\(02\)00657-4](https://doi.org/10.1016/S0096-3003(02)00657-4).
- [11] M. A. Latif, S. S. Dragomir, and E. Momoniat, *Weighted generalization of some integral inequalities for differentiable co-ordinated convex functions*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **78** (2016), no. 4, 197–210.
- [12] V. G. Miheşan, *A generalization of the convexity*, Seminar on Functional Equations, Approx. Convex, Cluj-Napoca, 1993. (Romania)
- [13] C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications: A Contemporary Approach*, CMS Books in Mathematics, Springer-Verlag, 2006; available online at <https://doi.org/10.1007/0-387-31077-0>.
- [14] C. P. Niculescu and L.-E. Persson, *Convex Functions and Their Applications: A Contemporary Approach*, Second edition, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2018; available online at <https://doi.org/10.1007/978-3-319-78337-6>.
- [15] M. A. Noor, M. U. Awan, K. I. Noor, and M. Postolache, *Some integral inequalities for p -convex functions*, Filomat **30** (2016), no. 9, 2435–2444; available online at <https://doi.org/10.2298/FIL1609435N>.
- [16] M. A. Noor, K. I. Noor, and M. U. Awan, *Some characterizations of harmonically log-convex functions*, Proc. Jangjeon Math. Soc. **17** (2014), no. 1, 51–61.
- [17] M. A. Noor, K. I. Noor, M. U. Awan, and S. Costache, *Some integral inequalities for harmonically h -convex functions*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **77** (2015), no. 1, 5–16.
- [18] M. E. Özdemir, M. A. Latif, and A. O. Akdemir, *On some Hadamard-type inequalities for product of two s -convex functions on the co-ordinates*, J.

- Inequal. Appl. **2012**, 2012:21, 13 pages; available online at <https://doi.org/10.1186/1029-242X-2012-21>.
- [19] C. E. M. Pearce and J. Pečarić, *Inequalities for differentiable mappings with application to special means and quadrature formulae*, Appl. Math. Lett. **13** (2000), no. 2, 51–55; available online at [https://doi.org/10.1016/S0893-9659\(99\)00164-0](https://doi.org/10.1016/S0893-9659(99)00164-0).
- [20] F. Qi, Z.-L. Wei, and Q. Yang, *Generalizations and refinements of Hermite-Hadamard's inequality*, Rocky Mountain J. Math. **35** (2005), no. 1, 235–251; available online at <https://doi.org/10.1216/rmjm/1181069779>.
- [21] F. Qi and B.-Y. Xi, *Some integral inequalities of Simpson type for GA- ε -convex functions*, Georgian Math. J. **20** (2013), no. 4, 775–788; available online at <https://doi.org/10.1515/gmj-2013-0043>.
- [22] M. Z. Sarikaya, E. Set, and M. E. Özdemir, *On new inequalities of Simpson's type for s-convex functions*, Comput. Math. Appl. **60** (2010), no. 8, 2191–2199; available online at <https://doi.org/10.1016/j.camwa.2010.07.033>.
- [23] J.-F. Tian and M.-H. Ha, *Properties of generalized sharp Hölder's inequalities*, J. Math. Inequal. **11** (2017), no. 2, 511–525; available online at <https://doi.org/10.7153/jmi-11-42>.
- [24] G. Toader, *Some generalizations of the convexity*, Proc. Colloq. Approx. Optim., Univ. Cluj-Napoca, Cluj-Napoca, 1985, 329–338.
- [25] S.-H. Wu, *On the weighted generalization of the Hermite-Hadamard inequality and its applications*, Rocky Mountain J. Math. **39** (2009), no. 5, 1741–1749; available online at <https://doi.org/10.1216/RMJ-2009-39-5-1741>.
- [26] B.-Y. Xi, R.-F. Bai, and F. Qi, *Hermite-Hadamard type inequalities for the m - and (α, m) -geometrically convex functions*, Aequationes Math. **84** (2012), no. 3, 261–269; available online at <https://doi.org/10.1007/s00010-011-0114-x>.
- [27] B.-Y. Xi, D.-D. Gao, and F. Qi, *Integral inequalities of Hermite-Hadamard type for (α, s) -convex and (α, s, m) -convex functions*, HAL preprint (2018), available online at <https://hal.archives-ouvertes.fr/hal-01761678>.
- [28] B.-Y. Xi and F. Qi, *Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means*, J. Nonlinear Convex Anal. **16** (2015), no. 5, 873–890.
- [29] B.-Y. Xi and F. Qi, *Some Hermite-Hadamard type inequalities for differentiable convex functions and applications*, Hacet. J. Math. Stat. **42** (2013), no. 3, 243–257.

- [30] B.-Y. Xi and F. Qi, *Some integral inequalities of Hermite–Hadamard type for s -logarithmically convex functions*, Acta Math. Sci. Ser. A Chin. Ed. **35** (2015), no. 3, 515–524. (Chinese)
- [31] B.-Y. Xi and F. Qi, *Hermite–Hadamard type inequalities for geometrically r -convex functions*, Studia Sci. Math. Hungar. **51** (2014), no. 4, 530–546; available online at <https://doi.org/10.1556/SScMath.51.2014.4.1294>.
- [32] B.-Y. Xi, F. Qi, and T.-Y. Zhang, *Some inequalities of Hermite–Hadamard type for m -harmonic-arithmetically convex functions*, ScienceAsia **41** (2015), no. 5, 357–361; available online at <https://doi.org/10.2306/scienceasia1513-1874.2015.41.357>.
- [33] B.-Y. Xi, Y. Wang, and F. Qi, *Some integral inequalities of Hermite–Hadamard type for extended (s, m) -convex functions*, Transylv. J. Math. Mechanics **5** (2013), no. 1, 69–84.

Accepted: October 31, 2019