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Integral inequalities of Hermite–Hadamard type for (α, s) -convex and (α, s, m) -convex functions

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Abstract. In the paper, the authors introduce the concepts “ (α, s) -convex function” and “ (α, s, m) -convex function” and establish some new integral inequalities of the Hermite–Hadamard type for (α, s) -convex and (α, s, m) -convex functions in terms of the classical Euler beta, gamma, and polygamma functions.

Keywords: integral inequality; (α, s) -convex function; (α, s, m) -convex function; Hermite–Hadamard type; beta function; gamma function; polygamma function

1. Introduction

We first recite some definitions of various convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2 ([24]). For $f : [0, b] \rightarrow \mathbb{R}_0 = [0, \infty)$ with $b > 0$ and some $m \in (0, 1]$, if

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)$$

for $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m -convex function on $[0, b]$.

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Definition 1.3 ([12]). For $f : [0, b] \rightarrow \mathbb{R}_0$ with $b > 0$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an (α, m) -convex function on $[0, b]$.

Definition 1.4 ([4, 9]). Let $s \in (0, 1]$ be a real number. A function $f : \mathbb{R} \rightarrow \mathbb{R}_0$ is said to be s -convex (in the second sense) if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.5 ([28]). For some $s \in [-1, 1]$, a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be extended s -convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$.

Definition 1.6 ([33]). For some $(s, m) \in [-1, 1] \times (0, 1]$, a function $f : [0, b] \rightarrow \mathbb{R}_0$ is said to be extended (s, m) -convex if

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$.

The famous Hermite–Hadamard integral inequality for convex functions and some of its diverse generalizations can be reformulated as follows.

Theorem 1.1. If $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

Theorem 1.2 ([5]). Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.$$

Theorem 1.3 ([10, Theorems 2.3 and 2.4]). Let $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^p$ is s -convex on $[a, b]$ for some $s \in (0, 1]$ and $p > 1$, then

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1} \right)^{1/p} (|f'(a)| + |f'(b)|)$$

and

$$\begin{aligned} \left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| &\leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{1/p} \left\{ [|f'(a)|^{p/(p-1)} \right. \\ &\quad \left. + 3|f'(b)|^{p/(p-1)}]^{1-1/p} + [3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}]^{1-1/p} \right\}. \end{aligned}$$

For more information on integral inequalities of the Hermite–Hadamard type for various kinds of convex functions, please refer to the monograph [6, 8, 13, 14], to recently published papers [2, 3, 5, 7, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, 31, 32], and to the closely related references therein.

In this paper, we introduce two new concepts “ (α, s) -convex function” and “ (α, s, m) -convex function” and present inequalities of Hermite–Hadamard type for functions whose twice differentiation are of (α, s, m) -convexity.

2. Two definitions and two lemmas

We introduce the notions of “ (α, s) -convex function” and “ (α, s, m) -convex function”.

Definition 2.1. For some $s \in [-1, 1]$ and $\alpha \in (0, 1]$, a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be (α, s) -convex if

$$f(tx + (1-t)y) \leq t^{\alpha s} f(x) + (1-t^\alpha)^s f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$.

Definition 2.2. For some $s \in [-1, 1]$ and $(\alpha, m) \in (0, 1]^2$, a function $f : [0, b] \rightarrow \mathbb{R}$ is said to be (α, s, m) -convex if

$$f(tx + m(1-t)y) \leq t^{\alpha s} f(x) + m(1-t^\alpha)^s f(y)$$

for all $x, y \in [0, b]$ and $t \in (0, 1)$.

Remark 2.1. By Definition 2.2, we see that

1. if $s = 1$, then $f(x)$ is an (α, m) -convex function on $(0, b]$;
2. if $\alpha = 1$, then $f(x)$ is an extended (s, m) -convex function on $(0, b]$;
3. if $\alpha = m = 1$, then $f(x)$ is an extended s -convex function on $(0, b]$;
4. if $\alpha = s = m = 1$, then $f(x)$ is a convex function on $(0, b]$.

This means that Definitions (2.1) and (2.2) are significant.

For establishing new integral inequalities of the Hermite–Hadamard type for (α, s) -convex and (α, s, m) -convex functions, we need the following lemmas.

Lemma 2.1. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I$ with $a < b$. If $f'' \in L_1([a, b])$, then

$$\begin{aligned} \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx &= \frac{(b-a)^2}{16} \\ &\times \int_0^1 t(1-t) \left[f''\left(at + (1-t)\frac{a+b}{2}\right) + f''\left(t\frac{a+b}{2} + (1-t)b\right) \right] dt. \end{aligned}$$

Proof. Integrating by parts and changing variable of integral give

$$\begin{aligned} &\int_0^1 t(1-t)f''\left(at + (1-t)\frac{a+b}{2}\right) dt \\ &= \frac{2}{b-a} \int_0^1 (1-2t)f'\left(at + (1-t)\frac{a+b}{2}\right) dt \\ &= \frac{4}{(b-a)^2} \left[f(a) + f\left(\frac{a+b}{2}\right) \right] - \frac{8}{(b-a)^2} \int_0^1 f\left(at + (1-t)\frac{a+b}{2}\right) dt \\ &= \frac{4}{(b-a)^2} \left[f(a) + f\left(\frac{a+b}{2}\right) \right] - \frac{16}{(b-a)^3} \int_a^{(a+b)/2} f(x) dx \end{aligned}$$

and

$$\begin{aligned} &\int_0^1 t(1-t)f''\left(t\frac{a+b}{2} + (1-t)b\right) dt \\ &= \frac{4}{(b-a)^2} \left[f\left(\frac{a+b}{2} + f(b)\right) \right] - \frac{16}{(b-a)^3} \int_{(a+b)/2}^b f(x) dx. \end{aligned}$$

The proof of Lemma 2.1 is complete. \square

Lemma 2.2. Let $\alpha \in (0, 1]$ and $s \in [-1, 1]$. Then

1. when $s \in (-1, 1]$, we have

$$\begin{aligned} M(\alpha, s) &\triangleq \int_0^1 t(1-t)(1-t^\alpha)^s dt \\ (2.1) \quad &= \frac{1}{\alpha} \left[B\left(s+1, \frac{2}{\alpha}\right) - B\left(s+1, \frac{3}{\alpha}\right) \right]; \end{aligned}$$

2. when $s = -1$, we have

$$\int_0^1 \frac{t(1-t)}{1-t^\alpha} dt = \frac{1}{\alpha} \left[\psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right) \right],$$

where $\Gamma(x)$, $B(x, y)$, and $\psi(x)$ are the classical Euler gamma, beta, and psi functions defined respectively by

$$(2.2) \quad \Gamma(x) = \int_0^1 t^{x-1} e^{-t} dt, \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

and

$$(2.3) \quad \psi(x) = \frac{d \ln \Gamma(x)}{dx} = \int_0^\infty \left[\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}} \right] dt$$

for $\Re(x), \Re(y) > 0$.

Proof. Let $u = t^\alpha$ for $t \in [0, 1]$. If $s \in (-1, 1]$, we have

$$\begin{aligned} M(\alpha, s) &= \int_0^1 t(1-t)(1-t^\alpha)^s dt \\ &= \frac{1}{\alpha} \int_0^1 (u^{2/\alpha-1} - u^{3/\alpha-1})(1-u)^s du = \frac{1}{\alpha} \left[B\left(s+1, \frac{2}{\alpha}\right) - B\left(s+1, \frac{3}{\alpha}\right) \right]. \end{aligned}$$

When $s = -1$, from the formulas

$$\psi(z) + \gamma = \int_0^1 \frac{1-t^{z-1}}{1-t} dt, \quad \text{and} \quad \gamma = \int_0^\infty \left(\frac{1}{1+t} - e^{-t} \right) \frac{dt}{t}$$

in [1, p. 259, 6.3.22], it is easily to deduce

$$\int_0^1 \frac{t(1-t)}{1-t^\alpha} dt = \frac{1}{\alpha} \int_0^1 \frac{u^{2/\alpha-1} - u^{3/\alpha-1}}{1-u} du = \frac{1}{\alpha} \left[\psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right) \right].$$

The proof of Lemma 2.2 is complete. \square

3. Integral inequalities of Hermite–Hadamard type

Now we start out to establish some new integral inequalities of the Hermite–Hadamard type for (α, s, m) -convex functions.

Theorem 3.1. For $(\alpha, m) \in (0, 1]^2$ and $s \in (-1, 1]$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $q \geq 1$, then

$$(3.1) \quad \begin{aligned} &\left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \\ &\times \left\{ \left[\frac{1}{(\alpha s+2)(\alpha s+3)} |f''(a)|^q + m M(\alpha, s) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ &\left. + \left[\frac{1}{(\alpha s+2)(\alpha s+3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + m M(\alpha, s) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $M(\alpha, s)$ is defined by (2.1).

Proof. By Lemma 2.1 and Hölder's integral inequality, we obtain

$$\begin{aligned}
(3.2) \quad & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{(b-a)^2}{16} \left[\int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right| dt \right. \\
& \quad \left. + \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \leq \frac{(b-a)^2}{16} \\
& \times \left\{ \left[\int_0^1 t(1-t) dt \right]^{1-1/q} \left[\int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \right]^{1/q} \right. \\
& \quad \left. + \left(\int_0^1 t(1-t) dt \right)^{1-1/q} \left[\int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \right]^{1/q} \right\}.
\end{aligned}$$

From the (α, s, m) -convexity of $|f''|^q$ and Lemma 2.2, we arrive at

$$\begin{aligned}
& \int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \\
& \leq \int_0^1 t(1-t) \left(t^{\alpha s} |f''(a)|^q + m(1-t^\alpha)^s \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right) dt \\
& = \frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + m M(\alpha, s) \left| f''\left(\frac{a+b}{2m}\right) \right|^q
\end{aligned}$$

and

$$\begin{aligned}
(3.3) \quad & \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \\
& \leq \frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + m M(\alpha, s) \left| f''\left(\frac{b}{m}\right) \right|^q.
\end{aligned}$$

By the inequalities between (3.2) and (3.3), we conclude the inequality (3.1). The proof of Theorem 3.1 is complete. \square

Corollary 3.1. *For $\alpha \in (0, 1]$ and $s \in (-1, 1]$, let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an (α, s) -convex function on I for $q \geq 1$, then*

$$\begin{aligned}
& \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + M(\alpha, s) \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\
& \quad \left. + \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + M(\alpha, s) |f''(b)|^q \right]^{1/q} \right\},
\end{aligned}$$

where $M(\alpha, s)$ is defined by (2.1).

Proof. This is a special case of Theorem 3.1 for $m = 1$. \square

Theorem 3.2. For $(\alpha, m) \in (0, 1]^2$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is $(\alpha, -1, m)$ -convex function on $(0, \frac{b^*}{m}]$ for $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{|f''(a)|^q}{(2-\alpha)(3-\alpha)} + \frac{m[\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})]}{\alpha} \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{(2-\alpha)(3-\alpha)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{m[\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})]}{\alpha} \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $\psi(x)$ is defined by (2.3).

Proof. This follows from similar argument to Theorem 3.1. \square

Corollary 3.2. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an $(\alpha, -1)$ -convex function on I for $\alpha \in (0, 1]$ and $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{1}{(2-\alpha)(3-\alpha)} |f''(a)|^q + \frac{\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})}{\alpha} \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{(2-\alpha)(3-\alpha)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})}{\alpha} |f''(b)|^q \right]^{1/q} \right\}, \end{aligned}$$

where $\psi(x)$ is defined by (2.3).

Proof. This is a special case of Theorem 3.2 for $m = 1$. \square

Theorem 3.3. For $(\alpha, m) \in (0, 1]^2$, $s \in (-1, 1]$, and $\ell \geq 0$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $q > 1$ and $q \geq \ell$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left[B\left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \\ & \quad \times \left\{ \left[\frac{1}{as+\ell+1} |f''(a)|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{as+\ell+1} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. By Lemma 2.1, Hölder's integral inequality, and the (α, s, m) -convexity of $|f''|^q$, one has

$$\begin{aligned}
& \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
& \leq \frac{(b-a)^2}{16} \left(\int_0^1 t^{(q-\ell)/(q-1)} (1-t)^{q/(q-1)} dt \right)^{1-1/q} \left\{ \left[\int_0^1 t^\ell \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \right]^{1/q} \right. \\
& \quad \left. + \left[\int_0^1 t^\ell \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \right]^{1/q} \right\} \\
& \leq \frac{(b-a)^2}{16} \left(\int_0^1 t^{(q-\ell)/(q-1)} (1-t)^{q/(q-1)} dt \right)^{1-1/q} \\
& \quad \times \left\{ \left[\int_0^1 t^\ell \left(t^{\alpha s} |f''(a)|^q + m(1-t^\alpha)^s |f''\left(\frac{a+b}{2m}\right)|^q \right) dt \right]^{1/q} \right. \\
& \quad \left. + \left[\int_0^1 t^\ell \left(t^{\alpha s} \left| f''\left(\frac{a+b}{2}\right) \right|^q + m(1-t^\alpha)^s \left| f''\left(\frac{b}{m}\right) \right|^q \right) dt \right]^{1/q} \right\} \\
& = \frac{(b-a)^2}{16} \left[B\left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \\
& \quad \times \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\
& \quad \left. + \left[\frac{1}{\alpha s + \ell + 1} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}.
\end{aligned}$$

The proof of Theorem 3.3 is thus complete. \square

Corollary 3.3. Under assumptions of Theorem 3.3, if $\ell = 0$, then

$$\begin{aligned}
& \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{16} \\
& \quad \times \left[B\left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \left\{ \left[\frac{1}{\alpha s + 1} |f''(a)|^q + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) \right. \right. \\
& \quad \left. \left. \times \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} + \left[\frac{|f''(\frac{a+b}{2})|^q}{\alpha s + 1} + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\},
\end{aligned}$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Corollary 3.4. For $\alpha \in (0, 1]$, $s \in (-1, 1]$, and $\ell \geq 0$, let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an (α, s) -convex function on I for $q \geq 1$ and $q \geq \ell$, then

$$\left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{16} \left[B\left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} + \left[\frac{|f''(\frac{a+b}{2})|^q}{\alpha s + 1} + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\},$$

$$\left[\frac{2q-1}{q-1} \right]^{1-1/q} \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{1}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\ \left. + \left[\frac{1}{\alpha s + \ell + 1} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{1}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) |f''(b)|^q \right]^{1/q} \right\},$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. This is a special case of Theorem 3.3 for $m = 1$. \square

Theorem 3.4. For $(\alpha, m) \in (0, 1]^2$ and $s \in (-1, 1]$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $b^* > 0$. If $f \in L_1([a, b])$ for $a, b \in (0, b^*]$ and $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + m(2^\alpha - 1)^s f\left(\frac{x}{m}\right)}{2^{\alpha s}} dx$$

and

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \max \left\{ \frac{1}{\alpha s + 1} f(a) + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{b}{m}\right), \right. \\ \left. \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{a}{m}\right) + \frac{1}{\alpha s + 1} f(b) \right\},$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. By the (α, s, m) -convexity of f , we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{\alpha s}} \int_0^1 \left[f(ta + (1-t)b) + m(2^\alpha - 1)^s f\left(\frac{(1-t)a + tb}{m}\right) \right] dt.$$

Letting $x = ta + (1-t)b$ or $x = (1-t)a + tb$ for all $t \in [0, 1]$ leads to

$$\int_0^1 f(c[ta + (1-t)b]) dt = \int_0^1 f(c[(1-t)a + tb]) dt = \frac{1}{b-a} \int_a^b f(cx) dx$$

for $c \in \mathbb{R}$.

On the other hand, when putting $x = ta + (1-t)b$ for all $t \in [0, 1]$, by the (α, s, m) -convexity of f , we can gain

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \int_0^1 \left[t^{\alpha s} f(a) + m(1-t^\alpha)^s f\left(\frac{b}{m}\right) \right] dt \\ = \frac{1}{\alpha s + 1} f(a) + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{b}{m}\right).$$

The proof of Theorem 3.4 is complete. \square

Corollary 3.5. *Under conditions of Theorem 3.4, if $m = 1$, then*

$$\begin{aligned} \frac{2^{\alpha s}}{(2^\alpha - 1)^s + 1} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \max \left\{ \frac{f(a)}{\alpha s + 1} + \frac{1}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f(b), \frac{1}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f(a) + \frac{f(b)}{\alpha s + 1} \right\}. \end{aligned}$$

Remark 3.1. This paper is a corrected and revised version of the preprint [27].

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