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► **To cite this version:**

Bo-Yan Xi, Dan-Dan Gao, Feng Qi. INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR (α, s) -CONVEX AND (α, s, m) -CONVEX FUNCTIONS. 2018. hal-01761678v1

HAL Id: hal-01761678

<https://hal.science/hal-01761678v1>

Preprint submitted on 9 Apr 2018 (v1), last revised 14 Nov 2020 (v2)

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INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE FOR (α, s) -CONVEX AND (α, s, m) -CONVEX FUNCTIONS

BO-YAN XI, DAN-DAN GAO, AND FENG QI

ABSTRACT. In the paper, the authors introduce the concepts “ (α, s) -convex function” and “ (α, s, m) -convex function” and establish some new integral inequalities of the Hermite–Hadamard type for (α, s) -convex and (α, s, m) -convex functions in terms of the classical Euler beta, gamma, and polygamma functions.

1. INTRODUCTION

We first recite some definitions of various convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all $x, y \in I$ and $\lambda \in [0, 1]$.

Definition 1.2 ([22]). For $f : [0, b] \rightarrow \mathbb{R}_0 = [0, \infty)$ with $b > 0$ and some $m \in (0, 1]$, if

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)$$

for $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that f is an m -convex function on $[0, b]$.

Definition 1.3 ([11]). For $f : [0, b] \rightarrow \mathbb{R}_0$ with $b > 0$ and $(\alpha, m) \in (0, 1]^2$, if

$$f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y)$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$, then we say that $f(x)$ is an (α, m) -convex function on $[0, b]$.

Definition 1.4 ([3, 8]). Let $s \in (0, 1]$ be a real number. A function $f : \mathbb{R} \rightarrow \mathbb{R}_0$ is said to be s -convex (in the second sense) if

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.5 ([25]). For some $s \in [-1, 1]$, a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be extended s -convex if

$$f(tx + (1 - t)y) \leq t^s f(x) + (1 - t)^s f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$.

Definition 1.6 ([30]). For some $(s, m) \in [-1, 1] \times (0, 1]$, a function $f : [0, b] \rightarrow \mathbb{R}_0$ is said to be extended (s, m) -convex if

$$f(tx + m(1 - t)y) \leq t^s f(x) + m(1 - t)^s f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$.

2010 *Mathematics Subject Classification.* Primary 26D15; Secondary 26A51, 33B15, 41A55.

Key words and phrases. integral inequality; (α, s) -convex function; (α, s, m) -convex function; Hermite–Hadamard type; beta function; gamma function; polygamma function.

This paper was typeset using $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$.

The famous Hermite–Hadamard integral inequality for convex functions and some of its diverse generalizations can be reformulated as follows.

Theorem 1.1. *If $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is a convex function on the interval I and $a, b \in I$ with $a < b$, then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{f(a)+f(b)}{2}.$$

Theorem 1.2 ([4]). *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a, b]$, then*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{(b-a)(|f'(a)|+|f'(b)|)}{8}.$$

Theorem 1.3 ([9, Theorems 2.3 and 2.4]). *Let $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$ be differentiable on I° and $a, b \in I$ with $a < b$. If $|f'(x)|^p$ is s -convex on $[a, b]$ for some $s \in (0, 1]$ and $p > 1$, then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} (|f'(a)|+|f'(b)|)$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{1/p} \left\{ [|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)}]^{1-1/p} + [3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}]^{1-1/p} \right\}.$$

For more information on integral inequalities of the Hermite–Hadamard type for various kinds of convex functions, please refer to the monograph [5, 7, 12], to recently published papers [1, 2, 4, 6, 10, 13, 14, 15, 16, 17, 18, 19, 20, 21, 23, 24, 25, 26, 27, 28, 29], and to the closely related references therein.

In this paper, we introduce two new concepts “ (α, s) -convex function” and “ (α, s, m) -convex function” and establish some integral inequalities of the Hermite–Hadamard type for functions whose twice differentiation are of (α, s, m) -convexity.

2. TWO DEFINITIONS AND TWO LEMMAS

We introduce the notions “ (α, s) -convex function” and “ (α, s, m) -convex function”.

Definition 2.1. For some $s \in [-1, 1]$ and $\alpha \in (0, 1]$, a function $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ is said to be (α, s) -convex if

$$f(tx + (1-t)y) \leq t^{\alpha s} f(x) + (1-t^\alpha)^s f(y)$$

for all $x, y \in I$ and $t \in (0, 1)$.

Definition 2.2. For some $s \in [-1, 1]$ and $(\alpha, m) \in (0, 1]^2$, a function $f : [0, b] \rightarrow \mathbb{R}$ is said to be (α, s, m) -convex if

$$f(tx + m(1-t)y) \leq t^{\alpha s} f(x) + m(1-t^\alpha)^s f(y)$$

for all $x, y \in [0, b]$ and $t \in (0, 1)$.

Remark 2.1. By Definition 2.2, we see that

- (1) if $s = 1$, then $f(x)$ is an (α, m) -convex function on $(0, b]$;
- (2) if $\alpha = 1$, then $f(x)$ is an extended (s, m) -convex function on $(0, b]$;
- (3) if $\alpha = m = 1$, then $f(x)$ is an extended s -convex function on $(0, b]$;
- (4) if $\alpha = s = m = 1$, then $f(x)$ is a convex function on $(0, b]$.

This means that Definitions (2.1) and (2.2) are significant.

For establishing new integral inequalities of the Hermite–Hadamard type for (α, s) -convex and (α, s, m) -convex functions, we need the following lemmas.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function on I° and $a, b \in I$ with $a < b$. If $f'' \in L_1([a, b])$, then*

$$\begin{aligned} & \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \\ &= \frac{(b-a)^2}{16} \int_0^1 t(1-t) \left[f''\left(at + (1-t)\frac{a+b}{2}\right) + f''\left(t\frac{a+b}{2} + (1-t)b\right) \right] dt. \end{aligned}$$

Proof. Integrating by part and changing variable of integral give

$$\begin{aligned} & \int_0^1 t(1-t) f''\left(at + (1-t)\frac{a+b}{2}\right) dt \\ &= \frac{2}{b-a} \int_0^1 (1-2t) f'\left(at + (1-t)\frac{a+b}{2}\right) dt \\ &= \frac{4}{(b-a)^2} \left[f(a) + f\left(\frac{a+b}{2}\right) \right] - \frac{8}{(b-a)^2} \int_0^1 f\left(at + (1-t)\frac{a+b}{2}\right) dt \\ &= \frac{4}{(b-a)^2} \left[f(a) + f\left(\frac{a+b}{2}\right) \right] - \frac{16}{(b-a)^3} \int_a^{\frac{a+b}{2}} f(x) \, dx \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 t(1-t) f''\left(t\frac{a+b}{2} + (1-t)b\right) dt \\ &= \frac{4}{(b-a)^2} \left[f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{16}{(b-a)^3} \int_{(a+b)/2}^b f(x) \, dx. \end{aligned}$$

The proof of Lemma 2.1 is complete. \square

Lemma 2.2. *Let $\alpha \in (0, 1]$ and $s \in [-1, 1]$. Then*

(1) *when $s \in (-1, 1]$, we have*

$$\begin{aligned} M(\alpha, s) &\triangleq \int_0^1 t(1-t)(1-t^\alpha)^s dt \\ &= \frac{1}{6} \left[3B\left(s+1, \frac{2}{\alpha}+1\right) - 2B\left(s+1, \frac{3}{\alpha}+1\right) \right]; \end{aligned} \quad (2.1)$$

(2) *when $s = -1$, we have*

$$\int_0^1 t(1-t)(1-t^\alpha)^{-1} dt = \frac{1}{\alpha} \left[\Psi_0\left(\frac{3}{\alpha}\right) - \Psi_0\left(\frac{2}{\alpha}\right) \right],$$

where $\Gamma(x)$, $B(x, y)$, and $\Psi_n(x)$ are the classical Euler gamma, beta, and polygamma functions defined respectively by

$$\Gamma(x) = \int_0^1 z^{x-1} e^{-z} dz, \quad B(x, y) = \int_0^1 z^{x-1} (1-z)^{y-1} dz, \quad (2.2)$$

and

$$\Psi_n(x) = \frac{d^{n+1} \ln \Gamma(x)}{dx^{n+1}} = (-1)^{n+1} \int_0^\infty \frac{z^n}{1-e^{-z}} e^{-xz} dz \quad (2.3)$$

for $\Re(x), \Re(y) > 0$ and $n \in \mathbb{N}_0$.

3. INTEGRAL INEQUALITIES OF HERMITE–HADAMARD TYPE

Now we start out to establish some new integral inequalities of the Hermite–Hadamard type for (α, s, m) -convex functions.

Theorem 3.1. *For $(\alpha, m) \in (0, 1]^2$ and $s \in (-1, 1]$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $q \geq 1$, then*

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + mM(\alpha, s) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + mM(\alpha, s) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \quad (3.1) \end{aligned}$$

where $M(\alpha, s)$ is defined by (2.1).

Proof. By Lemma 2.1 and Hölder's integral inequality, we obtain

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left[\int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right| dt \right. \\ & \quad \left. + \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \\ & \leq \frac{(b-a)^2}{16} \left\{ \left[\int_0^1 t(1-t) dt \right]^{1-1/q} \left[\int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \right]^{1/q} \right. \\ & \quad \left. + \left(\int_0^1 t(1-t) dt \right)^{1-1/q} \left[\int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \right]^{1/q} \right\}. \quad (3.2) \end{aligned}$$

From the (α, s, m) -convexity of $|f''|^q$ and Lemma 2.2, we arrive at

$$\begin{aligned} & \int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \\ & \leq \int_0^1 t(1-t) \left(t^{\alpha s} |f''(a)|^q + m(1-t)^s \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right) dt \\ & = \frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + mM(\alpha, s) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \\ & \leq \frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + mM(\alpha, s) \left| f''\left(\frac{b}{m}\right) \right|^q. \quad (3.3) \end{aligned}$$

By the inequalities between (3.2) and (3.3), we conclude the inequality (3.1). The proof of Theorem 3.1 is complete. \square

Corollary 3.1. *For $\alpha \in (0, 1]$ and $s \in (-1, 1]$, let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an*

(α, s) -convex function on I for $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + M(\alpha, s) \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + M(\alpha, s) |f''(b)|^q \right]^{1/q} \right\}, \end{aligned}$$

where $M(\alpha, s)$ is defined by (2.1).

Proof. This is a special case of Theorem 3.1 for $m = 1$. \square

Theorem 3.2. For $(\alpha, m) \in (0, 1]^2$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is $(\alpha, -1, m)$ -convex function on $(0, \frac{b^*}{m}]$ for $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{|f''(a)|^q}{(2-\alpha)(3-\alpha)} + \frac{m[\Psi_0(\frac{3}{\alpha}) - \Psi_0(\frac{2}{\alpha})]}{\alpha} \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{(2-\alpha)(3-\alpha)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{m[\Psi_0(\frac{3}{\alpha}) - \Psi_0(\frac{2}{\alpha})]}{\alpha} \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $\Psi(x)$ is defined by (2.3).

Proof. This follows from similar argument to Theorem 3.1. \square

Corollary 3.2. Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an $(\alpha, -1)$ -convex function on I for $\alpha \in (0, 1]$ and $q \geq 1$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[\frac{1}{(2-\alpha)(3-\alpha)} |f''(a)|^q + \frac{\Psi_0(\frac{3}{\alpha}) - \Psi_0(\frac{2}{\alpha})}{\alpha} \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{(2-\alpha)(3-\alpha)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{\Psi_0(\frac{3}{\alpha}) - \Psi_0(\frac{2}{\alpha})}{\alpha} |f''(b)|^q \right]^{1/q} \right\}, \end{aligned}$$

where $\Psi(x)$ is defined by (2.3).

Proof. This is a special case of Theorem 3.2 for $m = 1$. \square

Theorem 3.3. For $(\alpha, m) \in (0, 1]^2$, $s \in (-1, 1]$, and $\ell \geq 0$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in (0, b^*]$ with $a < b$. If $|f''|^q$ is an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $q > 1$ and $q \geq \ell$, then

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \\ & \leq \frac{(b-a)^2}{16} \left[B\left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \\ & \quad \times \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \end{aligned}$$

$$+ \left[\frac{1}{\alpha s + \ell + 1} \left| f'' \left(\frac{a+b}{2} \right) \right|^q + \frac{m}{\alpha} B \left(s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \Bigg\},$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. By Lemma 2.1, Hölder's integral inequality, and the (α, s, m) -convexity of $|f''|^q$, one has

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f \left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left(\int_0^1 t^{(q-\ell)/(q-1)} (1-t)^{q/(q-1)} dt \right)^{1-1/q} \\ & \quad \times \left\{ \left[\int_0^1 t^\ell \left| f'' \left(at + (1-t) \frac{a+b}{2} \right) \right|^q dt \right]^{1/q} \right. \\ & \quad \left. + \left[\int_0^1 t^\ell \left| f'' \left(t \frac{a+b}{2} + (1-t)b \right) \right|^q dt \right]^{1/q} \right\} \\ & \leq \frac{(b-a)^2}{16} \left(\int_0^1 t^{(q-\ell)/(q-1)} (1-t)^{q/(q-1)} dt \right)^{1-1/q} \\ & \quad \times \left\{ \left[\int_0^1 t^\ell \left(t^{\alpha s} |f''(a)|^q + m(1-t^\alpha)^s \left| f'' \left(\frac{a+b}{2m} \right) \right|^q \right) dt \right]^{1/q} \right. \\ & \quad \left. + \left[\int_0^1 t^\ell \left(t^{\alpha s} \left| f'' \left(\frac{a+b}{2} \right) \right|^q + m(1-t^\alpha)^s \left| f'' \left(\frac{b}{m} \right) \right|^q \right) dt \right]^{1/q} \right\} \\ & = \frac{(b-a)^2}{16} \left[B \left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q} \\ & \quad \times \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B \left(s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left(\frac{a+b}{2m} \right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{\alpha s + \ell + 1} \left| f'' \left(\frac{a+b}{2} \right) \right|^q + \frac{m}{\alpha} B \left(s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \right\}. \end{aligned}$$

The proof of Theorem 3.3 is thus complete. \square

Corollary 3.3. *Under assumptions of Theorem 3.3, if $\ell = 0$, then*

$$\begin{aligned} & \left| \frac{1}{4} \left[f(a) + 2f \left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left[B \left(\frac{2q-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q} \\ & \quad \times \left\{ \left[\frac{1}{\alpha s + 1} |f''(a)|^q + \frac{m}{\alpha} B \left(s+1, \frac{1}{\alpha} \right) \left| f'' \left(\frac{a+b}{2m} \right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[\frac{1}{\alpha s + 1} \left| f'' \left(\frac{a+b}{2} \right) \right|^q + \frac{m}{\alpha} B \left(s+1, \frac{1}{\alpha} \right) \left| f'' \left(\frac{b}{m} \right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Corollary 3.4. *For $\alpha \in (0, 1]$, $s \in (-1, 1]$, and $\ell \geq 0$, let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a twice differentiable function and $f'' \in L_1([a, b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an (α, s) -convex function on I for $q \geq 1$ and $q \geq \ell$, then*

$$\left| \frac{1}{4} \left[f(a) + 2f \left(\frac{a+b}{2} \right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right|$$

$$\begin{aligned} &\leq \frac{(b-a)^2}{16} \left[B\left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \\ &\quad \times \left\{ \left[\frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{1}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\ &\quad \left. + \left[\frac{1}{\alpha s + \ell + 1} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{1}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) |f''(b)|^q \right]^{1/q} \right\}, \end{aligned}$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. This is a special case of Theorem 3.3 for $m = 1$. \square

Theorem 3.4. For $(\alpha, m) \in (0, 1]^2$ and $s \in (-1, 1]$, let $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$ be an (α, s, m) -convex function on $(0, \frac{b^*}{m}]$ for $b^* > 0$. If $f \in L_1([a, b])$ for $a, b \in (0, b^*]$ and $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + m(2^\alpha - 1)^s f\left(\frac{x}{m}\right)}{2^{\alpha s}} dx$$

and

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) dx \leq \max \left\{ \frac{1}{\alpha s + 1} f(a) + mB\left(s+1, \frac{1}{\alpha} + 1\right) f\left(\frac{b}{m}\right), \right. \\ \left. mB\left(s+1, \frac{1}{\alpha} + 1\right) f\left(\frac{a}{m}\right) + \frac{1}{\alpha s + 1} f(b) \right\}, \end{aligned}$$

where $B(x, y)$ is the classical beta function defined in (2.2).

Proof. By the (α, s, m) -convexity of f , we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{\alpha s}} \int_0^1 \left[f(ta + (1-t)b) + m(2^\alpha - 1)^s f\left(\frac{(1-t)a + tb}{m}\right) \right] dt.$$

Letting $x = ta + (1-t)b$ or $x = (1-t)a + tb$ for all $t \in [0, 1]$ leads to

$$\int_0^1 f(c[ta + (1-t)b]) dt = \int_0^1 f(c[(1-t)a + tb]) dt = \frac{1}{b-a} \int_a^b f(cx) dx$$

for $c \in \mathbb{R}$.

On the other hand, when putting $x = ta + (1-t)b$ for all $t \in [0, 1]$, by the (α, s, m) -convexity of f , we can gain

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) dx &\leq \int_0^1 \left[t^{\alpha s} f(a) + m(1-t^\alpha)^s f\left(\frac{b}{m}\right) \right] dt \\ &= \frac{1}{\alpha s + 1} f(a) + mB\left(s+1, \frac{1}{\alpha} + 1\right) f\left(\frac{b}{m}\right). \end{aligned}$$

The proof of Theorem 3.4 is complete. \square

Corollary 3.5. Under conditions of Theorem 3.4, if $m = 1$, then

$$\begin{aligned} \frac{2^{\alpha s}}{(2^\alpha - 1)^s + 1} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \max \left\{ \frac{f(a)}{\alpha s + 1} + B\left(s+1, \frac{1}{\alpha} + 1\right) f(b), B\left(s+1, \frac{1}{\alpha} + 1\right) f(a) + \frac{f(b)}{\alpha s + 1} \right\}. \end{aligned}$$

Acknowledgements. This work was partially supported by the National Natural Science Foundation of China (Grant No. 11361038).

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