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## Integral inequalities of Hermite–Hadamard type for $(\alpha, s)$ -convex and $(\alpha, s, m)$ -convex functions

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**Abstract.** In the paper, the authors introduce the concepts “ $(\alpha, s)$ -convex function” and “ $(\alpha, s, m)$ -convex function” and establish some new integral inequalities of the Hermite–Hadamard type for  $(\alpha, s)$ -convex and  $(\alpha, s, m)$ -convex functions in terms of the classical Euler beta, gamma, and polygamma functions.

**Keywords:** integral inequality;  $(\alpha, s)$ -convex function;  $(\alpha, s, m)$ -convex function; Hermite–Hadamard type; beta function; gamma function; polygamma function

### 1. Introduction

We first recite some definitions of various convex functions.

**Definition 1.1.** A function  $f : I \subseteq \mathbb{R} = (-\infty, \infty) \rightarrow \mathbb{R}$  is said to be convex if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

for all  $x, y \in I$  and  $\lambda \in [0, 1]$ .

**Definition 1.2** ([24]). For  $f : [0, b] \rightarrow \mathbb{R}_0 = [0, \infty)$  with  $b > 0$  and some  $m \in (0, 1]$ , if

$$f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)$$

for  $x, y \in [0, b]$  and  $t \in [0, 1]$ , then we say that  $f$  is an  $m$ -convex function on  $[0, b]$ .

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**Definition 1.3** ([12]). For  $f : [0, b] \rightarrow \mathbb{R}_0$  with  $b > 0$  and  $(\alpha, m) \in (0, 1]^2$ , if

$$f(tx + m(1-t)y) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)$$

for all  $x, y \in [0, b]$  and  $t \in [0, 1]$ , then we say that  $f(x)$  is an  $(\alpha, m)$ -convex function on  $[0, b]$ .

**Definition 1.4** ([4, 9]). Let  $s \in (0, 1]$  be a real number. A function  $f : \mathbb{R} \rightarrow \mathbb{R}_0$  is said to be  $s$ -convex (in the second sense) if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all  $x, y \in I$  and  $t \in [0, 1]$ .

**Definition 1.5** ([28]). For some  $s \in [-1, 1]$ , a function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is said to be extended  $s$ -convex if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all  $x, y \in I$  and  $t \in (0, 1)$ .

**Definition 1.6** ([33]). For some  $(s, m) \in [-1, 1] \times (0, 1]$ , a function  $f : [0, b] \rightarrow \mathbb{R}_0$  is said to be extended  $(s, m)$ -convex if

$$f(tx + m(1-t)y) \leq t^s f(x) + m(1-t)^s f(y)$$

for all  $x, y \in I$  and  $t \in (0, 1)$ .

The famous Hermite–Hadamard integral inequality for convex functions and some of its diverse generalizations can be reformulated as follows.

**Theorem 1.1.** *If  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is a convex function on the interval  $I$  and  $a, b \in I$  with  $a < b$ , then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2}.$$

**Theorem 1.2** ([5]). *Let  $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a differentiable mapping on  $I^\circ$  and  $a, b \in I^\circ$  with  $a < b$ . If  $|f'(x)|$  is convex on  $[a, b]$ , then*

$$\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)(|f'(a)| + |f'(b)|)}{8}.$$

**Theorem 1.3** ([10, Theorems 2.3 and 2.4]). *Let  $f : I \subseteq \mathbb{R}_0 \rightarrow \mathbb{R}$  be differentiable on  $I^\circ$  and  $a, b \in I$  with  $a < b$ . If  $|f'(x)|^p$  is  $s$ -convex on  $[a, b]$  for some  $s \in (0, 1]$  and  $p > 1$ , then*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{b-a}{16} \left(\frac{4}{p+1}\right)^{1/p} (|f'(a)| + |f'(b)|)$$

and

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{4} \left(\frac{4}{p+1}\right)^{1/p} \left\{ [|f'(a)|^{p/(p-1)} + 3|f'(b)|^{p/(p-1)}]^{1-1/p} + [3|f'(a)|^{p/(p-1)} + |f'(b)|^{p/(p-1)}]^{1-1/p} \right\}.$$

For more information on integral inequalities of the Hermite–Hadamard type for various kinds of convex functions, please refer to the monograph [6, 8, 13, 14], to recently published papers [2, 3, 5, 7, 11, 15, 16, 17, 18, 19, 20, 21, 22, 23, 25, 26, 28, 29, 30, 31, 32], and to the closely related references therein.

In this paper, we introduce two new concepts “ $(\alpha, s)$ -convex function” and “ $(\alpha, s, m)$ -convex function” and present inequalities of Hermite–Hadamard type for functions whose twice differentiation are of  $(\alpha, s, m)$ -convexity.

### 2. Two definitions and two lemmas

We introduce the notions of “ $(\alpha, s)$ -convex function” and “ $(\alpha, s, m)$ -convex function”.

**Definition 2.1.** For some  $s \in [-1, 1]$  and  $\alpha \in (0, 1]$ , a function  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  is said to be  $(\alpha, s)$ -convex if

$$f(tx + (1-t)y) \leq t^{\alpha s} f(x) + (1-t^{\alpha})^s f(y)$$

for all  $x, y \in I$  and  $t \in (0, 1)$ .

**Definition 2.2.** For some  $s \in [-1, 1]$  and  $(\alpha, m) \in (0, 1]^2$ , a function  $f : [0, b] \rightarrow \mathbb{R}$  is said to be  $(\alpha, s, m)$ -convex if

$$f(tx + m(1-t)y) \leq t^{\alpha s} f(x) + m(1-t^{\alpha})^s f(y)$$

for all  $x, y \in [0, b]$  and  $t \in (0, 1)$ .

*Remark 2.1.* By Definition 2.2, we see that

1. if  $s = 1$ , then  $f(x)$  is an  $(\alpha, m)$ -convex function on  $(0, b]$ ;
2. if  $\alpha = 1$ , then  $f(x)$  is an extended  $(s, m)$ -convex function on  $(0, b]$ ;
3. if  $\alpha = m = 1$ , then  $f(x)$  is an extended  $s$ -convex function on  $(0, b]$ ;
4. if  $\alpha = s = m = 1$ , then  $f(x)$  is a convex function on  $(0, b]$ .

This means that Definitions (2.1) and (2.2) are significant.

For establishing new integral inequalities of the Hermite–Hadamard type for  $(\alpha, s)$ -convex and  $(\alpha, s, m)$ -convex functions, we need the following lemmas.

**Lemma 2.1.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function on  $I^\circ$  and  $a, b \in I$  with  $a < b$ . If  $f'' \in L_1([a, b])$ , then

$$\begin{aligned} & \frac{1}{4} \left[ f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx = \frac{(b-a)^2}{16} \\ & \quad \times \int_0^1 t(1-t) \left[ f''\left(at + (1-t)\frac{a+b}{2}\right) + f''\left(t\frac{a+b}{2} + (1-t)b\right) \right] dt. \end{aligned}$$

**Proof.** Integrating by parts and changing variable of integral give

$$\begin{aligned} & \int_0^1 t(1-t) f''\left(at + (1-t)\frac{a+b}{2}\right) dt \\ & = \frac{2}{b-a} \int_0^1 (1-2t) f'\left(at + (1-t)\frac{a+b}{2}\right) dt \\ & = \frac{4}{(b-a)^2} \left[ f(a) + f\left(\frac{a+b}{2}\right) \right] - \frac{8}{(b-a)^2} \int_0^1 f\left(at + (1-t)\frac{a+b}{2}\right) dt \\ & = \frac{4}{(b-a)^2} \left[ f(a) + f\left(\frac{a+b}{2}\right) \right] - \frac{16}{(b-a)^3} \int_a^{(a+b)/2} f(x) dx \end{aligned}$$

and

$$\begin{aligned} & \int_0^1 t(1-t) f''\left(t\frac{a+b}{2} + (1-t)b\right) dt \\ & = \frac{4}{(b-a)^2} \left[ f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{16}{(b-a)^3} \int_{(a+b)/2}^b f(x) dx. \end{aligned}$$

The proof of Lemma 2.1 is complete.  $\square$

**Lemma 2.2.** Let  $\alpha \in (0, 1]$  and  $s \in [-1, 1]$ . Then

1. when  $s \in (-1, 1]$ , we have

$$\begin{aligned} (2.1) \quad M(\alpha, s) & \triangleq \int_0^1 t(1-t)(1-t^\alpha)^s dt \\ & = \frac{1}{\alpha} \left[ B\left(s+1, \frac{2}{\alpha}\right) - B\left(s+1, \frac{3}{\alpha}\right) \right]; \end{aligned}$$

2. when  $s = -1$ , we have

$$\int_0^1 \frac{t(1-t)}{1-t^\alpha} dt = \frac{1}{\alpha} \left[ \psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right) \right],$$

where  $\Gamma(x)$ ,  $B(x, y)$ , and  $\psi(x)$  are the classical Euler gamma, beta, and psi functions defined respectively by

$$(2.2) \quad \Gamma(x) = \int_0^1 t^{x-1} e^{-t} dt, \quad B(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

and

$$(2.3) \quad \psi(x) = \frac{d \ln \Gamma(x)}{dx} = \int_0^\infty \left[ \frac{e^{-t}}{t} - \frac{e^{-xt}}{1 - e^{-t}} \right] dt$$

for  $\Re(x), \Re(y) > 0$ .

**Proof.** Let  $u = t^\alpha$  for  $t \in [0, 1]$ . If  $s \in (-1, 1]$ , we have

$$\begin{aligned} M(\alpha, s) &= \int_0^1 t(1-t)(1-t^\alpha)^s dt \\ &= \frac{1}{\alpha} \int_0^1 (u^{2/\alpha-1} - u^{3/\alpha-1})(1-u)^s du = \frac{1}{\alpha} \left[ B\left(s+1, \frac{2}{\alpha}\right) - B\left(s+1, \frac{3}{\alpha}\right) \right]. \end{aligned}$$

When  $s = -1$ , from the formulas

$$\psi(z) + \gamma = \int_0^1 \frac{1 - t^{z-1}}{1-t} dt, \quad \text{and} \quad \gamma = \int_0^\infty \left( \frac{1}{1+t} - e^{-t} \right) \frac{dt}{t}$$

in [1, p. 259, 6.3.22], it is easily to deduce

$$\int_0^1 \frac{t(1-t)}{1-t^\alpha} dt = \frac{1}{\alpha} \int_0^1 \frac{u^{2/\alpha-1} - u^{3/\alpha-1}}{1-u} du = \frac{1}{\alpha} \left[ \psi\left(\frac{3}{\alpha}\right) - \psi\left(\frac{2}{\alpha}\right) \right].$$

The proof of Lemma 2.2 is complete. □

### 3. Integral inequalities of Hermite–Hadamard type

Now we start out to establish some new integral inequalities of the Hermite–Hadamard type for  $(\alpha, s, m)$ -convex functions.

**Theorem 3.1.** For  $(\alpha, m) \in (0, 1]^2$  and  $s \in (-1, 1]$ , let  $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$  be a twice differentiable function and  $f'' \in L_1([a, b])$  for  $a, b \in (0, b^*]$  with  $a < b$ . If  $|f''|^q$  is an  $(\alpha, s, m)$ -convex function on  $(0, \frac{b^*}{m}]$  for  $q \geq 1$ , then

$$(3.1) \quad \begin{aligned} &\left| \frac{1}{4} \left[ f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \\ &\times \left\{ \left[ \frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + mM(\alpha, s) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ &\left. + \left[ \frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + mM(\alpha, s) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where  $M(\alpha, s)$  is defined by (2.1).

**Proof.** By Lemma 2.1 and Hölder’s integral inequality, we obtain

$$\begin{aligned}
 (3.2) \quad & \left| \frac{1}{4} \left[ f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{16} \left[ \int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right| dt \right. \\
 & \quad \left. + \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right| dt \right] \leq \frac{(b-a)^2}{16} \\
 & \times \left\{ \left[ \int_0^1 t(1-t) dt \right]^{1-1/q} \left[ \int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \right]^{1/q} \right. \\
 & \quad \left. + \left( \int_0^1 t(1-t) dt \right)^{1-1/q} \left[ \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \right]^{1/q} \right\}.
 \end{aligned}$$

From the  $(\alpha, s, m)$ -convexity of  $|f''|^q$  and Lemma 2.2, we arrive at

$$\begin{aligned}
 & \int_0^1 t(1-t) \left| f''\left(at + (1-t)\frac{a+b}{2}\right) \right|^q dt \\
 & \leq \int_0^1 t(1-t) \left( t^{\alpha s} |f''(a)|^q + m(1-t)^{\alpha s} \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right) dt \\
 & = \frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + mM(\alpha, s) \left| f''\left(\frac{a+b}{2m}\right) \right|^q
 \end{aligned}$$

and

$$\begin{aligned}
 (3.3) \quad & \int_0^1 t(1-t) \left| f''\left(t\frac{a+b}{2} + (1-t)b\right) \right|^q dt \\
 & \leq \frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + mM(\alpha, s) \left| f''\left(\frac{b}{m}\right) \right|^q.
 \end{aligned}$$

By the inequalities between (3.2) and (3.3), we conclude the inequality (3.1). The proof of Theorem 3.1 is complete.  $\square$

**Corollary 3.1.** For  $\alpha \in (0, 1]$  and  $s \in (-1, 1]$ , let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function and  $f'' \in L_1([a, b])$  for  $a, b \in I$  with  $a < b$ . If  $|f''|^q$  is an  $(\alpha, s)$ -convex function on  $I$  for  $q \geq 1$ , then

$$\begin{aligned}
 & \left| \frac{1}{4} \left[ f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\
 & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[ \frac{1}{(\alpha s + 2)(\alpha s + 3)} |f''(a)|^q + M(\alpha, s) \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\
 & \quad \left. + \left[ \frac{1}{(\alpha s + 2)(\alpha s + 3)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + M(\alpha, s) |f''(b)|^q \right]^{1/q} \right\},
 \end{aligned}$$

where  $M(\alpha, s)$  is defined by (2.1).

**Proof.** This is a special case of Theorem 3.1 for  $m = 1$ .  $\square$

**Theorem 3.2.** For  $(\alpha, m) \in (0, 1]^2$ , let  $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$  be a twice differentiable function and  $f'' \in L_1([a, b])$  for  $a, b \in (0, b^*]$  with  $a < b$ . If  $|f''|^q$  is  $(\alpha, -1, m)$ -convex function on  $(0, \frac{b^*}{m}]$  for  $q \geq 1$ , then

$$\begin{aligned} & \left| \frac{1}{4} \left[ f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[ \frac{|f''(a)|^q}{(2-\alpha)(3-\alpha)} + \frac{m[\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})]}{\alpha} \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[ \frac{1}{(2-\alpha)(3-\alpha)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{m[\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})]}{\alpha} \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where  $\psi(x)$  is defined by (2.3).

**Proof.** This follows from similar argument to Theorem 3.1. □

**Corollary 3.2.** Let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function and  $f'' \in L_1([a, b])$  for  $a, b \in I$  with  $a < b$ . If  $|f''|^q$  is an  $(\alpha, -1)$ -convex function on  $I$  for  $\alpha \in (0, 1]$  and  $q \geq 1$ , then

$$\begin{aligned} & \left| \frac{1}{4} \left[ f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16 \times 6^{1-1/q}} \left\{ \left[ \frac{1}{(2-\alpha)(3-\alpha)} |f''(a)|^q + \frac{\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})}{\alpha} \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[ \frac{1}{(2-\alpha)(3-\alpha)} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{\psi(\frac{3}{\alpha}) - \psi(\frac{2}{\alpha})}{\alpha} |f''(b)|^q \right]^{1/q} \right\}, \end{aligned}$$

where  $\psi(x)$  is defined by (2.3).

**Proof.** This is a special case of Theorem 3.2 for  $m = 1$ . □

**Theorem 3.3.** For  $(\alpha, m) \in (0, 1]^2$ ,  $s \in (-1, 1]$ , and  $\ell \geq 0$ , let  $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$  be a twice differentiable function and  $f'' \in L_1([a, b])$  for  $a, b \in (0, b^*]$  with  $a < b$ . If  $|f''|^q$  is an  $(\alpha, s, m)$ -convex function on  $(0, \frac{b^*}{m}]$  for  $q > 1$  and  $q \geq \ell$ , then

$$\begin{aligned} & \left| \frac{1}{4} \left[ f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \\ & \leq \frac{(b-a)^2}{16} \left[ B\left(\frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1}\right) \right]^{1-1/q} \\ & \quad \times \left\{ \left[ \frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2m}\right) \right|^q \right]^{1/q} \right. \\ & \quad \left. + \left[ \frac{1}{\alpha s + \ell + 1} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{m}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{b}{m}\right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where  $B(x, y)$  is the classical beta function defined in (2.2).



**Proof.** By Lemma 2.1, Hölder’s integral inequality, and the  $(\alpha, s, m)$ -convexity of  $|f''|^q$ , one has

$$\begin{aligned} & \left| \frac{1}{4} \left[ f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ & \leq \frac{(b-a)^2}{16} \left( \int_0^1 t^{(q-\ell)/(q-1)} (1-t)^{q/(q-1)} dt \right)^{1-1/q} \left\{ \left[ \int_0^1 t^\ell \left| f'' \left( at + (1-t)\frac{a+b}{2} \right) \right|^q dt \right]^{1/q} + \left[ \int_0^1 t^\ell \left| f'' \left( t\frac{a+b}{2} + (1-t)b \right) \right|^q dt \right]^{1/q} \right\} \\ & \leq \frac{(b-a)^2}{16} \left( \int_0^1 t^{(q-\ell)/(q-1)} (1-t)^{q/(q-1)} dt \right)^{1-1/q} \\ & \quad \times \left\{ \left[ \int_0^1 t^\ell \left( t^{\alpha s} |f''(a)|^q + m(1-t^\alpha)^s \left| f'' \left( \frac{a+b}{2m} \right) \right|^q \right) dt \right]^{1/q} + \left[ \int_0^1 t^\ell \left( t^{\alpha s} \left| f'' \left( \frac{a+b}{2} \right) \right|^q + m(1-t^\alpha)^s \left| f'' \left( \frac{b}{m} \right) \right|^q \right) dt \right]^{1/q} \right\} \\ & = \frac{(b-a)^2}{16} \left[ B \left( \frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q} \\ & \quad \times \left\{ \left[ \frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B \left( s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left( \frac{a+b}{2m} \right) \right|^q \right]^{1/q} + \left[ \frac{1}{\alpha s + \ell + 1} \left| f'' \left( \frac{a+b}{2} \right) \right|^q + \frac{m}{\alpha} B \left( s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left( \frac{b}{m} \right) \right|^q \right]^{1/q} \right\}. \end{aligned}$$

The proof of Theorem 3.3 is thus complete. □

**Corollary 3.3.** Under assumptions of Theorem 3.3, if  $\ell = 0$ , then

$$\begin{aligned} & \left| \frac{1}{4} \left[ f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{16} \\ & \quad \times \left[ B \left( \frac{2q-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q} \left\{ \left[ \frac{1}{\alpha s + 1} |f''(a)|^q + \frac{m}{\alpha} B \left( s+1, \frac{1}{\alpha} \right) \left| f'' \left( \frac{a+b}{2m} \right) \right|^q \right]^{1/q} + \left[ \frac{|f''(a+b/2)|^q}{\alpha s + 1} + \frac{m}{\alpha} B \left( s+1, \frac{1}{\alpha} \right) \left| f'' \left( \frac{b}{m} \right) \right|^q \right]^{1/q} \right\}, \end{aligned}$$

where  $B(x, y)$  is the classical beta function defined in (2.2).

**Corollary 3.4.** For  $\alpha \in (0, 1]$ ,  $s \in (-1, 1]$ , and  $\ell \geq 0$ , let  $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$  be a twice differentiable function and  $f'' \in L_1([a, b])$  for  $a, b \in I$  with  $a < b$ . If  $|f''|^q$  is an  $(\alpha, s)$ -convex function on  $I$  for  $q \geq 1$  and  $q \geq \ell$ , then

$$\left| \frac{1}{4} \left[ f(a) + 2f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{(b-a)^2}{16} \left[ B \left( \frac{2q-\ell-1}{q-1}, \frac{2q-1}{q-1} \right) \right]^{1-1/q} \left\{ \left[ \frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{m}{\alpha} B \left( s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left( \frac{a+b}{2m} \right) \right|^q \right]^{1/q} + \left[ \frac{1}{\alpha s + \ell + 1} \left| f'' \left( \frac{a+b}{2} \right) \right|^q + \frac{m}{\alpha} B \left( s+1, \frac{\ell+1}{\alpha} \right) \left| f'' \left( \frac{b}{m} \right) \right|^q \right]^{1/q} \right\}.$$

$$\frac{2q-1}{q-1} \Big)^{1-1/q} \left\{ \left[ \frac{1}{\alpha s + \ell + 1} |f''(a)|^q + \frac{1}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) \left| f''\left(\frac{a+b}{2}\right) \right|^q \right]^{1/q} + \left[ \frac{1}{\alpha s + \ell + 1} \left| f''\left(\frac{a+b}{2}\right) \right|^q + \frac{1}{\alpha} B\left(s+1, \frac{\ell+1}{\alpha}\right) |f''(b)|^q \right]^{1/q} \right\},$$

where  $B(x, y)$  is the classical beta function defined in (2.2).

**Proof.** This is a special case of Theorem 3.3 for  $m = 1$ . □

**Theorem 3.4.** For  $(\alpha, m) \in (0, 1]^2$  and  $s \in (-1, 1]$ , let  $f : (0, \frac{b^*}{m}] \rightarrow \mathbb{R}$  be an  $(\alpha, s, m)$ -convex function on  $(0, \frac{b^*}{m}]$  for  $b^* > 0$ . If  $f \in L_1([a, b])$  for  $a, b \in (0, b^*]$  and  $a < b$ , then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b \frac{f(x) + m(2^\alpha - 1)^s f\left(\frac{x}{m}\right)}{2^{\alpha s}} dx$$

and

$$\frac{1}{b-a} \int_a^b f(x) dx \leq \max \left\{ \frac{1}{\alpha s + 1} f(a) + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{b}{m}\right), \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{a}{m}\right) + \frac{1}{\alpha s + 1} f(b) \right\},$$

where  $B(x, y)$  is the classical beta function defined in (2.2).

**Proof.** By the  $(\alpha, s, m)$ -convexity of  $f$ , we obtain

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2^{\alpha s}} \int_0^1 \left[ f(ta + (1-t)b) + m(2^\alpha - 1)^s f\left(\frac{(1-t)a + tb}{m}\right) \right] dt.$$

Letting  $x = ta + (1-t)b$  or  $x = (1-t)a + tb$  for all  $t \in [0, 1]$  leads to

$$\int_0^1 f(c[ta + (1-t)b]) dt = \int_0^1 f(c[(1-t)a + tb]) dt = \frac{1}{b-a} \int_a^b f(cx) dx$$

for  $c \in \mathbb{R}$ .

On the other hand, when putting  $x = ta + (1-t)b$  for all  $t \in [0, 1]$ , by the  $(\alpha, s, m)$ -convexity of  $f$ , we can gain

$$\begin{aligned} \frac{1}{b-a} \int_a^b f(x) dx &\leq \int_0^1 \left[ t^{\alpha s} f(a) + m(1-t^\alpha)^s f\left(\frac{b}{m}\right) \right] dt \\ &= \frac{1}{\alpha s + 1} f(a) + \frac{m}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f\left(\frac{b}{m}\right). \end{aligned}$$

The proof of Theorem 3.4 is complete. □

**Corollary 3.5.** *Under conditions of Theorem 3.4, if  $m = 1$ , then*

$$\begin{aligned} \frac{2^{\alpha s}}{(2^\alpha - 1)^s + 1} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{b-a} \int_a^b f(x) dx \\ &\leq \max \left\{ \frac{f(a)}{\alpha s + 1} + \frac{1}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f(b), \frac{1}{\alpha} B\left(s+1, \frac{1}{\alpha}\right) f(a) + \frac{f(b)}{\alpha s + 1} \right\}. \end{aligned}$$

*Remark 3.1.* This paper is a corrected and revised version of the preprint [27].

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