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<sup>1</sup> **An effective streamflow process model for optimal**  
<sup>2</sup> **reservoir operation using Stochastic Dual Dynamic**  
<sup>3</sup> **Programming**

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4 **Abstract.** We present an innovative streamflow process model to be used  
5 for reservoir operational rule design in Stochastic Dual Dynamic Program-  
6 ming (SDDP). Model features, which can be applied independently, are: i)  
7 a non-linear multiplicative process model for the forward phase, which pro-  
8 duces positive streamflow values only, and its linearized version for the back-  
9 ward phase, and ii) a non-uniform time-step, which divides the hydrologi-  
10 cal period in time-steps of different length in order to have a process with  
11 approximately constant variance. Model identification is straightforward as  
12 for additive periodic autoregressive model generally used in SDDP. We ap-  
13 plied this model on the Senegal River for the optimal operation of Manan-  
14 tali reservoir, and evaluated the proposed solutions against streamflow pro-  
15 cess model currently used in the water management literature.

## 1. Introduction

16 In reservoir operation, present benefits must be balanced with future, uncertain ones  
17 [*Soncini-Sessa et al.*, 2007; *Castelletti et al.*, 2008]. After each release decision, new infor-  
18 mation becomes available and partially reduces uncertainty. Optimal reservoir operation  
19 can be framed as a Multistage Stochastic Programming problem [*Birge and Louveaux*,  
20 1997; *Shapiro and Andrzej*, 2003], which, for long horizon, is conveniently solved by  
21 Stochastic Dynamic Programming [*Bellman and Dreyfus*, 1966]. SDP, notwithstanding  
22 its elegance and potential, is affected by the so-called “curse of dimensionality”, limiting  
23 its application to systems made of few variables [*Stedinger et al.*, 1984; *Trezos and Yeh*,  
24 1987]. In the literature, some alternatives try to circumvent these limitations, for example  
25 by defining an optimal trajectory [*Turgeon*, 1980] or fixing the release policy family and  
26 find the parameters by evolutionary algorithm [*Nicklowsky et al.*, 2009; *Reed et al.*, 2013].  
27 These solutions, even if advantageous for some aspects, have rarely been tested over large  
28 systems, i.e. made of a large number of reservoirs. Stochastic Dual Dynamic Programming  
29 [*Pereira and Pinto*, 1991] (SDDP) is an approximation of SDP that largely attenuates the  
30 curse of dimensionality. SDDP, however, requires the optimization problem to be modeled  
31 as linear, since problem linearity ensures cost-to-go function convexity.

32 SDDP requires identifying a linear stochastic inflow model which reproduces the stream-  
33 flow process and its uncertainty. Streamflow process model identification is a critical step  
34 in dynamic programming problem setting, sometimes referred to as “curse of modelling”  
35 [*Tsitsiklis and Van Roy*, 1996; *Bertsekas and Tsitsiklis*, 1995], to stress that model iden-  
36 tification can be problematic, hence a limitation for the methodology. SDDP applications

37 generally use a standardized Periodic Autoregressive (PAR) model of lag 1 [*Tilmant et al.*,  
38 2008; *Tilmant and Kelman*, 2007; *Tilmant et al.*, 2007, 2010, 2012; *Tilmant and Kinzel-*  
39 *bach*, 2012; *Tilmant et al.*, 2009; *Goor et al.*, 2010; *Arjoon et al.*, 2014; *Marques and*  
40 *Tilmant*, 2013; *Gjelsvik et al.*, 2010], also known as Thomas-Fiering model [*Loucks*, 1992].  
41 The drawback of this additive model is the non-negligible probability of negative dis-  
42 charge values. This is to be avoided, because negative discharges have no physical sense.  
43 Existing solutions dealing with negative discharges [*Stedinger and Taylor*, 1982; *Pereira*  
44 *et al.*, 1984; *Bezerra et al.*, 2012] use non-linear transformations that make these models  
45 not usable in SDDP.

46 The monthly time-step preserves the process as markov, but it risks to underestimate  
47 the system adaptivity to changing conditions. Such a large time-step, in fact, may not  
48 take into account the adaptivity at a smaller time-step, and it can be a limitation to the  
49 analysis of system response to fast processes, such as flood, resulting in an underestimation  
50 of system capacity to react to this type of events.

51 Short-term system adaptation can be taken into account by a time decomposition ap-  
52 proach [*Karamouz et al.*, 2003]. In time decomposition, long-term policies are refined by  
53 optimizations at shorter-term windows, using results from the long-term optimization as  
54 boundary conditions. Even if time decomposition increases the accuracy of performance  
55 estimation, short-term optimization can use more information than the long-term opti-  
56 mization supposes, leading to an underestimation of the performance value in the long  
57 term planning [*Weijjs*, 2011]. Therefore, aggregating discharges at large time steps is  
58 an approximation with negative impact on performance, and time-step length must be a  
59 trade-off between i) calculation time and capacity to represent the process as markov, and

60 ii) accurate representation of the relevant processes. The former requires a long time-step,  
61 the latter a short one.

62 Time-step length depends on both the system characteristics and the hydrological pro-  
63 cess that we intend to model. For some hydrological systems, variability is not uniform  
64 along the year, depending on local climate. For example, where a rainy season is sepa-  
65 rated from a dry one, the latter has generally less variability than the first, for drought is  
66 a relatively slow process compared to flood.

67 In this paper we present an innovative stochastic streamflow model, to be used within  
68 SDDP, that avoids some important limitations of existing models. This paper is struc-  
69 tured as in the following. In Section 2 we introduce the methodology, from the original  
70 optimal control problem, until the SDDP as a way to solve a Multistage Stochastic Pro-  
71 gramming problem; Section 2.1 presents a procedure to estimate a streamflow model with  
72 a multiplicative stochastic component, and its linearized version. This model guarantees  
73 a negligible probability of negative discharge values, and its identification is straightfor-  
74 ward. Section 2.2 exposes a procedure to identify non-uniform time-step lengths, to take  
75 advantage of this hydrological variability to have better distributed decision instants. The  
76 proposed features are independent from each other, then each of them can be applied sep-  
77 arately. In Section 3 we test the proposed solutions for modeling the streamflow process  
78 on the Senegal river, West Africa. In Section 4 we draw the conclusions.

## 2. Methodology

79 Consider a water system composed of  $N_{\text{res}}$  reservoirs that is operated by  $N_{\text{dec}}$  discharge  
80 decisions. Discharge decisions are diversions from rivers and releases from reservoirs. A  
81 reservoir may have multiple releases (by different structures or for different users). The

82 system is influenced by  $N_{\text{scen}}$  scenarios, such as future inflows. Equations (1) define the  
 83 control problem.

$$\text{Find } \pi_t, \forall t \in \{1, \dots, H\}: \quad (1a)$$

$$\max_{\pi_t} \sum_{t=1}^H \mathbb{E}_{\mathbf{q}_t} [g_t(\mathbf{v}_t, \mathbf{r}_t, \mathbf{q}_t)] \quad (1b)$$

Subject to:

$$\mathbf{v}_t = \mathbf{v}_{t-1} + \Delta t \cdot (I \cdot [\mathbf{r}_t, \mathbf{q}_t] - O \cdot [\mathbf{r}_t, \mathbf{q}_t]) \quad (1c)$$

$$\mathbf{c}_t(\mathbf{v}_t, \mathbf{r}_t, \mathbf{q}_t) \leq 0 \quad (1d)$$

$$\mathbf{q}_t \sim f_{\mathbf{Q}_t} \quad (1e)$$

$$(1f)$$

84 In Problem (1), vectors  $\mathbf{v}_t \in \mathbb{R}^{N_{\text{res}}}$ ,  $\mathbf{r}_t \in \mathbb{R}^{N_{\text{dec}}}$ ,  $\mathbf{q}_t \in \mathbb{R}^{N_{\text{scen}}}$  represent reservoir volumes,  
 85 discharge decisions, and scenarios, at instant  $t$  for stocks and in the period  $[t - \Delta t, t)$  for  
 86 flows;  $\pi_t$  is the optimal release rule, that suggests the optimal release decision  $\mathbf{r}_t$  in function  
 87 of the occurring scenario, i.e. the realization of  $\mathbf{q}_t$ . In Expression (1b),  $g_t(\cdot)$  is a  $\mathbb{R}^N$  to  $\mathbb{R}$   
 88 function, representing the system objective at  $t$ , where  $N = N_{\text{res}} + N_{\text{dec}} + N_{\text{scen}}$ . Equation  
 89 (1c) is the continuity equation, represented by the reservoirs mass balance. In Equation  
 90 (1c),  $\Delta t$  is time-step length,  $I$  and  $O$  are the input and output matrix, of dimension  $N_{\text{res}} \times$   
 91  $(N_{\text{dec}} + N_{\text{scen}})$ , associating at each inflow and discharge decision to its reservoir.  $O(i, j)$   
 92 and  $I(i, j)$  is 1 if the  $i$  variable is input or output of reservoir  $j$ , 0 elsewhere. Scenarios  
 93  $q_t$ , in Expressions (1e), are either stochastic or deterministic scenarios. Deterministic  
 94 scenarios are a vector of values, stochastic scenarios are random variables distributed as  
 95  $f_{\mathbf{Q}_t}(\mathbf{q}_t)$ . Future inflows to the reservoirs are described as stochastic scenarios, while other

96 variables, such as evaporation, for which uncertainty can be neglected, are considered as  
 97 deterministic scenarios.

98  $\mathbf{c}_t$ , in Inequality (1d), defines other constraints that apply to the system, such as physical  
 99 constraints, or other legal or environmental requirements treated as constraints. For  
 100 example, discharge decisions can have a physical upper limit and be limited by water  
 101 availability.

102 Problem (1) is to be solved for an optimization horizon of  $H$  time-steps, from  $t = 1$ ,  
 103 where the initial condition,  $\mathbf{v}_0$ , is given. Decisions and realizations of stochastic variables  
 104 come in recursive mode, therefore, at each decision step, release can be adjusted thanks to  
 105 the new information on the occurring scenario. In this case the optimization problem is  
 106 set as Multistage Stochastic Programming [*Shapiro and Andrzej, 2003*], as in Expression  
 107 (2):

$$\begin{aligned} \max_{\mathbf{r}_1} g_1(\mathbf{v}_1, \mathbf{r}_1, \mathbf{q}_1) + \mathbb{E}_{q_{2,i}} \left[ \max_{\mathbf{r}_{2,i}} g_2(\mathbf{v}_{2,i}, \mathbf{r}_{2,i}, \mathbf{q}_{2,i}) + \mathbb{E}_{q_{3,i}} \left[ \dots \right. \right. \\ \left. \left. \dots + \mathbb{E}_{q_{H,i}} \left[ \max_{\mathbf{r}_{H,i}} g(\mathbf{v}_{H,i}, \mathbf{r}_{H,i}, \mathbf{q}_{H,i}) \right] \dots \right] \right] \end{aligned} \quad (2)$$

108 under conditions given by Equations (1c,??,1d), and with an initial condition  $\mathbf{v}_0$ . In  
 109 some case, a condition on final time-step  $\mathbf{c}_H(\mathbf{v}_H) \leq \mathbf{b}_H$  may be present, in the form of  
 110 hard or soft constraint [*van Overloop et al., 2008*].

111 By solving Problem (2), the optimization procedure finds an optimal release rule for a  
 112 future horizon  $H$ . In expression (2), the release rule  $\{\pi_t\}_{t=1}^H$  is a decision tree,  $\mathbf{r}_{t,i} \forall t, \forall i$ ,  
 113 made of multiple bifurcations at each time-steps,  $t \in [1 : H]$ , representing the optimal  
 114 decision strategy adapted to the  $i$  realisation of the stochastic variable,  $\mathbf{q}_{t,i}$ . The main  
 115 drawback of Multistage Stochastic Programming is its computational complexity, which  
 116 increases exponentially with  $H$ . Multistage Stochastic Programming can be applied when



117  $H$  is small, for example in short term management [*Raso et al.*, 2014]. In same case the  
 118 problem has been aggregated and reduced to seasonal decisions [*Seifi and Hipel*, 2001].  
 119 For long-term optimization, however, MSP can be considered as a theoretical, rather than  
 120 practical, method [*Mayne et al.*, 2000].

121 Stochastic Dynamic Programming (SDP) decomposes the MSP problem in step-by-  
 122 step optimal decision problems. Then, the optimization problem in Expression (2) can  
 123 be written as Bellman Chain, as in Equation (3). SDP can solve problems with a much  
 124 longer horizon, because problem complexity increases only linearly with  $H$ .

$$F_t(\mathbf{v}_t, \mathbf{q}_t) = \max_{\mathbf{r}_t} g_t(\mathbf{v}_t, \mathbf{r}_t, \mathbf{q}_t) + \mathbb{E}_{\mathbf{q}_{t+1}} \left[ F_{t+1}(\mathbf{v}_{t+1}, \mathbf{q}_{t+1}) \right] \quad (3)$$

125 In Equation (3),  $F_t$  is the cost-to-go function, the average cost for leaving the system in  
 126 the state  $[\mathbf{v}_t, \mathbf{q}_t]$ , which is the compromise of present and future benefits,  $g_t(\cdot)$  and  $F_{t+1}$ .  
 127 In SDP, Equation (3) is the release rule  $\pi_t$ , which maps the system state to the optimal  
 128 release decisions.

129 Equation (3) can be solved backwards, from  $t = H$  to the initial time-step. Condition  
 130 (1e) is substituted by Condition (4). The probability transition from  $\mathbf{q}_{t-1}$  to  $\mathbf{q}_t$ , together  
 131 with the continuity Equation (1c), makes up the transition equations, which describes the  
 132 system dynamic from one state to the next one.

133 SDP requires the stochastic transition to be expressed as a Markov process [*Rabiner*  
 134 *and Juang*, 1986], i.e. the probability of each event,  $f(\mathbf{q}_t)$ , depends only on the state at  
 135 previous instant,  $\mathbf{q}_{t-1}$ , as in Equation (4).

$$\mathbf{q}_t \sim f_{Q_t}(\mathbf{q}_t | \mathbf{q}_{t-1}) \quad (4)$$

136 When an autoregressive lag 1 process is not sufficient,  $\mathbf{q}_{t-1}$  can be enlarged to contain  
 137 all the informative variables [Turgeon, 1980]. Then, the process ‘memory’  $\mathbf{q}_{t-1}$  includes  
 138  $[\mathbf{q}_{t-1}, \dots, \mathbf{q}_{t-p}, \mathbf{e}_{t-1}, \dots, \mathbf{e}_{t-q}]$ , where  $\mathbf{e}_{t-j}$  is the difference between the observed value and  
 139 the expected value of the model forecast at  $\mathbf{q}_{t-j}$ . In SDP, this state variables augmentation  
 140 allows to represent the stochastic process by an Periodic ARMA( $p, q$ ) model. However,  
 141 hydrological processes representation with more than one variable has rarely been applied,  
 142 and SDP applications are often limited to strategic reservoir operation. In fact, SDP  
 143 suffers from the so-called ‘curse of dimensionality’, i.e. complexity increases exponentially  
 144 with the number of system variables.

145 Stochastic Dual Dynamic Programming (SDDP) [Shapiro, 2011] is an approximation  
 146 of the original SDP problem. SDDP attenuates the curse of dimensionality, and can be  
 147 applied to larger systems. SDDP does not require variables discretization, but the time-  
 148 step optimization problem must be linear: transition equations (1c, 4) and objective (1b)  
 149 must be linear, and equations defining the constraints space (1d) must be affine. SDDP is  
 150 solved iteratively forward and backward. In the backward stage, at each  $t = H, \dots, 2$ , the  
 151 optimization finds the minimum average cost to pass from  $[\mathbf{v}_{t-1}, \mathbf{q}_{t-1}]$  to  $[\mathbf{v}_t, \mathbf{q}_t]$ , adding  
 152 an extra cut  $l_k(\mathbf{v}_t, \mathbf{q}_t)$  to the approximation of the cost-to-go function  $\mathcal{F}_t(\mathbf{v}_t, \mathbf{q}_t)$ , such  
 153 that  $\mathcal{F}_t(\cdot) := \max\{\mathcal{F}_t(\cdot), l_k(\cdot)\}$ . In the forward stage, the approximate problem is solved  
 154 from  $t = 1$  to  $H$  to find the optimal trajectories to be used in the next backward phase.  
 155 By successive iterations,  $\mathcal{F}_t$  converges to the real cost-to-go function  $F_t$ , as demonstrated,  
 156 under mild conditions, by Philpott and Guan [2008] and Linowsky and Philpott [2005].

## 2.1. Multiplicative error model identification

157 In this section we propose a multiplicative error model to deal with the problem of  
 158 generating negative discharges.

159 In SDDP, the stochastic hydrological model must be linear, but no conditions are im-  
 160 posed on distribution  $f_t(\mathbf{q}_t)$ . There is not specific reason to prefer separate additive terms  
 161 for the predictive and the uncertain parts of the model [Koutsoyiannis, 2009]. In the  
 162 following, we show a procedure to estimate a linear model in  $\mathbf{q}_t$ , with a multiplicative  
 163 stochastic component.

164 We start from a multivariate signal of observed discharge, which we want to reproduce,  
 165  $\mathbf{q}_t$ , made of  $T$  samples for  $N_y$  years of data, where  $T$  is the number of time-steps per year.  
 166 The original signal is transformed into  $\mathbf{y}_t$  according to Equation (5).

$$\mathbf{y}_t = \log(\mathbf{q}_t) - \log(\bar{\mathbf{q}}_\tau) \quad (5)$$

167 In Equation (5),  $\tau = 1, \dots, T$  is the periodic time index. Equation (6) defines  $\bar{\mathbf{q}}_\tau$ , i.e.  
 168 the periodic geometric average of  $\mathbf{q}_t$  for all  $\mathbf{t} = \tau$ , where  $\mathbf{t} = (t - 1) \bmod T + 1$ , and  $\bmod$   
 169 is the modulus (or remainder) operator.

$$\bar{\mathbf{q}}_\tau = \left( \prod_{\mathbf{t}}^{N_y} \mathbf{q}_t \right)^{1/N_y} = \exp \left( \frac{1}{N_y} \sum_{\mathbf{t}}^{N_y} \log \mathbf{q}_t \right), \forall \mathbf{t} = \tau \quad (6)$$

170 Logarithm smooths extremes and long tails, making model identification easier. In  
 171 fact, logarithm is a particular case of the Box-Cox transformation, suggested in *Box et al.*  
 172 [1970] to deal with non-normal residuals. We use the  $\mathbf{y}_t$  signal to identify an ARMA  
 173 model on the  $N_{\text{stoch}}$  dimensional signal. Contemporaneous ARMA (CARMA) [*Camacho*  
 174 *et al.*, 1987] are an effective sub-class of multivariate ARMA model that proved effective  
 175 in hydrological applications. A CARMA model is identified as a set of  $N_{\text{stoch}}$  univariate

176 ARMA models, with correlated residuals. For simplicity of explanation and notation,  
 177 we describe the model identification procedure for an univariate ARMA model, recalling  
 178 that possible correlation among sub-catchments can be included in the residuals covariance  
 179 matrix [*Salas et al.*, 1985].

180 In hydrological systems, parameters are generally time-variant [*Salas*, 1980; *Hipel and*  
 181 *McLeod*, 1994], because of climatic effect and because different hydrological processes are  
 182 dominant at different periods in the years. Time-variant parameters can effectively dealt  
 183 with Periodic ARMA model (PARMA). Equation (7) defines the  $PARMA_{\tau}(p, q)$  model.

$$y_t = \sum_{i=1}^p \phi_{\tau,i} y_{t-i} + \sum_{j=1}^q \psi_{\tau,j} \varepsilon_{t-j} + \varepsilon_t \quad (7)$$

184 In Equation (7),  $\varepsilon_t$  is the stochastic process, extracted from  $\mathcal{N}(0, \sigma_{\tau}^2)$ , independent on  
 185 previous  $\varepsilon_{t-i}$ , where  $\varepsilon_{t-i} = y_{t-i} - \hat{y}_{t-i}$ ;  $y_t$  is observed value, and  $\hat{y}_t$  is expected value of  
 186 Model (7), i.e. when  $\varepsilon_t = 0$ . Identifying a  $PARMA_{\tau}(p, q)$  model is defining parameters  
 187  $\phi_{\tau,i}, \psi_{\tau,j}, \sigma_{\tau}$  of Equation (7)  $\forall i, \forall j, \forall \tau$ .

188 Equation (7) can be written as transition from  $q_{t-i}$  to  $q_t$ . By inversion of Equation (5)  
 189 and some rearrangement, one can obtain Equation (8).

$$q_t = \alpha_{\tau} \cdot \prod_{i=1}^p q_{t-i}^{\phi_{\tau,i}} \cdot \prod_{j=1}^q \xi_{t-j}^{\psi_{\tau,j}} \cdot \xi_t \quad (8)$$

190 where  $\xi_t \sim \ln \mathcal{N}(0, \sigma_{\tau})$ ,  $\alpha_{\tau} = \bar{q}_{\tau} / \prod_{i=1}^p \bar{q}_{\tau-i}^{\phi_{\tau,i}}$ , with  $\bar{q}_{\tau}$  defined as in Equation (6), and  
 191  $\xi_{t-j} = \frac{q_{t-j}}{\hat{q}_{t-j}}$ .

192 If in model (7) residuals are normal and additive, in model (8) residuals become log-  
 193 normal and multiplicative. For positive initial condition,  $q_0 > 0$ , multiplicative random

194 process ensures non-negative values of inflow process, offering a better representation of  
195 the hydrological process.

196 In Model (8), dependencies on  $q_{t-i}$  are non-linear. Model (8) can be directly employed  
197 in the forward phase of SDDP, also using a less parsimonious model than in backwards,  
198 which can produce, in some cases, considerably better results [Bartolini and Salas, 1993].  
199 SDDP forward phase requires in fact a streamflow time series, regardless of the method  
200 to generate it. In backward phase, instead, SDDP must solve a linear optimization,  
201 therefore transition from  $q_{t-1}$  to  $q_t$  must be linear. Model (8), to be applied in backward  
202 phase of SDDP, must be written as in Equation (9), which is Model (8) linearized by  
203 Taylor expansion on the argument median, i.e.  $q_{t-i} = \bar{q}_{\tau-i}$  and  $\xi_{\tau-j} = 1, \forall \tau, \forall i, \forall j$ .

$$q_t = \left[ \sum_{i=1}^p \rho_{\tau,i} q_{t-i} + \sum_{j=1}^q \omega_{\tau,i} \xi_{t-i} + \kappa_{\tau} \right] \cdot \xi_{\tau} \quad (9)$$

where parameters are defined in Equations (10) and derived in Appendix A.

$$\rho_{\tau,i} = \phi_{\tau,i} \cdot \frac{\bar{q}_{\tau}}{\bar{q}_{\tau-i}} \quad (10a)$$

$$\omega_{\tau,i} = \psi_{\tau,i} \cdot \bar{q}_{\tau} \quad (10b)$$

$$\kappa_{\tau} = \bar{q}_{\tau} \cdot \left( 1 - \sum_{i=1}^p \phi_{\tau,i} - \sum_{j=1}^q \psi_{\tau,i} \right) \quad (10c)$$

204 Linearisation introduces an approximation error that must be quantified. Equation (11)  
205 defines  $e_t$ , the error due to linearisation.

$$e_t = q_t^{\text{lin}} - q_t^{\text{nl}} \quad (11)$$

206 where  $q_t^{\text{lin}}$  and  $q_t^{\text{nl}}$  are the output of Model (9) and Model (8) for  $\xi_t = 1$ . Considering,  
207 for simplicity of notation, an univariate Multiplicative Periodic Autoregressive Model,

208  $PAR_{\tau}(1)$ , then  $e_t$  can be written as function of  $q_{t-1}$  only. Knowing the distribution  
 209  $f_{\tau}(q_{t-1})$  we can estimate that of  $e_t$ . Specifically, we are interested in the average and  
 210 extreme quantiles of  $e_t$ , defined as in Equations (12).

$$\mathbb{E}(e_{\tau}) = \int_0^{+\infty} e_t(q_{t-1}) \cdot f_{\tau}(q_{t-1}) \cdot dq_{t-1} \quad (12a)$$

$$q_p(e_{\tau}) = e_t(F_{q_{\tau-1}}^{-1}(p)) \quad (12b)$$

211 Where  $q_p(e_{\tau})$ , is the  $p$  quantile of  $e_{\tau}$ , and  $F_{q_{\tau-1}}(\cdot)$  is the Cumulative Density Function  
 212 of  $q_{\tau-1}$ . Distribution of  $q_{\tau-1}$  can be obtained from  $f(y_{\tau-1})$  through Equation (5). From  
 213 *Vecchia [1985]* *Bartolini et al. [1988]*, we know that  $f(y_{\tau-1})$  is normal with known average  
 214 and variance. Therefore  $f(q_{\tau-1})$  is lognormal, with parameters derived from  $f(y_{\tau-1})$   
 215 average and standard deviation.

216  $\mathbb{E}(e_{\tau})$ , from Equation (12a), can be used to correct results of linear Model (9) by shifting  
 217 the output value in order to have zero bias. In this case,  $-\mathbb{E}(e_{\tau})$  is to be added in Equation  
 218 (10c).

## 2.2. Non-uniform time-step length

219 In this section we propose a non-uniform discharge aggregation, which modulates the  
 220 time-step length to have a fine discharge representation only when needed.

221 Hydrological models in SDDP have so far always used a fixed time-step and periodi-  
 222 cally variable parameters. Predictive uncertainty, changing along the period, is included  
 223 in the process model by considering heteroscedastic residuals, i.e. residuals with different  
 224 variability. We consider here, instead, a variable time-step that divides the hydrological

225 period in time-steps of different length in order to maintain an approximately homoge-  
 226 neous variance.

227 In the following, we describe the proposed procedure to select the non-uniform time-step  
 228 length,  $\Delta k(\tau)$ , to aggregate data from the finest time-step,  $\tau_d = 1, \dots, T_d$  to the desired  
 229 level of aggregation,  $\tau = 1, \dots, T$ , such that the aggregated time-step is  $\Delta\tau = \Delta\tau_d \cdot \Delta k$ .

230 Starting from the data at the finest time aggregation of period  $T_d$  and time index  
 231  $\tau_d = 1, \dots, T_d$ , we identify a  $\text{PAR}_{T_d}(1)$  model on  $q_t$ , having parameters  $\phi_{\tau_d}, \sigma_{\tau_d}^2, \forall \tau_d \in$   
 232  $\{1, \dots, T_d\}$ , defined as in Equation (7). Even if such simple model may be not accurate  
 233 enough for prediction purposes, it is generally sufficient to catch the dominant dynamics.

234 Time-steps of homogeneous variability,  $\Delta k(\tau)$ , must be such that  $\text{VAR}(q_{\tau_0+\Delta k(\tau)}|q_{\tau_0})$  is  
 235 approximately homogeneous for all  $\tau$ , and  $\sum \Delta k(\tau) = T_d$ . Equation (13) defines variance  
 236 of  $q_{\tau_0+\Delta k(\tau)}$  conditional to  $q_{\tau_0}$  as function of time-step residual variance,  $\sigma_{\tau_d}^2$ .

$$\text{VAR}(q_{\tau_0+\Delta k(\tau)}|q_{\tau_0}) = \sum_{\tau_d=\tau_0+1}^{\tau_0+\Delta k(\tau)} \prod_{i=\tau_d+1}^{\tau_0+\Delta k(\tau)} \phi_i^2 \cdot \sigma_{\tau_d}^2 \quad (13)$$

237 where, by convention,  $\phi_i^2 = 1$  if  $i > \tau_0 + \Delta k(\tau)$ .

238 In Equation (13),  $\phi_{\tau_d}$  is generally close to one, especially for  $\tau_d$  where variability is  
 239 small, and time-steps  $\Delta k$  can be larger. Considering  $\prod \phi_{\tau_d}^2$  in Equation (13) as equal to  
 240 one,  $\text{VAR}(q_{\tau_0+\Delta k_i}|q_{\tau_0})$  can be written as proportional to the sum of variances only, which  
 241 we take as indicator of variability. This allows us to define a cumulated variability that  
 242 depends on  $\tau_d$  only.

243 The residual variance is used to define a Cumulative Variability in function of  $\tau$ , as in  
 244 Equation (14).

$$CV(\tau_d) = \frac{\sum_{t=1}^{\tau_d} \sigma_{\tau_d}^2}{\sum_{t=1}^{T_d} \sigma_{\tau_d}^2} \quad (14)$$

245 In Equation (14), the numerator is the cumulative variance until  $\tau_d$ , the denominator is  
 246 the cumulated variability for the entire period,  $T_d$ , to standardize  $CV$  between zero and  
 247 one. The non-uniform time-step  $\Delta k(\tau)$  is then chosen by splitting the hydrological period  
 248 in  $T$  time-steps having approximately homogenous variability, as in Equation (15).

$$\Delta k(\tau) = \left\lceil CV^{-1} \left( \frac{\tau}{T} \right) - CV^{-1} \left( \frac{\tau - 1}{T} \right) \right\rceil; \forall \tau = \{1, \dots, T\} \quad (15)$$

249 where operator  $\lceil \cdot \rceil$  returns the nearest integer. By convention,  $CV^{-1}(0) = 0$ .

250 The discharge signal is aggregated using the variable time-step, and a model is identified  
 251 on the aggregated signal according to procedure in Section 2.1, or others.

### 3. Application to the Senegal river

252 The Senegal River, West Africa, is a 1790 *km* long river. Its drainage basin extension  
 253 is 270.000 *km*<sup>2</sup>, over Guinea, Mali, Senegal and Mauritania. The river inflow is extremely  
 254 variable, following the tropical raining seasonality with a marked difference between the  
 255 dry season, in January-June, and the raining one, in July-October, when most of the  
 256 water falls in the upper part of the basin [*Albergel et al.*, 1997]. Figure 2 displays the  
 257 discharge at Soukoutali, inflow to Manantali for 64 years.

258 Manantali is an annual reservoir for hydropower, located in Mali and controlling about  
 259 50% of the total water flow of the Senegal River. Manantali was completed in 1987 and  
 260 started to produce electricity in 2002. Its benefits are shared among Mali, Mauritania,  
 261 and Senegal. These three countries participate with Guinea to the Organization for the



262 Valorization of Senegal River (Organisation pour la Mise en Valeur du fleuve Senegal,  
263 OMVS). Manantali reservoir volume is  $12 \times 10^9 m^3$ , its installed capacity is 205 MW, the  
264 average inflow is  $270 m^3/s$ , the average residence time is about one year.

265 The OMVS have gradually embraced the Integrated Water Resources Management  
266 paradigm, in which water allocation decisions are based on economic, social, technical  
267 and political factors, in accordance with stakeholders' interests. This led the OMVS  
268 to prepare a reservoir management optimization program [*Fraval et al.*, 2002]. Manantali  
269 operation was originally designed to satisfy different rival uses: energy production and low-  
270 flow augmentation on one hand; flood support on the other. Operating the reservoir for  
271 hydroelectric production reduces the annual streamflow variability, with negative effects  
272 on the ecological equilibrium and on some traditional activities in the valley, but with  
273 positive effects on water availability for irrigation and navigation, which will become a  
274 more important objective in the near future [*Bader et al.*, 2003]. This analysis, however,  
275 focuses on methodological aspects, so we consider energy production objective only.

276 In the following, at section 3.1, we define the non-uniform time-step, then we identify  
277 a streamflow process model as defined in Equations (8) and (9), and a classic Thomas-  
278 Fiering Model, for comparison. At section 3.2, we test the added value of non-uniform  
279 time-step in terms of reservoir operation.

### 3.1. Streamflow process model identification

280 The hydrological process on the Senegal river is characterised by a strong periodical  
281 component. Following the procedure described in section 2.2, we select  $\Delta k(\tau)$  using  
282  $T = 12$  time-steps, for comparison with monthly time-step. Figure 3 shows the  $CV(\tau_d)$   
283 function, defined in Equation (14), and the  $\Delta k(\tau)$  for  $T = 12$ , for the entire periodic year,

284 on plot (a), and a zoom on the period of high variability, on plot (b), for  $\tau_d$  between 135  
285 (May 15) and 285 (October 12).

286 The non-uniform time-step defined by  $\Delta k(\tau)$  is used to aggregate the daily inflow signal.  
287 The same inflow is also aggregated at the monthly time-step to be compared to the non-  
288 uniform aggregation. Figure 4 shows the aggregated observed inflow with non-uniform  
289 time-step and monthly time-step.

290 Non-uniform aggregation allocates only two time-steps for the entire recession curve,  
291 going from October to May, using the ten remaining time-steps for the raising part of the  
292 hydrography, against the seven and five time-steps used by monthly aggregation. A finer  
293 time-step during the raising part of the hydrograph should allow to better adapt to the  
294 incoming information on the inflow value.

295 Figure 5 shows the autocorrelation lag 1 on the de-trended logarithmic of discharge  
296 signal, equivalent to  $\phi_\tau$  of Model (7) for a PAR(1), being  $\phi_\tau = \frac{COV(y_t, y_{t-1})}{VAR(y_{t-1})}$  [Box et al.,  
297 1970]. In monthly aggregation,  $\phi_\tau$  stays close to 1 for different months during the dry  
298 period, reaching 0.5 at  $\tau = 8$ . The non-uniform aggregation autocorrelation is more  
299 regular: most of the cases lay between 0.6 and 0.8. This results suggest that the non-  
300 uniform aggregation can provide a more effective distribution of decision instants. This  
301 is further investigated in section 3.2.

302 From the non-uniform aggregated inflow signals, following the procedure described in  
303 2.1, we identify a multiplicative model as in Equations (9) and (8) and an additive Thomas-  
304 Fiering model. We refer to these models as multiplicative and additive model. A PAR(1)  
305 represents the process sufficiently well. The residuals autocorrelation is approximately  
306 zero for both the multiplicative and the additive model, being within the 95% confidence

307 band, for most of the lags larger than zero. This confirms the validity of the non-uniform  
308 aggregation, for it preserves the process as markov.

309 We test whether the residual sampling distribution adhere to the prescribed one, log-  
310 normal for the multiplicative case, normal for the additive. Both residuals must be stan-  
311 dardised by the time variant variance,  $\sigma_\tau$ . For the multiplicative residuals, we test their  
312 normality on  $\log(\Xi_t)$ . Figure 6 presents the normality plot for both the logarithm of resid-  
313 uals for the multiplicative model and the residuals for the additive model, showing that  
314 the multiplicative model follows the prescribed distribution, whereas the additive model  
315 suffers from a fat tail on higher values. A Kolmogorov-Smirnov test [*Dekking, 2005*]  
316 on the standardized logarithms of residuals does not reject the hypothesis of a normal  
317 distribution, with a p-value of 0.07. The same test on the additive model standardized  
318 residual gives a p-value of  $10^{-6}$ , leading to rejecting the assumption of residuals being nor-  
319 mally distributed with sufficient confidence. The positive results on the  $(\xi_t)$  distribution  
320 correctness gives us further confidence in the validity of the multiplicative model.

321 Figure 7 shows the observed discharge data and their match with the 95% confidence  
322 bands for both the multiplicative and the additive model. The confidence bands are  
323  $\bar{q}_\tau \cdot \exp(\pm 2 \cdot VAR(y_t))$  for the multiplicative model (bold continuous lines) and  $\mathbb{E}(q_t) \pm$   
324  $2 \cdot VAR(q_t)$  for the additive model (bold dashed lines). The signal variance  $VAR(q_t)$   
325 or  $VAR(y_t)$ , is derived from  $\{\sigma_\tau\}_{\tau=1}^T$  as described in *Bartolini et al. [1988]*. From Figure  
326 7 we see how the additive model has a non-negligible probability of producing negative  
327 inflow. Moreover, lower confidence band for the additive model is, for some time-steps,  
328 much lower than the lowest observed discharge. whereas the multiplicative model follows  
329 closely the observed signal variance for the entire period.

330 We quantify the approximation due to linearization using Equations (12) for error av-  
331 erage, and 5% and 95% quantile of  $q_{\tau-1}$ , examining both the average for all  $\tau$  and the  
332 largest value in  $\tau$ . The average error is  $0.6 \text{ m}^3/\text{s}$ , or 0.3% of median discharge, which  
333 we consider a relatively small value. Even for the largest error average, at  $\tau = 7$ , is  $23$   
334  $\text{m}^3/\text{s}$ , which is less than 3% of median discharge. Error at 5% quantile is, on average, 4%  
335 of median discharge, with a peak of 7% at  $\tau = 4$  ( $15 \text{ m}^3/\text{s}$  in absolute value). Error at  
336 95% quantile is 11% of median discharge, with a maximum absolute value of  $88 \text{ m}^3/\text{s}$  at  
337  $\tau = 7$ , which is 11% of median discharge. As a summary, we can state that error due to  
338 linearization is adequately small; its average is generally negligible, growing to about 5%  
339 and 10% of median discharge at 5% and 95% quantiles of  $q_{\tau-1}$ .

340 Figure 8 shows the detail of the error due to linearization for  $\tau = 7$ . Plot (a) compares  
341 output of nonlinear and linearized multiplicative model, i.e. model (8) and model (9)  
342 for  $\xi_t = 1$ , in function of  $q_{t-1}$ . The linear model is tangent to the nonlinear one at the  
343 linearisation point, i.e. at  $\bar{q}_{\tau-1}$ . Going further from the median the models diverge, even  
344 if the difference stays small. Plot (b) presents the error magnitude in function of  $q_{t-1}$  next  
345 to its probability of occurrence,  $f(q_{\tau-1})$ .

346 On a multiplicative  $PAR_{\tau}(1)$ ,  $\phi_{\tau} < 1$  for most of  $\tau$ , because stationarity condition  
347 requires that  $\prod_{\tau=1}^T \phi_{\tau} < 1$ . This implies that model (8) is a concave function in  $q_{t-1}$  and  
348  $e_t > 0, \forall q_{t-1}$ , i.e. Model (9) systematically overestimate its nonlinear version. A large  
349 number of time-steps per period results in a shorter time-steps length. As a consequence,  
350  $\phi_{\tau}$  values will be closer to one, for  $\phi_{\tau}$  is the autocorrelation lag 1 in a AR model [Box  
351 *et al.*, 1970], resulting in a smaller error due to linearization.

### 3.2. Effects of non-uniform aggregation on reservoir operation

352 We compare the system performances and behaviour of reservoir operation between  
353 monthly and non-uniform discharge aggregation to assess the advantage of the latter.

354 The system model is made of a reservoir and a hydrological component, as in Figure 9.  
355 Streamflow at Soukoutali, output of the hydrological model, is the input to the reservoir.  
356 The objective is energy production. Each element of the optimization problem is detailed  
357 in the following.

358 The reservoir is modelled as in Equation (1c). The reservoir input is the discharge from  
359 the hydrological component  $q_t$ . Reservoir outputs are: discharge through turbines  $r_{\text{turb},t}$ ,  
360 and discharge through spillways,  $r_{\text{spill},t}$ . Evaporation is not considered in this analysis.

361 Inequalities (16) are constraints on discharge decisions and reservoir volume.

$$v_{\min} \leq v_t \leq v_{\max} \quad (16a)$$

$$0 \leq r_{\text{turb},t} \leq r_{\text{turb},\max} \quad (16b)$$

$$r_{\text{spill},t} \geq m_{\text{safety}} \cdot (v_t - v_{\text{safety}}) \quad (16c)$$

362 Inequalities (16a) and (16b) are physical constraints, derived from the system charac-  
363 teristics. Inequality (16c) is a legal condition that forces to draw down the reservoir when  
364 its volume exceeds the safety threshold,  $v_{\text{safety}}$ . Constraint on release through spillages is  
365 large enough for never being active during the simulation period, as verified a posteriori,  
366 and therefore it is not included. Inequalities (16b) and (16c) are implemented as hard  
367 constraint, inequality (16a) is implemented as soft constraint to avoid non-feasibility.

368 The system objective is the yearly average energy production  $J_E = 1/N_{\text{years}} \sum_{t=1}^{T \times N_{\text{years}}} E_t$ ,  
369 composed of the sum of the daily energy production,  $E_t$ , as defined in Equation (17), for  
370 the entire simulation horizon  $T \times N_{\text{years}}$ , where  $N_{\text{years}} = 43$ , from 1970 to 2012.

$$E_t = \eta \cdot \Delta k(\tau) \cdot \Delta h_t \cdot r_{\text{turb},t} \quad (17)$$

In Equation (17),  $\Delta h_t$  is the hydraulic head [m],  $r_{\text{turb},t}$  is discharge trough turbines [m<sup>3</sup>/s].  $\eta$  is a multiplicative factor, such that  $\eta = \rho \cdot g \cdot \hat{\eta}(r_t, \Delta h) \cdot 24 \cdot 10^{-6}$ , where  $\rho$  is the water density, 1000 [kg/m<sup>3</sup>],  $g$  is the gravity acceleration, 9.8 [m/s<sup>2</sup>],  $\hat{\eta}$  is the efficiency coefficient, considered equal to 0.9, 24 and 10<sup>-6</sup> are unit transformation coefficients, [h/d] and [MW/W].  $E_t$  is expressed in [MWh].

Equation (17), to be employed in linear optimization within SDDP, is approximated by expressing it as linear function of  $r_t$  and  $v_t$ , linearized at an operational point, under the hypothesis of cylindric reservoir. Equation (18) defines the operational time-step objective indicator used in SDDP, being the weighted sum of releases and volume.

$$E_t^{\text{op}} = E_{0,t} + \eta \cdot \Delta k(\tau) \cdot \left\{ + \left[ \frac{r_{\text{turb},0}}{A_0} \right] \cdot v_t + [h_0 - h_0^v - m_v(1 + r_{\text{turb},0})] \cdot r_{\text{turb},t} + [-m_v] \cdot r_{\text{spill},t} \right\} \quad (18)$$

In Equation (18),  $E_{0,t} = \eta \cdot \Delta k(\tau) \cdot \left\{ [h_0 - h_0^v] \cdot r_{\text{turb},0} - \left[ \frac{r_{\text{turb},0}}{A_0} \right] \cdot v_0 - [h_0 - h_0^v - m_v(1 + r_{\text{turb},0})] \cdot r_{\text{turb},0} + m_v \cdot r_{\text{spill},0} \right\}$ . Parameters of Equation (18) and their derivation are described in Appendix B.

Equation (17) is linearized at a normal operational point, that is the reservoir state at which the reservoir is mostly operated, either historically observed or deduced from system characteristics. We infer the operational point from the reservoir characteristics,

386 considering  $v_0 = v_{\text{safety}}$ , i.e. the safety limit,  $r_{\text{turb},0} = \mathbb{E}(q_t)$ , i.e. the average inflow to the  
387 reservoir, and  $r_{\text{spill},0} = 0 \text{ m}^3/\text{s}$ , which considers no release through the spillages.

388 The optimization is performed using 25 extractions for the forward phase and 25 for the  
389 backward one until convergence, attained at an accuracy level of  $2 \times 10^7$ . This accuracy  
390 level lays within the  $\pm 2\sigma$  of forward simulation results. We can consider with sufficient  
391 confidence that the algorithm has converged to the optimum.

392 Performance results show a moderate improvement for non-uniform aggregation. Op-  
393 timal solution gives  $J_E$  equal to 930 GWh/year for the monthly aggregation, and 945  
394 GWh/year in the non-uniform case, equivalent to an improvement of one week of average  
395 energy production.

396 We analyse the reservoir operation behaviour. Figure 10 shows the reservoir volume and  
397 the release trough turbines for the monthly and the non-uniform aggregation in response to  
398 the 2005 inflow scenario. Inflow peak is larger than maximum discharge trough turbines;  
399 therefore, to avoid spillage, the reservoir must be drawn down before the high flow period,  
400 in order to create a buffer that stores part of the incoming water. SDDP optimal operating  
401 rules are the results of an optimal compromise between the objectives of keeping a high  
402 water level and avoiding spillages.

403 Figure 10 plot (b) shows how the reservoir operation using non-uniform steps adjusts  
404 decisions at higher frequency during high uncertainty periods, adapting more rapidly to  
405 the new observed discharge. Thanks to this rapid adaptation during the high discharge  
406 period, reservoir operation using non-uniform aggregation can draw down the reservoir  
407 less, as shown in Figure 10 plot (a).

#### 4. Conclusions and Discussion

408 SDDP, to be employed for reservoir operational rules design, requires the identification  
409 of a linear streamflow process model. Presently, models from the literature use almost  
410 always a periodic autoregressive model with monthly time-steps. In this study we proposed  
411 an innovative streamflow process model to be used in SDDP. Model features are i) a log-  
412 normal multiplicative stochastic component, which guarantees positive discharge values,  
413 and ii) non-uniform time-steps, which makes the process approximately homoscedastic  
414 i.e. having constant variability. The multiplicative non-linear model can be employed  
415 in the SDDP forward phase directly, whereas a linearized version must be used in the  
416 backward phase. We showed how to identify the multiplicative streamflow process model  
417 and its linearized version, and how to derive the non-uniform time-steps lengths from  
418 discharge data. Model identification for the proposed model is not more complex than  
419 for classic periodic autoregressive models with monthly time-step. The proposed features  
420 are independent from each other, then each of them can be applied separately. This work  
421 address specific problems encountered in SDDP, but some results may have a broader  
422 (potential) validity in time series modeling for synthetic streamflow sequences generation.

423 We applied the model to the streamflow process at Soukoutali, on the Senegal River,  
424 for the operational rules design of a single reservoir system. Model identification would  
425 not be different if the reservoirs were many. The proposed multiplicative model offers  
426 a better representation of the streamflow process both in the forward phase, where it  
427 correctly represents the streamflow dynamics and the discharge distribution, and in the  
428 backward phase, where it correctly represents the residual distribution, avoiding the fat-  
429 tail phenomenon, otherwise present in the classic Thomas-Fiering model. The model



430 using non-uniform time-steps has a relatively homogeneous variance. This brings in a  
 431 practical advantage: the non-uniform time-steps follow closer the changing hydrological  
 432 variability along the year, adapting the decision more frequently during high variability  
 433 periods, resulting in enhanced system performance evaluation. If time-step aggregation is  
 434 sufficiently fine, a non-uniform aggregation may even make time decomposition needless.

435 Model linearization, used in the backward phase, introduces an error. The analysis  
 436 on error due to linearisation show that the error average is negligible, growing to about  
 437 5%-10% of median discharge at 5%-95% quantiles of  $q_{\tau-1}$ , which we consider satisfactory;  
 438 this depends, however, on the specific test-case and, a priori, we cannot exclude it to be  
 439 a limitation for this model.

## Appendix A: Linear model parameters derivation

440 Parameters  $\rho_{\tau,i}$ ,  $\omega_{\tau,i}$ ,  $\kappa_{\tau}$  are derived by linerization of Model (8) on the median of its  
 441 deterministic inputs, at  $q_{t-i} = \bar{q}_{t-i}$ , with  $\bar{q}_{\tau}$  defined in Equation (6), and  $\xi_{t-j} = 1$ .

442 Equation A1 is model (8) for  $\xi = 1$  written as Taylor expansion on its deterministic  
 443 inputs.

$$\begin{aligned}
 q_t^{\text{nl}} \approx & \sum_{i=1}^p \frac{\partial q_t^{\text{nl}}}{\partial q_{t-i}} \cdot (q_{t-i}^{\text{nl}} - \bar{q}_{t-i}) + \\
 & \sum_{j=1}^q \frac{\partial q_t^{\text{nl}}}{\partial \xi_{t-j}} (\xi_{t-j} - \bar{\xi}_{t-j}) + \\
 & q_t^{\text{nl}}(q_{t-i}, \xi_{t-j})
 \end{aligned} \tag{A1}$$

444 Equations (A2) are the derivatives of  $q_t$  on inputs  $q_{t-i}$  and  $\xi_{t-j}$ .

$$\frac{\partial q_t^{\text{nl}}}{\partial q_{t-i}} = \alpha_\tau \cdot \left[ \phi_{\tau,i} \cdot q_{t-i}^{(\phi_{\tau,i}-1)} \cdot \prod_{k \in \{1, \dots, p\} \setminus i} q_{t-k}^{\phi_{\tau,k}} \cdot \prod_{j=1}^q \xi_{t-j}^{\psi_{\tau,i}} \right] \quad (\text{A2a})$$

$$\frac{\partial q_t^{\text{nl}}}{\partial \xi_{t-j}} = \alpha_\tau \cdot \left[ \psi \cdot \xi_{t-j}^{(\psi_{\tau,j}-1)} \cdot \prod_i^p q_{t-i}^{\phi_{\tau,i}} \cdot \prod_{k \in \{1, \dots, q\} \setminus j} \xi_{t-k}^{\psi_{\tau,k}} \right] \quad (\text{A2b})$$

445 Separating members of Equation (A1) by their inputs, and considering that  $\alpha_\tau =$   
 446  $\bar{q}_\tau / \prod_{i=1}^p \bar{q}_{\tau-i}^{\phi_{\tau,i}}$  we derive, in Equations (A3), parameters (10) of Model (9).

$$\begin{aligned} \rho_{\tau,i} &= \frac{\partial q_t^{\text{nl}}}{\partial q_{t-i}}(\bar{q}_{t-i}, \bar{\xi}_{t-j}) = \\ &= \frac{\bar{q}_\tau}{\prod_{i=1}^p \bar{q}_{\tau-i}^{\phi_{\tau,i}}} \cdot \left[ \phi_{\tau,i} \cdot \frac{\bar{q}_{t-i}^{\phi_{\tau,i}}}{\bar{q}_{t-i}} \cdot \prod_{k \in \{1, \dots, p\} \setminus i} \bar{q}_{t-k}^{\phi_{\tau,k}} \right] = \\ &= \phi_{\tau,i} \cdot \frac{\bar{q}_\tau}{\bar{q}_{\tau-i}} \end{aligned} \quad (\text{A3a})$$

$$\begin{aligned} \omega_{\tau,j} &= \frac{\partial q_t^{\text{nl}}}{\partial \xi_{t-j}}(\bar{q}_{t-i}, \bar{\xi}_{t-j}) = \\ &= \frac{\bar{q}_\tau}{\prod_{i=1}^p \bar{q}_{\tau-i}^{\phi_{\tau,i}}} \cdot \left[ \psi \cdot \prod_i^p \bar{q}_{t-i}^{\phi_{\tau,i}} \right] = \\ &= \psi_{\tau,i} \cdot \bar{q}_\tau \end{aligned} \quad (\text{A3b})$$

$$\begin{aligned} \kappa_\tau &= q_t^{\text{nl}}(\bar{q}_{t-i}, \bar{\xi}_{t-j}) - \sum_{i=1}^p \frac{\partial q_t^{\text{nl}}}{\partial q_{t-i}}(\bar{q}_{t-i}, \bar{\xi}_{t-j}) \cdot \bar{q}_{t-i} - \sum_{j=1}^q \frac{\partial q_t^{\text{nl}}}{\partial \xi_{t-j}}(\bar{q}_{t-i}, \bar{\xi}_{t-j}) \cdot \bar{\xi}_{t-j} = \\ &= \bar{q}_\tau \cdot \left( 1 - \sum_{i=1}^p \phi_{\tau,i} - \sum_{j=1}^q \psi_{\tau,i} \right) \end{aligned} \quad (\text{A3c})$$

## Appendix B: Energy objective linearization

447 Energy function, from Equation 17, is written in Equation (B1) as function of problem  
 448 variables.

$$E_t = \eta \cdot \Delta k(\tau) \cdot [R_1(v_t) - R_2(r_{\text{turb},t} + r_{\text{spill},t})] \cdot r_{\text{turb},t} \quad (\text{B1})$$

449 In Equation (B1), the hydraulic head  $\Delta h$ , is written as function of problem variables;

450  $h_t = R_1(v_t)$  is the stage-storage curve and  $h_t^v = R_2(r_{\text{turb},t} + r_{\text{spill},t})$  the rating-curve, where

451  $h_t$  is the water level in the reservoir and  $h_t^v$  the tailwater elevation, downstream of the

452 reservoir.

453 Equations (B2) are the partial derivatives of Equation (B1).

$$\begin{aligned} \frac{\partial E_t}{\partial v_t} &= \eta \cdot \Delta k(\tau) \cdot R_1'(v_t) \cdot r_{\text{turb},t} \\ \frac{\partial E_t}{\partial r_{\text{turb},t}} &= \eta \cdot \Delta k(\tau) \cdot \left[ R_1(v_t) - \right. \\ &\quad \left. R_2'(r_{\text{turb},t} + r_{\text{spill},t}) \cdot r_{\text{turb},t} - R_2(r_{\text{turb},t} + r_{\text{spill},t}) \right] \\ \frac{\partial E_t}{\partial r_{\text{spill},t}} &= -\eta \cdot \Delta k(\tau) \cdot R_2'(r_{\text{turb},t} + r_{\text{spill},t}) \end{aligned} \quad (\text{B2})$$

454 In Equations (B3), we consider a cylindrical reservoir in proximity of the operational

455 reservoir water level,  $h_0$ , and a linear rating curve in proximity of discharges  $r_{\text{turb},0} + r_{\text{spill},0}$ .

$$R_1(v_t) \approx h_0 + \frac{1}{A_0} \cdot (v_t - v_0) \quad (\text{B3a})$$

$$R_2(r_{\text{turb},t} + r_{\text{spill},t}) \approx h_0^v + m_v(r_{\text{turb},t} - r_{\text{turb},0} + r_{\text{spill},t} - r_{\text{spill},0}) \quad (\text{B3b})$$

456 where  $A_0$  is the reservoir surface corresponding to  $h_0$ ,  $h_0^v = R_2(r_{\text{turb},0} + r_{\text{spill},0})$ , and

457  $m_v = R_2'(r_{\text{turb},0} + r_{\text{spill},0})$ .

458 Considering Equations (B2) and (B3), we get, in Equation (B4), the linear approxima-

459 tion of Equation (B1).

$$\begin{aligned}
E_t &\approx E_t(v_0, r_{\text{turb},0}, r_{\text{spill},0}) + \\
&+ \frac{\partial E_t}{\partial v_t} \cdot (v_t - v_0) + \\
&+ \frac{\partial E_t}{\partial r_{\text{turb},t}} \cdot (r_{\text{turb},t} - r_{\text{turb},0}) + \\
&+ \frac{\partial E_t}{\partial r_{\text{spill},t}} (r_{\text{spill},t} - r_{\text{spill},0}) =
\end{aligned} \tag{B4a}$$

$$\begin{aligned}
&= \eta \cdot \Delta k(\tau) \cdot \left\{ [h_0 - h_0^v] \cdot r_{\text{turb},0} + \right. \\
&+ \left[ \frac{r_{\text{turb},0}}{A_0} \right] \cdot (v_t - v_0) + \\
&+ [h_0 - h_0^v - m_v(1 + r_{\text{turb},0})] (r_{\text{turb},t} - r_{\text{turb},0}) \\
&+ \left. [-m_v](r_{\text{spill},t} - r_{\text{spill},0}) \right\}
\end{aligned} \tag{B4b}$$

460 Equation (B4b) gives the weights for  $v_t$ ,  $r_{\text{turb},t}$ , and  $r_{\text{spill},t}$  that maximise energy, as in

461 Equation (18). For constant tailwater elevation, then  $m_v = 0$ , and Equations (B4b) and

462 (18) can be simplified.

### Appendix C: List of main variables

463 We use here the classic convention of representing vectors in bold.

464  $\mathbf{v}_t$  Reservoir volumes [ $m^3$ ]

465  $\mathbf{r}_t$  Discharge decision [ $m^3/s$ ]

466  $\mathbf{q}_t$  (Flow) scenarios [ $m^3/s$ ]

467  $N_{\text{res}}$  Number of reservoirs [—]

468  $N_{\text{dec}}$  Number of discharge decisions [—]

469  $N_{\text{scen}}$  Number of scenarios [—]

470  $I$  Input matrix [—]

471  $O$  Output matrix [—]

- 472  $\mathbf{c}_t(\cdot)$  Inequality constraints  $[-]$
- 473  $g(\cdot)$  Time-step objective function  $[-]$
- 474  $F(\cdot)$  Cost-to-go function  $[-]$
- 475  $\mathcal{F}(\cdot)$  cost-to-go function approximation by Bender's cuts  $[-]$
- 476  $\mathbb{E}$  Expected value  $[-]$
- 477  $f_{\mathbf{Q}_t}(\mathbf{q}_t)$ ; Probability density function of  $q_t$   $[-]$
- 478  $\alpha_{\tau,i}, \phi_{\tau,i}, \psi_{\tau,i}, \sigma_{\tau}^2$  Periodic ARMA parameters
- 479  $\rho_{\tau,i}, \omega_{\tau,i}, \kappa_{\tau}$  Linearized multiplicative model parameters
- 480  $\bar{q}_{\tau}$  Climatic average of  $Q_{\tau}$   $m^3/s$
- 481  $t$  Time-step index  $[-]$
- 482  $\tau$  Periodic time-step index  $[-]$
- 483  $\Delta t$  Daily time-step length  $[86400s]$
- 484  $\Delta k$  Number of daily time-steps  $[days]$
- 485  $T$  Period length  $[-]$
- 486  $H$  Optimization horizon  $[-]$
- 487  $e_t$  Error due to model linearization  $[m^3/s]$
- 488  $CV$  Cumulative variance  $[-]$
- 489  $v_{\min}$  Minimum reservoir volume  $[3.9 \times 10^9 m^3]$
- 490  $v_{\max}$  Maximum reservoir volume  $[1.5 \times 10^{10} m^3]$
- 491  $v_{\text{safety}}$  Reservoir volume safety limit  $[1.18 \times 10^{10} m^3]$
- 492  $r_{\text{spill,max}}$  Maximum discharge through turbines  $[500 m^3/s]$
- 493  $J_E$  Annual average energy production  $[GWh]$

- 494  $E$  Daily energy production [ $MWh$ ]
- 495  $E^{op}$  Linearized energy production objective [ $MWh$ ]
- 496  $v_0$  Operational volume [ $1.18 \times 10^{10} m^3$ ]
- 497  $r_{turb,0}$  Operational release through turbines [ $270 m^3/s$ ]
- 498  $r_{spill,0}$  Operational release through spillages [ $0 m^3/s$ ]

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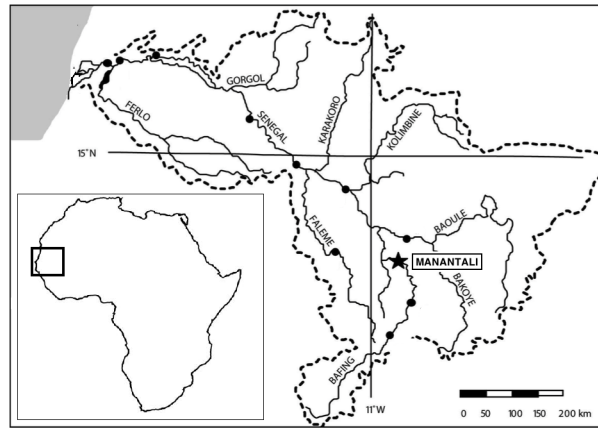
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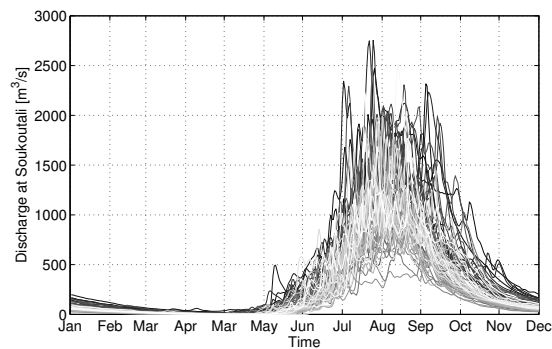
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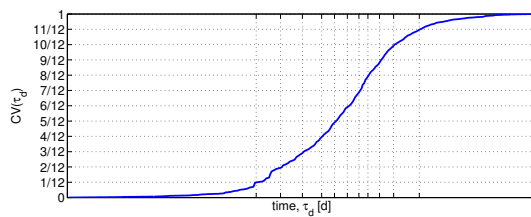
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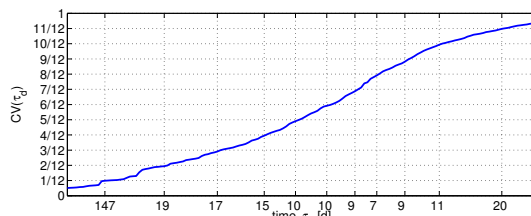
**Figure 1.** Map of the Senegal Basin.



**Figure 2.** Inflow at Soukoutali from 1 January 1950 to 31 December 2013, daily time-step.

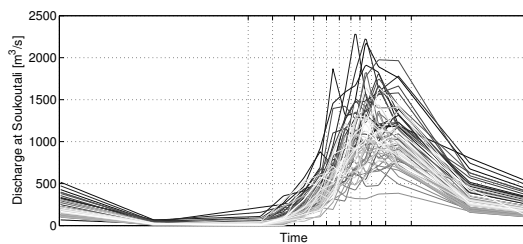


(a)  $\tau_d$  between 1 and 365

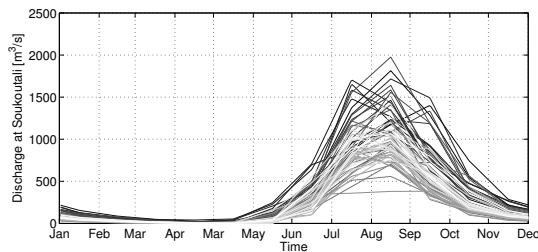


(b)  $\tau_d$  between 135 and 285

**Figure 3.** Cumulative Variance on  $\tau_d$ .



(a) Non-uniform aggregation



(b) Monthly aggregation

**Figure 4.** Inflow at Soukoutali from 1 January 1950 to 31 December 2013, aggregated observed discharge.

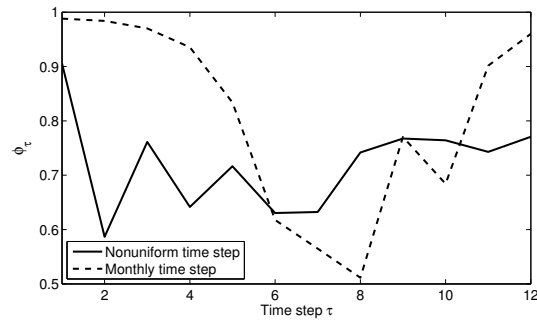


Figure 5.  $\phi_\tau$  for non-uniform and monthly time-step

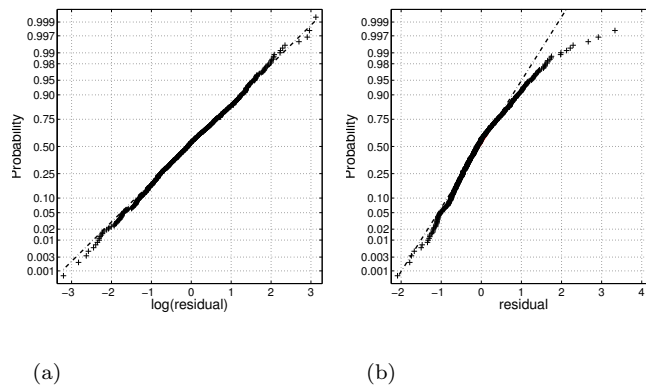
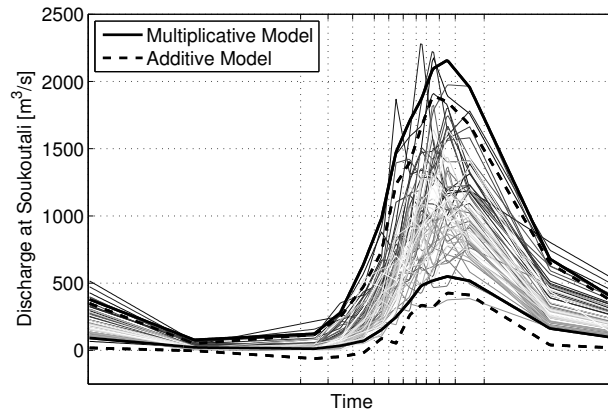
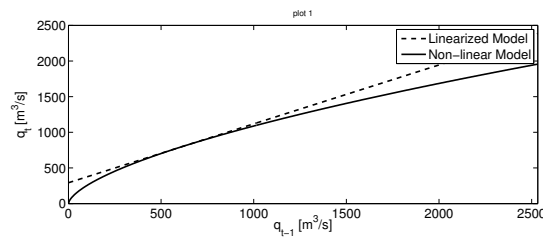


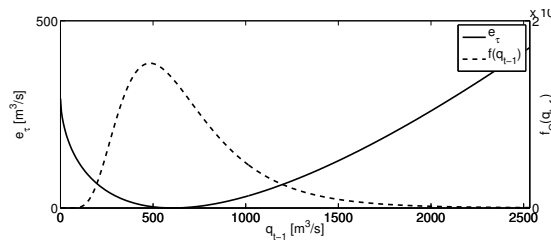
Figure 6. Normality plot for logarithms of model residuals of Model (9), Multiplicative Model (a), and residuals of Thomas-Fiering, Additive Model (b).



**Figure 7.** Observed discharge data and 95% confidence bands for Model (8), or multiplicative model (continuous bold lines) and Thomas-Fiering, or additive model (dashed bold lines).

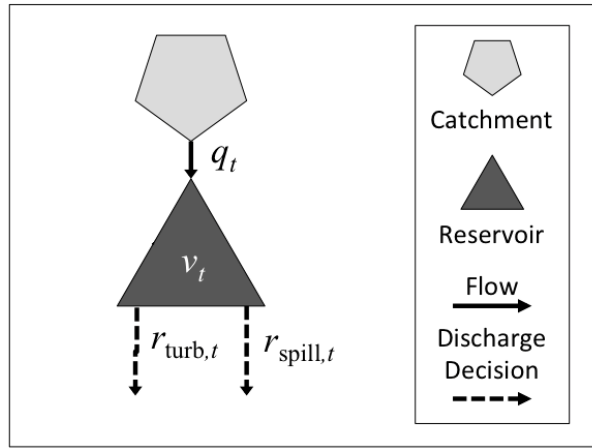


(a) Linear vs non-linear model.

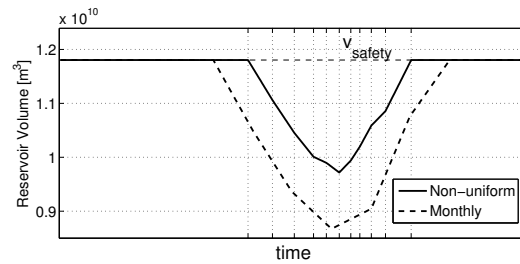


(b) error due to linearisation vs  $q_{t-1}$  distribution.

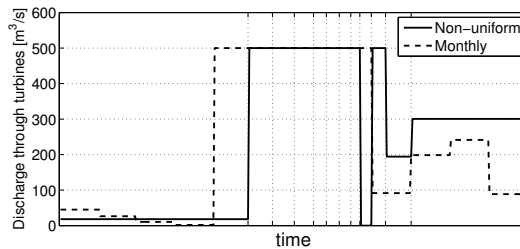
**Figure 8.** Error due to linearisation at  $\tau = 7$ .



**Figure 9.** Schema of the Senegal River System at Manantali, including: Manantali reservoir volume,  $v_t$ , inflows at Soukoutali,  $q_t$ , and discharge decisions,  $r_t$ .



(a)



(b)

**Figure 10.** Reservoir volume (a) and discharge trough turbines (b) for monthly (dashed line) and non-uniform (continuous line) aggregation, year 2005.