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Aymen Braghtha. Darboux systems with a cusp point and pseudo-abelian integrals. 2018. hal-01761229

HAL Id: hal-01761229

<https://hal.science/hal-01761229v1>

Preprint submitted on 8 Apr 2018

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DARBOUX SYSTEMS WITH A CUSP POINT AND PSEUDO-ABELIAN INTEGRALS

AYMEN BRAGHTHA

ABSTRACT. We study pseudo-abelian integrals associated with polynomial deformations of Darboux systems having a cuspidal singularity. Under some hypothesis we prove local boundedness of the number of their zeros.

1. INTRODUCTION AND MAIN RESULTS

The second part of Hilbert's 16th problem, asking for the maximum of the numbers of limit cycles and their relative positions for all planar polynomial differential systems of degree n . A weak version of this problem, proposed by Arnold, asking for the maximum of the numbers of isolated zeros of abelian integrals of all polynomial one-forms ω of degree n over algebraic ovals $\gamma(t) \in H_1(P^{-1}(t))$, where $P \in \mathbb{C}[x, y]$ of degree m . In [11, 8], Varchenko and Khovanskii prove the following result

Theorem 1.1. *There exist a uniform bound, depending only n and m , for the number of real zeros of abelian integrals.*

Varchenko and Khovanskii showed the existence of a uniform local bound on the number of zeros of abelian integrals when deforming the polynomial P and the polynomial form ω in their respective spaces. Next, the result is obtained using the fact that the space of parameters can be considered as being compact.

General explicit double exponential upper bound was achieved only in [2] by completely different methods. Exact upper bounds are still absent.

Arnold posed with insistence the analogous problem form more general polynomial deformations of integrable systems. In particular for deformations of system having a Darboux first integral. Then, instead of abelian integrals, one encounters pseudo-abelian integrals.

Pseudo-abelian integrals are integrals $I(t) = \int_{\gamma(t)} \frac{\eta}{M}$ of rational one-forms along cycles $\gamma(t) \subset \{H = t\}$, where

$$H = \prod_{i=1}^k P_i^{a_i}, \quad M = \prod_{i=1}^k P_i, \quad a_i \in \mathbb{R}_+^*, \quad P_i \in \mathbb{R}[x, y],$$

where M is an integrating factor. These integrals appear as principal part of the Poincaré displacement function of the deformation $\theta + \varepsilon\eta$ along $\gamma(t)$, where

$$\theta = M \frac{dH}{H}, \quad \eta = Rdx + Sdy, \quad R, S \in \mathbb{R}[x, y].$$

In [3, 10], Bobieński, Mardešić and Novikov prove the following result

Date: April 8, 2018.

2010 Mathematics Subject Classification. 34C07, 34C08.

Key words and phrases. Pseudo-abelian integrals, Limit cycles, Darboux integrability.

Theorem 1.2. *Let H, M, η be as above. Under some genericity hypothesis, there exists a uniform bound for number of zeros of pseudo-abelian integrals associated to Darboux integrable systems close to θ .*

Here we prove an analogous result in one of non-generic cases. Another non-generic cases was studied in [1], [4] and [5].

Consider Darboux integrable system $\omega_0 = M \frac{dH}{H}$, M is an integrating factor, where

$$H = P_0^a \prod_{i=1}^k P_i^{a_i}, \quad M = P_0 \prod_{i=1}^k P_i, \quad P_i \in \mathbb{R}[x, y], \quad a, a_i > 0. \quad (1)$$

Let $\omega_\varepsilon = M_\varepsilon \frac{dH_\varepsilon}{H_\varepsilon}$ be an unfolding of the form ω_0 , where ω_ε are one-forms with the Darboux first integral

$$H_\varepsilon = P_\varepsilon^a \prod_{i=1}^k P_i^{a_i}, \quad M_\varepsilon = P_\varepsilon \prod_{i=1}^k P_i, \quad P_\varepsilon(x, y) = y^2 - x^3 - \varepsilon x^2, \quad P_i(0, 0) \neq 0, i = 1, \dots, k. \quad (2)$$

Assume that the system $\omega_\varepsilon = M_\varepsilon \frac{dH_\varepsilon}{H_\varepsilon}$ has a family $\gamma_\varepsilon(h) \subset H_\varepsilon^{-1}(h)$ of cycles. Consider the polynomial deformation of the system ω_ε .

$$\theta_{\varepsilon, \varepsilon_1} = \omega_\varepsilon + \varepsilon_1 \eta, \quad \varepsilon_1 > 0, \quad \eta = Rdx + Sdy, \quad R, S \in \mathbb{R}[x, y]. \quad (3)$$

The linear part in deformation parameter ε_1 of Poincaré first return map is given by the pseudo-abelian integrals

$$I_\varepsilon(h) = \int_{\gamma_\varepsilon(h)} \frac{\eta}{M_\varepsilon}. \quad (4)$$

Assume that the levels curves $P_\varepsilon = 0$ and $P_i = 0$ intersect transversally and all level curves $P_i = 0, i = 1, \dots, k$ are smooth and together with the line at infinity intersect two by two transversally and no three of them intersect in the same point.

The Darboux first integral H_ε has two critical points $p_\varepsilon = (\frac{-2\varepsilon}{3}, 0)$, which is a center and a saddle point $p = (0, 0)$ for $\varepsilon \neq 0$, coincide with the cusp point $p_0 = (0, 0)$ for $\varepsilon = 0$. Then, for $\varepsilon \neq 0$, two singular points p_ε and p bifurcate from p_0 .

For $\varepsilon \neq 0$, we choose a compact region D which is bounded by $P_\varepsilon = 0$ and some separatrices $P_i = 0, i = 1, \dots, k$ such that the center p_ε is outside of D . Assume that the cycles $\gamma_\varepsilon(h) \subset H_\varepsilon^{-1}(h)$ filling D , see Figure 1.

The main result of this paper is the following theorem

Theorem 1.3. *Let $I_\varepsilon(h)$ be the family of pseudo-abelian integrals as defined above. Then there exists a bound for the number of isolated zeros of the pseudo-abelian integrals $I_\varepsilon(h)$. The bound is locally uniform with respect to all parameters, i.e. the parameters of η , the coefficients of the polynomials P_i , the exponents a, a_i and the parameter ε .*

The Darboux systems $\omega_\varepsilon = M_\varepsilon \frac{dH_\varepsilon}{H_\varepsilon}$ have a family of cycles in the basin of the center p_ε bifurcating from $p_0 = (0, 0)$. To give a ε -uniform estimate for the number of zeros of the pseudo-abelian integrals we make the blowing-up of the cusp point of the family in the product space (x, y, ε) of phase and parameter spaces. The family blowing-ups were introduced by Denkowska and Rousarie in [6].

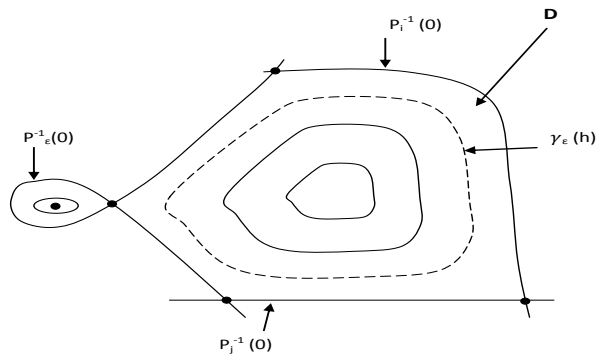


FIGURE 1. The real phase portrait of H_ε for $\varepsilon \neq 0$.

2. BLOWING-UP THE CUSP POINT

Let \mathbb{C}^3 be equipped with system of coordinates (x, y, ε) . Denote \mathcal{F} the one-dimensional foliation on \mathbb{C}^3 which is given by the 2-form $\omega_\varepsilon \wedge d\varepsilon, \omega_\varepsilon \in \Omega^1(\mathbb{C}^3)$. This foliation has a cuspidal singularity at the origin. We want our blow-up to simplify this singularity. This requirements leads to the quasi-homogeneous blow-up with weights $(2, 3, 2)$.

Recall the construction of the quasi-homogeneous blow-up. We define the weighted projective space $\mathbb{C}\mathbb{P}_{2:3:2}^2$ as factor space of \mathbb{C}^3 by the \mathbb{C}^* action $(x, y, \varepsilon) \mapsto (r^2X, r^3Y, r^2E)$. The quasi-homogeneous blow-up of \mathbb{C}^3 at the origin is defined as the incidence three dimensional manifold W . Let $q = (x, y, \varepsilon) \in \mathbb{C}^3, [(X, Y, E)] \in \mathbb{C}\mathbb{P}_{2:3:2}^2$. Then $W = \{(p, q) \in \mathbb{C}\mathbb{P}_{2:3:2}^2 \times \mathbb{C}^3 : \exists r \in \mathbb{C} : (x, y, \varepsilon) = (r^2X, r^3Y, r^2E)\}$ what means that the point q belongs to the closure of the equivalence class defined by $p \in \mathbb{C}\mathbb{P}_{2:3:2}^2$.

The blow-down $\sigma : W \rightarrow \mathbb{C}^3$ is just the restriction to W of the projection $\mathbb{C}\mathbb{P}_{2:3:2}^2 \times \mathbb{C}^3 \rightarrow \mathbb{C}^3$. For future applications we will need explicit formula for the blow-up in the standard affine charts of W . The projective space $\mathbb{C}\mathbb{P}_{2:3:2}^2$ is covered by three affine charts: $W_1 = \{x \neq 0\}$ with coordinates (Y_1, E_1) , $W_2 = \{y \neq 0\}$ with coordinates (X_2, E_2) and $W_3 = \{\varepsilon \neq 0\}$ with coordinates (X_3, Y_3) . The transition formula follow from the requirement that the points $(1, Y_1, E_1)$, $(X_2, 1, E_2)$ and $(X_3, Y_3, 1)$ lie on the same orbit of the action:

$$\begin{aligned}\Psi_{12} : (Y_1, E_1) &\mapsto \left(X_2 = \frac{1}{Y_1\sqrt{Y_1}}, E_2 = \frac{E_1}{Y_1\sqrt{Y_1}} \right) \\ \Psi_{13} : (Y_1, E_1) &\mapsto \left(X_3 = \frac{1}{E_1}, Y_3 = \frac{Y_1}{E_1\sqrt{E_1}} \right).\end{aligned}$$

These affine charts define affine charts on W , with coordinates (Y_1, E_1, r_1) , (X_2, E_2, r_2) and (X_3, Y_3, r_3) . The projection σ is written as

$$\sigma_1 : \quad x = r_1^2, \quad y = r_1^3 Y_1, \quad \varepsilon = r_1^2 E_1 \quad (5)$$

$$\sigma_2 : \quad x = r_2^2 X_2, \quad y = r_2^3, \quad \varepsilon = r_2^2 E_2 \quad (6)$$

$$\sigma_3 : \quad x = r_3^2 X_3, \quad y = r_3^3 Y_3, \quad \varepsilon = r_3^2. \quad (7)$$

We apply this blow-up σ to the one-dimensional foliation \mathcal{F} on \mathbb{C}^3 given by the intersection of $d\varepsilon = 0$ and $\omega_\varepsilon = 0$. This foliation has a cuspidal singularity at the origin. Denote by $\sigma^{-1}\mathcal{F}$ the lifting of the foliation \mathcal{F} to the complement of the exceptional divisor $\sigma^{-1}(0)$. This foliation is regular outside of the preimage of $\{P_\varepsilon = 0, \varepsilon = 0\}$.

Proposition 2.1. *The foliation $\sigma^{-1}\mathcal{F}$ can be extended analytically to the exceptional divisor $\sigma^{-1}(0)$. The resulting foliation $\sigma^*\mathcal{F}$ is regular outside of the strict transform of $\{P_\varepsilon = 0, \varepsilon = 0\}$.*

Proposition 2.2. *The singularities of the resulting foliation $\sigma^*\mathcal{F}$ on $\sigma^{-1}(0)$ are located at the points $p_1 = (0, 1, 0)$, $p_2 = (0, -1, 0)$, $p_3 = (0, 0, 0)$ and $p_c = (\frac{-2}{3}, 0, 0)$. The points p_1, p_2 and p_3 are linearisable saddles and the point p_c is a center.*

Proof. The quasi-homogeneous blow-up $\sigma : W \rightarrow \mathbb{C}^3$ is a biholomorphism outside the exceptional divisor $\sigma^{-1}(0)$, all singularities of the foliation $\sigma^*\mathcal{F}$ outside $\sigma^{-1}(0)$ correspond to singularities of \mathcal{F} . Let $\sigma_i^*\mathcal{F}$, $i = 1, 2, 3$ be the restriction of $\sigma^*\mathcal{F}$ to the chart W_i .

- (1) In the chart W_1 , on the exceptional divisor $\{r_1 = 0\}$ and in a suitable local coordinates (r, Y, E) , the foliation $\sigma_1^*\mathcal{F}$ is given by the first integral $F_1 = E^{-3}(Y^2 - 1) = t$. We have

$$\frac{\partial F_1}{\partial r} \Big|_{(0, \pm 1, 0)} = \frac{\partial F_1}{\partial Y} \Big|_{(0, \pm 1, 0)} = \frac{\partial F_1}{\partial E} \Big|_{(0, \pm 1, 0)} = 0,$$

then the singularities of the foliation $\sigma_1^*\mathcal{F}$ on the exceptional divisor are located at $p_1 = (0, 1, 0)$ and $p_2 = (0, -1, 0)$.

By a local change of coordinates $(\tilde{r}, \tilde{Y}, \tilde{E})$, the point $p_1 = (0, 1, 0)$ becomes $(0, 0, 0)$ and the foliation $\sigma_1^*\mathcal{F}$ is given by two first integrals $t = \tilde{Y}\tilde{E}^{-3}$ and $\varepsilon = \tilde{r}^2\tilde{E}$. Let $\mathcal{X} = \beta_1\tilde{r}\frac{\partial}{\partial \tilde{r}} + \beta_2\tilde{Y}\frac{\partial}{\partial \tilde{Y}} + \beta_3\tilde{E}\frac{\partial}{\partial \tilde{E}}$ be the vector field generating the foliation $\sigma_1^*\mathcal{F}$. To compute β_1, β_2 and β_3 we use that $\mathcal{X}(\tilde{Y}\tilde{E}^{-3}) = 0$ and $\mathcal{X}(\tilde{r}^2\tilde{E}) = 0$. Then β_1, β_2 and β_3 satisfy the following equations $\langle (\beta_1, \beta_2, \beta_3), (0, 1, -3) \rangle = 0$ and $\langle (\beta_1, \beta_2, \beta_3), (2, 0, 1) \rangle = 0$, where

\langle, \rangle is the usual scalar product on \mathbb{C}^3 . By short calculus we obtain

$$X(\tilde{r}, \tilde{Y}, \tilde{E}) = -\frac{1}{2}\tilde{r}\frac{\partial}{\partial\tilde{r}} + 3\tilde{Y}\frac{\partial}{\partial\tilde{Y}} + \tilde{E}\frac{\partial}{\partial\tilde{E}}.$$

The same calculus holds for the point $p_2 = (0, -1, 0)$.

- (2) In the chart W_3 , on the exceptional divisor $\{r_3 = 0\}$ and in a suitable local coordinates (X, Y, r) , the foliation $\sigma_3^*\mathcal{F}$ is given by the first integral $F_3 = Y^2 - X^3 - X^2 = t$. We have

$$\begin{aligned} \frac{\partial F_3}{\partial X}\Big|_{(0,0,0)} &= \frac{\partial F_3}{\partial Y}\Big|_{(0,0,0)} = \frac{\partial F_3}{\partial r}\Big|_{(0,0,0)} = 0, \\ \frac{\partial F_3}{\partial X}\Big|_{(\frac{-2}{3},0,0)} &= \frac{\partial F_3}{\partial Y}\Big|_{(\frac{-2}{3},0,0)} = \frac{\partial F_3}{\partial r}\Big|_{(\frac{-2}{3},0,0)} = 0, \end{aligned}$$

then the singularities of the foliation $\sigma_3^*\mathcal{F}$ on the exceptional divisor are located at the saddle point $p_3 = (0, 0, 0)$ and the center $p_c = (\frac{-2}{3}, 0, 0)$.

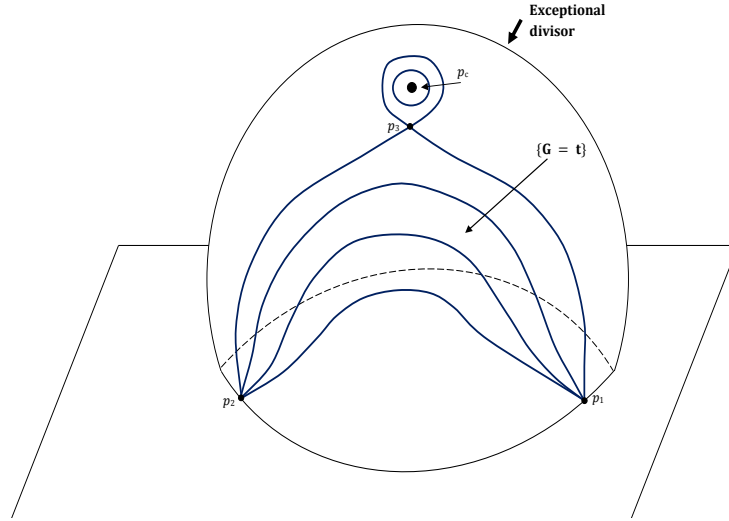


FIGURE 2. Singularities of $\sigma^*\mathcal{F}$ on the exceptional divisor.

□

3. PROOF OF THEOREM 1.3

In this section we first take benefit from the blowing-up in the family performed in the previous section to prove Theorem 1.3.

Let \mathcal{F} be the one-dimensional foliation on \mathbb{C}^3 introduced in the previous section and $\text{saddle}\mathcal{F} = \{p_4, \dots, p_k\}$ be the set of its saddles.

3.1. Hyperbolicity of polycycles. Let $T \subset \sigma^{-1}(0)$ be the manifold of dimension two with vertices saddles p_1, p_2 and p_3 . Let $t = \frac{\varepsilon^3}{h}$ and

$$G = \frac{(\sigma^* \varepsilon)^3}{\sigma^* H(\varepsilon, x, y)} = t. \quad (8)$$

After the making of the blowing-up we obtain a family of hyperbolic polycycles (i.e. each intersection of consecutive edges we have a saddle point), denote by

$$\delta = (\sigma^{-1}(\gamma_0(0) \setminus (0, 0, 0)) \cup (T \cap \{G = t\}))^{\mathbb{R}}, \quad (9)$$

where $(\dots)^{\mathbb{R}}$ denotes the real part of a complex analytic set. Let $\text{sad} \delta \text{le}(\sigma^* \mathcal{F}) = \{p_1, \dots, p_k\}$ be the set of saddles of the resulting foliation $\sigma^* \mathcal{F}$.

Let us fix a polycycle δ of family (9) and denote $\delta(\varepsilon, t)$ its corresponding cycle. We define

$$J(\varepsilon, t) = \int_{\delta(\varepsilon, t)} \sigma^* \frac{\eta}{M_\varepsilon}. \quad (10)$$

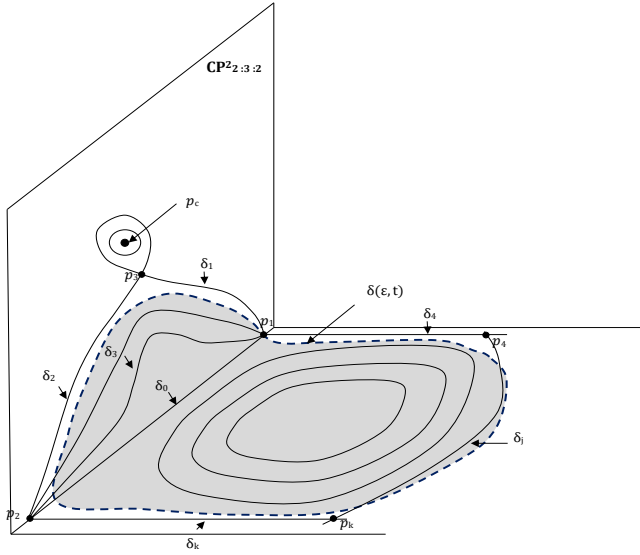


FIGURE 3. Real phase portrait of the foliation $\sigma^* \mathcal{F}$.

3.2. Variation relations. Define the variation operator $\mathcal{V}ar$ as the difference between counter-clockwise and clockwise continuation of F

$$\mathcal{V}ar_{(t,\beta)} F(t) = F(te^{i\beta\pi}) - F(te^{-i\beta\pi}).$$

The integral J admitting analytic continuation to the universal covering of $\mathbb{C}_t^* \times \mathbb{C}_\varepsilon^*$.

Proposition 3.1. *The integral $J(\varepsilon, t)$ satisfies the following rescaled iterated variations equation*

$$\mathcal{V}ar_{(t, \alpha_1)} \circ \cdots \circ \mathcal{V}ar_{(t, \alpha_k)} J(\varepsilon, t) = 0, \quad \alpha_i \in \mathbb{R}[a, a_1, \dots, a_k]. \quad (11)$$

Proof. Let us fix ε and some hyperbolic polycycle $\delta = \coprod_i \delta_i$ of family (9). As in [3], using a partition of unity multiplying the form $\sigma^* \frac{\eta}{M_\varepsilon}$ we can consider semilocal problem with a relative cycle close to one edge of the polycycle. Precisely, let δ_i be a some edge (connecting saddles p_j and p_k) of the polycycle δ with exponent α_i and two saddles points $p_j = \delta_i \cap \delta_j, p_k = \delta_i \cap \delta_k$, where δ_j, δ_k two separatrices of δ with exponents α_j and α_k , respectively. Let $\delta_i(\varepsilon, t)$ be its corresponding relative cycle. The respective iterated variations of the relative cycle $\delta_i(\varepsilon, t)$ is a closed loop which is either the commutator loop γ_{jk} if $\alpha_j \neq \alpha_k$ or the figure-eight loop γ_k if $\alpha_j = \alpha_k$. Finally, the variations commute so

$$\mathcal{V}ar_{(t, \alpha_1)} \circ \cdots \circ \mathcal{V}ar_{(t, \alpha_k)} J(\varepsilon, t) = 0. \quad \square$$

Remark. The above calculus of variations holds for any hyperbolic polycycle δ of family (9).

Proposition 3.2. *The rescaled variation of the integral $J(\varepsilon, t) = \int_{\delta(\varepsilon, t)} \sigma^* \frac{\eta}{M_\varepsilon}$ with respect to ε is an integral of the blown-up form $\sigma^* \frac{\eta}{M_\varepsilon}$ along the figure eight cycle.*

Proof. Let us fix t . Denote δ_0 the edge connecting two saddles p_1 and p_2 . Let (r, Y, E) be a suitable coordinate system near the edge δ_0 such that the foliation $\sigma^* \mathcal{F}$ is given by two first integrals $\varepsilon = r^2 E$ and $t = \frac{E^3}{(Y^2 - 1)}$. Let γ_+ and γ_- are two paths in Y -complex plane turning counterclockwise, respectively around $p_1 = (0, 1, 0)$ and $p_2 = (0, -1, 0)$ which are parametrized by $\tau_\pm : [0, 2\pi] \ni \phi \mapsto (r(\phi, \varepsilon, t), Y(\phi) = \pm 1 + \varepsilon e^{i\phi}, E(\phi, \varepsilon, t))$. Then, we have $F_\pm(\varepsilon, t) = \int_{\gamma_\pm} \sigma^* \frac{\eta}{M_\varepsilon} = \int_0^{2\pi} \tau_\pm^* \sigma^* \frac{\eta}{M_\varepsilon} d\phi$. Let $L_i(\varepsilon, t) = \int_{\ell_i} \sigma^* \frac{\eta}{M_\varepsilon}$, $i = 1, 2$, where $\ell_1 = [1 - \varepsilon, -1 + \varepsilon]$ and $\ell_2 = [-1 + \varepsilon, 1 - \varepsilon]$. Then, we obtain

$$\mathcal{V}ar_{(\varepsilon, 1)} J(\varepsilon, t) = F_+(\varepsilon, t) + L_2(\varepsilon, t) + F_-(\varepsilon, t) + L_1(\varepsilon, t) = \int_{\widetilde{\gamma_+ \ell_2 \gamma_- \ell_1}} \sigma^* \frac{\eta}{M_\varepsilon},$$

where the lifting $\widetilde{\gamma_+ \ell_2 \gamma_- \ell_1}$ is a closed path homotopic to a figure eight loop. \square

Corollary 3.3. *The integral J possess, near $\varepsilon = 0$, the following expansion*

$$J(\varepsilon, t) = J_1(\varepsilon, t) + J_2(\varepsilon, t) \log \varepsilon, \quad (12)$$

where $J_i(\varepsilon, t) = \sum_{k=-N_i}^{+\infty} \psi_{ik}(t) \varepsilon^k$, $N_i > 0$.

Let \mathcal{P} be the following finite-dimensional functional space

$$\mathcal{P}(m, M; \alpha_1, \dots, \alpha_k; \varepsilon) := \left\{ \sum \sum \phi_{jln}(\varepsilon) t^{\alpha_j n} \log^n t, \phi_{jln} \in \mathbb{C}, m \leq \alpha_j n \leq M, 0 \leq l \leq k \right\}.$$

Lemma 3.4. *We have*

(1) *The functions $J_i, i = 1, 2$ satisfy the following iterated variations equation*

$$\mathcal{V}ar_{(t, \alpha_1)} \circ \cdots \circ \mathcal{V}ar_{(t, \alpha_k)} J_i(\varepsilon, t) = 0. \quad (13)$$

(2) *There exists a meromorphic family of functions $P_i(\varepsilon, \cdot), i = 1, 2$ in $\mathcal{P}(\cdots)$ such that $|J_i(\varepsilon, t) - P_i(\varepsilon, t)| |t|^{-M} \rightarrow 0$ as $t \rightarrow 0$ uniformly in ε*

Proof. (1) The first item is a consequence of linearity of the variation operator $\mathcal{V}ar$.

(2) Applying Lemma 4.8 from [3] to the function $J_i, i = 1, 2$, we have the second item. \square

3.3. Proof of Theorem 1.3. The integral $J(\varepsilon, t)$ can be viewed as the pull-back of the pseudo-abelian integrals $I_\varepsilon(h) = I(\varepsilon, h)$ by the blowing-up map σ . The proof of Theorem 1.3 is now reduced to the proof of

Theorem 3.5. *The number $\#\{t \in [0, +\infty], J(\varepsilon, t) = 0\}$ is uniformly bounded in ε .*

Proof. Let $\varepsilon > 0$ be sufficiently small and $\alpha_i \in \{\alpha_1, \dots, \alpha_k\}$. We choose a slit annulus Γ_i with boundary $\partial\Gamma_i = C_R^{\alpha_i} \cup C_\varepsilon^{\alpha_i} \cup C_\pm^{\alpha_i}$ where $C_R^{\alpha_i} = \{|t| = R, |\arg(t)| \leq \alpha_i\pi\}$, $C_\varepsilon^{\alpha_i} = \{|t| = \varepsilon, |\arg(t)| \leq \alpha_i\pi\}$ and $C_\pm^{\alpha_i} = \{\varepsilon < |t| < R, |\arg(t)| = \pm\alpha_i\pi\}$.

We use the argument principle on $\partial\Gamma_i$:

$$\#\{t \in \Gamma_i : J(\varepsilon, t) = 0\} \leq \frac{1}{2\pi} \Delta \arg_{\partial\Gamma_i} J = \frac{1}{2\pi} (\Delta \arg_{C_R^{\alpha_i}} J + \Delta \arg_{C_\pm^{\alpha_i}} J + \Delta \arg_{C_\varepsilon^{\alpha_i}} J).$$

- (1) The estimation of the increment of argument $\Delta \arg_{C_R^{\alpha_i}} J$ of J along the big circular arc $C_R^{\alpha_i}$ is again due to Gabrielov's theorem [7].
- (2) The estimation of the increment of argument $\Delta \arg_{C_\pm^{\alpha_i}} J$ of J along $C_+^{\alpha_i}$ and $C_-^{\alpha_i}$ reduces to estimating zeros of $\mathcal{V}ar_{(t, \alpha_i)} J(\varepsilon, t)$. Moreover, we have

$$\mathcal{V}ar_{(t, \alpha_i)} J(\varepsilon, t) = \mathcal{K}(e^{\frac{\alpha_1 t}{\alpha_i}}, \dots, e^{\frac{\alpha_{i-1} t}{\alpha_i}}, e^{\frac{\alpha_{i+1} t}{\alpha_i}}, \dots, e^{\frac{\alpha_k t}{\alpha_i}}, \varepsilon, \log \varepsilon),$$

where \mathcal{K} is a meromorphic function. Hence, the function $\mathcal{V}ar_{(t, \alpha_i)} J(\varepsilon, t)$ is a logarithmico-analytic function of type 1 in variable ε (for definition see [9]). Then, using Lion-Rolin preparation theorem [9], the function $\mathcal{V}ar_{(t, \alpha_i)} J(\varepsilon, t)$ has the following representation

$$\mathcal{V}ar_{(t, \alpha_i)} J(\varepsilon, t) = \varepsilon_0^{\nu_0} \varepsilon_1^{\nu_1} K(t) \Delta(t, \varepsilon_0, \varepsilon_1), \quad \Delta(0) \neq 0,$$

where $\varepsilon_0 = \varepsilon - k_0(t)$, $\varepsilon_1 = \log \varepsilon_0 - k_1(t)$ and k_0, k_1, K, Δ are a logarithmico-exponential functions (for definition see [9]). As the number of zeros of a logarithmico-exponential function is bounded, so the number of zeros of $t \mapsto \mathcal{V}ar_{(t, \alpha_i)} J(\varepsilon, t)$ is uniformly bounded in ε .

- (3) To obtain an upper bound for increment of argument $\Delta \arg_{C_\varepsilon^{\alpha_i}} J$ of J along the small circular arc $C_\varepsilon^{\alpha_i}$ we investigate the leading term of J at $t = 0$.

The function J , at $t = 0$, admits the following expansion

$$J(\varepsilon, t) = \sum_{i=1}^k \sum_{j=0}^{k-1} \sum_{n=-M}^{+\infty} c_{ijn} \tilde{\varphi}_{ijn}(\varepsilon) t^{n\alpha_i} \log^j t, \quad M > 0, c_{ijn} \in \mathbb{C}. \quad (14)$$

Corollary 3.3 gives us that a some coefficients $\tilde{\varphi}_{ijn}(\varepsilon)$ of the form $\frac{\log \varepsilon}{\varepsilon^{\mu_{ijn}}}$, $\mu_{ijn} > 0$ which are not analytic (a priori are not meromorphic) and hence, we have a situation similar to that of remark 4.7 from [3]. We can no use an argument similar to those from [1, 3, 4, 10]. A second application of a blowing-up resolve the problem.

Blowing-up. Let \mathbb{C}^3 be equipped with system of coordinates $(J_1, J_2, \log \varepsilon)$. Denote \mathcal{A} the algebraic variety $J_1 + J_2 \log \varepsilon = 0$ in \mathbb{C}^3 . \mathcal{A} can be desingularized by a composite of a locally-finite sequence of blowings-up.

- (a) *First step.* We make a quasi-homogeneous blow-up with weights $(2, 1, 1)$. We replace the origin of \mathbb{C}^3 by the projective space $\mathbb{C}\mathbb{P}_{2:1:1}^2$ which is covered by three affine charts: $U_1 = \{J_1 \neq 0\}$ with coordinates (K_{11}, K_{12}) , $U_2 = \{J_2 \neq 0\}$ with coordinates

(K_{21}, K_{22}) and $U_3 = \{\log \varepsilon \neq 0\}$ with coordinates (K_{31}, K_{32}) . The blow-up π is written as

$$\begin{aligned}\pi_1 : \quad J_1 &= t_1^2, \quad J_2 = t_1 K_{11}, \quad \log \varepsilon = t_1 K_{12}, \\ \pi_2 : \quad J_1 &= t_2^2 K_{21}, \quad J_2 = t_2, \quad \log \varepsilon = t_2 K_{22}, \\ \pi_3 : \quad J_1 &= t_3^2 K_{31}, \quad J_2 = t_3 K_{32}, \quad \log \varepsilon = t_3.\end{aligned}$$

The total transform $\pi^*(J_1 + J_2 \log \varepsilon)$ is given, respectively in charts U_1, U_2 and U_3 by

$$\begin{aligned}\pi_1^*(J_1 + J_2 \log \varepsilon) &= t_1^2(1 + K_{11}K_{12}), \\ \pi_2^*(J_1 + J_2 \log \varepsilon) &= t_2^2(K_{21} + K_{22}), \\ \pi_3^*(J_1 + J_2 \log \varepsilon) &= t_3^2(K_{31} + K_{32}),\end{aligned}$$

where $\{t_1 = 0\}, \{t_2 = 0\}$ and $\{t_3 = 0\}$ are local equations of the exceptional divisor and $\{1 + K_{11}K_{12} = 0\}, \{K_{21} + K_{22} = 0\}$ and $\{K_{31} + K_{32} = 0\}$ are local equations of strict transform of \mathcal{A} . But we have not yet achieved a *embedded* resolution of singularities because the strict transform and the exceptional divisor do not have normal crossings at the origin except the chart U_1 . Further blowing-up is needed for resolution of singularities.

(b) *Second step.* We make a directional blow-up ϱ with center C such that $C|_{U_2} = \{K_{21} = K_{22} = 0\}$ and $C|_{U_3} = \{K_{31} = K_{32} = 0\}$. Then

(i) the exceptional divisor $\varrho^{-1}(C|_{U_2})$ is covered by two coordinates charts $U_{K_{21}}$ and $U_{K_{22}}$ with coordinates $(\tilde{K}_{21}, \tilde{K}_{22}, \tilde{t}_2)$ such that the explicit formula of the blow-up ϱ is given, respectively in $U_{K_{21}}$ and $U_{K_{22}}$ by

$$\begin{aligned}t_2 &= \tilde{t}_2, \quad K_{21} = \tilde{K}_{21}, \quad K_{22} = \tilde{K}_{21}\tilde{K}_{22}, \\ t_2 &= \tilde{t}_2, \quad K_{21} = \tilde{K}_{21}\tilde{K}_{22}, \quad K_{22} = \tilde{K}_{22},\end{aligned}$$

(ii) in the chart U_1 the blow-up ϱ is a biholomorphism,

(iii) the exceptional divisor $\varrho^{-1}(C|_{U_3})$ is covered by two coordinates charts $U_{K_{31}}$ and $U_{K_{32}}$ with coordinates $(\tilde{K}_{31}, \tilde{K}_{32}, \tilde{t}_3)$ such that the explicit formula of ϱ is given, respectively in $U_{K_{31}}$ and $U_{K_{32}}$ by

$$\begin{aligned}t_3 &= \tilde{t}_3, \quad K_{31} = \tilde{K}_{31}, \quad K_{32} = \tilde{K}_{31}\tilde{K}_{32}, \\ t_3 &= \tilde{t}_3, \quad K_{31} = \tilde{K}_{31}\tilde{K}_{32}, \quad K_{32} = \tilde{K}_{32}.\end{aligned}$$

The total transform $\varrho^*(\pi^*(J_1 + J_2 \log \varepsilon))$ of $\pi^*(J_1 + J_2 \log \varepsilon)$ is given, respectively in $U_{K_{21}}, U_{K_{22}}, U_{K_{31}}$ and $U_{K_{32}}$ by

$$\begin{aligned}\varrho^*(\pi^*(J_1 + J_2 \log \varepsilon)) &= \tilde{t}_2^2 \tilde{K}_{21}(1 + \tilde{K}_{22}), \\ \varrho^*(\pi^*(J_1 + J_2 \log \varepsilon)) &= \tilde{t}_2^2 \tilde{K}_{22}(\tilde{K}_{21} + 1), \\ \varrho^*(\pi^*(J_1 + J_2 \log \varepsilon)) &= \tilde{t}_3^2 \tilde{K}_{31}(1 + \tilde{K}_{32}), \\ \varrho^*(\pi^*(J_1 + J_2 \log \varepsilon)) &= \tilde{t}_3^2 \tilde{K}_{32}(\tilde{K}_{31} + 1).\end{aligned}$$

The strict transform and the exceptional divisor have a normal crossings at the origin. The algebraic variety \mathcal{A} is desingularized.

Estimation. Now we take benefit from the desingularization of \mathcal{A} to estimate the increment of argument $\Delta \arg_{C_\epsilon^{\alpha_i}} J$ of the function J along the small circular arc $C_\epsilon^{\alpha_i}$.

(a) In the chart $U_{K_{21}}$ i.e. for $|\frac{\log \epsilon}{J_1}| \ll 1$, we have

$$\varrho^*(\pi^*(J_1 + J_2 \log \epsilon)) = \tilde{t}_2^2 \tilde{K}_{21}(1 + \tilde{K}_{22}) \stackrel{0}{\sim} \tilde{t}_2^2 \tilde{K}_{21} = J_1.$$

By Lemma 3.4, there exist a meromorphic function $P_1(\epsilon, \cdot) \in \mathcal{P}(\dots)$ such that $|t|^{-M} |J_1(\epsilon, t) - P_1(\epsilon, t)| \rightarrow 0$ as $t \rightarrow 0$ uniformly in ϵ . For each ϵ , we choose the leading term of $P_1(\epsilon, \cdot)$. By Gabrielov's theorem, the increment of argument of this leading term is uniformly bounded in ϵ .

(b) In the chart $U_{K_{22}}$ i.e. for $|\frac{J_1}{\log \epsilon}| \ll 1$, we have

$$\varrho^*(\pi^*(J_1 + J_2 \log \epsilon)) = \tilde{t}_2^2 \tilde{K}_{22}(\tilde{K}_{21} + 1) \stackrel{0}{\sim} \tilde{t}_2^2 \tilde{K}_{22} = J_2 \log \epsilon.$$

As $\log \epsilon \in \mathbb{R}$, we have $\arg(J_2 \log \epsilon) = \arg J_2$. By Lemma 3.4, there exist a meromorphic function $P_2(\epsilon, \cdot) \in \mathcal{P}(\dots)$ such that $|t|^{-M} |J_2(\epsilon, t) - P_2(\epsilon, t)| \rightarrow 0$ as $t \rightarrow 0$ uniformly in ϵ . For each ϵ , we choose the leading term of $P_2(\epsilon, \cdot)$. By Gabrielov's theorem, the increment of argument of this leading term is uniformly bounded in ϵ .

(c) In the chart $U_{K_{31}}$ i.e. for $|\frac{J_2}{J_1}| \ll 1$, we have

$$\varrho^*(\pi^*(J_1 + J_2 \log \epsilon)) = \tilde{t}_3^2 \tilde{K}_{31}(1 + \tilde{K}_{32}) \stackrel{0}{\sim} \tilde{t}_3^2 \tilde{K}_{31} = J_1.$$

By Lemma 3.4, there exist a meromorphic function $P_1(\epsilon, \cdot) \in \mathcal{P}(\dots)$ such that $|t|^{-M} |J_1(\epsilon, t) - P_1(\epsilon, t)| \rightarrow 0$ as $t \rightarrow 0$ uniformly in ϵ . For each ϵ , we choose the leading term of $P_1(\epsilon, \cdot)$. By Gabrielov's theorem, the increment of argument of this leading term is uniformly bounded in ϵ .

(d) In the chart $U_{K_{32}}$ i.e. for $|\frac{J_1}{J_2}| \ll 1$, we have

$$\varrho^*(\pi^*(J_1 + J_2 \log \epsilon)) = \tilde{t}_3^2 \tilde{K}_{32}(\tilde{K}_{31} + 1) \stackrel{0}{\sim} \tilde{t}_3^2 \tilde{K}_{32} = J_2 \log \epsilon.$$

As $\log \epsilon \in \mathbb{R}$, we have $\arg(J_2 \log \epsilon) = \arg J_2$. By Lemma 3.4, there exist a meromorphic function $P_2(\epsilon, \cdot) \in \mathcal{P}(\dots)$ such that $|t|^{-M} |J_2(\epsilon, t) - P_2(\epsilon, t)| \rightarrow 0$ as $t \rightarrow 0$ uniformly in ϵ . For each ϵ , we choose the leading term of $P_2(\epsilon, \cdot)$. By Gabrielov's theorem, the increment of argument of this leading term is uniformly bounded in ϵ .

(e) In the chart U_1 , the function J is meromorphic, by Gabrielov's theorem the increment of argument of J along small arc $C_\epsilon^{\alpha_i}$ is uniformly bounded in ϵ .

All the above constructions depend analytically on parameters like coefficients of the polynomials P_i , exponents a, a_i and coefficients of the form η .

□

REFERENCES

- [1] Bobieński, Marcin *Pseudo-Abelian integrals along Darboux cycles a codimension one case. J. Differential Equations* 246 (2009), no. 3, 1264-1273.
- [2] Binyamini, Gal; Novikov, Dmitry; Yakovenko, Sergei *On the number of zeros of Abelian integrals A constructive solution of the infinitesimal Hilbert sixteenth problem. Inventiones mathematicae* 181, no.2 (2010), p.227-289
- [3] Bobieński, Marcin; Mardešić, Pavao *Pseudo-Abelian integrals along Darboux cycles. Proc. Lond. Math. Soc. (3)* 97 (2008), no. 3, 669-688.
- [4] Bobieński, Marcin; Mardešić, Pavao; Novikov, Dmitry *Pseudo-Abelian integrals: unfolding generic exponential. J. Differential Equations* 247 (2009), no. 12, 3357-3376.

- [5] Bobieński, Marcin; Mardešić, Pavao; Novikov, Dmitry *Pseudo-Abelian integrals on slow-fast Darboux systems. Ann. Inst. Fourier (Grenoble)*, 63(2):417-430, 2013.
- [6] Denkowska, Zofia; Roussarie, Robert; *A method of desingularization for analytic two-dimensional vector field families. Bol. Soc. Brasil. Mat. (N.S.)* 22 (1991), no. 1, 93-126.
- [7] Gabrièlov, A. M *Projections of semianalytic sets. (Russian) Funktsional. Anal. i Priložen.* 2 1968 no. 4, 18-30
- [8] Khovanskii, A. G. *Real analytic manifolds with property of finiteness, and complex abelian integrals, Funktsional. Ana. i Priložen.* 18 (2) (1984) 40-50.
- [9] Lion, Jean Marie; Rolin, Jean Phillipe *Théorème de préparation pour les fonctions logarithmico-exponentielles. Annales de l'institut de Fourier, tome 47, n3 (1997) p.859-884.*
- [10] Novikov, Dmitry *On limit cycles appearing by polynomial perturbation of Darbouxian integrable systems. Geom. Funct. Anal.* 18 (2009), no. 5, 1750-1773.
- [11] Varchenko, A. N. *Estimation of the number of zeros of an abelian integral depending on a parameter, and limit cycles. (Russian) Funktsional. Anal. i Priložen.* 18 (1984), no. 2, 14-25.

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