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Reliability indicators for hidden Markov renewal models

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Irene Votsi

Krantz Template



6

Reliability indicators for hidden Markov renewal models

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6.1 Introduction

Hidden Markov models (HMMs) are stochastic models that are widely used in numerous applications in reliability and DNA analysis [2], seismology [26], speech recognition [19] etc. They are mostly used to describe systems that are observed at discrete times. However, the observations are induced by an underlying (“hidden”) process that is unknown or unexplained. In the simplest case this underlying process is a first-order, homogeneous Markov chain whose state space is finite.

Since the underlying process is a Markov chain, the sojourn times (i.e. the times between the successive visited states) are considered to follow the geometric distribution. However, in practice there is no clear evidence that the sojourn times should follow this particular distribution. In other words, there is no evidence that favors the choice of a Markov chain as the underlying chain over the most general semi-Markov chain. If the underlying process is a semi-Markov chain then the corresponding model is a hidden semi-Markov model (HSMM) and therefore HSMMs constitute an important extension of HMMs ([4],[5],[7],[20]).

Contrary to HMMs, HSMMs enable us to describe systems whose hidden states evolve based on their last visited state (Markov property) and on the time elapsed since its last visit. As a result, popular “memory-full” distributions, such as the Weibull or the shifted Poisson distributions could be

employed to describe the sojourn times. HSMs were introduced in [9] towards an application in speech recognition and since then they have been extensively studied and applied in many scientific disciplines. For the non-parametric HSM, the maximum likelihood estimators (MLEs) and their asymptotic behaviour were studied in [3]. Later on, the asymptotic properties of the MLEs for HSMs defined in a general state space were studied in [22].

In the literature of semi-Markov models (SMMs) many reliability indicators have been introduced, including availability, maintainability, mean times to failure, hazard rates etc. For recent advances in the topic concerning discrete time SMMs, see [2], [11], [10] and [1]. For continuous time SMMs, we address the interested reader in [15] and [16]. For advances in estimation methods of nonparametric SMMs, see [22] and the references therein.

Here we concentrate on the failure occurrence rate (ROCOF), which is a fundamental reliability indicator for random systems that could experience multiple failures. In a continuous time context, the value of ROCOF at time $t \in \mathbb{R}^*$ represents the derivative of the mean number of failures that occurred up to time t , whereas in a discrete time context ($k \in \mathbb{N}^*$) it represents the probability that a failure occurs during the interval $[k - 1, k]$. The failure occurrence rate may be increasing or decreasing, which means that the system of interest may improve or degenerate, respectively. On the other hand constant ROCOF means that the state of the system does not change over time.

ROCOF was firstly introduced for first-order Markov processes defined in a finite state space in [27]. More recently the previous results were generalized for higher-order Markov processes [6]. In a continuous time semi-Markov context, ROCOF was studied in [18] for finite state space processes and in [14] for general state space processes, respectively. The discrete time counterpart of ROCOF, was evaluated for semi-Markov chains in [25]. The authors proposed an empirical estimator and applied their results to seismological data. In particular they estimated the time-varying probability of the occurrence of a strong earthquake in Aegean Sea. Later on ROCOF was studied for discrete hidden Markov renewal chains (HMRCs) in [24] in the case that failures are associated with specific observations rather than hidden states. Counting processes were used to define an empirical estimator whose asymptotic behavior was studied. In particular the estimator was shown to be consistent and its asymptotic normal distribution was obtained by means of delta method.

In the previous studies ROCOF was defined as a “global” reliability indicator in the sense that it was defined by means of all the starting functioning states and all the ending inoperative states. In this sense it does not distinguish neither on the starting functioning state nor on the ending inoperative state. It takes into account all the starting operational and all the ending defective states simultaneously. However, the study of the influence of each state of the system (functioning or unworkable) in the evaluation of ROCOF could be of special interest. To the best of our knowledge, no studies have been done on the impact of the starting functioning states or/and the ending unworkable states on the ROCOF. To address this short-coming in the context of HM-

RCs, we introduce the conditional counterparts of the failure occurrence rate. The study of these elementary rates could uplift the functioning states that mostly influence the ROCOF and give insight into the dynamics of failures. In this sense these indicators could enable us to identify opportunities for improving reliability performance, which is of special interest for real-life applications. We further present consistent and asymptotically normal empirical estimators of the conditional versions of ROCOF.

The chapter is organized as follows. In Section 1.2 we present the notation and preliminaries of HMRCs. Section 1.3 describes the definition, evaluation and statistical estimation of the conditional rates of failure occurrences. Finally, in Section 1.4, we give some concluding remarks.

6.2 Hidden Markov renewal models

First we denote by $\mathbf{J} = (J_n)_{n \in \mathbb{N}}$ a Markov chain (*embedded Markov chain* (EMC)) with finite state space $E = \{1, \dots, s\}$ and by $\mathbf{S} = (S_n)_{n \in \mathbb{N}}$ a sequence of jump times, i.e. times when the EMC visits different, successive states. We further denote by $\mathbf{X} = (X_n)_{n \in \mathbb{N}}$ the sequence of sojourn (or inter-arrival) times, where $X_0 = S_0 = 0$ a.s. and $X_n = S_n - S_{n-1}$ for all $n \in \mathbb{N}^*$. The semi-Markov chain $\mathbf{Z} = (Z_k)_{k \in \mathbb{N}}$ is defined by $Z_k = J_{N(k)}$, where $N(k) = \max\{n \geq 0 : S_n \leq k\}$, $k \in \mathbb{N}$. The stochastic process $(\mathbf{J}, \mathbf{S}) = (J_n, S_n)_{n \in \mathbb{N}}$, called Markov renewal chain (MRC), is considered to be (time) homogeneous and is characterized by the semi-Markov kernel (SMK) $\mathbf{q} = (q_{ij}(k); i, j \in E, k \in \mathbb{N})$, where

$$q_{ij}(k) = P(J_{n+1} = j, X_{n+1} = k | J_n = i)$$

for any $i, j \in E$ and $k, n \in \mathbb{N}$.

The evaluation of the conditional ROCOF requires additionally the definition of the ℓ -fold convolution of the SMK $\mathbf{q}^{(\ell)} = (q_{ij}^{(\ell)}(k); i, j \in E, k \in \mathbb{N})$, where

$$q_{ij}^{(\ell)}(k) = P(J_\ell = j, S_\ell = k | J_0 = i),$$

for any $i, j \in E$ and $\ell, k \in \mathbb{N}$. The ℓ -fold convolution of the semi-Markov kernel could further be obtained recursively by means of the lower order convolutions of the semi-Markov kernel [2].

We further denote by $\mathbf{a} = (a(i); i \in E)$, where $a(i) = P(J_0 = i)$, the initial law of the EMC. The transition probability matrix of the EMC is defined by $\mathbf{P} = (p_{ij}; i, j \in E)$, where $p_{ij} = P(J_{n+1} = j | J_n = i)$, $i, j \in E$, $n \in \mathbb{N}$, and the conditional distribution of the sojourn time in state $i \in E$ given that the next visited state is $j \in E$ is given by $f_{ij}(k) = P(X_{n+1} = k | J_n = i, J_{n+1} = j)$, $k \in \mathbb{N}$. Here we assume that there are neither instantaneous transitions nor self-transitions for the EMC. Moreover we define the survival function of the

sojourn time in state $i \in E$ by $\overline{H}_i(k) = P(X_{n+1} > k | J_n = i)$, $k \in \mathbb{N}$ and the corresponding mean sojourn time by $m_i = \mathbb{E}(S_1 | J_0 = i)$.

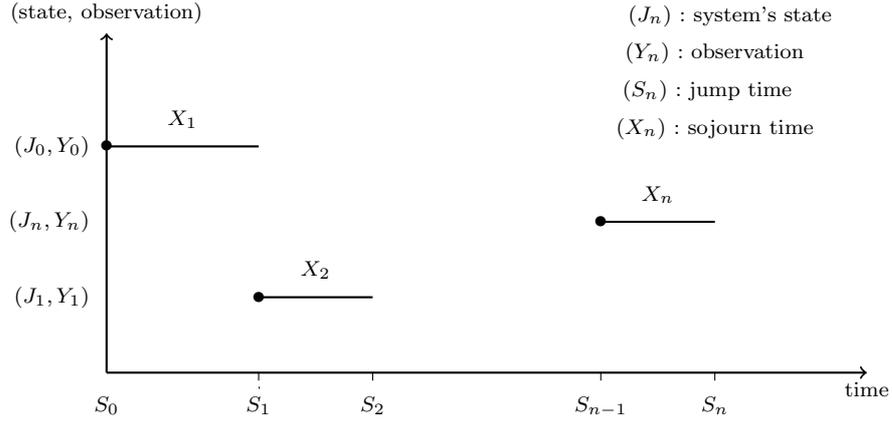


FIGURE 6.1

A representative sample path of the process $(J_n, S_n, Y_n)_{n \in \mathbb{N}}$.

In what follows we consider that the next assumptions are fulfilled:

- A1 The EMC \mathbf{J} is ergodic.
- A2 For any state $i \in E$, $m_i = \mathbb{E}(S_1 | J_0 = i)$, is finite.
- A3 The MRC (\mathbf{J}, \mathbf{S}) is aperiodic.

Second we denote by $\mathbf{Y} = (Y_n)_{n \in \mathbb{N}}$ an observation sequence that is defined in a finite state space A and is recorded only at jump times $n \in \mathbb{N}$. We assume that the observations are conditionally independent given the underlying states of the EMC, i.e.

$$P(Y_n = a | Y_0, \dots, Y_{n-1}, J_0, \dots, J_n = i) = P(Y_n = a | J_n = i),$$

for any $a \in A$, $i \in E$, $n \in \mathbb{N}$.

Here we consider that observations are recorded at jump times and call the process $(\mathbf{J}, \mathbf{S}, \mathbf{Y}) = (J_n, S_n, Y_n)_{n \in \mathbb{N}}$ a *hidden Markov renewal chain*. The state space of $(\mathbf{J}, \mathbf{S}, \mathbf{Y})$ is denoted by E^* ($|E^*| = d$) and its initial law by $\alpha = (\alpha(i, y); i \in E, y \in A)$, where $\alpha(i, y) = P(J_0 = i, S_0 = 0, Y_0 = y)$. On the contrary if we consider that observations are recorded at calendar times $k \in \mathbb{N}$ (and not just at jump times $n \in \mathbb{N}$), then the process $(\mathbf{Z}, \mathbf{Y}) = (Z_k, Y_k)_{k \in \mathbb{N}}$ is a *hidden semi-Markov chain* [2].

We consider that A is divided into two disjoint, non-empty subsets U and D such that $U, D \subset A$ ($U, D \neq A$). The subset $U = \{1, \dots, r\}$ represents the functioning (or up) states, whereas the subset $D = \{r+1, \dots, s\}$ stands for the unworkable (or down) states. We denote by $L = E \times \mathbb{N}$ ($|L| = \ell$) the

state space of the MRC and by $\boldsymbol{\pi} = (\pi(i, s); i \in E, s \in \mathbb{N})$, where $\pi(i, s) = \sum_{u \in E} \nu(u) q_{ui}(s)$, its stationary distribution. Since we assume stationarity, the stationary distribution of the EMC coincides with its initial distribution, \mathbf{a} . We further define $R_i(U) = P(Y_n \in U | J_n = i)$ and $R_i(D) = P(Y_n \in D | J_n = i)$, for any $i \in E, n \in \mathbb{N}$.

We turn now our attention to statistical estimation aspects with regards to HMRCs. We first consider a sample path of the HMRC up to fixed arbitrary time $M \in \mathbb{N}$, i.e.

$$\mathcal{H}(M) = (J_0, Y_0, S_1, \dots, J_{N(M)}, Y_{N(M)}, S_{N(M)}, U_M),$$

where $U_M = M - S_{N(M)}$. Then we denote $S_0^M = (S_0, \dots, S_{N(M)})$, $Y_0^M = (Y_0, \dots, Y_{N(M)})$, $J_0^M = (J_0, \dots, J_{N(M)})$ and $T_M = \{0, \dots, M\}$. We further denote by δ Kronecker's delta and define the (block diagonal) covariance matrix Δ , of dimension $(\ell d) \times (\ell d)$, by

$$\Delta = \begin{pmatrix} \Phi(1, 0) & \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \Phi(i, t) & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \dots & \dots & \dots & \Phi(s, S_{N(M)}) \end{pmatrix},$$

where

$$\Phi(i, t) = \frac{1}{\pi(i, t)} \left(\delta_{(j, t', m)(r, t'', q)} R_{j; m} q_{ij}(t' - t) - q_{ij}(t' - t) R_{j; m} R_{r; q} q_{ir}(t'' - t) \right),$$

for any $(i, t) \in L, (j, t', m), (r, t'', q) \in E^*$.

Second, we introduce the counting processes of transitions of $(J_n)_{n \in \mathbb{N}}$ up to time M

- $N_{ij}(M) = \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, J_n=j\}}$,
- $N_{ij}(k, M) = \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, X_n=k, J_n=j\}}$,

where $i, j \in E, k \in \mathbb{N}^*$. We further introduce the counting process of visits of $(J_n)_{n \in \mathbb{N}}$ and $(Y_n)_{n \in \mathbb{N}}$ up to time M

- $N_{i; a}(M) = \sum_{n=1}^{N(M)} \mathbf{1}_{\{J_{n-1}=i, Y_{n-1}=a\}}$,

where $i \in E, a \in A$.

Since we assume stationarity, the parameter set of the model is $\boldsymbol{\vartheta} = (\mathbf{q}, \mathbf{R})$. Moreover the complete likelihood function, i.e. the likelihood function which is based on the complete data, is denoted by $f_M(J_0^M, Y_0^M, S_0^M | \boldsymbol{\vartheta})$. Then the partial likelihood function, which is based on the incomplete data, is given by

$$g_M(Y_0^M | \boldsymbol{\vartheta}) = \sum_{J_0^M \in E^{N(M)+1}} \sum_{S_0^M \in T_M^{N(M)}} f_M(J_0^M, Y_0^M, S_0^M | \boldsymbol{\vartheta}).$$

The complete log-likelihood function is expressed in terms of the aforementioned counting processes as follows:

$$\begin{aligned}
\log f_M(J_0^M, Y_0^M, S_0^M | \boldsymbol{\vartheta}) &= \log \left(\prod_{i,j \in E} p_{ij}^{N_{ij}(M)} \prod_{i,j \in E} \prod_{k \in \mathbb{N}^*} f_{ij}^{N_{ij}(k,M)} \prod_{i \in E} \prod_{a \in A} R_{i;a}^{N_{i;a}(M)} \overline{H}_i(k) \right) \\
&= \sum_{i,j \in E} N_{ij}(M) \log p_{ij} + \sum_{i,j \in E} \sum_{k \in \mathbb{N}^*} N_{ij}(k, M) \log f_{ij}(k) \\
&\quad + \sum_{i \in E} \sum_{a \in A} N_{i;a}(M) \log R_{i;a} + \log \left(\sum_{k \geq M - S_N(M)} \overline{H}_i(k) \right).
\end{aligned}$$

The MLEs of the parameters

$$\boldsymbol{\vartheta}_{MLE} = \arg \max_{\boldsymbol{\vartheta}} g_M(Y_0^M | \boldsymbol{\vartheta}),$$

could be obtained by an adaptation of the EM algorithm [12] (as it is presented in [2]) in the case that observations are recorded at jump times. In the following we denote the MLEs of $q_{ij}(k)$, $q_{ij}^{(\ell)}(k)$, $R_{i;a}$ and $R_i(A_1)$ by $\widehat{q}_{ij}(k, M)$, $\widehat{q}_{ij}^{(\ell)}(k, M)$, $\widehat{R}_{i;a}(M)$ and $\widehat{R}_i(A_1, M)$, respectively, for any $i, j \in E$, $k, \ell \in \mathbb{N}$, $a \in A$, $A_1 \subset A$.

6.3 Conditional failure occurrence rates

Our main objective is to study reliability indicators that could be used to improve the performance of random, repairable systems that are encountered in real-life problems. These indicators could be taken into account in the development of maintenance strategies and in decision making. In particular, we focus on the conditional failure occurrence rates.

6.3.1 Full conditional ROCOF

First of all we are interested in studying the impact of both the starting up and the ending down states on the ROCOF, simultaneously. In order to do that, we define the full conditional ROCOF as the mean transition number of the HMRC to the set D at time k , starting in state $i \in U$ and ending in state $j \in D$, i.e.

$$r_{ij}^{\#}(k) = \mathbb{E}(N_{ij}^{\#}(k)) - \mathbb{E}(N_{ij}^{\#}(k-1)),$$

where

$$N_{ij}^{\#}(k) = \sum_{l=1}^k \mathbf{1}_{\{Y_{l-1}=i, Y_l=j\}}.$$

Proposition 1 *The full conditional ROCOF at time $k \in \mathbb{N}^*$ is given by*

$$r_{ij}^\#(k) = \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k a(\ell, y) R_{\ell; y} R_{s_0; i} R_{s_1; j} q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0),$$

for any fixed, arbitrary states $i \in U$, $j \in D$.

Proof 1 *For fixed $k \in \mathbb{N}^*$, $i \in U$, $j \in D$ we define*

$$\begin{aligned} r_{(\ell, y, i, j)}^\#(k) &= \mathbb{E}\left(N_{ij}^\#(k) - N_{ij}^\#(k-1) \mid J_0 = \ell, Y_0 = y\right) \\ &= \sum_{l=1}^k P(Y_{l-1} = i, Y_l = j, S_l = k \mid J_0 = \ell, Y_0 = y). \end{aligned} \quad (6.1)$$

Moreover,

$$\begin{aligned} P(Y_{l-1} = i, Y_l = j, S_l = k \mid J_0 = \ell, Y_0 = y) \\ = \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k R_{s_0; i} R_{s_1; j} q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0). \end{aligned} \quad (6.2)$$

Consequently, from (6.1) and (6.2), we get

$$\begin{aligned} r_{ij}^\#(k) &= \sum_{(\ell, y) \in E \times A} a(\ell, y) R_{\ell; y} r_{(\ell, y, i, j)}^\#(k) \\ &= \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k a(\ell, y) R_{\ell; y} R_{s_0; i} R_{s_1; j} q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0). \end{aligned}$$

For any fixed, arbitrary states $i \in U$, $j \in D$, the full conditional ROCOF at time $k \in \mathbb{N}^*$ could be estimated by the following plug-in type estimator

$$\begin{aligned} \widehat{r}_{ij}^\#(k, M) &= \\ &= \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k \widehat{a}(\ell, y) \widehat{R}_{\ell; y}(M) \widehat{R}_{s_0; i}(M) \widehat{R}_{s_1; j}(M) \widehat{q}_{i s_0}^{(l-1)}(k_0, M) \widehat{q}_{s_0 s_1}(k - k_0, M). \end{aligned}$$

Proposition 2 *For any $i \in U$, $j \in D$, the estimator of the full conditional ROCOF at time $k \in \mathbb{N}$ is strongly consistent, i.e.*

$$\lim_{M \rightarrow \infty} \widehat{r}_{ij}^\#(k, M) = r_{ij}^\#(k)$$

with probability 1.

Proof 2 Following [2], the estimators $\widehat{q}_{ij}(k, M)$, $\widehat{q}_{ij}^{(n)}(k, M)$ and $\widehat{R}_{i;a}(M)$ are strongly consistent. Moreover since the estimator $r_{ij}^\#(k, M)$ includes a finite number of terms, the result is straightforward.

Proposition 3 For any $k \in \mathbb{N}^*$, $i \in U$, $j \in D$, the random vector $\sqrt{M}(\widehat{r}_{ij}^\#(k, M) - r_{ij}^\#(k))$ is asymptotically normal, i.e.

$$\sqrt{M}(\widehat{r}_{ij}^\#(k, M) - r_{ij}^\#(k)) \xrightarrow[M \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Psi_1' \Delta \Psi_1'^\top),$$

where $\Psi_1 : \mathbb{R}^{\ell d} \rightarrow \mathbb{R}^+$ is defined by

$$\begin{aligned} & \Psi_1((R_{j';m'}, q_{i'j'}(k' - k'_0)); (i', k'_0) \in L, (i', j', k') \in E^*) \\ &= \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k a(\ell, y) R_{\ell;y} R_{s_0;i} R_{s_1;j} q_{is_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0), \end{aligned}$$

and $\Psi_1'^\top = \left(\left(\frac{\partial \Psi_1}{\partial R_{j';m'}}, \frac{\partial \Psi_1}{\partial q_{i'j'}(k' - k'_0)} \right); (i', k'_0) \in L, (i', j', k') \in E^* \right)$, is the column vector of derivatives of $r_{ij}^\#(k)$ with respect to $R_{j';m'}$ and $q_{i'j'}(k' - k'_0)$.

Proof 3 First we notice that

$$\begin{aligned} & \sqrt{M}(\widehat{r}_{ij}^\#(k, M) - r_{ij}^\#(k)) \\ &= \sqrt{M} \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k \left(\widehat{a}(\ell, y) \widehat{R}_{\ell;y}(M) \widehat{R}_{s_0;i}(M) \widehat{R}_{s_1;j}(M) \widehat{q}_{is_0}^{(l-1)}(k_0, M) \widehat{q}_{s_0 s_1}(k - k_0, M) \right. \\ & \quad \left. - a(\ell, y) R_{\ell;y} R_{s_0;i} R_{s_1;j} q_{is_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0) \right). \end{aligned}$$

Then since

$$\sqrt{M} \left(\widehat{q}_{i'j'}(k' - k'_0, M) \widehat{R}_{j';m'}(M) - q_{i'j'}(k' - k'_0) R_{j';m'} \right) \xrightarrow[M \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Delta),$$

delta method [23] leads directly to

$$\sqrt{M} \left(\Psi_1(\widehat{q}_{i'j'}(k' - k'_0, M), \widehat{R}_{j';m'}(M)) - \Psi_1(q_{i'j'}(k' - k'_0, M), R_{j';m'}) \right) \xrightarrow[M \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Psi_1' \Delta \Psi_1'^\top).$$

6.3.2 Right conditional ROCOF

We go one step further and aim at studying the impact of the ending down state on the ROCOF. In order to do that, we define the right conditional ROCOF as the mean transition number of the HMRC to the set D at time k , given that it ends in state $j \in D$, i.e.

$$\bar{r}_j(k) = \mathbb{E}(\overline{N}_j(k)) - \mathbb{E}(\overline{N}_j(k-1)),$$

where

$$\bar{N}_j(k) = \sum_{l=1}^k \mathbf{1}_{\{Y_{l-1} \in U, Y_l = j\}}.$$

Proposition 4 *The right conditional ROCOF at time $k \in \mathbb{N}^*$ is defined by*

$$\bar{r}_j(k) = \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k a(\ell, y) R_{\ell; y} R_{s_0}(U) R_{s_1; j} q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0),$$

for any fixed, arbitrary state $j \in D$.

Proof 4 *For fixed $k \in \mathbb{N}^*$, $j \in D$ we define*

$$\begin{aligned} \bar{r}_{(\ell, y, j)}(k) &= \mathbb{E}\left(\bar{N}_j(k) - \bar{N}_j(k-1) \mid J_0 = \ell, Y_0 = y\right) \\ &= \sum_{l=1}^k P(Y_{l-1} \in U, Y_l = j, S_l = k \mid J_0 = \ell, Y_0 = y). \end{aligned} \quad (6.3)$$

Moreover,

$$\begin{aligned} &P(Y_{l-1} \in U, Y_l = j, S_l = k \mid J_0 = \ell, Y_0 = y) \\ &= \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k R_{s_0}(U) R_{s_1; j} q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0). \end{aligned} \quad (6.4)$$

Consequently, from (6.3) and (6.4), we get

$$\begin{aligned} \bar{r}_j(k) &= \sum_{(\ell, y) \in E \times A} a(\ell, y) R_{\ell; y} \bar{r}_{(\ell, y, j)}(k) \\ &= \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k a(\ell, y) R_{\ell; y} R_{s_0}(U) R_{s_1; j} q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0). \end{aligned}$$

For any fixed, arbitrary state $j \in D$, the right conditional ROCOF at time $k \in \mathbb{N}^*$ could be estimated as follows:

$$\begin{aligned} \widehat{\bar{r}}_j(k, M) &= \\ &\sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k \widehat{\alpha}(\ell, y) \widehat{R}_{\ell; y}(M) \widehat{R}_{s_0}(U, M) \widehat{R}_{s_1; j}(M) \widehat{q}_{i s_0}^{(l-1)}(k_0, M) \widehat{q}_{s_0 s_1}(k - k_0, M). \end{aligned}$$

Proposition 5 *For any $j \in D$, $k \in \mathbb{N}$, the estimator of the right conditional ROCOF at time k is strongly consistent, i.e.*

$$\lim_{M \rightarrow \infty} \widehat{\bar{r}}_j(k, M) = \bar{r}_j(k)$$

with probability 1.

Proof 5 To prove the consistency, we work as in Proposition 2.

Proposition 6 For any $k \in \mathbb{N}^*$, $j \in D$, the random vector $\sqrt{M}(\widehat{r}_j(k, M) - \bar{r}_j(k))$ is asymptotically normal, i.e.

$$\sqrt{M}(\widehat{r}_j(k, M) - \bar{r}_j(k)) \xrightarrow[M \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Psi_2' \Delta \Psi_2'^{\top}),$$

where $\Psi_2 : \mathbb{R}^{\ell d} \rightarrow \mathbb{R}^+$ is the function

$$\begin{aligned} & \Psi_2((R_{j';m'}, q_{i'j'}(k' - k'_0)); (i', k'_0) \in L, (i', j', k') \in E^*) \\ &= \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k a(\ell, y) R_{\ell; y} R_{s_0}(U) R_{s_1; j} q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0) \end{aligned}$$

and $\Psi_2'^{\top} = \left(\left(\frac{\partial \Psi_2}{\partial R_{j';m'}}, \frac{\partial \Psi_2}{\partial q_{i'j'}(k' - k'_0)} \right); (i', k'_0) \in L, (i', j', k') \in E^* \right)$, is the column vector of derivatives of $\bar{r}_j(k)$ with respect to $R_{j';m'}$ and $q_{i'j'}(k' - k'_0)$.

Proof 6 We first notice that

$$\begin{aligned} & \sqrt{M}(\widehat{r}_j(k, M) - \bar{r}_j(k)) \\ &= \sqrt{M} \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k \left(\widehat{a}(\ell, y) \widehat{R}_{\ell; y}(M) \widehat{R}_{s_0}(U, M) \widehat{R}_{s_1; j}(M) \widehat{q}_{i s_0}^{(l-1)}(k_0, M) \widehat{q}_{s_0 s_1}(k - k_0, M) \right. \\ & \quad \left. - a(\ell, y) R_{\ell; y} R_{s_0}(U) R_{s_1; j} q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0) \right). \end{aligned}$$

Then since

$$\sqrt{M} \left(\widehat{q}_{i'j'}(k' - k'_0, M) \widehat{R}_{j';m'}(M) - q_{i'j'}(k' - k'_0) R_{j';m'} \right) \xrightarrow[M \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Delta),$$

we have that

$$\sqrt{M} \left(\Psi_2(\widehat{q}_{i'j'}(k' - k'_0), \widehat{R}_{j';m'}) - \Psi_2(q_{i'j'}(k' - k'_0), R_{j';m'}) \right) \xrightarrow[M \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Psi_2' \Delta \Psi_2'^{\top}).$$

6.3.3 Left conditional ROCOF

The left conditional ROCOF is defined as the mean transition number of the HMRC to the set D at time k , given that it starts from the fixed state $i \in U$, i.e.

$$\tilde{r}_i(k) = \mathbb{E}(\tilde{N}_i(k)) - \mathbb{E}(\tilde{N}_i(k-1)),$$

where

$$\tilde{N}_i(k) = \sum_{l=1}^k \mathbf{1}_{\{Y_{l-1}=i, Y_l \in D\}}.$$

Proposition 7 *The left conditional ROCOF at time $k \in \mathbb{N}^*$ is defined by*

$$\tilde{r}_i(k) = \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k a(\ell, y) R_{\ell; y} R_{s_0; i} R_{s_1}(D) q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0),$$

for any fixed, arbitrary state $i \in U$.

Proof 7 *For fixed $k \in \mathbb{N}^*$, $i \in U$ we define*

$$\begin{aligned} \tilde{r}_{(\ell, y, i)}(k) &= E\left(\tilde{N}_i(k) - \tilde{N}_i(k-1) | J_0 = \ell, Y_0 = y\right) \\ &= \sum_{l=1}^k P(Y_{l-1} = i, Y_l \in D, S_l = k | J_0 = \ell, Y_0 = y). \end{aligned} \quad (6.5)$$

Moreover,

$$\begin{aligned} &P(Y_{l-1} = i, Y_l \in D, S_l = k | J_0 = \ell, Y_0 = y) \\ &= \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k R_{s_0; i} R_{s_1}(D) q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0). \end{aligned} \quad (6.6)$$

Consequently, from (6.5) and (6.6), we get

$$\begin{aligned} \tilde{r}_i(k) &= \sum_{(\ell, y) \in E \times A} a(\ell, y) R_{\ell; y} \tilde{r}_{(\ell, y, i)}(k) \\ &= \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k a(\ell, y) R_{\ell; y} R_{s_0; i} R_{s_1}(D) q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0). \end{aligned}$$

For any fixed, arbitrary state $i \in U$, the left conditional ROCOF at time $k \in \mathbb{N}^*$ could be estimated by the following plug-in type, empirical estimator

$$\begin{aligned} \hat{\tilde{r}}_i(k, M) &= \\ &\sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k \hat{\alpha}(\ell, y) \hat{R}_{\ell; y}(M) \hat{R}_{s_0; i}(M) \hat{R}_{s_1}(D, M) \hat{q}_{i s_0}^{(l-1)}(k_0, M) \hat{q}_{s_0 s_1}(k - k_0, M). \end{aligned}$$

Proposition 8 *For any state $i \in U$ and any fixed arbitrary $k \in \mathbb{N}^*$, the estimator of the right conditional ROCOF at time k is strongly consistent in the sense that*

$$\lim_{M \rightarrow \infty} \hat{\tilde{r}}_i(k, M) = \tilde{r}_i(k)$$

with probability 1.

Proof 8 To prove the consistency, we work as in Proposition 2.

Proposition 9 For any $k \geq 1$, $j \in D$, the random vector $\sqrt{M}(\widehat{r}_i(k, M) - \widetilde{r}_i(k))$ is asymptotically normal, i.e.

$$\sqrt{M}(\widehat{r}_i(k, M) - \widetilde{r}_i(k)) \xrightarrow[M \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Psi_3 \Delta \Psi_3^\top),$$

where $\Psi_3 : \mathbb{R}^{\ell d} \rightarrow \mathbb{R}^+$ is the function

$$\begin{aligned} & \Psi_3((R_{j';m'}, q_{i'j'}(k' - k'_0)); (i', k'_0) \in L, (i', j', k') \in E^*) \\ &= \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k a(\ell, y) R_{\ell; y} R_{s_0; i} R_{s_1}(D) q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0) \end{aligned}$$

and $\Psi_3^\top = \left(\left(\frac{\partial \Psi_3}{\partial q_{i'j'}(k' - k'_0)}, \frac{\partial \Psi_3}{\partial R_{j';m'}} \right); (i', k'_0) \in L, (i', j', k') \in E^* \right)$, is the column vector of derivatives of $\widetilde{r}_i(k)$ with respect to $R_{j';m'}$ and $q_{i'j'}(k' - k'_0)$.

Proof 9 We first have that

$$\begin{aligned} & \sqrt{M}(\widehat{r}_i(k, M) - \widetilde{r}_i(k)) \\ &= \sqrt{M} \sum_{(\ell, y) \in E \times A} \sum_{l=1}^k \sum_{s_0, s_1 \in E} \sum_{k_0=0}^k \left(\widehat{a}(\ell, y) \widehat{R}_{\ell; y}(M) \widehat{R}_{s_0; i}(M) \widehat{R}_{s_1}(D, M) \widehat{q}_{i s_0}^{(l-1)}(k_0, M) \widehat{q}_{s_0 s_1}(k - k_0, M) \right. \\ & \quad \left. - a(\ell, y) R_{\ell; y} R_{s_0; i} R_{s_1}(D) q_{i s_0}^{(l-1)}(k_0) q_{s_0 s_1}(k - k_0) \right). \end{aligned}$$

Then since

$$\sqrt{M} \left(\widehat{q}_{i'j'}(k' - k'_0, M) \widehat{R}_{j';m'}(M) - q_{i'j'}(k' - k'_0) R_{j';m'} \right) \xrightarrow[M \rightarrow \infty]{\mathcal{L}} \mathcal{N}(0, \Delta),$$

we obtain directly the desired result.

The (unconditional) ROCOF for HMRCs was introduced in [24] as the mean transition number of the HMRC to the set D at time k , i.e.,

$$r(k) = \mathbb{E}(M(k) - M(k-1)),$$

where

$$M(k) = \sum_{l=1}^k \mathbf{1}_{\{Y_{l-1} \in U, Y_l \in D\}}.$$

Remark 1 The following relationship between the ROCOF and its conditional counterparts holds true

$$r(k) = \sum_{i \in U} \sum_{j \in D} r_{ij}^\#(k) = \sum_{j \in D} \bar{r}_j(k) = \sum_{i \in U} \widetilde{r}_i(k),$$

for any $k \in \mathbb{N}^*$.

6.4 Concluding Remarks

Failure occurrence rates are important risk indicators in the theory and applications of stochastic models. We encounter their distribution as reliability or survival function in mechanical systems or biology, respectively. Many stochastic models have been developed to analyze data associated with failure occurrences. Often these data are driven by hidden or unobservable mechanisms. Here we propose hidden Markov renewal models to describe systems that could experience multiple random failures. We concentrate on ROCOF which plays a critical role in the reliability analysis, since its increasing value signals the degradation of the system under study, whereas its decreasing value indicates its improvement.

We study the sensitivity of the ROCOF in the current up and ending down states by introducing its conditional counterparts. This is the first attempt to arise sensitivity issues of reliability indicators for hidden Markov renewal models. We provide plug-in type estimators and show that they have appealing asymptotic properties including consistency and asymptotic normality. Once the estimators are shown to be asymptotically normal, the estimation of the confidence interval is straightforward. It is worth noting that the estimation of the (conditional) ROCOF necessitates the estimation of the corresponding parameters of the HMRC. The maximum likelihood estimators could be obtained by minor modifications of the EM algorithm.

The obtained results could be applied in different scientific disciplines including seismology and mechanics. To be more precise let us present two possible applications of our results. In seismology, earthquake occurrences are governed by the actual level of the stress field, which is not observable. We could consider a hidden Markov renewal model where states correspond to stress levels and observations to earthquake magnitude classes. In this case the conditional counterparts of ROCOF could describe the impact of each magnitude class in the occurrence of the magnitude class which includes stronger earthquakes. On the other side, mechanical systems are subject to random loadings that could affect their integrity when they exceed certain thresholds. Semi-markov models could be used to describe loading data and conditional failure occurrence rates could be determined to allow the monitoring of mechanical systems. Since the real state of such systems is unknown the conditional failure occurrence rate could be studied in a hidden Markov renewal context. For recent advances in the applications of HSMMs in health monitoring we address readers to [8].

From a theoretical point of view, the current results could be extended for general HMRCs. A topic of interest for future research is the study of (conditional) ROCOF for (general) HSMMs. In this case the ROCOF for HMMs could be studied as a particular case. ROCOF could further be investigated when independent and identical copies of the process are observed, each over a

fixed duration. Finally, the consideration of models with a more refined structure [17] could improve the understanding of the functioning of a system and eventually provide feedback for the management of random systems.

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