



HAL
open science

Spectral theory of pseudo-differential operators of degree 0 and application to forced linear waves

Yves Colin de Verdière

► **To cite this version:**

Yves Colin de Verdière. Spectral theory of pseudo-differential operators of degree 0 and application to forced linear waves. 2018. hal-01761011v1

HAL Id: hal-01761011

<https://hal.science/hal-01761011v1>

Preprint submitted on 7 Apr 2018 (v1), last revised 13 May 2019 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Spectral theory of pseudo-differential operators of degree 0 and application to forced linear waves

Yves Colin de Verdière*

April 7, 2018

Introduction

This paper contains new developpements of some ideas already introduced in our paper [CSR-18] concerning the spectral theory of self-adjoint pseudo-differential operators of degree 0 on closed manifolds. The main motivation comes from the study of forced internal waves in physics, see [BFM-13, Br-16, GDDSV-06, MBSL-97, ML-95, Og-05, RV-10, RV-18] and many other works. In what follows, H is a classical self-adjoint scalar pseudo-differential operator on a compact manifold M of dimension n without boundary, f is a smooth function and the spectral parameter ω is a real number. The main object to study is the following linear forced wave equation:

$$\frac{1}{i} \frac{du}{dt} + Hu = fe^{-i\omega t}, \quad u(0) = 0. \quad (1)$$

We are interested in the behaviour of $u(t)$ as $t \rightarrow +\infty$. Thanks to the spectral theorem, we can relate this behaviour to the spectral theory of H and hence to the Hamiltonian dynamics of the principal symbol $h : T^*M \setminus 0$ which is a smooth function homogeneous of degree 0. The main tools that we use are already classical: they are, on one hand, the general theory of

*Université Grenoble Alpes, Institut Fourier, Unité mixte de recherche CNRS-UGA 5582, BP 74, 38402-Saint Martin d'Hères Cedex (France); yves.colin-de-verdiere@univ-grenoble-alpes.fr

pseudo-differential operators, culminating in the works of Lars Hörmander, Hans Duistermaat, Alan Weinstein and many others, in the beginning of the seventies, see [Du-11, Fo-89, We-71, We-75, DZ-17], and, on the other hand, the theory initiated by Eric Mourre in the beginning of the eighties in order to get a flexible way to have a limit absorption principle, see [Mo-81, Mo-83, JMP-84, Gé-08, Ca-05, CGH-06].

What will be the content, beyond that of [CSR-18]? After recalling general facts on the Hamiltonian dynamics of a homogeneous Hamiltonian h of degree 0 in Section 1 and on the spectral theory of H in Section 2, we give, in Section 3, a necessary and sufficient condition on the dynamics at infinity, which insures the existence of an escape function that will be the key input in order to apply Mourre’s theory. In Section 4, after recalling general facts that we got in [CSR-18] for the forced wave equation from Mourre’s theory, we use radial propagation estimates, going back to works of Melrose and Vasy, in order to locate the wavefrontset of the Schwartz distribution u_∞ which is the limit of $u(t)$ as $t \rightarrow +\infty$. In a quite independent Section 5, we propose to apply a “Fermi golden rule” in order to show that there is no embedded eigenvalues in the generic case. We consider then, in Section 6, the case where M is a surface ($n = 2$), extending our results of [CSR-18] to the generic case where the foliation is Morse-Smale and can have singular points (foci, nodes or saddles). Finally, we consider, in Section 7, the case where M is a 3D manifold with an S^1 -action leaving H invariant which is important for many applications. We end the paper with a short review of related problems in Section 8

Acknowledgments: I would first thank Laure Saint-Raymond for proposing to me to work with her on this problem and for sharing many ideas. Other people were helping us at different steps of this work: Thierry Dauxois, Semyon Dyatlov, Frédéric Faure, Vladimir Georgescu, Etienne Ghys, Johannes Sjöstrand, Michal Wrochna, Maciej Zworski, I thank all of them very warmly.

1 Hamiltonian of degree 0: classical dynamics

In what follows, we fix the following notations: M is a smooth compact manifold of dimension n without boundary, q is the generic point of M and $|dq|$ a smooth density on M . The Hamiltonian h is a smooth positively

homogeneous function $h : T^*M \setminus 0$. We denote by (q, p) some local canonical coordinates on T^*M and by extension a generic point of T^*M . The Hamiltonian vector field of h is denoted by \mathcal{X}_h and we fix the “symplectic” conventions so that

$$\mathcal{X}_h = \frac{\partial h}{\partial p} \partial_q - \frac{\partial h}{\partial q} \partial_p, \quad \mathcal{X}_h f = \{h, f\}$$

and denote by Φ_t the flow of \mathcal{X}_h . Because of the homogeneity of h , we have $pdq(\mathcal{X}_h) = 0$ and \mathcal{X}_h is homogeneous of degree -1 . Let us fix $\omega \in \mathbb{R}$ and define the energy shell $\Sigma_\omega := h^{-1}(\omega)$. We will assume that ω is not a critical value of h and hence Σ_ω is a smooth conic hypersurface in $T^*M \setminus 0$. We need to introduce $Z := \Sigma_\omega / \mathbb{R}^+$ which is a smooth closed manifold of dimension $2n - 2$ and can be seen as the boundary at infinity of T^*M . The vector field \mathcal{X}_h defines by projection a conformal class of vector fields on Z , which we will call an (oriented) foliation and denote by \mathcal{F} . This foliation admits in general singular points corresponding to the lines $\mathbb{R}^+ \cdot (q, p)$ where \mathcal{X}_h is parallel to $p\partial_p$. Note that we can and will often reduce ourselves to the case $\omega = 0$ by looking at the Hamiltonian $h - \omega$.

2 Hamiltonian of degree 0: spectral theory

Let us choose a self-adjoint pseudo-differential operator H of degree 0 acting on $L^2(M, |dq|)$ and of principal symbol h . Note that H is a bounded operator. In what follows, all pseudo-differential operators are “classical”, it means that the symbols do have full expansions in homogeneous functions with integer degrees. We are mainly interested by the spectral theory of H . As a warm up, we have the

Theorem 2.1 *The essential spectrum of H is the interval $J := [h_-, h_+]$ with $h_- := \min h$, $h_+ := \max h$.*

Proof.– If $\omega \in \mathbb{C} \setminus J$, $H - \omega$ is elliptic and hence admits an inverse $R(\omega)$ modulo compact operators which can be chosen holomorphic in ω by taking $R(\omega) := \text{Op}(h - \omega)^{-1}$ where Op is a fixed quantization on M :

$$R(\omega)(H - \omega) = \text{Id} + K(\omega)$$

with K compact and holomorphic in ω . On the other hand, H being bounded, $(H - \omega)$ is invertible for large values of ω . It follows from the Fredholm

analytic Theorem that the operator $H - \omega$ is invertible outside a discrete set where the kernels are finite dimensional.

On the other hand, if $\omega \in J$, with $h(q_0, p_0) = \omega$ and $\epsilon > 0$ is fixed, choose a small neighborhood U of q_0 so that, if $q \in U$, $|h(q, p_0) - \omega| \leq \epsilon$. Pick then $\phi \in C^\infty(U)$ with $\int_M |\phi|^2(q) |dq| = 1$. Let us check that, for t large enough,

$$\|(H - \omega) (\phi e^{itqp_0})\|_{L^2(M)} \leq 2\epsilon \quad (2)$$

It follows from the general properties of the principal symbols that

$$H (\phi(q) e^{itqp_0}) = h(q, p_0) \phi(q) e^{itqp_0} + O\left(\frac{1}{t}\right)$$

Take t so that the L^2 norm of the remainder is smaller than ϵ . We get inequality (2) by applying the triangular inequality. Hence

$$\|(H - \omega) (\phi e^{itqp_0})\|_{L^2(M)} \leq 2\epsilon \|\phi e^{itqp_0}\|_{L^2(M)}$$

which proves that $\sigma(H) \cap [\omega - 2\epsilon, \omega + 2\epsilon] \neq \emptyset$.

□

3 Escape functions

The key object of this paper is an escape function:

Definition 3.1 *A smooth function $k : \Sigma_0 \rightarrow \mathbb{R}$, positively homogeneous of degree 1, is called an escape function if there exists $\delta > 0$ so that the Poisson bracket $\{h, k\} = \mathcal{X}_h k$ is larger than δ on Σ_0 .*

We first give a general dynamical assumption on the foliation \mathcal{F} which turns out to be equivalent to the existence of a global *escape function*. We always assume that $Z = K^+ \cup K^- \cup \Omega$ as a disjoint union where:

- K^+ is an attractive compact invariant set of the oriented foliation \mathcal{F} , the *sink*
- K^- is a repulsive compact invariant set of the oriented foliation \mathcal{F} , the *source*
- Ω is an open set of wandering points. We do not assume that Ω is the set of all wandering points. Equivalently, all (oriented) leaves of points in Ω converge to K^+ at “ $+\infty$ ” and to K^- at “ $-\infty$ ”.

Definition 3.2 We say that a compact invariant set K^+ is weakly hyperbolic, denoted **(WH)**, if there exists, in some neighborhood of K^+ , a vector field W generating \mathcal{F} and a density $d\mu$ so that $\operatorname{div}_{d\mu}(W) < 0$. Similarly for K^- , $\operatorname{div}_{d\mu}(W) > 0$.

Our main result in this section is

Theorem 3.1 If K^+ and of K^- satisfy **(WH)** and $\Omega = Z \setminus (K^+ \cup K^-)$ is wandering, then there exists an escape function. The converse is true: the existence of an escape function implies that there exists a decomposition $Z = K^+ \cup K^- \cup \Omega$ which is independent of the choice of the escape function so that K^+ and of K^- satisfy **(WH)** and Ω is wandering.

3.1 Dynamical assumptions implying weak hyperbolicity

Let us choose a vector field W generating \mathcal{F} , whose flow is denoted by ϕ_t , $t \in \mathbb{R}$, and equip Z with a smooth density $d\mu$.

Let us describe properties of closed invariant sets of \mathcal{F} from which we can deduce **(WH)**:

1. If some component of K^+ is an isolated point a , the assumption **(WH)** says that the trace of the linearized vector field of W at the point a is non positive. This is independent of the choice of W . The case where the singular point is hyperbolic is studied in the work of Guillemin and Schaeffer [GS-77]. They show that, in the generic situation, there exists a pseudo-differential normal form for such points. Independently, the classical part of this normal form is also described in dimension 2 in the works of Davydov and co-authors [Da-85, Ar-88, DIIS-03].
2. If some component of K^+ is a closed curve γ , the assumption **(WH)** says that the modulus of the determinant of the linearized Poincaré map is < 1 . In dimension $n = 2$, this is equivalent to our assumption (M2) in [CSR-18].

3.2 Construction of an escape function

We construct an escape function assuming that all components of K^\pm satisfy **(WH)**.

3.2.1 Escape function on Γ^+

Let Γ^\pm be the sub-cones of Σ_0 generated by the sets K^\pm . We will first construct an escape function k^+ in the conic set generated by the basin of K^+ . A similar construction can be done on the basin of Γ^- .

Let us first construct “polar coordinates” (ρ, ω) on Σ_0 where $\rho \in \mathbb{R}^+ \setminus 0$, $\omega \in Z$ and the dilations on Σ_0 act by $\lambda.(\rho, \omega) = (\lambda\rho, \omega)$:

Lemma 3.1 *If W is a given vector field on Z generating \mathcal{F} , there exist polar coordinates $(\rho, \omega) \in (\mathbb{R}^+ \setminus 0) \times Z$ on Σ_0 so that*

$$\mathcal{X}_h = a(\omega)\partial_\rho + \frac{1}{\rho}W .$$

Proof. – We start with arbitrary polar coordinates (ρ_1, ω) : for example identify Z with the co-sphere bundle S_1^* for some Riemannian metric on M and define $\rho_1(q, p)$ so that $(q, p/\rho_1(q, p)) \in S_1^*$. We get, using the homogeneity of \mathcal{X}_h and the fact that W span \mathcal{F} ,

$$\mathcal{X}_h = a_1(\omega)\partial_{\rho_1} + \frac{1}{\rho}W$$

with $\rho = A(\omega)\rho_1$ and hence $\partial_{\rho_1} = A(\omega)\partial_\rho$. \square

The Liouville measure $dL_0 := |dqdp/dh|$ on Σ_0 , being homogeneous of degree $n + 1$, w.r. to dilations, writes $dL_0 = \rho^n |d\rho|d\mu$ where $d\mu$ is a smooth measure on Z .

The fact that

$$\operatorname{div}_{dL_0}(\mathcal{X}_h) = 0$$

rewrites

$$na + \operatorname{div}_\mu(W) = 0 \tag{3}$$

The assumption **(WH)** implies that we have a smooth > 0 function F defined near K^+ so that

$$\operatorname{div}_{F\mu}(W) = \frac{dF(W)}{F} + \operatorname{div}_\mu(W) \leq -c < 0$$

Then, if $k^+ := F^{-1/n}\rho$, we get

$$dk^+(\mathcal{X}_h) = -\frac{1}{n}F^{-1/n} \left(\frac{dF(W)}{F} - na \right)$$

which is equal to

$$dk^+(\mathcal{X}_h) = -\frac{1}{n}F^{-1/n}\operatorname{div}_{F\mu}(W)$$

and we get that the function k^+ is an escape function in some conical neighborhood of Γ^+ . \square

Note that k_+ tends to $+\infty$ as z tends to K_+ . Hence $k_+ \sim \langle p \rangle$ being homogeneous of degree 1.

Similarly, the function k_- defined near Γ_- tends to $-\infty$.

3.2.2 Extension to Σ_0

We choose a non negative function m on Σ_0 which is smooth, homogeneous of degree 0 and equal to $\{h, k_\pm\}$ in some conical neighbourhoods of Γ_\pm .

This allows to get some new escape functions l_\pm in the basins of Γ_\pm so that we have $\{h, l_\pm\} = m$. Let C_0 be the cone $C_0 := \{l_+ = 0\}$ which is smooth and transversal to \mathcal{X}_h . On C_0 we have now two functions l_\pm . The difference $\delta(z) = l_+(z) - l_-(z)$ is homogeneous of degree 1 and is constant along the flow lines. We define now k on the Hamiltonian trajectories $t \rightarrow \Phi_t(z)$ starting from $z \in \{\rho = 1\} \cap C_0$, so that k is l_+ for $t \gg 1$, l_- for $t \ll -1$ and the derivative of k along the flow is > 0 . We extend then k by homogeneity.

3.3 Deriving (WH) from the existence of an escape function

In what follows, we assume only the existence of an escape function k .

Let us present a construction of Γ^\pm using only the dynamics of \mathcal{X}_h . We will see that these sets are defined independently of the choice of k : Γ^+ is the set of points $(q, p) \in \Sigma_0$ so that there exists $t_0 < 0$ with $\Phi_t(q, p) \rightarrow 0$ as $t \rightarrow t_0^+$, i.e. the trajectory of \mathcal{X}_h is not complete as time $t \rightarrow -\infty$. Similarly for Γ^- . We define K_\pm so that they generate the cones Γ_\pm .

Proposition 3.1 *The picture of the dynamics is as follows:*

- if $(q, p) \in \Sigma_0 \setminus (\Gamma^+ \cup \Gamma^-)$, $\Phi_t(q, p)$ is defined for all $t \in \mathbb{R}$, $\Phi_t(q, p) \rightarrow K^+$ as $t \rightarrow +\infty$ and $\Phi_t(q, p) \rightarrow K^-$ as $t \rightarrow -\infty$
- if $(q, p) \in \Gamma^+$, $\Phi_t(q, p)$ is defined for all $t > t_0(q, p)$, $\Phi_t(q, p) \rightarrow K^+$ as $t \rightarrow +\infty$ and $\Phi_t(q, p) \rightarrow 0$ as $t \rightarrow t_0(q, p)$

- if $(q, p) \in \Gamma^-$, $\Phi_t(q, p)$ is defined for all $t < t_0(q, p)$, $\Phi_t(q, p) \rightarrow K^-$ as $t \rightarrow -\infty$ and $\Phi_t(q, p) \rightarrow 0$ as $t \rightarrow t_0(q, p)$.

Proof. – Let us choose a metric g on M and consider the set $C_0 := k^{-1}(0) \cap (g^*)^{-1}(1)$. If $(q, p) \in C_0$, the trajectory $t \rightarrow \Phi_t(q, p)$ is well defined, because $k(\Phi_t(q, p))$ is strictly monotonic and hence does not tend to the zero section where $k = 0$ at $\pm\infty$. This set is a generating set for the cone $k^{-1}(0)$. This way we get a smooth submanifold C of Σ_0 generating $\Sigma_0 \setminus (\Gamma^+ \cup \Gamma^-)$:

$$C := \{\Phi_t(q, p) \mid (q, p) \in C_0, t \in \mathbb{R}\}$$

The \mathcal{X}_h dynamics restricted to C is complete and the function k has a derivative $\geq \delta > 0$ along the trajectories. Hence all points of C are non wandering and the trajectories have to converge to K^+ as $t \rightarrow +\infty$ and to K^- as $t \rightarrow -\infty$. \square

Let us show that the existence of an escape function implies that K^+ satisfy **(WH)**: we choose polar coordinates (ρ, ω) near Γ^+ with the $\rho = k$ and we have, from the equations derived in section 3.2.1, that $dk(\mathcal{X}_h) = a > 0$ and hence $\operatorname{div}_\mu W = -a/n < 0$: all components of K^+ satisfy **(WH)**. A similar argument works for K^- .

Let us recall the following

Definition 3.3 ([DZ-17], **definition E.52**) *K^+ is a radial sink if there exists a conic neighborhood U of Γ^+ so that, for $\langle k \rangle \geq 1$ and $(q, p) \in U$, $\Phi_t(q, p) \rightarrow K^+$ as $t \rightarrow +\infty$ uniformly and, for $t \geq 0$, $k(\Phi_t(q, p)) \geq Ce^{\theta|t|}$ for some $\theta > 0$.*

A similar definition holds for a radial source K^- by changing t into $-t$.

It follows that

Proposition 3.2 *If there exists an escape function, K^+ is a radial sink and K^- a radial source as defined in [DZ-17].*

Proof.– Let us check that K^+ is a radial sink (def. E.52): we have near Γ^+ on Σ_0 with $l = \langle k \rangle h$, $\mathcal{X}_l = \langle k \rangle \mathcal{X}_h$ and hence, there exists $c > 0$ so that

$$\mathcal{X}_l(\langle k \rangle) \geq c \langle k \rangle$$

This estimate is still true in a conic neighbourhood of Γ^+ . This proves the exponential estimates (E.5.31) of [DZ-17].

We need also to prove uniform convergence to K^+ : the restriction of k restricted to the trajectories passing by $k^{-1}(0) \cap (g^*)^{-1}(1)$ is a Liapounov function for K^+ and for K^+ . \square

4 Applying Mourre's theory

4.1 Results from [CSR-18]

Let us first recall some results of [CSR-18]. Let us fix $\omega = 0$ for simplicity and assume that there exists an escape function k on the energy shell Σ_0 . Then k can be extended to $T^*M \setminus 0$ as an escape function in the cone $|h| \leq a$ with some $a > 0$. Let K be a self-adjoint operator of degree 1 of principal symbol k . Using the “sharp Gårding's inequality” (see [Fo-89] pp 129–136), one gets that K is a conjugate operator in the sense of Mourre: if J is as small enough open interval containing 0 and π_J is the spectral projector of H associated to the interval J , then

$$i\pi_J[K, H]\pi_J \geq c\pi_J + K$$

where $c > 0$ and K compact. Moreover the operator H is K -smooth, ie the map $t \rightarrow e^{itK} H e^{-itK}$ is smooth with values in bounded self-adjoint operators. It follows then from Mourre theory [Mo-81, Mo-83, JMP-84, Gé-08] that

Theorem 4.1 (Mourre) *The operator H has a finite number of eigenvalues in J , they have finite multiplicity. Assuming that 0 is not an eigenvalue, the resolvent $(H - z)^{-1}$ defined for $\Im z > 0$ admits a boundary value $\omega \rightarrow (H - \omega - i0)^{-1}$ for ω real close to 0 which, for any $\epsilon > 0$, is Hölder continuous for some non negative Hölder exponent from the Sobolev space $\mathcal{H}^{-\frac{1}{2}-\epsilon}$ into $\mathcal{H}^{\frac{1}{2}+\epsilon}$ for all $\epsilon > 0$.*

Moreover, if Π_- is the spectral projector on the negative part of the spectrum of K , then, if $f \in \mathcal{H}^{1+\epsilon}$, then $\Pi_- ((H - i0)^{-1} f) \in L^2$.

It follows then in our context:

Theorem 4.2 ([CSR-18]) *Assuming the existence of an escape function at $\omega = 0$ and that 0 is not an eigenvalue of H , then the solution $u(t)$ of the forced wave equation (1) with a smooth forcing f can be written as*

$$u(t) = u_\infty + \eta(t) + r(t)$$

where

- $u_\infty = (H - i0)^{-1}(f)$ belongs to $\mathcal{H}^{-\frac{1}{2}-\epsilon}$ for all $\epsilon > 0$
- u_∞ is microlocally L^2 near Γ^-

- $\eta(t) \rightarrow 0$ in $\mathcal{H}^{-\frac{1}{2}-\epsilon}$ for all $\epsilon > 0$
- The function $t \rightarrow r(t)$ is bounded in L^2 has a Fourier transform vanishing near 0
- $\|u(t)\|_{L^2}^2 \sim ct$ as $t \rightarrow +\infty$ with in general $c > 0$

If 0 is an eigenvalue of H , the corresponding eigenfunctions are smooth.

4.2 Wavefront set of u_∞

We will now derive results on the distribution u_∞ using the radial propagation estimates of Dyatlov-Zworski, based on earlier ideas of Richard Melrose [Me-94] and Andras Vasy [Va-13], and get

Theorem 4.3 *The wavefront set of u_∞ is contained in the cone Γ^+ .*

We will use results contained in Appendix E of [DZ-17], in particular Exercice 33, in order to prove smoothness of u_∞ outside the attracting cone.

This work as follows: we want to apply Theorem E.54 of [DZ-17] in our situation with $L = K^-$ and $\hbar = 1$. Let us restate the Theorem E.54 of [DZ-17] in our specific context:

Theorem 4.4 *Let $P \in \Psi^0(M)$ with principal symbol h and with a radial source K^- , assume the threshold condition (E.5.36)*

$$\sigma_{\text{sub}}(\mathfrak{S}P) + \left(s + \frac{1}{2}\right) \frac{\{h, k_-\}}{k_-} < 0$$

near K^- , then there exists $A \in \Psi^0$, elliptic near K^- , so that, if $Au \in \mathcal{H}^s$ and $f = Hu$ is smooth,

$$\|Au\|_s \leq c_1 \|f\|_{s+1} + c_2 \|u\|_{-N}$$

We want to deduce that u_∞ is smooth near Γ^- inspired by Exercice E.33 of [DZ-17]. We choose a positive invertible self-adjoint elliptic operator E of degree $m > 0$ on X . We can assume that the principal symbol of E is $|k_-|^m$ near K^- , and define, for $0 < \epsilon \leq 1$, $P_\epsilon := (1 + \epsilon E)^{-1} H (1 + \epsilon E)$ and $u_\epsilon := (1 + \epsilon E)^{-1} u$. We will apply Theorem 4.4 with $P = P_\epsilon$, " u " = u_ϵ and $s = m$. We have " f " = $(1 + \epsilon E)^{-1} f$. The distribution Au_ϵ is in \mathcal{H}^m , because Au is in L^2 (Mourre). We have to check that the assumptions of

Theorem E.54 are satisfied uniformly for P_ϵ (in particular that P_ϵ is uniformly bounded in Ψ^0). This follows from the fact that the sub-principal symbol of P_ϵ is $-i|k_-|^{-m}\{h, |k_-|^m\}$. The quantity (E.5.36) is then independent of ϵ and given by

$$\frac{1}{|k_-|} \left((m + \frac{1}{2})\{h, |k_-|\} - m\{h, |k_-|\} \right)$$

which is non positive near K^- . We get that $\|A(1 + \epsilon E)^{-1}u\|_m$ is uniformly bounded for $0 < \epsilon \leq 1$ and this implies that Au is in \mathcal{H}^m .

From this, we get that u_∞ is smooth near Γ^- and, by propagation of singularities applied to the equation $Hu_\infty \in C^\infty$, this proves Theorem 4.3.

Remark 4.1 *In the case $n = 2$, not all closed conical invariants subsets of Γ^+ can be wavefront sets of some u_∞ . If the wavefront set contains the line generated by a (ws)saddle point, it contains also one of the 2 branches of the associated unstable manifold and hence, being closed, also an attractive invariant set. This is proved in the paper [GS-77] at least for generic cases.*

5 Embedded eigenvalues

In [Ca-05, CGH-06], a *Fermi golden rule* is proved in the context of Mourre theory. This rule gives a condition implying that an eigenvalue ω , necessary of finite multiplicity, embedded in the continuous spectrum, “dissolves” in the continuum for a small bounded perturbation ϵV . Let us define the quadratic form Γ on the eigenspace $E_\omega := \ker_{L^2}(H - \omega)$ by

$$\Gamma(\phi) := -\Im \langle \bar{P}V\phi | (\omega + i0 - \bar{P}H\bar{P})^{-1} \bar{P}V\phi \rangle$$

where \bar{P} is the orthogonal projection of L^2 onto the orthogonal of E_ω . The form Γ is always ≥ 0 . The result is the following one:

Theorem 5.1 (Fermi golden rule) *If $\Gamma(\phi) > 0$ for any $\phi \in E_\omega \setminus 0$ and $a > 0$ so that H has no eigenvalues, except ω , in $I := [\omega - a, \omega + a]$, then, for $\epsilon \neq 0$ small enough, $H + \epsilon V$ has no eigenvalues in I .*

I was not able to prove the generic absence of embedded eigenvalues, however the following weak result is already an indication

Theorem 5.2 *If H is given and assuming the existence of an escape function for $\omega \in J$ where J is a closed interval, then the spectrum is a.c. with a finite number of embedded eigenvalues. There exists a smoothing operator R so that $H + R$ has no eigenvalue in J .*

Proof.– Recall that all the corresponding eigenfunctions are smooth. Let us denote by $E \subset C^\infty(M)$ the subspace generated by the eigenfunctions associated to the eigenvalues in J . We decompose $L^2 = E^\perp \oplus E$ and choose a self-adjoint linear map K from E into E with no eigenvalues in J , then, if

$$H_1 = H|_{E^\perp} \oplus K ,$$

$R = H_1 - H$ is a finite rank smoothing operator and has no eigenvalues in J . \square

6 The 2D case

In this Section $n = 2$.

6.1 Morse-Smale foliation

Definition 6.1 *A hyperbolic singular point of \mathcal{F} is said weakly stable if the trace of the linearization of any smooth vector field generating \mathcal{F} is < 0 . We define similarly weakly unstable hyperbolic singular points. We denote these properties respectively (ws) and (wu).*

Note that if $dh \neq 0$ on Σ_0 all saddle points are either weakly stable or weakly instable depending on the fact that \mathcal{X}_h is pointing to the infinity or not, this follows from equation (3) where $a \neq 0$.

Let us recall that a vector field on a surface is *Morse-Smale* if the non wandering points are singular hyperbolic points and closed hyperbolic cycles and there is no saddle connection, i.e. there is no leave whose both limit points are saddle points. We extend this definition to oriented foliations of surfaces by choosing any vector field generating the foliation.

Theorem 6.1 *Let n be equal to 2. Let us assume that the foliation \mathcal{F} is Morse-Smale (this implies that all saddle points are weakly stable or weakly unstable). Then there exists an escape function. The set K^+ is the union of all the attracting cycles and points and all the unstable manifolds of the ws-saddle points. The set K^- is constructed in a similar way.*

Remark 6.1 *Any generic foliation of a closed surface satisfies the previous properties: Mauricio Peixoto proved in the sixties that Morse-Smale vector fields on surfaces are generic, see [PdM-82], Chapter 4, for a detailed proof. As pointed out to me by Sylvain Courte, this genericity property extends to our context, i.e. to singular foliations of a surface embedded in a contact manifold, as it is proved in the PhD thesis of Emmanuel Giroux [Gi-91], Lemme 1.3.*

Proof-. Note first that K^+ and K^- are compact and disjoint because there is no saddle connection.

Let us prove that, if K^\pm are defined as before and $z \in \Omega$, the leave of z tends to K^+ at $+\infty$ and to K^- at $-\infty$. The leave of z tends to some critical point at $+\infty$ along the stable manifold of that point. If it is a saddle point, it will be (ws) because $z \notin K^-$.

Let us fix a density $d\mu$ on Z and construct a vector field W generating \mathcal{F} near K^+ whose divergence is non positive on K^+ . First, we construct a vector field W_b with $\text{div}(W_b) < 0$ in some neighborhood U_b of each (ws) saddle point b . We construct also (see Appendix A.2) a vector field W_a in the basin of each attractive cycle or point a with non positive divergence. Let us choose a positive function l_a tending to $+\infty$ at the boundary of the basin of a . Then, for L_a large enough the set $\{l_a \geq L_a\}$ intersects the unstable manifolds Y_j of each (ws) saddle point b_j inside U_{b_j} . We choose $\chi_a \in C_0^\infty(\mathbb{R}, [0, 1])$ so that $\chi_a(s) = 1$ for $0 \leq s \leq L_a$ and $\chi_a'(s) \leq 0$ for $s \geq 0$. Then we take,

$$W = \sum_a \left((\chi_a \circ l_a) W_a + C \sum_{b_j(\text{ws})} \psi_j W_{b_j} \right)$$

where ψ_j satisfies

- $\psi_j \in C_0^\infty(U_{b_j}, \mathbb{R}^+)$
- $\psi_j = 1$ on $\{l_a \geq L_a\} \cap Y_j$
- $d\psi_j(W) \leq 0$ on $Y_j \cap U_{b_j}$

and $C \gg 1$. This vector field is smooth well defined near K^+ and has non positive divergence on K^+ . \square

6.2 Lagrangian distributions associated to hyperbolic closed leaves

Let Γ be a component of Γ^+ generated by a closed cycle γ of the foliation \mathcal{F} . The cone Γ is a conic Lagrangian submanifold of $T^*X \setminus 0$: $\omega(\mathcal{X}_h, p\partial_p) = 0$ because of Euler identity. A theorem of Alan Weinstein [We-71] implies that there is an homogeneous canonical transformation χ defined in a conic neighborhood C of Γ whose image is a conic neighborhood of the zero section of $T^*\Gamma$ and so that $\chi(\Gamma)$ is the zero section of $T^*\Gamma$. Taking polar coordinates $(x, \eta) \in (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}^+ \setminus 0)$, the cotangent bundle of Γ admits coordinates $(x, \eta; \xi, y)$ with the symplectic form $d\xi \wedge dx + dy \wedge d\eta$. Changing y into $-y$, identifies $T^*\Gamma$ with T^*X_0 with $X_0 := (\mathbb{R}/2\pi\mathbb{Z})_x \times \mathbb{R}_y$. With this identification, Γ is moved into $\Gamma_0 = \{y = 0, \xi = 0\}$. The Hamiltonian vector field \mathcal{X}_0 of $h_0 := h \circ \chi^{-1}$ preserves Γ_0 : along Γ_0 , it is then given by $\mathcal{X}_0 = \partial_\xi h_0 \partial_x - \partial_y h_0 \partial_\eta$ and there $\partial_x h_0 = \partial_\eta h_0 = 0$. Because the foliation is non singular along γ_0 , we have $\partial_\xi h_0 \neq 0$. Hence the image of the energy shell Σ_0 is given by $\xi/\eta = F(x, y)$. The projection $\pi : Z_0 \rightarrow X_0$ is a local diffeomorphism. Because it is a diffeo on the cycle, it is even a global diffeo near γ_0 .

Using the tools introduced by Alan Weinstein in [We-75], we can build a FIO microlocally unitary $U : L^2(X) \rightarrow L^2(X_0, M)$ with M a flat bundle, called the Maslov bundle, so that $UHU^* - K$ is smoothing in C and $\sigma_p(K) = h \circ \chi^{-1}$, $\text{sub}(K) = 0$. We are then reduced to the non-singular case.

This proves, following then [CSR-18], the

Theorem 6.2 *If Γ is a component of Γ^+ generated by a closed hyperbolic stable cycle of \mathcal{F} , the distribution u_∞ is microlocally near Γ a Lagrangian distribution.*

7 The 3D case with S^1 invariance

The case where M is a 3-manifold with an S^1 -action commuting with H is frequent in physical applications: internal waves in some canal [ML-95], inertial waves inside the earth or some stars [RV-18], ... Let us start with the simple situation where M is a product $N \times S^1$ and examine then the case where there is only a quasi-free action of S^1 onto M .

7.1 Products $N \times S^1$

Let $M_{q,\theta} := N_q \times (\mathbb{R}/2\pi\mathbb{Z})_\theta$ with N_q a 2D compact manifold and denote by $(q, p; \theta, \tau)$ some local canonical coordinates on T^*M . Let us assume that N is equipped with a smooth density $|dq|$ and M with $|dq d\theta|$. Let us give a smooth Hamiltonian $h = h(q, p, \tau)$, homogeneous of degree 0, on $T^*M \setminus 0$ and a self-adjoint pseudo-differential operator of degree 0, H , of principal symbol h , acting on $L^2(M, |dq d\theta|)$. We assume that H commutes with the translations in θ . The operator H is then a direct sum of operators on M :

$$H = \bigoplus_{n \in \mathbb{Z}} H_n$$

where H_n acts on $L^2(N, |dq|)$ as a self-adjoint pseudo-differential operator of principal symbol $h_n(q, p) := h(q, p, n)$ which is also equal to $h(q, p/n, 1)$ if $n \neq 0$.

The spectrum of H is clearly the union of the spectra of the H_n 's.

7.1.1 Spectra of H and the H_n 's

Let us define $h_0(q, p) := h(q, p, 0)$ and $h_1(q, p) = h(q, p, 1)$. Note that h_1 is a smooth symbol of degree 0 on T^*N which is asymptotic to h_0 at infinity. The essential spectrum of H is the interval $I_\infty := [a_\infty, b_\infty]$ where $a_\infty = \inf h_1$ and $b_\infty = \sup h_1$. The essential spectrum of the H_n 's is quite different: from the identities

$$h(q, p, n) = h\left(q, \frac{p}{|p|}, \frac{n}{|p|}\right) = h_0(q, p) + O\left(\frac{1}{|p|}\right),$$

one gets that the principal symbol of H_n is h_0 . Hence the essential spectrum of any of the H_n 's is $I_0 := [a_0, b_0]$ where $a_0 = \inf h_0$ and $b_0 = \sup h_0$. Note that we have $I_0 \subset I_\infty$ and they are often identical in the applications to physical problems.

We are interested in more precise properties of the spectra: we claim that, in $I_\infty \setminus I_0$, the spectrum of H is pure point dense, ie there is a basis of L^2 pairwise orthogonal eigenfunctions. Moreover the eigenvalues of H_n obey a Weyl rule when $n \rightarrow \infty$. One expects (see section 5) that the spectrum has no embedded eigenvalues in the interior of I_0 . But quasi-modes of the type "well in an island" are possible if the dynamics of h_1 has stable bounded invariant sets (see Section 7.1.2).

Theorem 7.1 (Weyl law) *The spectra $\sigma(H_n)$ of the operators H_n in $I_\infty \setminus I_0$ are discrete. For any compact interval J included in $I_\infty \setminus I_0$, we have*

$$\#\{\sigma(H_n) \cap J\} \sim_{n \rightarrow \infty} \frac{n^2}{4\pi^2} \text{vol}(\{(q, p) | h_1(q, p) \in J\})$$

where the volume is defined with the Liouville measure on T^*N and the eigenvalues of H_n in J are counted with multiplicities.

Proof. – The full symbol of H writes

$$\tilde{h} = h(q, p, \tau) + \sum_{j=1}^{\infty} k_j(q, p, \tau)$$

with k_j homogeneous of degree j . Hence H_n can be viewed as a semi-classical pseudo-differential operator on N of semi-classical symbol

$$\tilde{h}_n = h_1(q, \hbar p) + \sum_{j=1}^{\infty} \hbar^j k_j(q, \hbar p, 1)$$

with $\hbar = 1/n$. The Theorem follows hence from the semi-classical Weyl asymptotics. \square

7.1.2 Classical dynamics

We will assume that the frequency $\omega = 0$ is fixed and the 2D Hamiltonian $h_0(q, p) := h(q, p, 0)$ admits an escape function. We will look at the dynamics of $h_1 := h(q, p, 1)$. Note that the dynamics of h reduces on each set $\tau = a$ with $a \neq 0$ to that of h_1 by some simple rescaling of the time. Moreover

$$\lim_{p \rightarrow \infty} h_1(q, p) = h_0(q, p)$$

Near infinity the dynamics still admits an escape (Liapounov function) and hence the orbits, if they come close enough to infinity, will converge to K_+ at $+\infty$ and K_- at $-\infty$. The dynamics $t \rightarrow \phi_t$ of h_1 is hence complete. We split the phase space into 3 pieces: $T^*M = \Omega \cup C_+ \cup C_-$ where

- Ω is the set of (q, p) so that $\phi_t(q, p) \rightarrow K_\pm$ as $t \rightarrow \pm\infty$
- C_+ is the set of (q, p) so that $\phi_t(q, p)$ stays bounded for $t \geq 0$

- C_- is the set of (q, p) so that $\phi_t(q, p)$ stays bounded for $t \leq 0$

Finally, we define $C := C_+ \cap C_-$ the set (q, p) so that $\phi_t(q, p)$ stays bounded for $t \in \mathbb{R}$. In the literature, C is called the *trapped set*.

It could happen that C supports some quasi-modes associated to the semi-classical parameter $1/n$. Generically, these quasi-modes are not close to true L^2 -eigenfunctions because they do not exist. There are still visible in the wave dynamics for a very long time...

7.2 Quasi-free action

Let M be a closed 3-manifold with a smooth S^1 -action and an invariant density $|dq|$. We say that the action of S^1 on M^3 is *quasi-free* if $M = M_0 \cup Y$ where Y is a 1D-submanifold of M so that all points of Y are fixed by the action and all points of M_0 have trivial isotropy groups. The quotient $N := M/S^1$ is then a surface with a boundary identified to Y and M_0 is a principal S^1 -bundle over the surface $N_0 := M_0/S^1$. This is the case for the spherical shells studied in [RV-10, RV-18], but there the manifold M itself has a boundary. Let h be an Hamiltonian on M homogeneous of degree 0 and invariant by the action of S^1 and H a self-adjoint pseudo-differential operator of principal symbol h on $L^2(M, |dq|)$ commuting with the action of S^1 . Again $H = \bigoplus_{n \in \mathbb{Z}} H_n$ where H_n acts on the space \mathcal{H}_n of functions u on M satisfying $u(\theta, q) = e^{in\theta} u(q)$. The spectrum of H is then the union of the spectra of the H_n 's.

Let us introduce the two functions h_0 and h_1 which reduce to the functions h_0 and h_1 of the previous section in the case where $M = N \times S^1$. We denote by τ the principal symbol of the infinitesimal action ∂_θ , sometimes called the momentum map, given by $\tau(q, p) = pdq(\partial_\theta)$ where ∂_θ is the vector field generating the S^1 -action. Note that τ vanishes on T_Y^* , the restriction to Y of the cotangent bundle.

The function h_1 is defined on the cotangent space T^*N_0 of $N_0 := M_0/S^1$, which is the symplectic reduction of T^*M obtained from the smooth hypersurface $\tau^{-1}(1) \subset T^*M_0$: h_1 is obtained by taking first the restriction of h to $\tau^{-1}(1)$ and then passing to the quotient by the action of S^1 . Note that h_1 is smooth. The function h_0 is defined similarly by taking the function induced by h on the reduction of $\tau^{-1}(0) \cap (T^*M_0 \setminus 0)$ which is identified to $T^*N_0 \setminus 0$. Note that h_0 is homogeneous of degree 0. Moreover $h_1(q, p) \rightarrow h_0(q, p)$ as $p \rightarrow \infty$.

The essential spectrum of H is the image I_∞ of h which is also equal to the closure of the image of h_1 . The essential spectrum of any of the H_n is the closure of the image of h_0 .

We will show that Theorem 7.1 still holds with “ N ” replaced by “ N_0 ”.

Proof. –

The operator H_n acts on \mathcal{H}_n which can be interpreted as a space of sections of a line bundle $L_1^{\otimes n}$ on N_0 with principal symbol h_0 : this can be seen by reducing locally where the principal bundle is trivial to the previous section. In order to avoid the problem of the singularity Y , it is simpler to work directly on M .

Let J be a closed interval whose intersection with I_0 is empty and $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ so that $\chi \equiv 1$ on J and, if $K := \text{Support}(\chi)$, $K \cap I_0 = \emptyset$. We will look at the spectral properties of $\chi(H) = \bigoplus_{n \in \mathbb{Z}} \chi(H_n)$. Let us observe that, if $L := \{q \in M \mid \exists (q, p) \in T^*M, \chi \circ h \neq 0\}$, we have $\bar{L} \cap Y = \emptyset$. Let us take $\psi_0 \in C_0^\infty(M)$ so that $\psi_0 \equiv 1$ on Y and $\text{Support}(\psi_0) \cap \bar{L} = \emptyset$. Then $\psi_0 \chi(H)$ is smoothing, because its full symbol vanishes. In particular $\text{Trace}(\psi_0 \chi(H_n)) \rightarrow 0$ as $n \rightarrow \infty$. What remains to do, using a partition of unity (ψ_0, ψ_j) so that that the bundle $M_0 \rightarrow N_0$ is trivial on the support of each ψ_j for $j \geq 1$, is to evaluate the traces of the operators $\psi_j \chi(H_n)$. This is done by reducing to the situation of Section 7.1. \square

Note also, that the problem is quite similar to what we did in [CdV-79] if one introduces the two commuting self-adjoint operators $P_1 = -i\partial_\theta$, $P_2 = P_1 H$. They are however not jointly elliptic near Y . This is the reason for which we have first to eliminate the contribution of a neighborhood of Y to the spectral counting. In other words, to show that the eigenfunctions of H_n with eigenvalues in J are rapidly decaying in n near Y .

8 Open problems

There are still many open problems. Let us describe a few of them:

- How does the spectral picture extend to the boundaries of the intervals with a.c. spectra? This problem is already not solved in the simple case where Z is a 2-torus, assuming the existence of a global transversal to the foliation, and the Poincaré map loses its hyperbolicity in a generic way.

- More generally, can we study what happens at the critical values of h assuming that this function is Morse or even Morse-Bott on S^*M ?
- What can we do in the case of a manifold with boundary? In particular, can we say something in the case of a polygon which is studied in the experiments of the Thierry Dauxois's team [Br-16].
- Prove the generic absence of embedded eigenvalues.
- Study the “small viscosity” limit $\sigma \rightarrow 0$? In particular, do the limits $\sigma \rightarrow 0$ and $t \rightarrow +\infty$ commute?
- Is there a discrete analogue of Mourre's theory for unitary maps? If yes, what can be said from the spectral theory of the unitary action of a diffeomorphism of a closed manifold on half-densities? For example, what is the spectral theory of a diffeomorphism of the circle with irrational rotation number which is not C^1 -conjugated to a rotation?

Appendices

A Divergences

A.1 Formulae

Let us give a smooth vector field W and a smooth density $d\mu$. The divergence of W with respect to $d\mu$ is the function defined by

$$\operatorname{div}_{d\mu}(W) := \frac{\mathcal{L}_W d\mu}{d\mu}$$

where the Lie derivative $\mathcal{L}_W d\mu$ is defined by $\mathcal{L}_W d\mu := \frac{d}{dt}|_{t=0} \phi_t^*(d\mu)$. Cartan's formula gives

$$\operatorname{div}_{d\mu}(W) = \frac{d(\iota(W)d\mu)}{d\mu}$$

In particular, we get the usefull formulae

$$\operatorname{div}_{d\mu}(fW) = df(W) + f \operatorname{div}_{d\mu}(W)$$

$$\operatorname{div}_{gd\mu}(W) = \frac{dg(W)}{g} + \operatorname{div}_{d\mu}(W)$$

A.2 Extending vector fields with non positive divergence

Lemma A.1 *Let us assume that the invariant compact a admits a smooth (Liapounov) function l_a defined in the basin B_a of a with $dl_a(W) < 0$ outside a and $l_a(a) = 0$ and $l_a \rightarrow +\infty$ at the boundary of B_a (this is the case in particular if the attractor K is hyperbolic). If the vector field W satisfies $\operatorname{div}_{d\mu}(W) < 0$ in some open neighborhood V of a , then there exists a vector field $W_1 = FW$ in B_a , so that $F > 0$ and $\operatorname{div}_{d\mu}(W_1) < 0$ in B_a .*

Proof. Let us choose $r > 0$ so that $\{l \leq r\} \subset V$. It is enough to take $F = 1$ in $\{l \leq r\}$ and, for any x so that $l(x) = r$ and any $t \geq 0$,

$$F(\phi_t(x)) := e^{\int_0^t \Phi(\phi_s(x)) ds}$$

with Φ smooth, $\Phi = 0$ near $l(y) = r$ and, for all y with $l(y) > r$, $\Phi(y) < -\operatorname{div}_{d\mu}(W)(y)$. \square

References

- [Ar-88] V.I. Arnold. *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer, Grundlehren der mathematischen Wissenschaften **250** (2d ed.) (1988).
- [BFM-13] J. Bajars, J. Frank, L. R. M. Maas, *On the appearance of internal wave attractors due to an initial or parametrically excited disturbance*. J. Fluid Mech. **714**:283–311 (2013).
- [Br-16] C. Brouzet, *Internal wave attractors: from geometrical focusing to non-linear energy cascade and mixing*. PhD thesis <https://hal.archives-ouvertes.fr/tel-01361201/document> (2016).
- [Ca-05] L. Cattaneo. *Mourre's inequality and embedded bound states*. Bull. Sci. Math. **129**:591–614 (2005).
- [CGH-06] L. Cattaneo, G.M. Graf & W. Hunziker. *A general resonance theory based on Mourre's inequality*. Ann. Henri Poincaré **7**:583–601 (2006).

- [CdV-79] Y. Colin de Verdière. *Spectre conjoint d'opérateurs pseudo-différentiels qui commutent I. Le cas non intégrable*. Duke Math. Journal **46**:169–182 (1979).
- [CSR-18] Y. Colin de Verdière & L. Saint-Raymond. *Attractors for two dimensional waves with homogeneous Hamiltonians of degree 0*. <https://arxiv.org/pdf/1801.05582.pdf>, submitted (2018).
- [Da-85] A. A. Davydov *Normal form of a differential equation, not solvable for the derivative, in a neighborhood of a singular point*. Functional Analysis and Its Applications **19**:81–89 (1985).
- [DIIS-03] A.A. Davydov, G. Ishikawa, S. Izumiya & W. Sun. *Generic singularities of implicit systems of first order differential equations on the plane*. Japanese Journal of Mathematics **3**:93–119 (2008).
- [DvS-93] W. de Melo, S. van Strien, *One-Dimensional Dynamics*. A series of modern surveys in mathematics, Springer (1993).
- [Du-11] H. Duistermaat. *Fourier Integral Operators*. Birkhäuser (2011).
- [DZ-17] S. Dyatlov & M. Zworski. *Mathematical theory of scattering resonances Version 0.1*. Book in preparation (October 19, 2017). http://math.mit.edu/~dyatlov/res/res_20171019.pdf
- [Fo-89] G. Folland. *Harmonic Analysis in phase space*. Annals of Maths Studies **122**, Princeton (1989).
- [Gé-08] C. Gérard. *A proof of the abstract limiting absorption principle by energy estimates*. J. Funct. Anal. **254**:2707–2725 (2008).
- [Gi-91] E. Giroux. *Convexité en topologie de contact*. Comment. Math. Helv. **66**:637–677 (1991).
- [GDDSV-06] L. Gostiaux, T. Dauxois, H. Didelle, J. Sommeria & S. Viboux. *Quantitative laboratory observations of internal wave reflection on ascending slopes*. Physics of Fluids **18**, 056602 (2006).
- [GS-77] V. Guillemin & D. Schaeffer. *On a certain class of Fuchsian partial differential equations*. Duke Math. Jour. **44**:157–199 (1977).

- [JMP-84] A. Jensen, E. Mourre & P. Perry. *Multiple commutator estimates and resolvent smoothness in quantum scattering theory*. Ann. Inst. Poincaré, **41**(2):207–225 (1984).
- [MBSL-97] L. R. M. Maas, D. Benielli, J. Sommeria & F.-P. A. Lam, *Observation of an internal wave attractor in a confined stably stratified fluid*. Nature, London **388**:557 (1997).
- [Me-94] R. Melrose. *Spectral and Scattering theory for the Laplacian on asymptotic Euclidian spaces*. in Spectral en scattering theory (M. Ikawa ed.), Marcel Dekker (1994).
- [ML-95] L. R. M. Maas & F.-P. A. Lam, *Geometric focusing of internal waves*. J. Fluid Mech. **300**, 1 (1995).
- [Mo-81] E. Mourre. *Absence of Singular Continuous Spectrum for Certain Self-Adjoint Operators*. Comm. Math. Phys. **78**:391–408 (1981).
- [Mo-83] E. Mourre. *Opérateurs conjugués et propriétés de propagation*. Comm. Math. Phys. **91**:279–300 (1983).
- [Og-05] G. Ogilvie. *Wave attractors and the asymptotic dissipation rate of tidal disturbances*. J. Fluid Mech. **543**:19–44 (2005).
- [PdM-82] J. Palis & W. de Melo. *Geometric Theory of Dynamical Systems. An Introduction*. Springer-Verlag (1982).
- [RV-10] M. Rieutord & L. Valdettaro. *Viscous dissipation by tidally forced inertial modes in a rotating spherical shell*. J. Fluid Mech. **643**:363–394 (2010).
- [RV-18] M. Rieutord & L. Valdettaro. *Axisymmetric inertial modes in a spherical shell at low Ekman numbers*. <https://arxiv.org/pdf/1802.07582.pdf>, to appear in J. Fluid Mech. (2018).
- [Va-13] A. Vasy. *Microlocal analysis of asymptotically hyperbolic and Kerr-de Sitter spaces*. with an Appendix of Semyon Dyatlov. Inventiones Math. **194**:381–513 (2013).
- [We-71] A. Weinstein, *Symplectic manifolds and their Lagrangian submanifolds*. Adv. Math. **6**:329–346 (1971).

- [We-75] A. Weinstein, *Fourier integral operators, quantization and the spectra of Riemannian manifolds* in Géométrie symplectique et physique mathématique, Colloques Internationaux CNRS **237**:289–298 (1975).