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THE DISCRETE LAPLACIAN ACTING ON 2-FORMS AND APPLICATION

HATEM BALOUDI, SAYDA BELGACEM, AND AREF JERIBI

ABSTRACT. [In the current paper](#), we study the discrete Laplacian acting on 2-forms [which](#) was introduced and investigated [by Chebbi \(2018\)](#). We establish a new criterion of essential self-adjointness using the Nelson lemma. Moreover, we give an upper bound on the infimum of the essential spectrum. Furthermore, we establish a link between the adjacency matrix and the discrete Laplacian on 2-forms.

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1. INTRODUCTION

Spectral graph theory [represents](#) an active area of research. In the last few years, the questions of the essential self-adjointness of discrete Laplacian

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operators on infinite graphs have attracted a lot of interest, see [11, 18, 22, 17]. There exist other definitions of the discrete Laplacian, e.g., [24, 16, 4, 2]. The one we are studying here is the discrete Laplacian acting on 2-forms and denoted by $\mathcal{L}_{2,\text{skew}}$, where skew stands for skew-symmetric. This operator was introduced by Chebbi in [7]. The author shows the relation between the χ -completeness geometric hypothesis for the graph and essentially self-adjointness for the discrete Laplacian $\mathcal{L}_{2,\text{skew}}$. More specifically, the author has proved that $\mathcal{L}_{2,\text{skew}}$ is essential self-adjoint, when the triangulation (we refer to Section 2 for a precise definition) is χ -complete.

The current study has two major aims. It first aims to discuss the question of essential self-adjointness for $\mathcal{L}_{2,\text{skew}}$. It is worth noting that this operator depends on the weight \mathcal{R} on oriented triangular faces and the weight \mathcal{E} on oriented edges, see Section 4 for more details. In the setting of electrical networks, the weight \mathcal{E} correspond to the conductance. We establish a hypothesis on the weights and involve essential self-adjointness by using the Nelson commutator theorem. The technique of the proof are inspired from [3]. Moreover, we give an upper bound on the infimum of the essential spectrum $\sigma_{\text{ess}}(\mathcal{L}_{2,\text{skew}}^{\mathcal{F}})$, where $\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}$ is the Friedrichs extension of $\mathcal{L}_{2,\text{skew}}$. Secondly, the paper aims to identify the link between the adjacency matrix and the discrete Laplacian $\mathcal{L}_{2,\text{skew}}$. To achieve this goal, we analyze the structure of $\mathcal{L}_{2,\text{skew}}$. Note that this discrete Laplacian was introduced on a skew-symmetric statistic on the space of 2-forms. We can define the discrete Laplacian $\mathcal{L}_{2,\text{sym}}$ in the symmetric case by the same expression of $\mathcal{L}_{2,\text{skew}}$. In the case of a tri-partite graph, we prove that the two operators are unitarily equivalent. Furthermore, $\mathcal{L}_{2,\text{sym}}$ is unitarily equivalent to the adjacency matrix of the triangular graph, see Section 7 for more details. We recall that the spectral theory of adjacency matrix acting on graphs is useful for the study of some gelling polymers, of some electrical networks, and in number theory, see [14, 13, 23].

As for the rest of this paper, it is structured as follows: The next section is devoted to some definitions and notations for graph. We find the definitions of two different Hilbert structures on the set of faces, in Section 3. Both definitions have their own interest. This permits to define two different types of discrete Laplacian associated to faces. The relation between these two operators is clearly presented in Section 4.3. In Section 5, we discuss the question of essential self-adjointness for the discrete Laplacian $\mathcal{L}_{2,\text{skew}}$. We establish a new criterion of essential self-adjointness using the Nelson lemma. In Section 6, we give an upper bound on the infimum of the essential spectrum. The obtained findings from the previous sections are presented in Section 7 for the purpose of investigating the questions of boundedness and essential self-adjointness for the adjacency matrix.

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comments on the text. They would like to thank also the anonymous referee for their numerous relevant remarks and useful suggestions.

2. GENERALITIES ABOUT GRAPHS

We start with some definitions to fix notations for graphs and refer to [9, 10, 24] for surveys on the matter. Let \mathcal{V} be a countable set. We equip \mathcal{V} with the discrete topology. Let $\mathcal{E} : \mathcal{V} \times \mathcal{V} \rightarrow [0, +\infty)$ and assume that \mathcal{E} is symmetric (i.e., $\mathcal{E}(x, y) = \mathcal{E}(y, x)$, for all $x, y \in \mathcal{V}$). Let $m : \mathcal{V} \rightarrow (0, +\infty)$. We say that $\mathcal{G} = (\mathcal{V}, m, \mathcal{E})$ is a weighted *graph* with *vertices* \mathcal{V} , weight of vertices m and weight of *edges* \mathcal{E} . In the setting of electrical networks, the weights correspond to the conductances. We say that x, y are *neighbors* if $\mathcal{E}(x, y) \neq 0$ and we denote it by $x \sim y$. A graph \mathcal{G} is *simple* if it has no loops (i.e., $\mathcal{E}(x, x) = 0$), $m=1$ and \mathcal{E} has values in $\{0, 1\}$. The set of neighbors of $x \in \mathcal{V}$ is denoted by

$$\mathcal{N}_{\mathcal{G}}(x) := \{y \in \mathcal{V} : \mathcal{E}(x, y) \neq 0\}.$$

A graph is *locally finite* if $\#\mathcal{N}_{\mathcal{G}}(x)$ is finite for all $x \in \mathcal{V}$. The *weighted degree* of vertices is given by

$$d_{\mathcal{V}}(x) := \frac{1}{m(x)} \sum_{y \sim x} \mathcal{E}(x, y).$$

When \mathcal{G} is simple, $d_{\mathcal{V}}(x) = \#\mathcal{N}_{\mathcal{G}}(x)$. A graph \mathcal{G} is *connected*, if for all $x, y \in \mathcal{V}$, there exists an $x-y$ -path, i.e., there is a sequence $(x_1, \dots, x_{N+1}) \in \mathcal{V}^{N+1}$ such that $x_1 = x$, $x_{N+1} = y$ and $\mathcal{E}(x_n, x_{n+1}) > 0$ for all $n \in \{1, \dots, N\}$. If no vertices appear more than once in (x_1, \dots, x_N) , the path (x_1, \dots, x_{N+1}) is called a *simple path*. The path is called a *cycle* or *closed* when the origin and the end are identical, i.e., $x_1 = x_{N+1}$. An n -*cycle* is a cycle with n vertices.

In the sequel, we shall always consider graphs \mathcal{G} , which are locally finite, connected and have no loop.

The set of cyclic permutations of $(x, y, z) \in \mathcal{V}^3$ is denoted by

$$\circlearrowleft(x, y, z) := \{(x, y, z), (y, z, x), (z, x, y)\}.$$

Let Tr the set of all simple 3-cycles

$$\mathcal{F} = Tr / \cong$$

where $\varpi_1 \cong \varpi_2$ if and only if ϖ_1 is a cyclic permutation of ϖ_2 . The elements of \mathcal{F} are called *triangular faces*. A *triangulation* is a couple $(\mathcal{G}, \mathcal{F})$ where \mathcal{G} is a graph and \mathcal{F} is the set of all triangular faces.

In the sequel, we represent the triangular faces by their vertices. For a triangular face $\varpi = (x, y, z)$, we have

$$\varpi = (x, y, z) = (y, z, x) = (z, x, y).$$

Let $(\mathcal{G} = (\mathcal{V}, \mathcal{E}, m), \mathcal{F})$ be a triangulation and let $\mathcal{R} : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \longrightarrow [0, +\infty)$ such that

$$(x, y, z) \in \mathcal{F} \iff \mathcal{R}(x, y, z) > 0.$$

We assume that \mathcal{R} is symmetric, i.e.

$$\mathcal{R}(x, y, z) = \mathcal{R}(\sigma(x, y, z))$$

For all $(x, y, z) \in \mathcal{V} \times \mathcal{V} \times \mathcal{V}$ and for any permutation $\sigma(x, y, z)$ of (x, y, z) . We say that $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ is a weighted triangulation with weight of triangular faces \mathcal{R} . We say that \mathcal{T} is simple if $\mathcal{G} := (\mathcal{V}, m, \mathcal{E})$ is simple and the weights of the faces equal 1. Choosing an orientation of triangulation consists of defining a partition of \mathcal{F} :

$$\mathcal{F} = \mathcal{F}^+ \sqcup \mathcal{F}^-,$$

$$(x_1, x_2, x_3) \in \mathcal{F}^+ \iff (x_3, x_2, x_1) \in \mathcal{F}^-.$$

For $(x_1, x_2, x_3) \in \mathcal{F}$, we denote

$$-(x_1, x_2, x_3) = (x_3, x_2, x_1).$$

The set of neighbors of the edge (x, y) is given by

$$\mathcal{F}_{(x,y)} := \mathcal{N}_{\mathcal{G}}(x) \cap \mathcal{N}_{\mathcal{G}}(y).$$

The weighted degree of edges is given by:

$$d_{\mathcal{E}}(x, y) := \frac{1}{\mathcal{E}(x, y)} \sum_{z \in \mathcal{F}_{(x,y)}} \mathcal{R}(x, y, z).$$

When \mathcal{T} is simple, $d_{\mathcal{E}}(x, y) = \#\mathcal{F}_{(x,y)}$.

3. THE SYMMETRIC AND SKEW-SYMMETRIC SPACES

3.1. Hilbert structures on the set of edges. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. Let $\mathcal{E} := \{(x, y) \in \mathcal{V} \times \mathcal{V} : \mathcal{E}(x, y) > 0\}$. The set of 1-cochains (or 1-forms) is given by:

$$\mathcal{C}_{\text{skew}}(\mathcal{E}) := \left\{ f : \mathcal{E} \rightarrow \mathbb{C}, f(x, y) = -f(y, x) \text{ for all } x, y \in \mathcal{V} \right\},$$

where skew stands for skew-symmetric. This corresponds to fermionic statistics. The set of functions with finite support is denoted by $\mathcal{C}_{\text{skew}}^c(\mathcal{E})$. Concerning bosonic statistics, we define:

$$\mathcal{C}_{\text{sym}}(\mathcal{E}) := \left\{ f : \mathcal{E} \rightarrow \mathbb{C}, f(x, y) = f(y, x) \text{ for all } x, y \in \mathcal{V} \right\}.$$

The set of functions with finite support is denoted by $\mathcal{C}_{\text{sym}}^c(\mathcal{E})$.

We turn to the Hilbert structures.

$$\ell_{\text{skew}}^2(\mathcal{E}) := \left\{ f \in \mathcal{C}_{\text{skew}}(\mathcal{E}) \text{ such that } \|f\|^2 := \frac{1}{2} \sum_{x,y \in \mathcal{V}} \mathcal{E}(x, y) |f(x, y)|^2 < \infty \right\}$$

and

$$\ell_{\text{sym}}^2(\mathcal{E}) := \left\{ f \in \mathcal{C}_{\text{sym}}(\mathcal{E}) \text{ such that } \|f\|^2 := \frac{1}{2} \sum_{x,y \in \mathcal{V}} \mathcal{E}(x,y) |f(x,y)|^2 < \infty \right\}.$$

The associated scalar product is given by

$$\langle f, g \rangle := \frac{1}{2} \sum_{(x,y) \in \mathcal{E}} \mathcal{E}(x,y) \overline{f(x,y)} g(x,y),$$

when f and g are both in $\ell_{\text{skew}}^2(\mathcal{E})$ or in $\ell_{\text{sym}}^2(\mathcal{E})$.

3.2. Hilbert structures on the set of faces. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. The set of 2-cochains or 2-forms is given by

$$\mathcal{C}_{\text{skew}}(\mathcal{R}) = \left\{ f : \mathcal{F} \rightarrow \mathbb{C} : f(x,y,z) = -f(z,y,x) \right\}.$$

The set of functions with finite support is denoted by $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$. Concerning the case symmetric, we define

$$\mathcal{C}_{\text{sym}}(\mathcal{R}) = \left\{ f : \mathcal{F} \rightarrow \mathbb{C} : f(x,y,z) = f(z,y,x) \right\}.$$

The set of functions with finite support is denoted by $\mathcal{C}_{\text{sym}}^c(\mathcal{R})$. Let us define the Hilbert spaces $\ell_{\text{skew}}^2(\mathcal{R})$ and $\ell_{\text{sym}}^2(\mathcal{R})$ as the sets of cochains with finite norm, we have

$$\ell_{\text{skew}}^2(\mathcal{R}) := \left\{ f \in \mathcal{C}_{\text{skew}}(\mathcal{R}) : \|f\|^2 = \frac{1}{2} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) |f(x,y,z)|^2 < \infty \right\}$$

and

$$\ell_{\text{sym}}^2(\mathcal{R}) := \left\{ f \in \mathcal{C}_{\text{sym}}(\mathcal{R}) : \|f\|^2 = \frac{1}{2} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) |f(x,y,z)|^2 < \infty \right\}.$$

The associated scalar product is given by

$$\begin{aligned} \langle f, g \rangle &:= \frac{1}{2} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) \overline{f(x,y,z)} g(x,y,z) \\ &:= \frac{1}{6} \sum_{(x,y) \in \mathcal{E}} \sum_{z \in \mathcal{F}(x,y)} \mathcal{R}(x,y,z) \overline{f(x,y,z)} g(x,y,z) \end{aligned}$$

when f and g are both in $\ell_{\text{skew}}^2(\mathcal{R})$ or in $\ell_{\text{sym}}^2(\mathcal{R})$.

4. OPERATORS

In this section, we recall the concept of exterior derivative operator associated to a faces space, we refer to [7, 8] for more details. This permits to define the discrete Laplacian acting on 2-forms.

4.1. Skew-symmetric case. We start **with** defining the operators in the skew-symmetric case. The *skew-symmetric exterior operator* is the operator $d_{\text{skew}}^1 : \mathcal{C}_{\text{skew}}^c(\mathcal{E}) \rightarrow \mathcal{C}_{\text{skew}}^c(\mathcal{R})$, given by

$$d_{\text{skew}}^1(f)(x, y, z) = f(x, y) + f(y, z) + f(z, x).$$

The *skew-symmetric co-exterior derivative operator* is the formal adjoint of d_{skew}^1 , i.e. it is the operator $\delta_{\text{skew}}^1 : \mathcal{C}_{\text{skew}}^c(\mathcal{R}) \rightarrow \mathcal{C}_{\text{skew}}^c(\mathcal{E})$, given by

$$\forall f \in \mathcal{C}_{\text{skew}}^c(\mathcal{R}), \quad \delta_{\text{skew}}^1(f)(x, y) = \frac{1}{\mathcal{E}(x, y)} \sum_{z \in \mathcal{F}(x, y)} \mathcal{R}(x, y, z) f(x, y, z).$$

Both operators are closable (see [7, Lemme 3.1]). We denote their closure by the same symbol. The skew-symmetric discrete Laplacian operator acting on 2-forms is given by

$$\begin{aligned} \mathcal{L}_{2, \text{skew}}(f)(x, y, z) &= d_{\text{skew}}^1 \delta_{\text{skew}}^1(f)(x, y, z) \\ &= \frac{1}{\mathcal{E}(x, y)} \sum_{t \in \mathcal{F}(x, y)} \mathcal{R}(x, y, t) f(x, y, t) \\ &\quad + \frac{1}{\mathcal{E}(y, z)} \sum_{t \in \mathcal{F}(y, z)} \mathcal{R}(y, z, t) f(y, z, t) \\ &\quad + \frac{1}{\mathcal{E}(z, x)} \sum_{t \in \mathcal{F}(z, x)} \mathcal{R}(z, x, t) f(z, x, t), \end{aligned}$$

with $f \in \mathcal{C}_{\text{skew}}^c(\mathcal{R})$.

4.2. Symmetric case. We turn to the symmetric case. The *symmetric exterior operator* is the operator $d_{\text{sym}}^1 : \mathcal{C}_{\text{sym}}^c(\mathcal{E}) \rightarrow \mathcal{C}_{\text{sym}}^c(\mathcal{R})$, given by

$$\forall f \in \mathcal{C}_{\text{sym}}^c(\mathcal{E}), \quad d_{\text{sym}}^1(f)(x, y, z) = f(x, y) + f(y, z) + f(z, x).$$

The *symmetric co-exterior derivative operator* is the formal adjoint of d_{sym}^1 , i.e. it is the operator $\delta_{\text{sym}}^1 : \mathcal{C}_{\text{sym}}^c(\mathcal{R}) \rightarrow \mathcal{C}_{\text{sym}}^c(\mathcal{E})$, given by

$$\forall f \in \mathcal{C}_{\text{sym}}^c(\mathcal{R}), \quad \delta_{\text{sym}}^1(f)(x, y) := \frac{1}{\mathcal{E}(x, y)} \sum_{z \in \mathcal{F}(x, y)} \mathcal{R}(x, y, z) f(x, y, z).$$

Indeed, for $f \in \mathcal{C}_{\text{sym}}^c(\mathcal{E})$ and $g \in \mathcal{C}_{\text{sym}}^c(\mathcal{R})$ we get:

$$\begin{aligned}
 \langle d_{\text{sym}}^1 f, g \rangle &= \frac{1}{2} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) d_{\text{sym}}^1 f(x,y,z) \overline{g(x,y,z)} \\
 &= \frac{1}{6} \sum_{x \sim y} \sum_{z \in \mathcal{F}(x,y)} \mathcal{R}(x,y,z) (f(x,y) + f(y,z) + f(z,x)) \overline{g(x,y,z)} \\
 &= \frac{1}{2} \sum_{x \sim y} \sum_{z \in \mathcal{F}(x,y)} \mathcal{R}(x,y,z) f(x,y) \overline{g(x,y,z)} \\
 &= \frac{1}{2} \sum_{x \sim y} \mathcal{E}(x,y) f(x,y) \overline{\left(\frac{1}{\mathcal{E}(x,y)} \sum_{z \in \mathcal{F}(x,y)} \mathcal{R}(x,y,z) g(x,y,z) \right)} \\
 &= \langle f, \delta_{\text{sym}}^1 g \rangle.
 \end{aligned}$$

The operators d_{sym}^1 and δ_{sym}^1 are closable. Indeed, since $\delta_{\text{sym}}^1 : \ell_{\text{sym}}^2(\mathcal{R}) \rightarrow \ell_{\text{sym}}^2(\mathcal{E})$ (resp. $d_{\text{sym}}^1 : \ell_{\text{sym}}^2(\mathcal{E}) \rightarrow \ell_{\text{sym}}^2(\mathcal{R})$) is with dense domain then δ_{sym}^1 (resp. d_{sym}^1) is closable. We denote their closure by the same symbol. The symmetric discrete Laplacian operator acting on 2-forms is the operator $\mathcal{L}_{2,\text{sym}} = d_{\text{sym}}^1 \delta_{\text{sym}}^1$, given by the same expression of $\mathcal{L}_{2,\text{skew}}$.

4.3. Relationship between $\mathcal{L}_{2,\text{skew}}$ and $\mathcal{L}_{2,\text{sym}}$. The two operators $\mathcal{L}_{2,\text{skew}}$ and $\mathcal{L}_{2,\text{sym}}$ have the same expression. However, they do not act on the same spaces. Namely, when \mathcal{T} is tri-partite, we shall prove that the two operators are unitarily equivalent.

Definition 4.1. *A tri-partite graph is a graph whose vertices can be partitioned into 3 disjoint sets so that there are no two vertices within the same set are adjacent. A tri-partite triangulation is a triangulation $\mathcal{T} = (\mathcal{G}, \mathcal{F})$ such that \mathcal{G} is tri-partite.*

Theorem 4.2. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a tri-partite weighted triangulation. Then, $\mathcal{L}_{2,\text{skew}}$ and $\mathcal{L}_{2,\text{sym}}$ are unitarily equivalent.*

Proof. We consider the tri-partite decomposition $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\}$. Set

$$\circlearrowleft \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3 = \{\circlearrowleft (x,y,z) : (x,y,z) \in \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3\}.$$

Let $\mathcal{U} : \ell_{\text{skew}}^2(\mathcal{R}) \rightarrow \ell_{\text{sym}}^2(\mathcal{R})$ be the unitary map given by

$$\mathcal{U}(f)(x,y,z) = S(x,y,z) f(x,y,z),$$

where

$$S(x,y,z) := \begin{cases} 1, & \text{if } (x,y,z) \in \circlearrowleft \mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3, \\ -1, & \text{if } (x,y,z) \in \circlearrowleft \mathcal{V}_3 \times \mathcal{V}_2 \times \mathcal{V}_1. \end{cases}$$

Let \mathcal{W} be the following mapping from $\ell_{\text{sym}}^2(\mathcal{R})$ into $\ell_{\text{skew}}^2(\mathcal{R})$: $\mathcal{W}(f)(x, y, z) = S(x, y, z)f(x, y, z)$. Then

$$\langle \mathcal{U}f, g \rangle = \langle f, \mathcal{W}g \rangle, \quad \mathcal{U}(\mathcal{W}(g)) = g \quad \text{and} \quad \mathcal{W}(\mathcal{U}(f)) = f$$

for all $f \in \ell_{\text{skew}}^2(\mathcal{F})$ and $g \in \ell_{\text{sym}}^2(\mathcal{F})$. So we have

$$\mathcal{W}(f) = \mathcal{U}^{-1}(f) = \mathcal{U}^*(f)$$

for all $f \in \ell_{\text{sym}}^2(\mathcal{R})$. Therefore,

$$\mathcal{U}\mathcal{L}_{2,\text{skew}}\mathcal{U}^{-1}(f)(x, y, z) = \mathcal{L}_{2,\text{sym}}(f)(x, y, z)$$

for all $f \in \mathcal{C}_{\text{sym}}^c(\mathcal{R})$. □

5. A NELSON CRITERIUM

For the general theory of unbounded Hermitian operators and their extensions, we refer the reader to [25, 20, 27]. Let $\mathcal{L}_{\text{skew}}$ be the following mapping from $\mathcal{C}_{\text{skew}}(\mathcal{R})$ into itself:

$$\begin{aligned} \mathcal{L}_{\text{skew}}(f)(x, y, z) &= \frac{1}{\mathcal{E}(x, y)} \sum_{t \in \mathcal{F}(x, y)} \mathcal{R}(x, y, t)f(x, y, t) \\ &\quad + \frac{1}{\mathcal{E}(y, z)} \sum_{t \in \mathcal{F}(y, z)} \mathcal{R}(y, z, t)f(y, z, t) \\ &\quad + \frac{1}{\mathcal{E}(z, x)} \sum_{t \in \mathcal{F}(z, x)} \mathcal{R}(z, x, t)f(z, x, t). \end{aligned}$$

Let $\mathcal{L}_{2,\text{max,skew}}$ be the restrictions of $\mathcal{L}_{\text{skew}}$ to

$$\mathcal{D}(\mathcal{L}_{2,\text{max,skew}}) := \left\{ f \in \ell_{\text{skew}}^2(\mathcal{R}) \text{ such that } \mathcal{L}_{\text{skew}}f \in \ell_{\text{skew}}^2(\mathcal{R}) \right\}.$$

Lemma 5.1. $\mathcal{L}_{2,\text{skew}}^* = \mathcal{L}_{2,\text{max,skew}}$.

Proof. Let $f \in \mathcal{C}_{\text{skew}}^c(\mathcal{R})$ and let $g \in \mathcal{C}_{\text{skew}}(\mathcal{R})$. Let \mathcal{F}_0 the support of f and set

$$\mathcal{F}_0 = \left\{ (x, y, z) \in \mathcal{F} : \exists u \in \mathcal{V}, \{(x, y, u), (y, z, u), (z, x, u)\} \cap \mathcal{F}_0 \neq \emptyset \right\}$$

which is a finite set. Then, $\text{supp}(\mathcal{L}_{2,\text{skew}}) \subset \mathcal{F}_0$ and the following relation holds:

$$\begin{aligned}
 \frac{1}{2} \sum_{(x,y,z) \in \mathcal{F}_0} \mathcal{R}(x,y,z) \mathcal{L}_{2,\text{skew}}(f)(x,y,z) \overline{g(x,y,z)} &= \\
 \frac{1}{2} \sum_{(x,y,z) \in \mathcal{F}_0} \mathcal{R}(x,y,z) &\left(\frac{1}{\mathcal{E}(x,y)} \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x,y,t) f(x,y,t) \right. \\
 &+ \frac{1}{\mathcal{E}(y,z)} \sum_{t \in \mathcal{F}(y,z)} \mathcal{R}(y,z,t) f(y,z,t) \\
 &+ \left. \frac{1}{\mathcal{E}(z,x)} \sum_{t \in \mathcal{F}(z,x)} \mathcal{R}(z,x,t) f(z,x,t) \right) \overline{g(x,y,z)} \\
 = \frac{1}{2} \sum_{(x,y,z) \in \mathcal{F}_0} \mathcal{R}(x,y,z) f(x,y,z) &\overline{\left(\frac{1}{\mathcal{E}(x,y)} \sum_{u \in \mathcal{V}} \mathcal{R}(x,y,u) g(x,y,u) \right.} \\
 &+ \frac{1}{\mathcal{E}(y,z)} \sum_{u \in \mathcal{V}} \mathcal{R}(y,z,u) g(y,z,u) \\
 &+ \left. \frac{1}{\mathcal{E}(z,x)} \sum_{u \in \mathcal{V}} \mathcal{R}(z,x,u) g(z,x,u) \right)} \\
 (1) \quad &= \frac{1}{2} \sum_{x,y,z \in \mathcal{V}} \mathcal{R}(x,y,z) f(x,y,z) \overline{\mathcal{L}_{2,\text{skew}} g(x,y,z)}.
 \end{aligned}$$

Let $g \in \mathcal{D}(\mathcal{L}_{2,\text{max,skew}})$. It follows from (1) that

$$\langle \mathcal{L}_{2,\text{skew}} f, g \rangle = \langle f, \mathcal{L}_{2,\text{max,skew}} g \rangle$$

for all $f \in \mathcal{C}_{\text{skew}}^c(\mathcal{R})$, which implies that $g \in \mathcal{D}(\mathcal{L}_{2,\text{skew}}^*)$. Now let $g \in \mathcal{D}(\mathcal{L}_{2,\text{skew}}^*)$. Let $(x,y,z) \in \mathcal{F}$ and let

$$f = \frac{1}{\mathcal{R}(x,y,z)} \left(\mathbf{1}_{\circlearrowleft(x,y,z)} - \mathbf{1}_{\circlearrowleft(z,y,x)} \right).$$

Then, $f \in \mathcal{C}_{\text{skew}}^c(\mathcal{R})$ and we obtain from (1):

$$\begin{aligned}
 \overline{(\mathcal{L}_{2,\text{skew}}^* g)(x,y,z)} &= \langle f, \mathcal{L}_{2,\text{skew}}^* g \rangle \\
 &= \langle \mathcal{L}_{2,\text{skew}} f, g \rangle \\
 &= \frac{1}{2} \sum_{(u,v,w) \in \mathcal{F}} \mathcal{R}(u,v,w) \mathcal{L}_{2,\text{skew}} f(u,v,w) \overline{g(u,v,w)} \\
 &= \frac{1}{2} \sum_{(u,v,w) \in \mathcal{F}} \mathcal{R}(u,v,w) f(u,v,w) \overline{\mathcal{L}_{2,\text{max,skew}} g(u,v,w)} \\
 &= \overline{(\mathcal{L}_{2,\text{max,skew}} g)(x,y,z)}.
 \end{aligned}$$

which implies that $\mathcal{L}_{2,\text{skew}}g = \mathcal{L}_{2,\text{max,skew}}g \in \ell_{\text{skew}}^2(\mathcal{R})$ by the definition of the adjoint, it follows that $g \in \mathcal{D}(\mathcal{L}_{2,\text{max,skew}})$. Hence $\mathcal{L}_{2,\text{skew}}^* = \mathcal{L}_{2,\text{max,skew}}$. \square

Remark 5.2. Let \mathcal{L}_{sym} be the mapping from $\mathcal{C}_{\text{sym}}(\mathcal{R})$ into itself given by the same expression of $\mathcal{L}_{2,\text{sym}}$. Then, $\mathcal{L}_{2,\text{sym}}^* = \mathcal{L}_{2,\text{max,sym}}$ where $\mathcal{L}_{2,\text{max,sym}}$ is the restrictions of \mathcal{L}_{sym} to

$$\mathcal{D}(\mathcal{L}_{2,\text{max,sym}}) := \left\{ f \in \ell_{\text{sym}}^2(\mathcal{R}) \text{ such that } \mathcal{L}_{\text{sym}}f \in \ell_{\text{sym}}^2(\mathcal{R}) \right\}.$$

Using the Nelson commutator theorem, we prove the criterium of essential self-adjointness for $\mathcal{L}_{2,\text{skew}}$ and $\mathcal{L}_{2,\text{sym}}$.

Theorem 5.3. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. Set

$$\mathcal{N}(x, y, z) = 1 + d_{\mathcal{E}}(x, y) + d_{\mathcal{E}}(y, z) + d_{\mathcal{E}}(z, x).$$

Suppose that

$$\sup_{x \sim y, z \in \mathcal{F}(x,y)} \sum_{r \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x,y)} \mathcal{R}(x, y, r) | \mathcal{N}(x, y, r) - \mathcal{N}(x, y, z) |^2 < \infty.$$

Then $\mathcal{L}_{2,\text{skew}}$ is essentially self-adjoint on $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$ and $\mathcal{L}_{2,\text{sym}}$ is essentially self-adjoint on $\mathcal{C}_{\text{sym}}^c(\mathcal{R})$.

Proof. Let \mathcal{N} be the operator of multiplication by $\mathcal{N}(\cdot, \cdot, \cdot)$ and take $f \in \mathcal{C}_{\text{skew}}^c(\mathcal{F})$. Going over the same techniques of the proof of [3, Theorem 5.13], we obtain:

$$\begin{aligned} \|\mathcal{L}_{2,\text{skew}}f\|^2 &\leq \frac{2}{3} \sum_{x \sim y, z \in \mathcal{F}(x,y)} \mathcal{R}(x, y, z) \left(\frac{1}{\mathcal{E}^2(x,y)} \left| \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x, y, t) f(x, y, t) \right|^2 \right. \\ &\quad \left. + \frac{1}{\mathcal{E}^2(y,z)} \left| \sum_{t \in \mathcal{F}(y,z)} \mathcal{R}(y, z, t) f(y, z, t) \right|^2 \right. \\ &\quad \left. + \frac{1}{\mathcal{E}^2(z,x)} \left| \sum_{t \in \mathcal{F}(z,x)} \mathcal{R}(z, x, t) f(z, x, t) \right|^2 \right) \\ &\leq 2 \sum_{x \sim y, z \in \mathcal{F}(x,y)} \mathcal{R}(x, y, z) \frac{1}{\mathcal{E}^2(x,y)} \left| \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x, y, t) f(x, y, t) \right|^2 \\ &\leq 2 \sum_{x \sim y, z \in \mathcal{F}(x,y)} \mathcal{R}(x, y, z) \frac{1}{\mathcal{E}^2(x,y)} \left(\sum_{r \in \mathcal{F}(x,y)} \mathcal{R}(x, y, r) \right) \times \\ &\quad \left(\sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x, y, t) | f(x, y, t) |^2 \right) \\ &= 2 \sum_{x \sim y, z \in \mathcal{F}(x,y)} \mathcal{R}(x, y, z) \left(\frac{1}{\mathcal{E}(x,y)} \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x, y, t) \right)^2 | f(x, y, z) |^2 \\ &\leq 12 \|\mathcal{N}(f)\|^2. \end{aligned}$$

Moreover, we notice that $\mathcal{N}(\cdot, \cdot, \cdot)$ is symmetric and $f(x, y, z) = -f(z, y, x)$ and let $J = |\langle f, [\mathcal{L}_{2,\text{skew}}, \mathcal{N}]f \rangle|$. We get:

$$\begin{aligned}
 J &\leq \frac{1}{12} \sum_{x \sim y, z \in \mathcal{F}(x,y)} \mathcal{R}(x, y, z) \left(|f(x, y, z)|^2 \right. \\
 &\quad \left. + |[\mathcal{L}_{2,\text{skew}}, \mathcal{N}](f)(x, y, z)|^2 \right) \\
 &\leq \frac{1}{2} \|\mathcal{N}^{\frac{1}{2}}(f)\|^2 + \sum_{x \sim y, z \in \mathcal{F}(x,y)} \mathcal{R}(x, y, z) \left| \sum_{t \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x, y)} \mathcal{R}(x, y, t) \times \right. \\
 &\quad \left. (\mathcal{N}(x, y, t) - \mathcal{N}(x, y, z)) f(x, y, t) \right|^2 \\
 &\leq \frac{1}{2} \|\mathcal{N}^{\frac{1}{2}}(f)\|^2 + \sum_{x \sim y, z \in \mathcal{F}(x,y)} \mathcal{R}(x, y, z) \left(\sum_{t \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x, y)} \mathcal{R}(x, y, t) \right) \times \\
 &\quad \left(\sum_{r \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x, y)} \mathcal{R}(x, y, r) |\mathcal{N}(x, y, r) - \mathcal{N}(x, y, z)| |f(x, y, r)|^2 \right) \\
 &= \frac{1}{2} \|\mathcal{N}^{\frac{1}{2}}(f)\|^2 + \sum_{x \sim y, z \in \mathcal{F}(x,y)} \mathcal{R}(x, y, z) \left(\sum_{t \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x, y)} \mathcal{R}(x, y, t) \right) \times \\
 &\quad \underbrace{\sum_{r \in \mathcal{V}} \frac{1}{\mathcal{E}(x, y)} \mathcal{F}(x, y, r) |\mathcal{N}(x, y, r) - \mathcal{N}(x, y, z)| |f(x, y, z)|^2}_{\leq C} \\
 &\leq \left(\frac{1 + 12C}{2} \right) \|\mathcal{N}^{\frac{1}{2}}(f)\|.
 \end{aligned}$$

Applying [26, Theorem X.37], the result follows. The proof of $\mathcal{L}_{2,\text{sym}}$ may be checked in the same way as the proof of $\mathcal{L}_{2,\text{skew}}$. \square

Corollary 5.4. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a simple triangulation. Assume that*

$$\sup_{x \sim y, z \in \mathcal{F}(x,y)} \sum_{r \in \mathcal{F}(x,y)} |\#\mathcal{F}(y,z) + \#\mathcal{F}(z,x) - \#\mathcal{F}(y,r) - \#\mathcal{F}(r,x)|^2 < \infty.$$

Then $\mathcal{L}_{2,\text{skew}}$ is essentially self-adjoint on $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$ and $\mathcal{L}_{2,\text{sym}}$ is essentially self-adjoint on $\mathcal{C}_{\text{sym}}^c(\mathcal{R})$.

6. ESSENTIAL SPECTRUM

Let A be a closed, densely defined linear operator on a Banach space X , and let $\sigma(A)$ denote the spectrum of A . We denote by $\mathcal{K}(X)$ the set of compact operators on X to itself. We define the essential spectrum of the operator A by

$$\sigma_{\text{ess}}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K).$$

It is well known that if A is a self-adjoint operator on a Hilbert space, the essential spectrum of A is the set of limit points of the spectrum of A , i.e., all points of the spectrum except isolated eigenvalues of finite multiplicity, see [30]. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. Note that $\mathcal{L}_{2,\text{skew}}$ is non-negative symmetric operator on $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$. We consider the quadratic form

$$q(f, g) = \langle f, \mathcal{L}_{2,\text{skew}}g \rangle + \langle f, g \rangle$$

on $\mathcal{C}_{\text{skew}}^c(\mathcal{R}) \times \mathcal{C}_{\text{skew}}^c(\mathcal{R})$. Let \mathcal{H}_1 be the completion of $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$ under the norm

$$\|f\|_q = \sqrt{\langle \mathcal{L}_{2,\text{skew}}f, f \rangle + \|f\|^2}.$$

We define the Friedrichs extension $\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}$ of $\mathcal{L}_{2,\text{skew}}$ by:

- i) A vector f is in domain $\mathcal{D}(\mathcal{L}_{2,\text{skew}}^{\mathcal{F}})$ if and only if $f \in \mathcal{H}_1$ and $\mathcal{C}_{\text{skew}}^c(\mathcal{R}) \ni g \mapsto \langle f, \mathcal{L}_{2,\text{skew}}g \rangle + \langle f, g \rangle$ extends to a norm continuous function on $\ell_{\text{skew}}^2(\mathcal{R})$.
- (ii) For each $f \in \mathcal{D}(\mathcal{L}_{2,\text{skew}}^{\mathcal{F}})$, there is a unique u_f such that $\langle f, \mathcal{L}_{2,\text{skew}}g \rangle + \langle f, g \rangle = \langle u_f, g \rangle$ by Riesz' Theorem. The Friedrichs extension of $\mathcal{L}_{2,\text{skew}}$, is given by $\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}f = u_f - f$. It is a self-adjoint extension of $\mathcal{L}_{2,\text{skew}}$, e.g. see [26, Theorem X.23]. Note that $\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}$ is bounded if and only if $d_{\mathcal{E}}(\cdot)$ is bounded, e.g. see [8].

Theorem 6.1. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation and let $\mathcal{F}_0 = \{\mathcal{K} \subset \mathcal{F} : \mathcal{K} \text{ finite}\}$. Then,*

$$\inf \sigma(\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}) \leq \inf_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x, y, z) \left(\frac{1}{\mathcal{E}(x, y)} + \frac{1}{\mathcal{E}(y, z)} + \frac{1}{\mathcal{E}(z, x)} \right).$$

and

$$\inf \sigma_{\text{ess}}(\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}) \leq \sup_{\mathcal{K} \subset \mathcal{F}_0} \inf_{(x,y,z) \in \mathcal{K}^c} \mathcal{R}(x, y, z) \left(\frac{1}{\mathcal{E}(x, y)} + \frac{1}{\mathcal{E}(y, z)} + \frac{1}{\mathcal{E}(z, x)} \right).$$

In particular, if \mathcal{T} is a simple triangulation then $\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}$ is not with compact resolvent.

Proof. Let $(x_0, y_0, z_0) \in \mathcal{F}$ and let

$$f = \frac{\mathbf{1}_{\circlearrowleft(x_0, y_0, z_0)} - \mathbf{1}_{\circlearrowright(z_0, y_0, x_0)}}{\sqrt{\mathcal{R}(x_0, y_0, z_0)}}$$

where $\mathbf{1}_{\circlearrowleft(x_0, y_0, z_0)}$ denotes the indicator function of $\circlearrowleft(x_0, y_0, z_0)$. Then $\|f\| = 1$ and

$$\langle f, \mathcal{L}_{2,\text{skew}}f \rangle = \mathcal{R}(x_0, y_0, z_0) \left(\frac{1}{\mathcal{E}(x_0, y_0)} + \frac{1}{\mathcal{E}(y_0, z_0)} + \frac{1}{\mathcal{E}(z_0, x_0)} \right)$$

Applying [21, Proposition 3], the result follows. \square

7. APPLICATION TO THE STUDY OF THE ADJACENCY MATRIX

7.1. Adjacency matrix. The adjacency matrix has important implications. For example, it uses the semi-boundedness in order to give meaning to the heat equation, see [5]. Let $\mathcal{G} = (\mathcal{V}, m, \mathcal{E})$ be a weighted graph. We define the set of 0-cochains on \mathcal{V} by

$$\mathcal{C}(\mathcal{V}) = \{f : \mathcal{V} \rightarrow \mathbb{C}\}.$$

We denote by $\mathcal{C}^c(\mathcal{V})$ the 0-cochains with finite support in \mathcal{V} . We associate a Hilbert space to \mathcal{V} :

$$\ell^2(\mathcal{V}) = \left\{ f \in \mathcal{C}(\mathcal{V}) \text{ such that } \|f\|^2 = \sum_{x \in \mathcal{V}} m(x) |f(x)|^2 < \infty \right\}.$$

The associated scalar product is given by

$$\langle f, g \rangle = \sum_{x \in \mathcal{V}} m(x) \overline{f(x)} g(x), \text{ for } f, g \in \ell^2(\mathcal{V}).$$

We define the adjacency matrix:

$$\mathcal{A}_{\mathcal{G}}(f)(x) = \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) f(y), \text{ } f \in \mathcal{C}^c(\mathcal{V}).$$

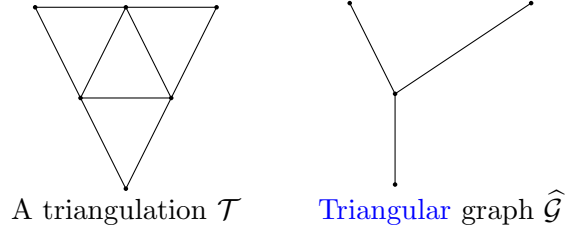
It is symmetric and thus closable. We denote its closure by the same symbol. When \mathcal{G} is simple, we have that $\mathcal{A}_{\mathcal{G}}$ is unbounded if and only if it is unbounded from above and if and only if the degree is unbounded, see [15].

7.2. Triangular graph. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. Set $\widehat{\mathcal{V}} = \mathcal{F} / \sim$, where $\varpi \sim -\varpi$.

Definition 7.1. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. Set $\widehat{\mathcal{V}} = \mathcal{F} / \sim$, where $\varpi \sim -\varpi$. The triangular graph of \mathcal{T} is the graph $\widehat{\mathcal{G}} = (\widehat{\mathcal{V}}, \widehat{m}, \widehat{\mathcal{E}})$ where $\widehat{m} = 1$ and

$$\begin{aligned} \widehat{\mathcal{E}}((x_0, y_0, z_0), (x, y, z)) = & \sqrt{\mathcal{R}(x_0, y_0, z_0)} \left(\frac{\sqrt{\mathcal{R}(x, y, z)}}{\mathcal{E}(x, y)} 1_{x=x_0, y=y_0} \right. \\ & \left. + \frac{\sqrt{\mathcal{R}(x, y, z)}}{\mathcal{E}(y, z)} 1_{y=y_0, z=z_0} + \frac{\sqrt{\mathcal{R}(x, y, z)}}{\mathcal{E}(z, x)} 1_{z=z_0, x=x_0} \right) \end{aligned}$$

if $(x_0, y_0, z_0) \neq (x, y, z)$ and 0 otherwise.



Remark 7.2. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a triangulation. The adjacency matrix on $\widehat{\mathcal{G}}$ is given by

$$\begin{aligned} \mathcal{A}_{\widehat{\mathcal{G}}}(f)(x_0, y_0, z_0) &:= \sum_{z \in \mathcal{V}, z \neq z_0} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(x_0, y_0, z)}}{\mathcal{E}(x_0, y_0)} f(x_0, y_0, z) \\ &+ \sum_{x \in \mathcal{V}, x \neq x_0} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(x, y_0, z_0)}}{\mathcal{E}(y_0, z_0)} f(x, y_0, z_0) \\ &+ \sum_{y \in \mathcal{V}, y \neq y_0} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(x_0, y, z_0)}}{\mathcal{E}(z_0, x_0)} f(x_0, y, z_0) \end{aligned}$$

for all $f \in \mathcal{C}^c(\widehat{\mathcal{V}})$.

Proposition 7.3. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. Then $\mathcal{L}_{2, \text{sym}}$ is unitarily equivalent to

$$\mathcal{A}_{\widehat{\mathcal{G}}} + \mathcal{Q}(V)$$

where

$$V(x_0, y_0, z_0) = \frac{\mathcal{R}(x_0, y_0, z_0)}{\mathcal{R}(x_0, y_0)} + \frac{\mathcal{R}(x_0, y_0, z_0)}{\mathcal{E}(y_0, z_0)} + \frac{\mathcal{R}(x_0, y_0, z_0)}{\mathcal{E}(z_0, x_0)}$$

and $\mathcal{Q}(V)$ be the operator of multiplication by V .

Proof. Set $U : \ell^2_{\text{sym}}(\mathcal{R}) \rightarrow \ell^2(\widehat{\mathcal{V}})$ as the operator given by

$$U(f)(x, y, z) = \sqrt{\mathcal{R}(x, y, z)} f(x, y, z).$$

Notice that

$$U^{-1}(f)(x, y, z) = U^*(f)(x, y, z) = \frac{1}{\sqrt{\mathcal{R}(x, y, z)}} f(x, y, z)$$

for all $f \in \ell^2(\widehat{\mathcal{V}})$. Notice now that on $\mathcal{C}_c(\widehat{\mathcal{V}})$

$$\begin{aligned} U\mathcal{L}_{2, \text{sym}}U^{-1}(f)(x_0, y_0, z_0) &= \sum_{t \in \mathcal{V}} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(x_0, y_0, t)}}{\mathcal{E}(x_0, y_0)} f(x_0, y_0, t) \\ &+ \sum_{t \in \mathcal{V}} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(y_0, z_0, t)}}{\mathcal{E}(y_0, z_0)} f(y_0, z_0, t) \\ &+ \sum_{t \in \mathcal{V}} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(z_0, x_0, t)}}{\mathcal{E}(z_0, x_0)} f(z_0, x_0, t). \end{aligned}$$

Using Remark 7.2, we obtain the result. \square

Corollary 7.4. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted 3-partite triangulation. If V is bounded on \mathcal{F} , then*

$$\mathcal{A}_{\widehat{\mathcal{G}}} \geq \sup_{(x,y,z) \in \mathcal{F}} V(x, y, z).$$

In particular, $\mathcal{A}_{\widehat{\mathcal{G}}} \geq -3$ when \mathcal{T} is simple.

Proof. Since $\mathcal{L}_{2,\text{sym}}$ is non-negative operator, then

$$\langle \mathcal{A}_{\widehat{\mathcal{G}}} f, f \rangle \geq -3 \|f\|^2$$

for all $f \in \mathcal{C}^c(\widehat{\mathcal{V}})$ □

Corollary 7.5. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a 3-partite weighted triangulation. Set*

$$\mathcal{N}(x, y, z) = 1 + \sum_{r \in \mathcal{V}} \left(d_{\mathcal{E}}(x, y) + d_{\mathcal{E}}(y, z) + d_{\mathcal{E}}(z, x) \right).$$

Suppose that

$$\sup_{(x,y,z) \in \mathcal{F}} \sum_{r \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x, y)} \mathcal{R}(x, y, r) | \mathcal{N}(x, y, r) - \mathcal{N}(x, y, z) |^2 < \infty.$$

and V is bounded. Then, $\mathcal{A}_{\widehat{\mathcal{G}}}$ is essentially self-adjoint on $\mathcal{C}^c(\widehat{\mathcal{V}})$.

Proof. Combine Proposition 7.3, Theorem 5.3 and Theorem 4.2. □

7.3. Geometric Hypothesis. We recall the two following definitions:

Definition 7.6. [1, Definition 8] *The graph $\mathcal{G} := (\mathcal{V}, m, \mathcal{E})$ is χ -complete if there exists an increasing sequence of finite set $(\mathcal{V}_n)_n$ such that $\mathcal{V} = \cup_n \mathcal{V}_n$ and there exist related functions χ_n satisfying the following three conditions:*

- 1) $\chi_n \in \mathcal{C}^c(\mathcal{V})$, $0 \leq \chi_n \leq 1$,
- 2) $\chi_n(x) = 1$ if $x \in \mathcal{V}_n$,
- 3) $\exists C > 0$, $\forall n \in \mathbb{N}$, $x \in \mathcal{V}$,

$$\frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) |\chi_n(x) - \chi_n(y)|^2 \leq C.$$

Definition 7.7. [7, Definition 4.2] *A weighted triangulation $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ is χ -complete, if*

- 1) $\mathcal{G} = (\mathcal{V}, m, \mathcal{E})$ is χ -complete.
- 2) $\exists M > 0$, $\forall n \in \mathbb{N}$, $(x, y) \in \mathcal{E}$

$$\frac{1}{\mathcal{E}(x, y)} \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x, y, t) |2\chi_n(t) - \chi_n(x) - \chi_n(y)|^2 \leq M.$$

We recall the criterion obtained in [7].

Theorem 7.8. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a χ -complete weighted triangulation then $\mathcal{L}_{2,\text{skew}}$ is essentially self-adjoint on $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$.*

Corollary 7.9. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a χ -complete weighted triangulation. If \mathcal{T} is tri-partite then $\mathcal{A}_{\widehat{\mathcal{G}}}$ is essentially self-adjoint on $\mathcal{C}^c(\widehat{\mathcal{V}})$.*

Proof. Combine Proposition 7.3, Theorem 4.2 and Theorem 7.8.

7.4. Book-like triangulation. We recall the definition of 1-dimensional decomposition given in [6] for the case of graphs.

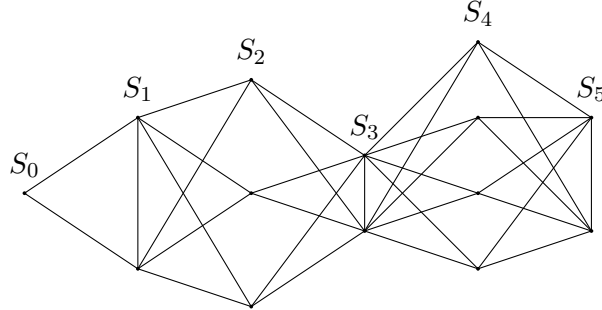
Definition 7.10. [6] *A 1-dimensional decomposition of the graph $\mathcal{G} = (\mathcal{V}, m, \mathcal{E})$ is a family of finite sets $(S_n)_{n \in \mathbb{N}}$ which forms a partitions of \mathcal{V} , that is $\mathcal{V} = \cup_{n \in \mathbb{N}} S_n$, and such that for all $x \in S_n, y \in S_m$,*

$$\mathcal{E}(x, y) > 0 \implies |n - m| \leq 1$$

The following definition is introduced in [7].

Definition 7.11. *Let $\mathcal{T} := (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation and $(S_n)_{n \in \mathbb{N}}$ a 1-dimensional decomposition of the graph $\mathcal{G} = (\mathcal{V}, m, \mathcal{E})$. We say that \mathcal{T} is a book-like triangulation if*

- 1) $\#S_0 = 1, \#S_{2n+1} = 2$ and $\#(S_{2n+1}^2 \cap \mathcal{E}) = 1$, for all $n \in \mathbb{N}$.
- 2) $x, y \in S_{2n+2} \implies \mathcal{E}(x, y) = 0$,
- 3) $\forall x \in S_{2n+1}, \mathcal{N}_{\mathcal{G}}(x) = S_{2n} \cup S_{2n+2}$.



A book-like triangulation

We recall [7, Proposition 6.5]:

Proposition 7.12. *Let \mathcal{T} be a simple book-like triangulation. Assume that*

$$(2) \quad n \mapsto \frac{\#S_{2n}}{\#S_{2(n+1)}} \in \ell^1(\mathbb{N}).$$

Then, $\mathcal{L}_{2, \text{skew}}$ is not essentially self-adjoint on $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$.

Corollary 7.13. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a simple book-like triangulation satisfying (2). Then, $\mathcal{A}_{\widehat{\mathcal{G}}}$ is not essentially self-adjoint on $\mathcal{C}^c(\widehat{\mathcal{V}})$.*

Proof. Set $S_0 = \{x_0^0\}$, $S_{2n+1} = \{x_n^0, x_n^1\}$, $\mathcal{V}_0 = \cup_n S_{2n}$, $\mathcal{V}_1 = \cup_n \{x_n^0 : n \in \mathbb{N}\}$ and $\mathcal{V}_2 = \cup_n \{x_n^1 : n \in \mathbb{N}\}$. Then,

$$\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \text{ and } \mathcal{E} \cap (\mathcal{V}_0^2 \cup \mathcal{V}_1^2 \cup \mathcal{V}_2^2) = \emptyset.$$

So, \mathcal{T} is tri-partite. Using Theorem 4.2, Proposition 7.3 and Proposition 7.12, the result holds. \square

7.5. Triangular anti-tree. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. The sphere of radius $n \in \mathbb{N}$ around a vertex $v \in \mathcal{V}$ is the set

$$\mathcal{S}_n(v) := \left\{ w \in \mathcal{V} : d_{\mathcal{V}}(v, w) = n \right\}.$$

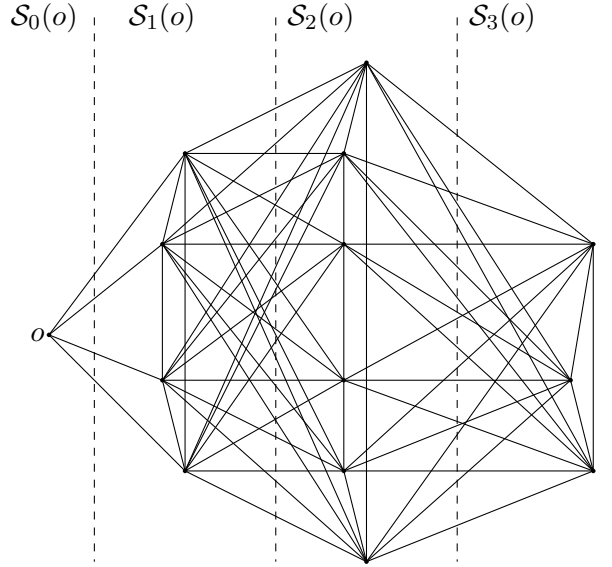
We recall that \mathcal{C}_n denotes the n -cycle graph, i.e. $\mathcal{V} = \mathbb{Z}/n\mathbb{Z}$, where $\mathcal{E}(x, y) > 0$ if and only if $|x - y| = 1$. Let $\mathcal{G}(\mathcal{S}_n(o)) = (\mathcal{S}_n(o), \mathcal{E}')$ where $\mathcal{E}' = \mathcal{E} \upharpoonright_{\mathcal{S}_n(o) \times \mathcal{S}_n(o)}$.

Definition 7.14. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. We say that \mathcal{T} is anti-tree if there exists a vertex $o \in \mathcal{V}$ such that

1) For all $n \in \mathbb{N}^*$ and $v \in \mathcal{S}_n(o)$, we have

$$\mathcal{N}_{\mathcal{G}}(v) \setminus \mathcal{S}_n(o) = \mathcal{S}_{n-1}(o) \cup \mathcal{S}_{n+1}(o).$$

2) For all $n \in \mathbb{N}^*$, $\mathcal{G}(\mathcal{S}_n(o)) \simeq \mathcal{C}_{\sharp \mathcal{S}_n(o)}$.



An triangular anti-tree with spheres $\mathcal{S}_0(o)$, $\mathcal{S}_1(o)$, $\mathcal{S}_2(o)$, $\mathcal{S}_3(o)$.

Theorem 7.15. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a simple triangular anti-tree whose root is o . Set $s_n = \sharp \mathcal{S}_n(o)$ and assume that

$$(3) \quad n \mapsto \frac{s_n^2}{s_{n+2}} \in \ell^1(\mathbb{N}).$$

Then $\mathcal{L}_{2, \text{sym}}$ does not essentially self-adjoint on $\mathcal{C}_{\text{sym}}^c(\mathcal{R})$.

Proof. Set $f \in \ell_{\text{sym}}^2(\mathcal{F}) \setminus \{0\}$ such that $f \in \ker(\mathcal{L}_{2,\text{sym}}^* + i)$ and such that f is constant on $\mathcal{S}_n \times \mathcal{S}_{n+1}^2 \cup \mathcal{S}_n^2 \times \mathcal{S}_{n+1}$, $n \in \mathbb{N}^*$. We denote the constant value by C_n . It takes the value 0 on \mathcal{S}_n^2 . We have the following equation:

$$(s_n + 4 + i)C_n + s_{n+2}C_{n+1} = 0.$$

Therefore,

$$\begin{aligned} \|f|_{\mathcal{S}_{n+1} \times \mathcal{S}_{n+2}^2}\|^2 &= \|f|_{\mathcal{S}_{n+1}^2 \times \mathcal{S}_{n+2}}\|^2 \\ &= \frac{1}{2} s_{n+2} s_{n+1} |C_{n+1}|^2 \\ &\leq \frac{|s_n + 4 + i|^2}{s_n s_{n+2}} \|f|_{\mathcal{S}_n \times \mathcal{S}_{n+1}^2}\|^2. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{s_n^2}{s_{n+2}} = 0$, we get by induction:

$$M := \sup_{n \in \mathbb{N}^*} \|f|_{\mathcal{S}_n \times \mathcal{S}_{n+1}^2}\|^2 < \infty.$$

Then, we have

$$\|f|_{\mathcal{S}_{n+1} \times \mathcal{S}_{n+2}^2}\|^2 \leq M \frac{|i + 4 + s_n|}{s_n s_{n+2}}.$$

From Eq. (3), we infer that $f \in \ell_{\text{sym}}^2(\mathcal{F})$. Using [26, Theorem X.36], we conclude that $\mathcal{L}_{2,\text{sym}}$ is not essentially self-adjoint. \square

Corollary 7.16. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a simple triangular anti-tree satisfying (3). Then, $\mathcal{A}_{\widehat{\mathcal{G}}}$ is not essentially self-adjoint on $\mathcal{C}^c(\widehat{\mathcal{V}})$.*

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