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THE DISCRETE LAPLACIAN ACTING ON 2-FORMS AND APPLICATION

HATEM BALOUDI, SAYDA BELGACEM, AND AREF JERIBI

ABSTRACT. In this paper, we study the discrete Laplacian acting on 2-forms that was introduced and investigated in [7]. We establish a new criterion of essential self-adjointness using the Nelson lemma. Moreover, we give an upper bound on the infimum of the essential spectrum. Furthermore, we establish a link between the adjacency matrix and the discrete Laplacian on 2-forms.

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1. INTRODUCTION

Spectral graph theory is an active area of research. In the last few years, the questions of the essential self-adjointness of discrete Laplacian operators on infinite graphs have attracted a lot of interest, see [10, 17, 21, 16]. There exist other definitions of the discrete Laplacian, e.g., [23, 15, 4, 2]. The one we are studying here is the discrete Laplacian acting on 2-forms and denoted by $\mathcal{L}_{2,\text{skew}}$, where skew stands for skew-symmetric. This operator was studied by Chebbi in [7]. The author gives a relation between χ -completeness geometric hypothesis for the graph and essential self-adjointness for the discrete Laplacian $\mathcal{L}_{2,\text{skew}}$. More specifically, the author has proved that $\mathcal{L}_{2,\text{skew}}$ is essential self-adjoint, when the triangulation (we refer to Section 2 for a precise definition) is χ -complete. The aim of this article is to continue to discuss the question of essential self-adjointness for $\mathcal{L}_{2,\text{skew}}$. Note that this operator depends on the weight \mathcal{R} on oriented triangular face and the weight \mathcal{E} on oriented edges, see Section 4 for more details. In the setting of electrical networks, the weight \mathcal{E} correspond to the conductance. We establish a hypothesis on the weights \mathcal{R} and \mathcal{E} involving essential self-adjointness. Our technique is based on the Nelson commutator theorem, see [25, Theorem X.37]. Moreover, we give a upper bound on the infimum of the essential spectrum $\sigma_{\text{ess}}(\mathcal{L}_{2,\text{skew}}^{\mathcal{F}})$, where $\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}$ is the Friedrich extension of $\mathcal{L}_{2,\text{skew}}$. Finally, we find a link between the adjacency matrix and the discrete Laplacian $\mathcal{L}_{2,\text{skew}}$. In the case of 3-partite graph, we prove that $\mathcal{L}_{2,\text{skew}}$ is unitarily equivalent to the adjacency matrix of the triangular graph, see Section 7 for more details.

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2. GENERALITIES ABOUT GRAPHS

We start with some definitions to fix notations for graphs and refer to [8, 9, 23] for surveys on the matter. Let \mathcal{V} be a countable set. We equip \mathcal{V} with the discrete topology. Let $\mathcal{E} : \mathcal{V} \times \mathcal{V} \rightarrow [0, +\infty)$ and assume that \mathcal{E} is symmetric (i.e, $\mathcal{E}(x, y) = \mathcal{E}(y, x)$, for all $x, y \in \mathcal{V}$). Let $m : \mathcal{V} \rightarrow (0, +\infty)$. We say that $\mathcal{G} = (\mathcal{V}, m, \mathcal{E})$ is a weighted graph with vertices \mathcal{V} , weight of vertices m and weight of edges \mathcal{E} . In the setting of electrical networks, the weights correspond of the conductances. We say that x, y are neighbors if $\mathcal{E}(x, y) \neq 0$ and we denote it by $x \sim y$. A graph \mathcal{G} is simple if it has no loops and \mathcal{E} has values in $\{0, 1\}$. The set of neighbors of $x \in \mathcal{V}$ is denoted by

$$\mathcal{N}_{\mathcal{G}}(x) := \{y \in \mathcal{V} : \mathcal{E}(x, y) \neq 0\}.$$

A graph is locally finite if $\#\mathcal{N}_{\mathcal{G}}(x)$ is finite for all $x \in \mathcal{V}$. The weighted degree of vertices is given by

$$d_{\mathcal{V}}(x) := \frac{1}{m(x)} \sum_{y \sim x} \mathcal{E}(x, y).$$

When \mathcal{G} is simple, $d_{\mathcal{V}}(x) = \#\mathcal{N}_{\mathcal{G}}(x)$. A graph \mathcal{G} is connected, if for all $x, y \in \mathcal{V}$, there exists an $x-y$ -path, i.e., there is a sequence $(x_1, \dots, x_{N+1}) \in \mathcal{V}^{N+1}$ such that $x_1 = x$, $x_{N+1} = y$ and $\mathcal{E}(x_n, x_{n+1}) > 0$ for all $n \in \{1, \dots, N\}$. In this case, we endow \mathcal{V} with the metric $d_{\mathcal{V}}$ defined by

$$d_{\mathcal{V}}(x, y) := \inf\{N \in \mathbb{N} : \text{there exists an } x-y \text{ path of length } N\}.$$

In the sequel, we shall always consider graphs \mathcal{G} , which are locally finite, connected and have no loop. The path $x-y$ is called a cycle or closed when the origin and the end are identical, i.e $x_0 = x_n$ with $n \geq 3$. If no cycle appear more than once in a path, the path is called a simple path. The weighted triangulations is a quartet $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ where $\mathcal{R} : \mathcal{V} \times \mathcal{V} \times \mathcal{V} \rightarrow (0, +\infty)$ is symmetric (i.e $\mathcal{R}(x, y, z) = \mathcal{R}(z, y, z)$ for all $(x, y, z) \in \mathcal{V} \times \mathcal{V} \times \mathcal{V}$) and

$$(x, y, z) \in \mathcal{F} \iff \mathcal{R}(x, y, z) > 0.$$

\mathcal{R} is said the weight of faces. We say that \mathcal{T} is simple if $\mathcal{G} := (\mathcal{V}, m, \mathcal{E})$ is simple and the weights of the faces equal 1. The set of vertices belonging to the edge (x, y) is given by

$$\mathcal{F}_{(x,y)} := \mathcal{N}_{\mathcal{G}}(x) \cap \mathcal{N}_{\mathcal{G}}(y).$$

The weighted degree of edges is given by:

$$d_{\mathcal{E}}(x, y) := \frac{1}{\mathcal{E}(x, y)} \sum_{z \in \mathcal{F}_{(x,y)}} \mathcal{R}(x, y, z).$$

When \mathcal{T} is simple, $d_{\mathcal{E}}(x, y) = \#\mathcal{F}_{(x,y)}$.

3. THE SYMMETRIC AND ANTISYMMETRIC SPACES

3.1. Hilbert structures on the set of edges. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. Let $\mathcal{E} := \{(x, y) \in \mathcal{V} \times \mathcal{V} : \mathcal{E}(x, y) > 0\}$. The set of 1-cochains (or 1-forms) is given by:

$$\mathcal{C}_{\text{skew}}(\mathcal{E}) := \left\{ f : \mathcal{E} \rightarrow \mathbb{C}, f(x, y) = -f(y, x) \text{ for all } x, y \in \mathcal{V} \right\},$$

where skew stands for skew-symmetric. This corresponds to fermionic statistics. The set of functions with finite support is denoted by $\mathcal{C}_{\text{skew}}^c(\mathcal{E})$. Concerning bosonic statistics, we define:

$$\mathcal{C}_{\text{sym}}(\mathcal{E}) := \left\{ f : \mathcal{E} \rightarrow \mathbb{C}, f(x, y) = f(y, x) \text{ for all } x, y \in \mathcal{V} \right\}.$$

The sets of functions with finite support is denoted by $\mathcal{C}_{\text{sym}}^c(\mathcal{E})$.

We turn to the Hilbert structures.

$$\ell_{\text{skew}}^2(\mathcal{E}) := \left\{ f \in \mathcal{C}_{\text{skew}}(\mathcal{E}) \text{ such that } \|f\|^2 := \frac{1}{2} \sum_{x,y \in \mathcal{V}} \mathcal{E}(x, y) |f(x, y)|^2 < \infty \right\}$$

and

$$\ell_{\text{sym}}^2(\mathcal{E}) := \left\{ f \in \mathcal{C}_{\text{sym}}(\mathcal{E}) \text{ such that } \|f\|^2 := \frac{1}{2} \sum_{x,y \in \mathcal{V}} \mathcal{E}(x,y) |f(x,y)|^2 < \infty \right\}.$$

The associated scalar product is given by

$$\langle f, g \rangle := \frac{1}{2} \sum_{(x,y) \in \mathcal{E}} \mathcal{E}(x,y) \overline{f(x,y)} g(x,y),$$

when f and g are both in $\ell_{\text{skew}}^2(\mathcal{E})$ or in $\ell_{\text{sym}}^2(\mathcal{E})$.

3.2. Hilbert structures on the set of faces. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. The set of 2-cochains or 2-forms is given by

$$\mathcal{C}_{\text{skew}}(\mathcal{R}) = \left\{ f : \mathcal{F} \longrightarrow \mathbb{C} : f(x,y,z) = -f(z,y,x) \right\}.$$

The sets of functions with finite support is denoted by $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$. Concerning the case symmetric, we define

$$\mathcal{C}_{\text{sym}}(\mathcal{R}) = \left\{ f : \mathcal{F} \longrightarrow \mathbb{C} : f(x,y,z) = f(z,y,x) \right\}.$$

The sets of functions with finite support is denoted by $\mathcal{C}_{\text{sym}}^c(\mathcal{R})$. Let us define the Hilbert spaces $\ell_{\text{skew}}^2(\mathcal{R})$ and $\ell_{\text{sym}}^2(\mathcal{R})$ as the sets of cochains with finite norm, we have

$$\ell_{\text{skew}}^2(\mathcal{F}) := \left\{ f \in \mathcal{C}_{\text{skew}}(\mathcal{R}) : \|f\| = \frac{1}{6} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) |f(x,y,z)|^2 \right\}$$

and

$$\ell_{\text{sym}}^2(\mathcal{R}) := \left\{ f \in \mathcal{C}_{\text{sym}}(\mathcal{R}) : \|f\| = \frac{1}{6} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) |f(x,y,z)|^2 \right\}.$$

The associated scalar product is given by

$$\langle f, g \rangle := \frac{1}{2} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) \overline{f(x,y,z)} g(x,y,z)$$

when f and g are both in $\ell_{\text{skew}}^2(\mathcal{R})$ or in $\ell_{\text{sym}}^2(\mathcal{R})$.

4. OPERATORS

In this section, we recall the concept of exterior derivative operator associated to a faces space, we refer to [7] for more detail. This permits to define the discrete Laplacian acting on 2-forms.

4.1. Skew-symmetric case. We start by defining the operators in the skew-symmetric case. The skew-symmetric exterior operator is the operator $d_{\text{skew}}^1 : \mathcal{C}_{\text{skew}}^c(\mathcal{E}) \rightarrow \mathcal{C}_{\text{skew}}^c(\mathcal{R})$, given by

$$d_{\text{skew}}^1(f)(x, y, z) = f(x, y) + f(y, z) + f(z, x).$$

The skew-symmetric co-exterior derivative operator is the formal adjoint of d_{skew}^1 , i.e. it is the operator $\delta_{\text{skew}}^1 : \mathcal{C}_{\text{skew}}^c(\mathcal{R}) \rightarrow \mathcal{C}_{\text{skew}}^c(\mathcal{E})$, given by

$$\forall f \in \mathcal{C}_{\text{skew}}^c(\mathcal{R}), \quad \delta_{\text{skew}}^1(f)(x, y) = \frac{1}{\mathcal{E}(x, y)} \sum_{z \in \mathcal{F}(x, y)} \mathcal{R}(x, y, z) f(x, y, z).$$

both operators are closable (see [7, Lemme 3.1]). We denote their closure by the same symbol. The skew-symmetric discrete Laplacian operator acting on 2-forms is given by

$$\begin{aligned} \mathcal{L}_{2, \text{skew}}(f)(x, y, z) &= d_{\text{skew}}^1 \delta_{\text{skew}}^1(f)(x, y, z) \\ &= \frac{1}{\mathcal{E}(x, y)} \sum_{t \in \mathcal{F}(x, y)} \mathcal{R}(x, y, t) f(x, y, t) \\ &\quad + \frac{1}{\mathcal{E}(y, z)} \sum_{t \in \mathcal{F}(y, z)} \mathcal{R}(y, z, t) f(y, z, t) \\ &\quad + \frac{1}{\mathcal{E}(z, x)} \sum_{t \in \mathcal{F}(z, x)} \mathcal{R}(z, x, t) f(z, x, t), \end{aligned}$$

with $f \in \mathcal{C}_{\text{skew}}^c(\mathcal{R})$.

4.2. Symmetric case. We turn to the symmetric case. The symmetric exterior derivative in the symmetric case, we define $d_{\text{sym}}^1 : \mathcal{C}_{\text{sym}}^c(\mathcal{E}) \rightarrow \mathcal{C}_{\text{sym}}^c(\mathcal{R})$ by

$$\forall f \in \mathcal{C}_{\text{sym}}^c(\mathcal{E}), \quad d_{\text{skew}}^1(f)(x, y, z) = f(x, y) + f(y, z) + f(z, x).$$

The symmetric co-exterior derivative operator is the formal adjoint of d_{sym}^1 , i.e. it is the operator $\delta_{\text{sym}}^1 : \mathcal{C}_{\text{sym}}^c(\mathcal{R}) \rightarrow \mathcal{C}_{\text{sym}}^c(\mathcal{E})$, given by

$$\forall f \in \mathcal{C}_{\text{sym}}^c(\mathcal{R}), \quad \delta_{\text{sym}}^1(f)(x, y) := \frac{1}{\mathcal{E}(x, y)} \sum_{z \in \mathcal{F}(x, y)} \mathcal{R}(x, y, z) f(x, y, z).$$

The symmetric discrete Laplacian operator acting on 2-forms is the operator $\mathcal{L}_{2, \text{sym}} = d_{\text{sym}}^1 \delta_{\text{sym}}^1$, given by the same expression of $\mathcal{L}_{2, \text{skew}}$.

4.3. Relationship between $\mathcal{L}_{2, \text{skew}}$ and $\mathcal{L}_{2, \text{sym}}$. The two operators $\mathcal{L}_{2, \text{skew}}$ and $\mathcal{L}_{2, \text{sym}}$ have the same expression. However they do not act on the same spaces. Namely, when \mathcal{T} is tri-partite, we shall prove that the two operators are unitarily equivalent.

Definition 4.1. A triangulation $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ is tri-partite if there are \mathcal{V}_1 , \mathcal{V}_2 and \mathcal{V}_3 such that $\mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3 = \emptyset$ and such this implication hold

$$\mathcal{E}(x, y) = 0 \implies (x, y) \in \mathcal{V}_1^2 \cup \mathcal{V}_2^2 \cup \mathcal{V}_3^2.$$

Theorem 4.2. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a tri-partite weighted triangulation. Then $\mathcal{L}_{2,\text{skew}}$ and $\mathcal{L}_{2,\text{sym}}$ are unitarily equivalent.

Proof. We consider the tri-partite decomposition $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3\}$. Let $\mathcal{U} : \ell_{\text{skew}}^2(\mathcal{R}) \longrightarrow \ell_{\text{sym}}^2(\mathcal{R})$ be the unitary map given by

$$\mathcal{U}(f)(x, y, z) = S(x, y, z)f(x, y, z),$$

where

$$S = 1_{\mathcal{V}_1 \times \mathcal{V}_2 \times \mathcal{V}_3} + 1_{\mathcal{V}_2 \times \mathcal{V}_3 \times \mathcal{V}_1} + 1_{\mathcal{V}_3 \times \mathcal{V}_1 \times \mathcal{V}_2} - 1_{\mathcal{V}_3 \times \mathcal{V}_2 \times \mathcal{V}_1} - 1_{\mathcal{V}_1 \times \mathcal{V}_3 \times \mathcal{V}_2}.$$

Note that

$$\mathcal{U}^{-1}(f) = \mathcal{U}^*(f)$$

for all $f \in \ell_{\text{sym}}^2(\mathcal{R})$. Therefore,

$$\mathcal{U}\mathcal{L}_{2,\text{skew}}\mathcal{U}^{-1}(f)(x, y, z) = \mathcal{L}_{2,\text{sym}}(f)(x, y, z)$$

for all $f \in \mathcal{C}_{\text{sym}}^c(\mathcal{R})$. □

5. A NELSON CRITERIUM

For the general theory of unbounded Hermitian operators and their extensions, we refer the reader to [24, 19, 26]. Let $\mathcal{L}_{2,\text{max,skew}}$ be the following mapping from $\mathcal{C}_{\text{skew}}(\mathcal{R})$ into itself:

$$\begin{aligned} \mathcal{L}_{2,\text{max,skew}}(f)(x, y, z) &= \frac{1}{\mathcal{E}(x, y)} \sum_{t \in \mathcal{F}_{(x, y)}} \mathcal{R}(x, y, t) f(x, y, t) \\ &+ \frac{1}{\mathcal{E}(y, z)} \sum_{t \in \mathcal{F}_{(y, z)}} \mathcal{R}(y, z, t) f(y, z, t) \\ &+ \frac{1}{\mathcal{E}(z, x)} \sum_{t \in \mathcal{F}_{(z, x)}} \mathcal{R}(z, x, t) f(z, x, t). \end{aligned}$$

Let $\mathcal{L}_{2,\text{min,skew}}$ be the restrictions of $\mathcal{L}_{2,\text{max,skew}}$ to

$$\mathcal{D}(\mathcal{L}_{2,\text{min,skew}}) := \left\{ f \in \ell_{\text{skew}}^2(\mathcal{R}) \text{ such that } \mathcal{L}_{2,\text{max,skew}}f \in \ell_{\text{skew}}^2(\mathcal{R}) \right\}.$$

Lemma 5.1. $\mathcal{L}_{2,\text{skew}}^* = \mathcal{L}_{2,\text{min,skew}}$.

Proof. Let $f \in \mathcal{C}_{\text{skew}}^c(\mathcal{R})$ and let $g \in \mathcal{C}_{\text{skew}}(\mathcal{R})$. Let \mathcal{F}_0 the support of f and set

$$\mathcal{F}_0 = \left\{ (x, y, z) \in \mathcal{F} : \exists u \in \mathcal{V}, \{(x, y, u), (y, z, u), (z, x, u)\} \cap \mathcal{F}_0 \neq \emptyset \right\}$$

which is a finite set. Then $\text{supp}(\mathcal{L}_{2,\text{skew}}) \subset \mathcal{F}_0$ and the following relation holds:

$$\begin{aligned}
 \langle \mathcal{L}_{2,\text{skew}} f, g \rangle &= \frac{1}{6} \sum_{(x,y,z) \in \mathcal{F}_0} \mathcal{R}(x,y,z) \mathcal{L}_{2,\text{skew}}(f)(x,y,z) \overline{g(x,y,z)} \\
 &= \frac{1}{6} \sum_{(x,y,z) \in \mathcal{F}_0} \mathcal{R}(x,y,z) \left(\frac{1}{\mathcal{E}(x,y)} \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x,y,t) f(x,y,t) \right. \\
 &\quad \left. + \frac{1}{\mathcal{E}(y,z)} \sum_{t \in \mathcal{F}(y,z)} \mathcal{R}(y,z,t) f(y,z,t) \right. \\
 &\quad \left. + \frac{1}{\mathcal{E}(z,x)} \sum_{t \in \mathcal{F}(z,x)} \mathcal{R}(z,x,t) f(z,x,t) \right) \overline{g(x,y,z)} \\
 &= \frac{1}{6} \sum_{(x,y,z) \in \mathcal{F}_0} \mathcal{R}(x,y,z) f(x,y,z) \overline{\left(\frac{1}{\mathcal{E}(x,y)} \sum_{u \in \mathcal{V}} \mathcal{R}(x,y,u) g(x,y,u) \right.} \\
 &\quad \left. + \frac{1}{\mathcal{E}(y,z)} \sum_{u \in \mathcal{V}} \mathcal{R}(y,z,u) g(y,z,u) \right.} \\
 &\quad \left. + \frac{1}{\mathcal{E}(z,x)} \sum_{u \in \mathcal{V}} \mathcal{R}(z,x,u) g(z,x,u) \right) \\
 (1) \quad &= \frac{1}{6} \sum_{x,y,z \in \mathcal{V}} \mathcal{R}(x,y,z) f(x,y,z) \overline{\mathcal{L}_2 g(x,y,z)}.
 \end{aligned}$$

Let $g \in \mathcal{D}(\mathcal{L}_{2,\text{min,skew}})$. It flows from (1) that

$$\langle \mathcal{L}_{2,\text{skew}} f, g \rangle = \langle f, \mathcal{L}_{2,\text{min,skew}} g \rangle$$

for all $f \in \mathcal{C}_{\text{skew}}^c(\mathcal{R})$, which implies that $g \in \mathcal{D}(\mathcal{L}_{2,\text{skew}}^*)$. Now let $g \in \mathcal{D}(\mathcal{L}_{2,\text{skew}}^*)$. Let $(x,y,z) \in \mathcal{F}$ and let

$$f = \frac{1}{\mathcal{R}(x,y,z)} \left(\delta_{x,y,z} + \delta_{y,z,x} + \delta_{z,x,y} \right) - \frac{1}{\mathcal{R}(z,y,x)} \left(\delta_{z,y,x} + \delta_{x,z,y} + \delta_{y,x,z} \right).$$

Then $f \in \mathcal{C}_{\text{skew}}^c(\mathcal{R})$ and we obtain from (1):

$$\begin{aligned}
 \overline{(\mathcal{L}_{2,\text{skew}}^* g)(x,y,z)} &= \langle f, \mathcal{L}_{2,\text{skew}}^* g \rangle \\
 &= \langle \mathcal{L}_{2,\text{skew}} f, g \rangle \\
 &= \frac{1}{6} \sum_{(u,v,w) \in \mathcal{F}} \mathcal{R}(u,v,w) \mathcal{L}_{2,\text{skew}} f(u,v,w) \overline{g(u,v,w)} \\
 &= \frac{1}{6} \sum_{(u,v,w) \in \mathcal{F}} \mathcal{R}(u,v,w) f(u,v,w) \overline{\mathcal{L}_{2,\text{min,skew}} g(u,v,w)} \\
 &= \overline{(\mathcal{L}_{2,\text{min,skew}} g)(x,y,z)}.
 \end{aligned}$$

which implies that $\mathcal{L}_{2,\text{skew}}g = \mathcal{L}_{2,\text{min,skew}}g \in \ell_{\text{skew}}^2(\mathcal{F})$ by the definition of the adjoint, it follows that $g \in \mathcal{D}(\mathcal{L}_{2,\text{min,skew}})$. Hence $\mathcal{L}_{2,\text{skew}}^* = \mathcal{L}_{2,\text{min,skew}}$. \square

Remark 5.2. Let $\mathcal{L}_{2,\text{max,sym}}$ be the mapping from $\mathcal{C}_c(\mathcal{R})$ into itself given by the same expression of $\mathcal{L}_{2,\text{max,sym}}$. Then $\mathcal{L}_{2,\text{sym}}^* = \mathcal{L}_{2,\text{min,sym}}$ where $\mathcal{L}_{2,\text{min,sym}}$ is the restrictions of $\mathcal{L}_{2,\text{max,sym}}$ to

$$\mathcal{D}(\mathcal{L}_{2,\text{min,sym}}) := \left\{ f \in \ell_{\text{sym}}^2(\mathcal{R}) \text{ such that } \mathcal{L}_{2,\text{max,sym}}f \in \ell_{\text{sym}}^2(\mathcal{R}) \right\}.$$

Using the Nelson commutator theorem, we prove the criterium of essential self-adjointness for $\mathcal{L}_{1,\text{skew}}$ and $\mathcal{L}_{1,\text{sym}}$.

Theorem 5.3. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. Set

$$\mathcal{N}(x, y, z) = 1 + d_{\mathcal{E}}(x, y) + d_{\mathcal{E}}(y, z) + d_{\mathcal{E}}(z, x).$$

Suppose that

$$\sup_{(x,y,z) \in \mathcal{F}} \sum_{r \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x,y)} \mathcal{R}(x,y,r) | \mathcal{N}(x,y,r) - \mathcal{N}(x,y,z) |^2 < \infty.$$

Then $\mathcal{L}_{2,\text{skew}}$ is essentially self-adjoint on $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$ and $\mathcal{L}_{2,\text{sym}}$ is essentially self-adjoint on $\mathcal{C}_{\text{sym}}^c(\mathcal{R})$.

Proof. Let \mathcal{N} be the operator of multiplication by $\mathcal{N}(\cdot, \cdot, \cdot)$ and take $f \in \mathcal{C}_{\text{skew}}^c(\mathcal{F})$. We have:

$$\begin{aligned} \|\mathcal{L}_{2,\text{skew}}f\|^2 &\leq \frac{2}{3} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) \left(\frac{1}{\mathcal{E}^2(x,y)} \left| \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x,y,t) f(x,y,t) \right|^2 \right. \\ &\quad \left. + \frac{1}{\mathcal{E}^2(y,z)} \left| \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(y,z,t) f(y,z,t) \right|^2 \right. \\ &\quad \left. + \frac{1}{\mathcal{E}^2(y,z)} \left| \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(y,z,t) f(y,z,t) \right|^2 \right) \\ &\leq 2 \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) \frac{1}{\mathcal{E}^2(x,y)} \left| \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x,y,t) f(x,y,t) \right|^2 \\ &\leq 2 \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) \frac{1}{\mathcal{E}^2(x,y)} \left(\sum_{r \in \mathcal{F}(x,y)} \mathcal{R}(x,y,r) \right) \times \\ &\quad \left(\sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x,y,t) | f(x,y,t) |^2 \right) \\ &= 2 \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x,y,z) \left(\frac{1}{\mathcal{E}(x,y)} \sum_{t \in \mathcal{F}(x,y)} \mathcal{R}(x,y,t) \right)^2 | f(x,y,z) |^2 \\ &\leq 12 \|\mathcal{N}(f)\|^2. \end{aligned}$$

Moreover, noticing that $\mathcal{N}(\cdot, \cdot, \cdot)$ is symmetric and $f(x, y, z) = -f(z, y, x)$ and let $J = |\langle f, [\mathcal{L}_{2,\text{skew}}, \mathcal{N}]f \rangle|$. We get:

$$\begin{aligned}
 J &\leq \frac{1}{12} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x, y, z) \left(|f(x, y, z)|^2 \right. \\
 &\quad \left. + |[\mathcal{L}_{2,\text{skew}}, \mathcal{N}](f)(x, y, z)|^2 \right) \\
 &\leq \frac{1}{2} \|\mathcal{N}^{\frac{1}{2}}(f)\|^2 + \frac{1}{4} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x, y, z) \left| \sum_{t \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x, y)} \mathcal{R}(x, y, t) \times \right. \\
 &\quad \left. \left(\mathcal{N}(x, y, t) - \mathcal{N}(x, y, z) \right) f(x, y, t) \right|^2 \\
 &\leq \frac{1}{2} \|\mathcal{N}^{\frac{1}{2}}(f)\|^2 + \frac{1}{4} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x, y, z) \left(\sum_{t \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x, y)} \mathcal{R}(x, y, t) \right) \times \\
 &\quad \left(\sum_{r \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x, y)} \mathcal{R}(x, y, r) \left| \mathcal{N}(x, y, r) - \mathcal{N}(x, y, z) \right|^2 |f(x, y, r)|^2 \right) \\
 &= \frac{1}{2} \|\mathcal{N}^{\frac{1}{2}}(f)\|^2 + \frac{1}{4} \sum_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x, y, z) \left(\sum_{t \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x, y)} \mathcal{R}(x, y, t) \right) \times \\
 &\quad \underbrace{\sum_{r \in \mathcal{V}} \frac{1}{\mathcal{E}(x, y)} \mathcal{F}(x, y, r) \left| \mathcal{N}(x, y, r) - \mathcal{N}(x, y, z) \right|^2 |f(x, y, z)|^2}_{\leq C} \\
 &\leq \left(\frac{1+3C}{2} \right) \|\mathcal{N}^{\frac{1}{2}}(f)\|.
 \end{aligned}$$

Applying [25, Theorem X.37], the result follows. The proof of $\mathcal{L}_{2,\text{sym}}$ may be checked in the same way as the proof of $\mathcal{L}_{2,\text{skew}}$. \square

Corollary 5.4. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a simple triangulation. Assume that*

$$\sup_{(x,y,z) \in \mathcal{F}} \sum_{r \in \mathcal{F}(x,y)} |\#\mathcal{F}(y,z) + \#\mathcal{F}(z,x) - \#\mathcal{F}(y,r) - \#\mathcal{F}(r,x)|^2 < \infty.$$

Then $\mathcal{L}_{2,\text{skew}}$ is essentially self-adjoint on $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$ and $\mathcal{L}_{2,\text{sym}}$ is essentially self-adjoint on $\mathcal{C}_{\text{sym}}^c(\mathcal{R})$.

6. ESSENTIAL SPECTRUM

Let A be a closed, densely defined linear operator on a Banach space X , and let $\sigma(A)$ denote the spectrum of A . We denote by $\mathcal{K}(X)$ the set of compact operators on X to itself. We define the essential spectrum of the operator A by

$$\sigma_{\text{ess}}(A) = \bigcap_{K \in \mathcal{K}(X)} \sigma(A + K).$$

It is well known that if A is a self-adjoint operator on a Hilbert space, the essential spectrum of A is the set of limit points of the spectrum of A , i.e., all points of the spectrum except isolated eigenvalues of finite multiplicity, see [29, 30]. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. Note that $\mathcal{L}_{2,\text{skew}}$ is non-negative symmetric operator on $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$. We consider the quadratic form

$$q(f, g) = \langle f, \mathcal{L}_{2,\text{skew}}g \rangle + \langle f, g \rangle$$

on $\mathcal{C}_{\text{skew}}^c(\mathcal{R}) \times \mathcal{C}_{\text{skew}}^c(\mathcal{R})$. Let \mathcal{H}_1 be the completion of $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$ under the norm

$$\|f\|_q = \sqrt{\langle \mathcal{L}_{2,\text{skew}}f, f \rangle + \|f\|^2}.$$

We define the Friedrichs extension $\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}$ of $\mathcal{L}_{2,\text{skew}}$ by:

- i*) A vector f is in domain $\mathcal{D}(\mathcal{L}_{2,\text{skew}}^{\mathcal{F}})$ if and only if $f \in \mathcal{H}_1$ and $\mathcal{C}_{\text{skew}}^c(\mathcal{R}) \ni g \mapsto \langle f, \mathcal{L}_{2,\text{skew}}g \rangle + \langle f, g \rangle$ extends to a norm continuous function on $\ell_{\text{skew}}^2(\mathcal{R})$.
- ii*) For each $f \in \mathcal{D}(\mathcal{L}_{2,\text{skew}}^{\mathcal{F}})$ there is a unique u_f such that $\langle f, \mathcal{L}_{2,\text{skew}}g \rangle + \langle f, g \rangle = \langle u_f, g \rangle$ by Riesz' Theorem. The Friedrichs extension of $\mathcal{L}_{2,\text{skew}}$, is given by $\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}f = u_f - f$. It is self-adjoint extension of $\mathcal{L}_{2,\text{skew}}$, e.g. see [25, Theorem X.23].

Theorem 6.1. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation and let $\mathcal{F}_0 = \{\mathcal{K} \subset \mathcal{F} : \mathcal{K} \text{ finite}\}$. Then,*

$$\inf \sigma(\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}) \leq \inf_{(x,y,z) \in \mathcal{F}} \mathcal{R}(x, y, z) \left(\frac{1}{\mathcal{E}(x, y)} + \frac{1}{\mathcal{E}(y, z)} + \frac{1}{\mathcal{E}(z, x)} \right).$$

and

$$\inf \sigma_{\text{ess}}(\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}) \leq \sup_{\mathcal{K} \subset \mathcal{F}_0} \inf_{(x,y,z) \in \mathcal{K}^c} \mathcal{R}(x, y, z) \left(\frac{1}{\mathcal{E}(x, y)} + \frac{1}{\mathcal{E}(y, z)} + \frac{1}{\mathcal{E}(z, x)} \right).$$

In particular, if \mathcal{T} is a simple triangulation then $\mathcal{L}_{2,\text{skew}}^{\mathcal{F}}$ is not with compact resolvent.

Proof. Let $(x_0, y_0, z_0) \in \mathcal{F}$ and let

$$f = \frac{\delta_{x_0, y_0, z_0} + \delta_{y_0, z_0, x_0} + \delta_{z_0, x_0, y_0} - \delta_{z_0, y_0, x_0} - \delta_{x_0, z_0, y_0} - \delta_{y_0, x_0, z_0}}{\sqrt{\mathcal{R}(x_0, y_0, z_0)}}.$$

Then $\|f\| = 1$ and

$$\langle f, \mathcal{L}_{2,\text{skew}}f \rangle = \mathcal{R}(x_0, y_0, z_0) \left(\frac{1}{\mathcal{E}(x_0, y_0)} + \frac{1}{\mathcal{E}(y_0, z_0)} + \frac{1}{\mathcal{E}(z_0, x_0)} \right)$$

Applying [20, Proposition 3], the result follows. \square

7. APPLICATION TO THE STUDY OF THE ADJACENCY MATRIX

7.1. Adjacency matrix. The adjacency matrix has some important implications for example it solves the equation of heat, see [5]. Let $\mathcal{G} = (\mathcal{V}, m, \mathcal{E})$ be a weighted graph. We define the set of 0-cochains on \mathcal{V} by

$$\mathcal{C}(\mathcal{V}) = \{f : \mathcal{V} \longrightarrow \mathbb{C}\}.$$

We denote by $\mathcal{C}^c(\mathcal{V})$ the 0-cochains with finite support in \mathcal{V} . We associate a Hilbert space to \mathcal{V} :

$$\ell^2(\mathcal{V}) = \left\{ f \in \mathcal{C}(\mathcal{V}) \text{ such that } \|f\|^2 = \sum_{x \in \mathcal{V}} m(x) |f(x)|^2 < \infty \right\}.$$

The associated scalar product is given by

$$\langle f, g \rangle = \sum_{x \in \mathcal{V}} m(x) \overline{f(x)} g(x), \text{ for } f, g \in \ell^2(\mathcal{V}).$$

We define the adjacency matrix:

$$\mathcal{A}_{\mathcal{G}}(f)(x) = \frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) f(y), \quad f \in \mathcal{C}^c(\mathcal{V}).$$

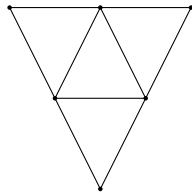
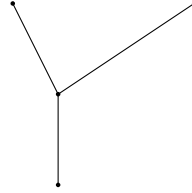
It is symmetric and thus closable. We denote its closure by the same symbol. When $m=1$, we have that $\mathcal{A}_{\mathcal{G}}$ is unbounded if and only if it is unbounded from above and if and only if the degree is unbounded, see [14].

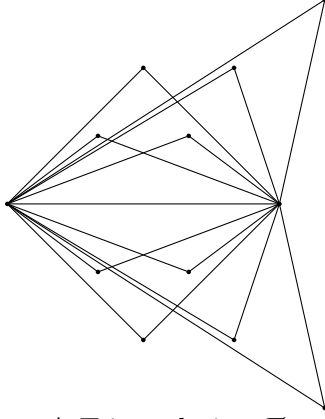
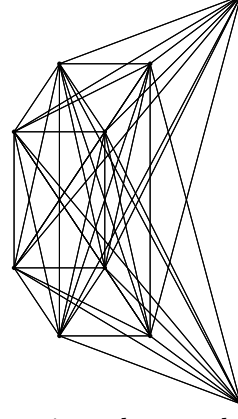
7.2. Triangular graph.

Definition 7.1. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. The triangular graph of \mathcal{T} is the graph $\widehat{\mathcal{T}} = (\widehat{\mathcal{V}}, \widehat{m}, \widehat{\mathcal{E}})$ where $\widehat{\mathcal{V}} = \mathcal{F}$, $\widehat{m} = 1$ and

$$\begin{aligned} \widehat{\mathcal{E}}((x_0, y_0, z_0), (x, y, z)) &= \sqrt{\mathcal{R}(x_0, y_0, z_0)} \left(\frac{\sqrt{\mathcal{R}(x, y, z)}}{\mathcal{E}(x, y)} 1_{x=x_0, y=y_0} \right. \\ &\quad \left. + \frac{\sqrt{\mathcal{R}(x, y, z)}}{\mathcal{E}(y, z)} 1_{y=y_0, z=z_0} + \frac{\sqrt{\mathcal{R}(x, y, z)}}{\mathcal{E}(z, x)} 1_{z=z_0, x=x_0} \right) \end{aligned}$$

if $(x_0, y_0, z_0) \neq (x, y, z)$ and 0 otherwise.


 A triangulation \mathcal{T}

 Triangular graph $\widehat{\mathcal{G}}$

A Triangulation \mathcal{T} Complete triangular graph $\widehat{\mathcal{G}}$

Remark 7.2. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a triangulation. The adjacency matrix on $\widehat{\mathcal{G}}$ is given by

$$\begin{aligned} \mathcal{A}_{\widehat{\mathcal{G}}}(f)(x_0, y_0, z_0) &:= \sum_{z \in \mathcal{V}, z \neq z_0} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(x_0, y_0, z)}}{\mathcal{E}(x_0, y_0)} f(x_0, y_0, z) \\ &+ \sum_{x \in \mathcal{V}, x \neq x_0} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(x, y_0, z_0)}}{\mathcal{E}(y_0, z_0)} f(x, y_0, z_0) \\ &+ \sum_{y \in \mathcal{V}, y \neq y_0} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(x_0, y, z_0)}}{\mathcal{E}(z_0, x_0)} f(x_0, y, z_0) \end{aligned}$$

for all $f \in \mathcal{C}^c(\widehat{\mathcal{V}})$.

Proposition 7.3. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. Then $\mathcal{L}_{2, \text{sym}}$ is unitarily equivalent to

$$\mathcal{A}_{\widehat{\mathcal{G}}} + \mathcal{Q}(V)$$

where

$$V(x_0, y_0, z_0) = \frac{\mathcal{E}(x_0, y_0, z_0)}{\mathcal{R}(x_0, y_0)} + \frac{\mathcal{R}(x_0, y_0, z_0)}{\mathcal{E}(y_0, z_0)} + \frac{\mathcal{R}(x_0, y_0, z_0)}{\mathcal{E}(z_0, x_0)}$$

and $\mathcal{Q}(V)$ be the operator of multiplication by V .

Proof. Set $U : \ell_{\text{sym}}^2(\mathcal{R}) \rightarrow \ell^2(\widehat{\mathcal{V}})$ as the operator given by

$$U(f)(x, y, z) = \sqrt{\mathcal{R}(x, y, z)} f(x, y, z).$$

Notice that

$$U^{-1}(f)(x, y, z) = U^*(f)(x, y, z) = \frac{1}{\sqrt{\mathcal{R}(x, y, z)}} f(x, y, z)$$

for all $f \in \ell^2(\widehat{\mathcal{V}})$. Notice now that on $\mathcal{C}_c(\widehat{\mathcal{V}})$

$$\begin{aligned} U\mathcal{L}_{2,\text{sym}}U^{-1}(f)(x_0, y_0, z_0) &= \sum_{t \in \mathcal{V}} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(x_0, y_0, t)}}{\mathcal{E}(x_0, y_0)} f(x_0, y_0, t) \\ &+ \sum_{t \in \mathcal{V}} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(y_0, z_0, t)}}{\mathcal{E}(y_0, z_0)} f(y_0, z_0, t) \\ &+ \sum_{t \in \mathcal{V}} \frac{\sqrt{\mathcal{R}(x_0, y_0, z_0)\mathcal{R}(z_0, x_0, t)}}{\mathcal{E}(z_0, x_0)} f(z_0, x_0, t). \end{aligned}$$

Using Remark 7.2, we obtain the result. \square

Corollary 7.4. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted 3-partite triangulation. If V is bounded on \mathcal{F} , then*

$$\mathcal{A}_{\widehat{\mathcal{G}}} \geq \sup_{(x,y,z) \in \mathcal{F}} V(x, y, z).$$

In particular, $\mathcal{A}_{\widehat{\mathcal{G}}} \geq -3$ when \mathcal{T} is simple.

Proof. Since $\mathcal{L}_{2,\text{sym}}$ is non-negative operator, then

$$\langle \mathcal{A}_{\widehat{\mathcal{G}}} f, f \rangle \geq -3 \|f\|^2$$

for all $f \in \mathcal{C}^c(\widehat{\mathcal{V}})$ \square

Corollary 7.5. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a 3-partite weighted triangulation. Set*

$$\mathcal{N}(x, y, z) = 1 + \sum_{r \in \mathcal{V}} \left(d_{\mathcal{E}}(x, y) + d_{\mathcal{E}}(y, z) + d_{\mathcal{E}}(z, x) \right).$$

Suppose that

$$\sup_{(x,y,z) \in \mathcal{F}} \sum_{r \in \mathcal{F}(x,y)} \frac{1}{\mathcal{E}(x, y)} \mathcal{R}(x, y, r) | \mathcal{N}(x, y, r) - \mathcal{N}(x, y, z) |^2 < \infty.$$

and V is bounded. Then, $\mathcal{A}_{\widehat{\mathcal{G}}}$ is essentially self-adjoint on $\mathcal{C}^c(\widehat{\mathcal{V}})$.

Proof. Combine Proposition 7.3, Theorem 5.3 and Theorem 4.2. \square

7.3. Geometric Hypothesis. We recall the two following definitions:

Definition 7.6. [1, Definition 8] *The graph $\mathcal{G} := (\mathcal{V}, m, \mathcal{E})$ is χ -complete if there exists a increasing sequence of finite set $(\mathcal{V}_n)_n$ such that $\mathcal{V} = \cup_n \mathcal{V}_n$ and there exist related functions χ_n satisfying the following three conditions:*

- 1) $\chi_n \in \mathcal{C}^c(\mathcal{V})$, $0 \leq \chi_n \leq 1$,
- 2) $\chi_n(x) = 1$ if $x \in \mathcal{V}_n$,
- 3) $\exists C > 0$, $\forall n \in \mathbb{N}$, $x \in \mathcal{V}$,

$$\frac{1}{m(x)} \sum_{y \in \mathcal{V}} \mathcal{E}(x, y) |\chi_n(x) - \chi_n(y)|^2 \leq C.$$

Definition 7.7. [7, Definition 4.2] A weighted triangulation $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ is χ -complete, if

- 1) $\mathcal{G} = (\mathcal{V}, m, \mathcal{E})$ is χ -complete.
- 2) $\exists M > 0, \forall n \in \mathbb{N}, (x, y) \in \mathcal{E}$

$$\frac{1}{\mathcal{E}(x, y)} \sum_{t \in \mathcal{F}(x, y)} \mathcal{R}(x, y, t) |2\chi_n(t) - \chi_n(x) - \chi_n(y)|^2 \leq M.$$

We recall the criterion obtained in [7].

Theorem 7.8. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a χ -complete weighted triangulation then $\mathcal{L}_{2, \text{skew}}$ is essentially self-adjoint on $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$.

Corollary 7.9. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a χ -complete weighted triangulation. If \mathcal{T} is tri-partite then $\mathcal{A}_{\widehat{\mathcal{G}}}$ is essentially self-adjoint on $\mathcal{C}^c(\widehat{\mathcal{V}})$.

Proof. Combine Proposition 7.3, Theorem 4.2 and Theorem 7.8.

8. EXAMPLES

8.1. Book-like triangulation. We recall the definition of 1-dimensional decomposition given in [6] for the case of graphs.

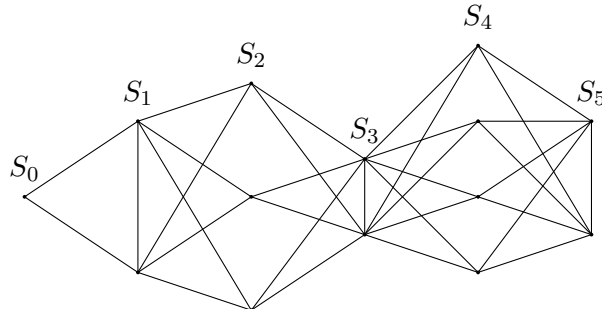
Definition 8.1. A 1-dimensional decomposition of the graph $\mathcal{G} = (\mathcal{V}, m, \mathcal{E})$ is a family of finite sets $(S_n)_{n \in \mathbb{N}}$ which forms a partitions of \mathcal{V} , that is $\mathcal{V} = \cup_{n \in \mathbb{N}} S_n$, and such that for all $x \in S_n, y \in S_m$,

$$\mathcal{E}(x, y) > 0 \implies |n - m| \leq 1$$

The following definition is introduced in [7].

Definition 8.2. Let $\mathcal{T} := (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation and $(S_n)_{n \in \mathbb{N}}$ a 1-dimensional decomposition of the graph $\mathcal{G} = (\mathcal{V}, m, \mathcal{E})$. We say that \mathcal{T} is a book-like triangulation if

- 1) $\#S_0 = 1, \#S_{2n+1} = 2$ and $\#(S_{2n+1}^2 \cap \mathcal{E}) = 1$, for all $n \in \mathbb{N}$.
- 2) $x, y \in S_{2n+2} \implies \mathcal{E}(x, y) = 0$,
- 3) $\forall x \in S_{2n+1}, \mathcal{N}_{\mathcal{G}}(x) = S_{2n} \cup S_{2n+2}$.



A book-like triangulation

We recall [7, Proposition 6.5]:

Proposition 8.3. *Let \mathcal{T} be a simple book-like triangulation. Assume that*

$$(2) \quad n \mapsto \frac{\#\mathcal{S}_{2n}}{\#\mathcal{S}_{2(n+1)}} \in \ell^1(\mathbb{N}).$$

Then, $\mathcal{L}_{2,\text{skew}}$ is not essentially self-adjoint on $\mathcal{C}_{\text{skew}}^c(\mathcal{R})$.

Corollary 8.4. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a simple book-like triangulation satisfying (2). Then, $\mathcal{A}_{\widehat{\mathcal{G}}}$ is not essentially self-adjoint on $\mathcal{C}^c(\widehat{\mathcal{V}})$.*

Proof. Set $\mathcal{S}_0 = \{x_0^0\}$, $\mathcal{S}_{2n+1} = \{x_n^0, x_n^1\}$, $\mathcal{V}_0 = \cup_n \mathcal{S}_{2n}$, $\mathcal{V}_1 = \cup_n \{x_n^0 : n \in \mathbb{N}\}$ and $\mathcal{V}_2 = \cup_n \{x_n^1 : n \in \mathbb{N}\}$. Then,

$$\mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \cup \mathcal{V}_2 \text{ and } \mathcal{E} \cap (\mathcal{V}_0^2 \cup \mathcal{V}_1^2 \cup \mathcal{V}_2^2) = \emptyset.$$

So, \mathcal{T} is tri-partite. Using Theorem 4.2, Proposition 7.3 and Proposition 8.3, the result holds. \square

8.2. Triangular anti-tree. Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. The sphere of radius $n \in \mathbb{N}$ around a vertex $v \in \mathcal{V}$ is the set

$$\mathcal{S}_n(v) := \left\{ w \in \mathcal{V} : d_{\mathcal{V}}(v, w) = n \right\}.$$

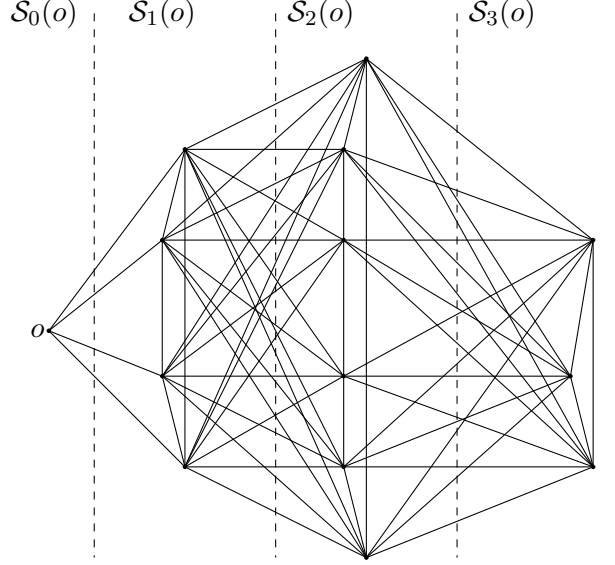
We recall that \mathcal{C}_n denotes the n -cycle graph, i.e. $\mathcal{V} = \mathbb{Z}/n\mathbb{Z}$, where $\mathcal{E}(x, y) > 0$ if and only if $|x - y| = 1$. Let $\mathcal{G}(\mathcal{S}_n(o)) = (\mathcal{S}_n(v), \mathcal{E}')$ where $\mathcal{E}' = \mathcal{E} \upharpoonright_{\mathcal{S}_n(v) \times \mathcal{S}_n(v)}$.

Definition 8.5. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a weighted triangulation. We say that \mathcal{T} is anti-tree if there exists a vertex $o \in \mathcal{V}$ such that*

1) *For all $n \in \mathbb{N}^*$ and $v \in \mathcal{S}_n(o)$, we have*

$$\mathcal{N}_{\mathcal{G}}(v) \setminus \mathcal{S}_n(o) = \mathcal{S}_{n-1}(o) \cup \mathcal{S}_{n+1}(o).$$

2) *For all $n \in \mathbb{N}^*$, $\mathcal{G}(\mathcal{S}_n(o)) \simeq \mathcal{C}_{\#\mathcal{S}_n(o)}$.*



An triangular anti-tree with spheres $\mathcal{S}_0(o)$, $\mathcal{S}_1(o)$, $\mathcal{S}_2(o)$, $\mathcal{S}_3(o)$.

Theorem 8.6. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a simple triangular anti-tree whose root in o . Set $s_n = \#\mathcal{S}_n(o)$ and assume that*

$$(3) \quad n \mapsto \frac{s_n^2}{s_{n+2}} \in \ell^1(\mathbb{N}).$$

Then $\mathcal{L}_{2,\text{sym}}$ does not essentially self-adjoint on $\mathcal{C}_{\text{sym}}^c(\mathcal{R})$.

Proof. Set $f \in \ell_{\text{sym}}^2(\mathcal{F}) \setminus \{0\}$ such that $f \in \ker(\mathcal{L}_{2,\text{sym}}^* + i)$ and such that f is constant on $\mathcal{S}_n \times \mathcal{S}_{n+1}^2 \cup \mathcal{S}_n^2 \times \mathcal{S}_{n+1}$, $n \in \mathbb{N}^*$. We denote the constant value by C_n . It takes the value 0 on \mathcal{S}_n^2 . We have the following equation:

$$(s_n + 4 + i)C_n + s_{n+2}C_{n+1} = 0.$$

Therefore,

$$\begin{aligned} \|f|_{\mathcal{S}_{n+1} \times \mathcal{S}_{n+2}^2}\|^2 &= \|f|_{\mathcal{S}_{n+1}^2 \times \mathcal{S}_{n+2}}\|^2 \\ &= \frac{1}{6} s_{n+2} s_{n+1} |C_{n+1}|^2 \\ &\leq \frac{|s_n + 4 + i|^2}{s_n s_{n+2}} \|f|_{\mathcal{S}_n \times \mathcal{S}_{n+1}^2}\|^2. \end{aligned}$$

Since $\lim_{n \rightarrow \infty} \frac{s_n^2}{s_{n+2}} = 0$, we get by induction:

$$M := \sup_{n \in \mathbb{N}^*} \|f|_{\mathcal{S}_n \times \mathcal{S}_{n+1}^2}\|^2 < \infty.$$

Then, we have

$$\|f|_{\mathcal{S}_{n+1} \times \mathcal{S}_{n+2}^2}\|^2 \leq M \frac{|i+4+s_n|}{s_n s_{n+2}}.$$

From Eq. (3), we conclude that $f \in \ell_{\text{sym}}^2(\mathcal{F})$. Using [25, Theorem X.36], we conclude that $\mathcal{L}_{2,\text{sym}}$ is not essentially self-adjoint. \square

Corollary 8.7. *Let $\mathcal{T} = (\mathcal{V}, m, \mathcal{E}, \mathcal{R})$ be a simple triangular anti-tree satisfying (3). Then, $\mathcal{A}_{\widehat{\mathcal{G}}}$ is not essentially self-adjoint on $\mathcal{C}^c(\widehat{\mathcal{V}})$.*

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