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Multiple front standing waves in the FitzHugh-Nagumo equations

Chao-Nien Chen ^{*} Éric Séré[†]

Abstract: There have been several existence results for the standing waves of FitzHugh-Nagumo equations. Such waves are the connecting orbits of an autonomous second-order Lagrangian system and the corresponding kinetic energy is an indefinite quadratic form in the velocity terms. When the system has two stable hyperbolic equilibria, there exist two stable standing fronts, which will be used in this paper as building blocks, to construct stable standing waves with multiple fronts in case the equilibria are of saddle-focus type. The idea to prove existence is somewhat close in spirit to [6]; however several differences are required in the argument: facing a strongly indefinite functional, we need to perform a nonlocal Lyapunov-Schmidt reduction; in order to justify the stability of multiple front standing waves, we rely on a more precise variational characterization of such critical points. Based on this approach, both stable and unstable standing waves are established.

Key words: reaction-diffusion system, FitzHugh-Nagumo equations, standing wave, stability, Hamiltonian system, connecting orbit.

AMS subject classification: 34C37, 35J50, 35K57.

1 Introduction

Following a fascinating idea of Turing [47], reaction-diffusion systems [1, 8, 14, 26, 37] serve as models for studying pattern formation and wave propagation. Significant progress [8, 18, 16, 19, 17, 23, 38, 42, 43, 48] on the self-organized

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patterns has been made for the system of FitzHugh-Nagumo equations

$$u_t - du_{xx} = f(u) - v, \quad (1.1)$$

$$\tau v_t - v_{xx} = u - \gamma v. \quad (1.2)$$

Here $f(\xi) = \xi(\xi - \beta)(1 - \xi)$, $\beta \in (0, 1/2)$ and $d, \tau, \gamma \in (0, \infty)$. Historically the original model [26, 37] was derived as a simplification of the Hodgkin-Huxley equations [28] for nerve impulse propagation. In recent years (1.1)-(1.2) has been extensively studied as a paradigmatic activator-inhibitor system. Such systems are of great interest to the scientific community as breeding grounds for studying the generation of localized structures.

The standing wave solutions of (1.1)-(1.2) are the connecting orbits of a second order Lagrangian system

$$-du'' = f(u) - v, \quad (1.3)$$

$$-v'' = u - \gamma v. \quad (1.4)$$

Associated with (1.3)-(1.4), the Lagrangian is

$$L(u', v', u, v) = \frac{d}{2}(u')^2 - \frac{1}{2}(v')^2 + uv - \frac{\gamma}{2}v^2 - \int_0^u f(\xi)d\xi. \quad (1.5)$$

Since (1.3)-(1.4) is an autonomous Lagrangian system, the associated energy

$$E(u', v', u, v) := \frac{d}{2}(u')^2 - \frac{1}{2}(v')^2 - uv + \frac{\gamma}{2}v^2 + \int_0^u f(\xi)d\xi \quad (1.6)$$

is constant along any solution. Moreover (1.3)-(1.4) can be rewritten as a first-order Hamiltonian system of Hamiltonian

$$H(p, q, u, v) := \frac{1}{2d}p^2 - \frac{1}{2}q^2 - uv + \frac{\gamma}{2}v^2 + \int_0^u f(\xi)d\xi \quad (1.7)$$

in the phase space \mathbb{R}^4 . System (1.7) will be referred to as (HS).

As (1.4) is a linear equation, v can be solved from u ; for a given $\phi \in H^1(\mathbb{R})$, let $\mathcal{L}\phi$ denote the unique solution, in $H^1(\mathbb{R})$, of the equation

$$-g'' + \gamma g = \phi. \quad (1.8)$$

Direct calculation shows that \mathcal{L} is a self-adjoint operator from $L^2(\mathbb{R})$ to itself. System (1.3)-(1.4) has two stable equilibria $(u_-, v_-) = (0, 0)$ and (u_+, v_+) with $u_+ > 0$. Since we are interested in standing front solutions joining such two equilibria, they must have the same energy E , and this imposes the condition $\gamma = 9(2\beta^2 - 5\beta + 2)^{-1}$, as to be a basic assumption of the paper. Under this

assumption, it is easy to see that $(u_+, v_+) = (2(\beta + 1)/3, 2(\beta + 1)/3\gamma)$ and the symmetry S with respect to the center $(u_+/2, v_+/2)$ preserves the Lagrangian.

Recall from [18] that \hat{v} is a C^∞ -function satisfying

$$\hat{v}(x) = \begin{cases} v_+ & \text{for } x \geq 1 \\ 0 & \text{for } x \leq -1. \end{cases} \quad (1.9)$$

To show the existence of standing front solutions of (1.3)-(1.4), we work with affine functional spaces of the form $H_w = w + H^1(\mathbb{R})$, with $w = \hat{v}, \hat{u}$, where $\hat{u} = (\gamma - \partial^2/\partial x^2)\hat{v}$. Then for $u \in \hat{u} + H^1(\mathbb{R})$, $\mathcal{L}u$ is the unique solution, in $\hat{v} + H^1(\mathbb{R})$, of the equation

$$-v'' + \gamma v = u.$$

In what follows, $L(u, v)$ stands for the Lagrangian defined by (1.5). For $u \in \hat{u} + H^1(\mathbb{R})$, define

$$J(u) = \int_{-\infty}^{\infty} L(u, \mathcal{L}u) dx, \quad (1.10)$$

a variational formulation with a nonlocal term involved. It is not difficult to verify that

$$J(u) = \max_{v \in \hat{v} + H^1(\mathbb{R})} \int_{-\infty}^{\infty} L(u, v) dx. \quad (1.11)$$

In [18] the action functional J has been employed, through a minimization argument, to obtain a basic type standing front solution of (1.1)-(1.2) as follows.

Theorem 1.1 *If $\gamma = 9(2\beta^2 - 5\beta + 2)^{-1}$ and $d > \gamma^{-2}$, there exists a standing front solution (u^*, v^*) of (1.3)-(1.4) with asymptotic behavior $(u^*, v^*) \rightarrow (u_-, v_-)$ as $x \rightarrow -\infty$ and $(u^*, v^*) \rightarrow (u_+, v_+)$ as $x \rightarrow \infty$. Moreover, u^* is a minimizer of J over $\hat{u} + H^1(\mathbb{R})$.*

Clearly (u_*, v_*) is also a heteroclinic orbit of (1.3)-(1.4) if we define $(u_*(x), v_*(x)) = (u^*(-x), v^*(-x))$. As a remark, such existence results can be extended to more general nonlinearities; that is, f is not necessarily a cubic polynomial.

The goal of this paper is to construct multiple front solutions using (u^*, v^*) together with the reverse orbit (u_*, v_*) . We only deal with the case when the equilibria are of saddle-focus type; that is, the linearization of the Hamiltonian system associated with (1.3)-(1.4) at (u_-, v_-) , as well as (u_+, v_+) , has eigenvalues $\pm\lambda \pm i\omega$. As to be seen in the Appendix, the values of the parameters have to satisfy

$$\beta \in (0, \frac{7 - \sqrt{45}}{2}), \quad \gamma = 9/(2\beta^2 - 5\beta + 2) \text{ and } \frac{1}{\gamma} < \sqrt{d} < \frac{2}{\gamma}. \quad (1.12)$$

We now state the main existence result of the paper.

Theorem 1.2 *Assume that (1.12) is satisfied. Then there are two real numbers κ_+ , κ_- , and, for each sufficiently small $\sigma > 0$, a large constant $D_\sigma > 0$, such that for any positive integer N and any sequence of positive integers $\mathbf{n} = (n_i)_{1 \leq i \leq N}$ with $n_i \geq D_\sigma$ for every i , there exist positive numbers X_1, \dots, X_N and a solution $(\hat{u}_{\mathbf{n}}, \hat{v}_{\mathbf{n}})$ of (1.3)-(1.4) satisfying the following properties:*

(a) *For i odd in $[1, N]$,*

$$\|(\hat{u}_{\mathbf{n}}, \hat{v}_{\mathbf{n}})(\cdot + C_i) - (u_*, v_*)\|_{H^1(-A_i, A_{i+1})} \leq \sigma, \quad |X_i - 2\pi n_i/\omega - \kappa_+| < \sigma.$$

(b) *For i even in $[0, N]$,*

$$\|(\hat{u}_{\mathbf{n}}, \hat{v}_{\mathbf{n}})(\cdot + C_i) - (u^*, v^*)\|_{H^1(-A_i, A_{i+1})} \leq \sigma, \quad |X_i - 2\pi n_i/\omega - \kappa_-| < \sigma.$$

Here, $A_0 = -\infty$, $C_0 = 0$, $C_i = C_{i-1} + X_i$, $A_i = X_i/2$ for $1 \leq i \leq N$, and $A_{N+1} = +\infty$.

Let us remark that if N is odd, $(\hat{u}_{\mathbf{n}}, \hat{v}_{\mathbf{n}})$ is homoclinic to (u_-, v_-) while for N even, it is a heteroclinic connection between (u_-, v_-) and (u_+, v_+) . Such orbits are the standing waves of (1.1)-(1.2) with multiple fronts; for the Hamiltonian system they are often called multi-bump solutions.

As already mentioned, the range of parameters under consideration is such that the basic heteroclinics (u_*, v_*) and (u^*, v^*) connect two equilibria of saddle-focus type. In this situation, multi-bump solutions are known to exist provided the stable and unstable manifolds intersect transversally, as was proved by Devaney [24] by constructing a Smale horseshoe. Transversality condition in general is difficult to check for a given Hamiltonian although it is generically true. Instead of verifying transversality, we follow a strategy introduced in [6]. We first prove that any critical point of J is isolated up to translation invariance in the spatial variable, by solving an auxiliary boundary value problem. Then we invoke this property to show the existence of multi-bump solutions by a variational argument.

The variational construction for multi-bump and chaotic solutions has a long history and the comments below are not exhaustive. To our knowledge, the earliest results were established by Bolotin [2, 3, 4] in the context of nonautonomous second order Lagrangian systems, the connecting orbits being minimizers of the action. In the case of twist maps on the annulus (also corresponding to nonautonomous Lagrangian systems), Mather [36] constructed chaotic connecting orbits by a minimization method in the region between two invariant circles. For non-autonomous first order Hamiltonian systems, multi-bump solutions were found by min-max methods [44, 45] under the assumption that critical points are isolated. This construction was extended to second order systems and elliptic PDEs in [21, 22]. We refer to [40] and references therein for more recent

development and related results in this direction. For autonomous problems of saddle-focus type a class of multi-bump solutions were obtained, in the special case of a fourth order equation related to water wave theory, by Buffoni [5] using a shooting argument. Subsequently a larger set of multi-bump solutions was constructed [6] by variational and degree arguments. This method was then adapted for studying the extended Fisher-Kolmogorov equations (of fourth order) [30]. In subsequent works [31, 32], the authors introduced a refined but more specific argument to obtain more precise results on the F-K model. As already mentioned, the present work is close in spirit to [6]. Note, however, that our system of autonomous second order Lagrangian equations is associated with a strongly indefinite variational problem and to our knowledge, it cannot be reduced to a fourth order equation which would allow a simpler variational interpretation. Instead, we use a nonlocal Lyapunov-Schmidt reduction. Moreover our approach is purely variational, contrary to [6] where degree theory was employed. Another novelty is our proof that *all* critical points are isolated up to translations in x , while in [6] the first step just consisted in showing that the basic one-bump solution is isolated. The purely variational construction and the stronger isolatedness property are needed for the sake of stability analysis, as always an important issue in considering pattern formation as well as wave propagation.

For the stationary solutions of (1.1)-(1.2), stability questions have been studied in [15, 16, 18, 17, 38, 49] by various methods. In conjunction with strongly indefinite variational structure, the Maslov index [14, 16] and relative Morse index [15] provide useful information to determine the stability of such solutions, obtained as the critical points of the action functional. Let $\mathbb{C}^- = \{\zeta \mid \zeta \in \mathbb{C} \text{ and } \operatorname{Re}\zeta < 0\}$, where $\operatorname{Re}\zeta$ denotes the real part of ζ . Denote by Λ the linearization of (1.1)-(1.2) at a standing wave solution (u, v) . A standing wave (u, v) is said to be non-degenerate if zero is a simple eigenvalue of Λ .

Definition: A non-degenerate standing wave (u, v) of (1.1)-(1.2) is spectrally stable if all the non-zero eigenvalues of Λ are in \mathbb{C}^- .

The following result follows immediately from an index method developed in [10]:

Theorem 1.3 Let (u, v) be a non-degenerate standing wave of (1.1)-(1.2). Suppose u is a local minimizer of J then (u, v) is spectrally stable, provided that $\tau < \gamma^2$.

Note, however, that the non-degeneracy of a standing wave is equivalent to the transversality of the stable and unstable manifolds, and we are unable to prove such a property. Fortunately, we can go beyond the spectral stability analysis, thanks to a Lyapunov functional introduced in [17] in a slightly different context. In Section 5 we shall give an extension of this Lyapunov functional, which can be applied to the standing waves of (1.1)-(1.2). Let us remark

that the standing waves are in affine subspaces of $H_{loc}^1(\mathbb{R}) \times H_{loc}^1(\mathbb{R})$ having $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ as underlying vector space. The norm of $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ induces the natural metric on such affine spaces, and we shall study the dynamical stability of the standing waves for this metric.

Theorem 1.4 *Let $\tau < \gamma^2$. Under the flow generated by (1.1)-(1.2) on the affine space $(\hat{u}_n + H^1(\mathbb{R})) \times (\hat{v}_n + H^1(\mathbb{R}))$, the standing wave (\hat{u}_n, \hat{v}_n) is asymptotically stable for the $H^1(\mathbb{R}) \times H^1(\mathbb{R})$ metric, up to a phase shift in spatial variable. More precisely, there is $\rho_n > 0$ such that if $(u(x, t), v(x, t))$ is a solution of (1.1)-(1.2) and*

$$\|u(\cdot, 0) - \hat{u}_n\|_{H^1(\mathbb{R})} + \|v(\cdot, 0) - \hat{v}_n\|_{H^1(\mathbb{R})} < \rho_n,$$

then

$$\inf_{y \in \mathbb{R}} \{ \|u(\cdot, t) - \hat{u}_n(\cdot - y)\|_{H^1(\mathbb{R})} + \|v(\cdot, t) - \hat{v}_n(\cdot - y)\|_{H^1(\mathbb{R})} \} \xrightarrow{t \rightarrow +\infty} 0.$$

As a final remark, there are plenty of unstable standing waves; however we do not attempt to describe them all. We just state a result in the two-bump case.

Theorem 1.5 *As in Theorem 1.2, assume (1.12) and take sufficiently small σ and large D_σ . For any positive integer $n \geq D_\sigma$ there exists a solution $(\check{u}_n, \check{v}_n)$ of (1.3)-(1.4) such that, for some $X \in \mathbb{R}$ with $|X - \pi(2n + 1)/\omega - \kappa_+| < \sigma$ and κ_+ as in Theorem 1.2, the following properties hold.*

(i) $\|(\check{u}_n, \check{v}_n) - (u^*, v^*)\|_{H^1(-\infty, X/2)} \leq \sigma.$

(ii) $\|(\check{u}_n, \check{v}_n) - (u_*, v_*)(\cdot - X)\|_{H^1(X/2, +\infty)} \leq \sigma.$

(iii) $(\check{u}_n, \check{v}_n)$ is unstable in the following sense: for some $\epsilon_0 > 0$ and for any $\rho > 0$, one can find $\tau_*(\rho) > 0$ and a solution $(u(x, t), v(x, t))$ of (1.1)-(1.2) such that

$$\|u(\cdot, 0) - \check{u}_n\|_{H^1(\mathbb{R})} + \|v(\cdot, 0) - \check{v}_n\|_{H^1(\mathbb{R})} < \rho,$$

while if $t \geq \tau_*(\rho)$ then

$$\inf_{y \in \mathbb{R}} \{ \|u(\cdot, t) - \check{u}_n(\cdot - y)\|_{H^1(\mathbb{R})} + \|v(\cdot, t) - \check{v}_n(\cdot - y)\|_{H^1(\mathbb{R})} \} \geq \epsilon_0.$$

The solution $(\check{u}_n, \check{v}_n)$ will be found in Section 6 by a mountain-pass type mini-max method. In the proof of (iii), the fact that $(\check{u}_n, \check{v}_n)$ is an isolated standing wave will be crucial.

2 Preliminaries

In this section we recall the variational setting [18] used to study (u^*, v^*) and discuss related properties, including a reduced functional J which is bounded from below. In the sequel, we work with affine functional spaces of the form $H_w = w + H^1(\mathbb{R})$, with $w = 0, u_+, v_+, u_*, u^*, v_*$ or v^* . For $a_u = 0, u_+, u_*, u^*$ respectively, and $u \in a_u + H^1(\mathbb{R})$, we also denote $\mathcal{L}u := a_v + \mathcal{L}(u - a_u)$, with $a_v = 0, v_+, v_*, v^*$ respectively. Let us remark that $\mathcal{L}u$ is the unique solution, in $a_v + H^1(\mathbb{R})$, of the equation

$$-v'' + \gamma v = u.$$

Recall from (1.5) that the Lagrangian associated with (1.3)-(1.4) is $L(u, v)$. Note that the main difference with [6] is that the present system does not seem to be reducible to an almost linear, fourth-order system having a simple variational interpretation. So one has to deal with an indefinite Lagrangian (1.5). Fortunately, this Lagrangian is concave in v . We exploit this property as follows: Observe that

$$\begin{aligned} \left\| \frac{d}{dx}(v - a_v) \right\|_{L^2(\mathbb{R})}^2 + \gamma \|v - a_v\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\infty} (u - a_u)(v - a_v) dx \\ &\leq \|u - a_u\|_{L^2(\mathbb{R})} \|v - a_v\|_{L^2(\mathbb{R})}. \end{aligned}$$

Hence there is a $C_0 > 0$ such that

$$\|v - a_v\|_{H^1(\mathbb{R})} \leq C_0 \|u - a_u\|_{L^2(\mathbb{R})}. \quad (2.1)$$

Given $\phi \in H^1(\mathbb{R})$, define, for all $\psi \in H^1(\mathbb{R})$,

$$I(\psi) = \int_{-\infty}^{\infty} \left(\frac{1}{2} |\psi'|^2 + \frac{\gamma}{2} \psi^2 - \phi \psi \right) dx$$

Lemma 2.1 *Let $\phi \in H^1(\mathbb{R})$. Then*

$$I(\psi) - I(\mathcal{L}\phi) = \int_{-\infty}^{\infty} \frac{1}{2} (\psi' - (\mathcal{L}\phi)')^2 + \frac{\gamma}{2} (\psi - (\mathcal{L}\phi))^2 dx$$

for all $\psi \in H^1(\mathbb{R})$.

Proof. It follows from straightforward calculation, by making use of

$$\int_{-\infty}^{\infty} ((\mathcal{L}\phi)')^2 + \gamma (\mathcal{L}\phi)^2 dx = \int_{-\infty}^{\infty} \phi \mathcal{L}\phi dx$$

and

$$\int_{-\infty}^{\infty} \psi' \mathcal{L}\phi' + \gamma\psi \mathcal{L}\phi dx = \int_{-\infty}^{\infty} \phi\psi dx.$$

■

For $w \in \{0, u_+, u^*, u_*\}$ and all $u \in H_w$, define

$$J(u) = \int_{-\infty}^{\infty} L(u, \mathcal{L}u) dx. \quad (2.2)$$

The next lemma is an immediate consequence of Lemma 2.1.

Lemma 2.2 *Taking $a_u = 0, u_+, u^*, u_*$ respectively and $a_v = 0, v_+, v^*, v_*$ respectively, if $u \in H_{a_u}$ then:*

$$J(u) = \max_{v \in H_{a_v}} \int_{-\infty}^{\infty} L(u, v) dx. \quad (2.3)$$

The operator \mathcal{L} has a good control in terms of local estimates:

Lemma 2.3 *There is a constant M such that, if $B - A \geq 1$ and $\phi \in H^1(\mathbb{R})$, then*

$$\|\mathcal{L}\phi\|_{H^2(A,B)} \leq M(\|\phi\|_{L^2(A,B)} + |\mathcal{L}\phi(A)| + |\mathcal{L}\phi(B)|).$$

Proof. Let ψ be the restriction of $\mathcal{L}\phi$ to the interval $[A, B]$. Then ψ solves the boundary value problem

$$-\psi'' + \gamma\psi = \phi \text{ in } (A, B), \quad (2.4)$$

$$\psi(A) = \mathcal{L}\phi(A), \quad \psi(B) = \mathcal{L}\phi(B).$$

Since $G(x) = -\frac{1}{2\sqrt{\gamma}}e^{-\sqrt{\gamma}|x|}$ satisfies of $-G'' + \gamma G = \delta_0$, we give an integral expression of ψ :

$$\forall x \in [A, B], \quad \psi(x) = b_1 e^{-\sqrt{\gamma}(x-A)} + b_2 e^{\sqrt{\gamma}(x-B)} + \int_A^B \phi(t)G(x-t)dt.$$

Here b_1 and b_2 are uniquely determined by the boundary conditions $\psi(A) = \mathcal{L}\phi(A)$, $\psi(B) = \mathcal{L}\phi(B)$, from which we are led to solving

$$\begin{cases} b_1 + e^{-\sqrt{\gamma}(B-A)}b_2 = \alpha_1 \\ b_1 e^{-\sqrt{\gamma}(B-A)} + b_2 = \alpha_2 \end{cases} \quad (2.5)$$

with

$$\alpha_1 = \mathcal{L}\phi(A) - \int_A^B \phi(t)G(A-t)dt$$

and

$$\alpha_2 = \mathcal{L}\phi(B) - \int_A^B \phi(t)G(B-t)dt.$$

The solution of (2.5) is

$$b_1 = \frac{\alpha_1 - e^{-\sqrt{\gamma}(B-A)}\alpha_2}{1 - e^{-2\sqrt{\gamma}(B-A)}}, \quad b_2 = \frac{\alpha_2 - e^{-\sqrt{\gamma}(B-A)}\alpha_1}{1 - e^{-2\sqrt{\gamma}(B-A)}}.$$

Using the integral formula for ψ , we get

$$\|\psi\|_{L^2(A,B)} \leq (|b_1| + |b_2|)\|e^{-\sqrt{\gamma}t}\|_{L^2(0,\infty)} + \|\phi\|_{L^2(A,B)}\|G\|_{L^1(\mathbb{R})}. \quad (2.6)$$

Since $B - A \geq 1$, we obtain

$$|b_1| + |b_2| \leq \frac{1 + e^{-\sqrt{\gamma}}}{1 - e^{-2\sqrt{\gamma}}}(|\alpha_1| + |\alpha_2|).$$

Moreover

$$\begin{aligned} |\alpha_1| &\leq |\mathcal{L}\phi(A)| + \|\phi\|_{L^2(A,B)}\|G\|_{L^2(-\infty,0)}, \\ |\alpha_2| &\leq |\mathcal{L}\phi(B)| + \|\phi\|_{L^2(A,B)}\|G\|_{L^2(0,\infty)}, \end{aligned}$$

and straightforward calculation gives

$$\|G\|_{L^1(\mathbb{R})} = \frac{1}{\gamma}, \quad \|e^{-\sqrt{\gamma}t}\|_{L^2(0,\infty)} = 2^{-1/2}\gamma^{-1/4}, \quad \|G\|_{L^2(\mathbb{R}_{\pm})} = 2^{-3/2}\gamma^{-3/4}.$$

Combining with (2.6) yields

$$\|\psi\|_{L^2(A,B)} \leq C_1(\gamma) (|\mathcal{L}\phi(A)| + |\mathcal{L}\phi(B)|) + C_2(\gamma) \|\phi\|_{L^2(A,B)}$$

with $C_1(\gamma)$, $C_2(\gamma)$ independent of A , B . Similarly, using the integral formula

$$\forall x \in [A, B], \quad \psi'(x) = -b_1\sqrt{\gamma}e^{-\sqrt{\gamma}(x-A)} + b_2\sqrt{\gamma}e^{\sqrt{\gamma}(x-B)} + \int_A^B \phi(t)G'(x-t)dt$$

with $G'(x) = -\frac{\text{sign}(x)}{2}e^{-\sqrt{\gamma}|x|}$, $\|G'\|_{L^1(\mathbb{R})} = \frac{1}{\sqrt{\gamma}}$, we get

$$\|\psi'\|_{L^2(A,B)} \leq C_3(\gamma) (|\mathcal{L}\phi(A)| + |\mathcal{L}\phi(B)|) + C_4(\gamma) \|\phi\|_{L^2(A,B)}.$$

Invoking (2.4) gives $\|\psi''\|_{L^2(A,B)} \leq \gamma\|\psi\|_{L^2(A,B)} + \|\phi\|_{L^2(A,B)}$. ■

The following proposition, which was proved in [18], implies that all the critical values of J on H_{u^*} must be positive, since $d - \frac{1}{\gamma^2} > 0$.

Proposition 2.4 *If $u \in H_w$ (with $w = u_-, u_+, u_*$ or u^*), then*

$$J(u) = \int_{-\infty}^{\infty} \left\{ \frac{1}{2} \left[\left(d - \frac{1}{\gamma^2} \right) (u')^2 + \left(v' - \frac{u'}{\gamma} \right)^2 + \gamma \left(v - \frac{u}{\gamma} \right)^2 \right] + \frac{1}{4} u^2 (u - u_+)^2 \right\} dx.$$

If $J(u) \leq C$ then Proposition 2.4 gives an upper bound on $\|u - w\|_{H^1}$. If, in addition, u is a critical point of J , then, invoking (1.3)-(1.4), we obtain an L^∞ bound for $(u, \mathcal{L}u, u', \mathcal{L}u')$:

Corollary 2.5 *Let $u \in H_w$ (with $w = u_-, u_+, u_*$ or u^*) be a critical point of J and $J(u) \leq C$. Then there is a $C' > 0$, depending only on C , such that*

$$\|u - w\|_{H^1} + \|(u, \mathcal{L}u, u', \mathcal{L}u')\|_{L^\infty} \leq C'.$$

The choice of parameters made in this paper implies that $z_\pm = (u_\pm, v_\pm, 0, 0)$ are equilibria of saddle-focus type: the linearized Hamiltonian system at these points possesses four complex eigenvalues $\pm\lambda \pm \omega$. Hence there exist invertible matrices P_- and P_+ such that if $Y_\pm = P_\pm(u - u_\pm, v - v_\pm)^T$ then (1.3)-(1.4) becomes $Y_\pm'' = \mu^2 R_{2\nu} Y_\pm + O(|Y_\pm|^2)$, where $\lambda = \mu \cos \nu$, $\omega = \mu \sin \nu$, $\mu > 0$ and $R_{2\nu}$ is the matrix of a rotation with angle 2ν in \mathbb{R}^2 , and the notation T means transposition.

With $z_\pm := (u_\pm, v_\pm, 0, 0)$ being hyperbolic equilibria, a non-constant solution of (1.3)-(1.4) cannot stay near them for all values of x . This together with Proposition 2.4 gives a positive lower bound ℓ for the value of J at critical points. Let $|\cdot|$ be the euclidean norm in \mathbb{R}^2 . We list a number of properties for the Hamiltonian system (1.7), denoted by (HS).

Proposition 2.6 *Suppose $z(x) = (u(x), v(x), p(x), q(x))$ is a non-constant solution of (HS), then there exist positive numbers ρ_0 and ℓ such that the following properties hold.*

(i) *If $|P_\pm(u(x) - u_\pm, v(x) - v_\pm)^T| < \rho_0$ for all $x \leq 0$, then $z(0) \in W^u(z_\pm)$, and there is an $a_1 > 0$ such that $|P_\pm(u(a_1) - u_\pm, v(a_1) - v_\pm)^T| = \rho_0$,*

$$\frac{d}{dx} |P_\pm(u(x) - u_\pm, v(x) - v_\pm)^T| \geq \frac{\lambda}{2} |P_\pm(u(x) - u_\pm, v(x) - v_\pm)^T| \quad (2.7)$$

for all $x \leq a_1$.

Similarly, if $|P_\pm(u(x) - u_\pm, v(x) - v_\pm)^T| < \rho_0$ for all $x \geq 0$, then $z(0) \in W^s(z_\pm)$, and there is an $a_2 < 0$ such that $|P_\pm(u(a_2) - u_\pm, v(a_2) - v_\pm)^T| = \rho_0$,

$$\frac{d}{dx} |P_\pm(u(x) - u_\pm, v(x) - v_\pm)^T| \leq -\frac{\lambda}{2} |P_\pm(u(x) - u_\pm, v(x) - v_\pm)^T| \quad (2.8)$$

for all $x \geq a_2$. Here both a_1 and a_2 depend continuously on the initial datum $z(0)$.

(ii) Let u_c be a non-constant critical point of J and $u_{s_{\pm}} = \lim_{x \rightarrow \pm\infty} u_c(x)$, with $s_{\pm} \in \{-, +\}$. If $\rho \in (0, \rho_0)$ there exist $x_1 < x_2$ such that for any $x \in (-\infty, x_1)$, $P_{s_-}(u_c, \mathcal{L}u_c)^{\text{T}}(x)$ lies in the open disk D_- of center $P_{s_-}(u_{s_-}, v_{s_-})^{\text{T}}$ with radius ρ and $P_{s_-}(u_c, \mathcal{L}u_c)^{\text{T}}(x_1)$ sits on the boundary of D_- , while for any $x \in (x_2, \infty)$, $P_{s_+}(u_c, \mathcal{L}u_c)^{\text{T}}(x)$ lies in the open disk D_+ of center $P_{s_+}(u_{s_+}, v_{s_+})^{\text{T}}$ with radius ρ and $P_{s_+}(u_c, \mathcal{L}u_c)^{\text{T}}(x_2)$ sits on the boundary of D_+ .

(iii) If u_c is a non-constant critical point of J in H_w (with $w = u_-, u_+, u_*$ or u^*) then $J(u_c) \geq \ell$. Moreover $\|u - u_{\pm}\|_{H^1} \geq \ell$ in case $w = u_{\pm}$.

Proof. We prove (i) when $|P_{\pm}(u(x) - u_{\pm}, v(x) - v_{\pm})^{\text{T}}| < \rho_0$ for all $x \leq 0$; the other case immediately follows from time reversal. With $z_{\pm} = (u_{\pm}, v_{\pm}, 0, 0)$ being hyperbolic equilibria, if ρ_0 is small enough, then, by the Hartman-Grossman Theorem, $z(0)$ must lie in the local unstable manifold $W_{\text{loc}}^u(z_{\pm})$, which is an embedded submanifold tangent to the unstable space $E^u(z_{\pm})$, by the Stable Manifold Theorem (see e.g. [33]). For any solution $(\dot{u}(x), \dot{v}(x), \dot{p}(x), \dot{q}(x))$ of the linearization of (HS) at z_{\pm} , the equality $\frac{d}{dx}|P_{\pm}(\dot{u}(x), \dot{v}(x))^{\text{T}}| = \lambda|P_{\pm}(\dot{u}(x), \dot{v}(x))^{\text{T}}|$ holds, provided that this solution lies in $E^u(z_{\pm})$. Then, for the nonlinear flow of (HS), (2.7) follows for small ρ_0 . Note that in (2.7), $\frac{\lambda}{2}$ could be replaced by any number $\lambda' < \lambda$, at the expense of choosing a smaller ρ_0 when λ' is closer to λ .

Next we prove (ii). For fix $0 < \rho \leq \rho_0$, let $X \in \mathbb{R}$ be such that $|P_{s_-}(u_c(x) - u_{s_-}, \mathcal{L}u_c(x) - v_{s_-})^{\text{T}}| < \rho$ for all $x \leq -X$ and $z(x) := (u_c(x + X), \mathcal{L}u_c(x + X), du'_c(x + X), -\mathcal{L}u'_c(x + X))$. Since z satisfies the assumptions of (i), set

$$x_1 = \max\{x \leq a_1 - X : |P_{s_-}(u_c(y) - u_{s_-}, \mathcal{L}u_c(y) - v_{s_-})^{\text{T}}| < \rho \forall y < x\},$$

where a_1 was defined by (i). Clearly x_1 satisfies the required properties and x_2 can be treated in the same manner. The continuity of x_1, x_2 , as the functions of $z(0)$, follow from (2.7)-(2.8); indeed, using these properties together with the real-analyticity of the system, we obtain a more precise information: x_1, x_2 are real-analytic functions of $z(0)$, as a consequence of the implicit function theorem in the version of analytic class.

Finally if u_c is a heteroclinic solution, (iii) follows from Proposition 2.4. For a non-constant critical point u_c of J in $H_{u_{\pm}}$, $\|u - u_{\pm}\|_{H^1}$ cannot be too small either, for otherwise $P_{s_-}(u_c(x_1), \mathcal{L}u_c(x_1))^{\text{T}}$ cannot reach the boundary of D_- . ■

Next we analyze the behavior of Palais-Smale sequences of J . Recall the classical notion of Palais-Smale sequence.

Definition. A sequence $\{u_m\}$ in H_w , with $w \in \{u_{\pm}, u_*, u^*\}$, is called a Palais-Smale sequence for J if $\{J(u_m)\}$ is bounded and $J'(u_m) \rightarrow 0$ in $H^{-1}(\mathbf{R})$

as $m \rightarrow \infty$.

Due to translation invariance in x , a Palais-Smale sequence does not necessarily have a convergent subsequence for the $H^1(\mathbf{R})$ metric. However, adapting the arguments of [18], we obtain the following result in the spirit of the concentration-compactness theory by Pierre-Louis Lions [35].

Proposition 2.7 *Let ρ_0 be the number introduced in Proposition 2.6, $w \in \{u_\pm, u_*, u^*\}$ and $u_{s_-} := \lim_{x \rightarrow -\infty} w(x)$, $s_- \in \{-, +\}$. Suppose $\{u_m\}$ is a Palais-Smale sequence for J in H_w such that $\liminf_{m \rightarrow \infty} J(u_m) > 0$ then there is a $\rho_1 \in (0, \rho_0)$ such that, for any fixed $\rho \in (0, \rho_1)$, the following properties hold.*

- (i) *There exists $\{x_m\} \subset \mathbf{R}$ such that, for all $x \in (-\infty, x_m)$, $P_{s_-}(u_m, \mathcal{L}u_m)^\mathbb{T}(x)$ lies in the open disk D_- of center $P_{s_-}(u_{s_-}, \mathcal{L}u_{s_-})^\mathbb{T}$ with radius ρ and $P_{s_-}(u_m, \mathcal{L}u_m)^\mathbb{T}(x_m)$ sits on the boundary of D_- . Moreover, after extraction, $\tilde{u}_m := u_m(\cdot - x_m)$ converges in H_{loc}^1 to a non-constant critical point $u^{(1)}$ of J , with $\lim_{x \rightarrow -\infty} u^{(1)}(x) = u_{s_-}$ and $u_{s_+} = \lim_{x \rightarrow +\infty} u^{(1)}(x)$ may differ from $\lim_{x \rightarrow +\infty} w(x)$.*
- (ii) *If $\liminf \|\tilde{u}_m - u^{(1)}\|_{H^1} > 0$, i.e. the convergence of $\{\tilde{u}_m\}$ does not hold for the $H^1(\mathbf{R})$ metric, there exist two sequences $\underline{t}_m \rightarrow \underline{t} \in \mathbf{R}$ and $\bar{t}_m \rightarrow +\infty$ such that for all $x \in (\underline{t}_m, \bar{t}_m)$, $P_{s_+}(\tilde{u}_m, \mathcal{L}\tilde{u}_m)^\mathbb{T}(x)$ lies in the open disk \tilde{D} of center $P_{s_+}(u_{s_+}, v_{s_+})^\mathbb{T}$ with radius ρ , $P_{s_+}(\tilde{u}_m, \mathcal{L}\tilde{u}_m)^\mathbb{T}(\underline{t}_m)$ and $P_{s_-}(\tilde{u}_m, \mathcal{L}\tilde{u}_m)^\mathbb{T}(\bar{t}_m)$ both sit on the boundary of \tilde{D} . Furthermore along a subsequence, $\{\tilde{u}_m(\cdot - \bar{t}_m)\}$ converges in H_{loc}^1 to a non-constant critical point $u^{(2)}$ of J and $u_{s_+} = \lim_{x \rightarrow +\infty} u^{(1)}(x) = \lim_{x \rightarrow -\infty} u^{(2)}(x)$.*
- (iii) $\ell \leq J(u^{(1)}) \leq \limsup J(u_m) - \ell$ and $\ell \leq J(u^{(2)}) \leq \limsup J(u_m) - \ell$.

Proof. Since $f(\xi) + \beta\xi = o(\xi)$ as $\xi \rightarrow 0$, there exists $\rho_1 \in (0, \rho_0)$ such that $|P_-(u, v)^\mathbb{T}| \leq \rho_1$ implies $-f(u)u \geq \frac{\beta u^2}{2}$. From the symmetry property of the null-clines, $|P_-(u - u_+, v - v_+)^\mathbb{T}| \leq \rho_1$ implies $-f(u)(u - u_+) \geq \frac{\beta(u - u_+)^2}{2}$.

To show the existence of $\{x_m\}$, we assume by contradiction that for all $x \in \mathbb{R}$, $P_{s_-}(u_m, \mathcal{L}u_m)^\mathbb{T}(x)$ lies in D_- . This is possible only if $w = u_{s_-}$. As $e_m := -du_m'' - f(u_m) + \mathcal{L}u_m \rightarrow 0$ in H^{-1} and $\hat{e}_m := -(\mathcal{L}u_m)'' + \gamma\mathcal{L}u_m - u_m = 0$, it follows that $\|u_m\|_{H^1} + \|\mathcal{L}u_m\|_{H^1} = O(1)$ and

$$\epsilon_m := \langle e_m, u_m - u_{s_-} \rangle + \langle \hat{e}_m, \mathcal{L}u_m - v_{s_-} \rangle = o(1). \quad (2.9)$$

By direct calculation

$$\epsilon_m = \int_{-\infty}^{\infty} \{d(u_m')^2 + (\mathcal{L}u_m')^2 + \gamma(\mathcal{L}u_m - v_{s_-})^2 - (u_m - u_{s_-})f(u_m)\}. \quad (2.10)$$

With $P_{s_-}(u_m, \mathcal{L}u_m)^\top(x)$ lies in D_- for all $x \in \mathbb{R}$, invoking (2.9)-(2.10) yields

$$\int_{-\infty}^{\infty} \{d(u'_m)^2 + (\mathcal{L}u'_m)^2 + \gamma(\mathcal{L}u_m - v_{s_-})^2 + \frac{\beta}{2}(u_m - u_{s_-})^2\} \leq o(1).$$

This implies $\|u_m - u_{s_-}\|_{H^1} \rightarrow 0$ and consequently $J(u_m) \rightarrow 0$, which violates $\liminf_{m \rightarrow \infty} J(u_m) > 0$.

Next, set $\tilde{u}_m := u_m(\cdot - x_m)$. Since $J(\tilde{u}_m)$ is bounded and $J'(\tilde{u}_m) \rightarrow 0$, using an argument in [18] yields a critical point $u^{(1)}$ of J and a subsequence, still denoted by $\{(u_m, \mathcal{L}u_m)\}$, converges to $u^{(1)}$ in $H^1_{\text{loc}}(\mathbf{R})$. Note that $u^{(1)}$ satisfies $|P_{s_-}(u^{(1)}(x) - u_{s_-}, \mathcal{L}u^{(1)}(x) - v_{s_-})^\top| \leq \rho_1$ for $x \in (-\infty, 0)$ and $|P_{s_-}(u^{(1)}(0) - u_{s_-}, \mathcal{L}u^{(1)}(0) - v_{s_-})^\top| = \rho_1$. By Proposition 2.6(i), it is clear that $(u^{(1)}, \mathcal{L}u^{(1)})(x)$ converges to (u_{s_-}, v_{s_-}) as $x \rightarrow -\infty$.

Assuming that the convergence of \tilde{u}_m to $u^{(1)}$ does not hold for the $H^1(\mathbf{R})$ metric, we now prove (ii) and set $u_{s_+} = \lim_{x \rightarrow +\infty} u^{(1)}(x)$ ($s_+ \in \{-, +\}$). Let $b_1 > 0$ such that

$$|P_{s_+}(u^{(1)}(x) - u_{s_+}, \mathcal{L}u^{(1)}(x) - v_{s_+})^\top| < \rho/2 \text{ for all } x \geq b_1. \quad (2.11)$$

By Proposition 2.6(ii), there is $b_2 < b_1$ such that $|P_{s_+}(u^{(1)}(b_2) - u_{s_+}, \mathcal{L}u^{(1)}(b_2) - v_{s_+})^\top| = \rho_0$. Since $\rho_0 > \rho_1 > \rho$ and $\{(\tilde{u}_m, \mathcal{L}\tilde{u}_m)\}$ uniformly converges to $u^{(1)}$ on the interval $[b_2, b_1]$, it follows that $|P_{s_+}(\tilde{u}_m(b_2) - u_{s_+}, \mathcal{L}\tilde{u}_m(b_2) - v_{s_+})^\top| > \rho$ and $|P_{s_+}(\tilde{u}_m(b_1) - u_{s_+}, \mathcal{L}\tilde{u}_m(b_1) - v_{s_+})^\top| < \rho$ for m large enough. Set

$$\underline{t}_m := \max\{x \leq b_1 : |P_{s_+}(\tilde{u}_m(x) - u_{s_+}, \mathcal{L}\tilde{u}_m(x) - v_{s_+})^\top| \geq \rho\}.$$

Then $b_2 < \underline{t}_m < b_1$ and we may assert that $\underline{t}_m \rightarrow \underline{t}$ after extracting a subsequence. By uniform convergence of $\{(\tilde{u}_m, \mathcal{L}\tilde{u}_m)\}$ on compact sets, we conclude that $|P_{s_+}(u^{(1)}(\underline{t}) - u_{s_+}, \mathcal{L}u^{(1)}(\underline{t}) - v_{s_+})^\top| = \rho$ and $|P_{s_+}(u^{(1)}(x) - u_{s_+}, \mathcal{L}u^{(1)}(x) - v_{s_+})^\top| \leq \rho$ for all $x \geq \underline{t}$.

Now we define $\bar{t}_m := \sup\{x \in \mathbb{R} : |P_{s_+}(\tilde{u}_m(x) - u_{s_+}, \mathcal{L}\tilde{u}_m(x) - v_{s_+})^\top| < \rho\}$. We claim $\bar{t}_m \rightarrow \infty$. Indeed, $\{\tilde{u}_m\}$ uniformly converges to $u^{(1)}$ on $[b_1, b_1 + \bar{b}]$ for any $\bar{b} > 0$. This together with (2.11) verifies the claim. Moreover we may pick any ρ in $(0, \rho_1)$.

Set $u_m^{(2)}(x) := \tilde{u}_m(x) - u^{(1)}(x) + u_{s_+}$. Then $\{u_m^{(2)}\}$ is a Palais-Smale sequence for J , and $\liminf J(u_m^{(2)}) > 0$ follows from the fact that $\{\tilde{u}_m\}$ fails to converge for the $H^1(\mathbf{R})$ metric. Let \hat{D} be the disk of center $P_{s_+}(u_{s_+}, v_{s_+})^\top$ with radius $(\rho_1 + \rho)/2$. As in the above argument, if m is sufficiently large $P_{s_+}(u_m^{(2)}, \mathcal{L}u_m^{(2)})^\top(x)$ has to exit from \hat{D} , from which we know $\bar{t}_m < \infty$ since $\rho < (\rho_1 + \rho)/2$. Then the same argument as above shows that, along a subsequence, $\{u_m^{(2)}(\cdot - \bar{t}_m)\}$ converges in $H^1_{\text{loc}}(\mathbf{R})$ to a non-constant critical point $u^{(2)}$. By Proposition 2.6(i), $\lim_{x \rightarrow -\infty} (u^{(2)}, \mathcal{L}u^{(2)})(x) = (u_{s_+}, v_{s_+})$, since $P_{s_+}(u^{(2)}, \mathcal{L}u^{(2)})^\top(x)$

stays in the closed disk of center $P_{s_+}(u_{s_+}, v_{s_+})^\top$ with radius ρ for all $x \leq 0$. This complete the proof of (ii).

In view of the construction of $u_m^{(2)}$, $J(u^{(1)}) + \limsup J(u_m^{(2)}) = \limsup J(u_m)$. Thus $J(u^{(1)}) + J(u^{(2)}) \leq \limsup J(u_m)$ and the proof is complete. \blacksquare

A consequence of Proposition 2.7 gives a local Palais-Smale compactness property as follows:

Corollary 2.8 *Let ℓ be the number as defined in Proposition 2.6. Suppose a Palais-Smale sequence satisfies $\|u_m - u_n\|_{H^1} \leq \ell/2$ for all m, n , then it has a convergent subsequence for the $H^1(\mathbf{R})$ metric.*

Proof. For the Palais-Smale sequence $\{u_m\}$, we pick out the point x_m from each u_m as in Proposition 2.7(i). We argue indirectly by assuming that the assertion of the corollary is false. If $\{x_m\}$ has a convergent subsequence then along this subsequence, a consequence of Proposition 2.7(i) tells that $\{\tilde{u}_m(\cdot - x_m)\}$ converges in $H_{\text{loc}}^1(\mathbf{R})$ to a non-constant critical point $u^{(1)}$ of J but the convergence for the $H^1(\mathbf{R})$ metric fails. Hence $\liminf \|\tilde{u}_m - u^{(1)}\|_{H^1} > 0$. Applying Proposition 2.7(ii) yields a non-constant critical point $u^{(2)}$ of J and $\lim_{x \rightarrow +\infty} u^{(1)}(x) = \lim_{x \rightarrow -\infty} u^{(2)}(x) = u_{s_+}$. Moreover $\limsup_{m \rightarrow \infty} \|u^{(1)} - u_m\|_{H^1} = +\infty$ if $(u^{(2)}, \mathcal{L}u^{(2)})$ is a heteroclinic solution and $\limsup_{m \rightarrow \infty} \|u^{(1)} - u_m\|_{H^1} \geq \|u^{(2)} - u_{s_+}\|_{H^1}$ when $(u^{(2)}, \mathcal{L}u^{(2)})$ is homoclinic to (u_{\pm}, v_{\pm}) .

Next we turn to the case when $\{x_m\}$ has no convergent subsequence. Again $\{\tilde{u}_m(\cdot - x_m)\}$ converges in $H_{\text{loc}}^1(\mathbf{R})$ to a non-constant critical point $u^{(1)}$ of J . Moreover $\limsup_{m \rightarrow \infty} \|u^{(1)} - u_m\|_{H^1} = +\infty$ if $(u^{(1)}, \mathcal{L}u^{(1)})$ is a heteroclinic solution and $\limsup_{m \rightarrow \infty} \|u^{(1)} - u_m\|_{H^1} \geq \|u^{(1)} - u_{s_-}\|_{H^1}$ when $(u^{(1)}, \mathcal{L}u^{(1)})$ is homoclinic to (u_{\pm}, v_{\pm}) . Together with Proposition 2.6 yields $\limsup_{m \rightarrow \infty} \|u^{(1)} - u_m\|_{H^1} \geq \ell$ in all cases. This leads to a contradiction and completes the proof. \blacksquare

In the construction of multi-front solutions, the trajectories between two fronts will need to be in good control. Such trajectories are very close to one of the two stable equilibria with asymptotical behavior being dominated by the linearized equations. Recall from (1.6) that if (u, v) is a solution of (1.3)-(1.4) then $E(u', v', u, v)$ is constant along the trajectory. In particular, for any critical point u of J , the energy $E(u', \mathcal{L}u', u, \mathcal{L}u)$ is identically zero. We now state a lemma in the same spirit as Lemmas 3.1, A.1, A.2 and A.3 of [6].

Lemma 2.9 *Let $\nu \in (0, \frac{\pi}{2\omega})$ and δ be small positive numbers. There exists a small radius $\bar{r} > 0$ such that for any given points (η_1, η_2) and $(\zeta_1, \zeta_2) \in \mathbb{R}^2$, if $P_+(\eta_1, \eta_2)^\top$ and $P_+(\zeta_1, \zeta_2)^\top$ are within a distance less than \bar{r} from $P_+(u_+, v_+)^\top$,*

then the boundary value problem

$$-du'' = f(u) - v, \quad (2.12)$$

$$-v'' = u - \gamma v, \quad (2.13)$$

$$(u(0), v(0)) = (\eta_1, \eta_2), \quad (u(T), v(T)) = (\zeta_1, \zeta_2), \quad (2.14)$$

has a solution, denoted by $(\bar{U}, \bar{V})_{T, \eta_1, \eta_2, \zeta_1, \zeta_2}(\cdot)$, staying in a small neighborhood of (u_+, v_+) and it is the only one having this property. Moreover, if $\pm\lambda \pm i\omega$ are the eigenvalues of the linearization of (HS) at $(u_+, v_+, 0, 0)$ and $E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(T)$ denotes the associated energy for the solution $(\bar{U}, \bar{V})_{T, \eta_1, \eta_2, \zeta_1, \zeta_2}$, then the function $E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(\cdot)$ has the following properties:

(i) Let (r_η, θ_η) and (r_ζ, θ_ζ) be the polar coordinates of $P_+(\eta_1 - u_+, \eta_2 - v_+)^T$ and $P_+(\zeta_1 - u_+, \zeta_2 - v_+)^T$, respectively. Then, for T large enough, $E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(T)$ is positive if $\cos(\theta_\zeta - \theta_\eta - T\omega + \varphi_+) > \delta$ while $E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(T)$ is negative if $\cos(\theta_\zeta - \theta_\eta - T\omega + \varphi_+) < -\delta$. Here, φ_+ is a phase independent of the parameters.

(ii) There is a real number κ_+ and, for each $r \leq \bar{r}/2$, a smaller radius $\epsilon(r)$ proportional to r , such that, if $|(u^*(z), v^*(z)) - (u_+, v_+)| = r$, $|(\eta_1, \eta_2) - (u^*(z), v^*(z))| < \epsilon$, $|(\zeta_1, \zeta_2) - (u^*(z), v^*(z))| < \epsilon$ and $\tilde{n} \geq 1/\epsilon$ with \tilde{n} an integer, then

$$E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(\kappa_+ - 2z + 2\pi\tilde{n}/\omega - \nu) > 0,$$

$$E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(\kappa_+ - 2z + 2\pi\tilde{n}/\omega + \nu) < 0.$$

(iii) Similar assertions hold when replacing (u_+, v_+) by (u_-, v_-) , P_+ by P_- , (u^*, v^*) by (u_*, v_*) and φ_+, κ_+ by possibly different phases φ_-, κ_- .

We refer to [6] for a proof; there the existence and local uniqueness of (\bar{U}, \bar{V}) follow from Lemma A.3. The sign property (i) of the energy is a consequence of Lemma A.2, and see Lemma 3.1 for the detail. Finally, (ii) is a consequence of (i). Note that the formula for the sign of E in (i) or (ii) is correct for a suitable choice of P_+ , for other choices the sign in front of $T\omega$ has to be changed.

3 Isolated critical points

The aim of this section is to show that all critical points of J are isolated, up to translations in x . As already mentioned in the introduction, for the construction of multi-front solutions we would just need to know that the basic heteroclinic u^* is isolated up to translations, as in [6]. Showing that all critical points are isolated is considerably more difficult, however this stronger property will be needed for working out the stability analysis in the last part of this paper.

Proposition 3.1 *Any critical point u_c of J is an isolated critical point for the $H^1(\mathbf{R})$ metric, up to translations in x .*

Proposition 3.1 will follow from two facts. The first one is an alternative. Its proof is delicate and relies crucially on the results of Section 2 and the real-analyticity of the Hamiltonian:

Lemma 3.2 *Either the stable and unstable manifolds of the hyperbolic points $(u_-, v_-, 0, 0)$ and $(u_+, v_+, 0, 0)$ are bounded, or every critical point of J is isolated up to translation in x .*

Note that the four manifolds mentioned in Lemma 3.2 are all bounded (resp. unbounded) if at least one of them is bounded (resp. unbounded), since the Lagrangian $L(u', v', u, v)$ is invariant under time reversal and under the symmetry of center $(u_+/2, v_+/2)$.

The second fact is stated in the next proposition, in which we find trajectories which are either on the unstable manifold of $(u_-, v_-, 0, 0)$ or on the stable manifold of $(u_+, v_+, 0, 0)$, and that reach a point of arbitrarily large size. This shows that the stable and unstable manifolds mentioned in Lemma 3.2 are unbounded, and thus Proposition 3.1 is established:

Proposition 3.3 *For any $b \in \mathbf{R}$, there is a solution (u, v) of (1.3)-(1.4) which satisfies $u(0) = \frac{\gamma}{2}v(0) = b$ and one of the following conditions:*

(i) $\lim_{x \rightarrow -\infty} (u(x), v(x)) = (u_-, v_-)$ and

$$\int_{-\infty}^0 L(u, v) dx \geq \int_{-\infty}^0 \frac{1}{2} \left(d - \frac{1}{\gamma^2} \right) (u')^2 + \frac{1}{4} (u - u_-)^2 (u - u_+)^2 dx. \quad (3.1)$$

(ii) $\lim_{x \rightarrow \infty} (u(x), v(x)) = (u_+, v_+)$ and

$$\int_0^{\infty} L(u, v) dx \geq \int_0^{\infty} \frac{1}{2} \left(d - \frac{1}{\gamma^2} \right) (u')^2 + \frac{1}{4} (u - u_-)^2 (u - u_+)^2 dx. \quad (3.2)$$

Proof. Let $F(u) = -\int_0^u f(\xi) d\xi$. Suppose that (u, v) satisfies (1.3)-(1.4), $u(0) = \frac{\gamma}{2}v(0) = b$ and $\lim_{x \rightarrow -\infty} (u(x), v(x)) = (u_-, v_-)$. Multiplying (1.4) by v and integrating over $(-\infty, 0)$, we get

$$-vv' \Big|_{-\infty}^0 + \int_{-\infty}^0 (v')^2 + \gamma v^2 dx = \int_{-\infty}^0 uv dx.$$

By direct calculation

$$\begin{aligned}
& \int_{-\infty}^0 \left\{ \frac{d}{2}(u')^2 + \frac{1}{2}uv + F(u) \right\} dx \\
&= \int_{-\infty}^0 \left\{ \frac{d}{2}(u')^2 + \frac{1}{2}(v')^2 + \frac{1}{2}\gamma v^2 + F(u) \right\} dx - \frac{1}{2}vv' \Big|_{-\infty}^0 \\
&= \int_{-\infty}^0 \left\{ \frac{d}{2}(u')^2 + \frac{1}{2}(v')^2 + \frac{1}{2}\gamma \left(v - \frac{u}{\gamma} \right)^2 + \left(uv - \frac{u^2}{\gamma} \right) \right\} dx \\
&\quad + \int_{-\infty}^0 \left(F(u) + \frac{1}{2\gamma}u^2 \right) dx - \frac{1}{2}vv' \Big|_{-\infty}^0. \tag{3.3}
\end{aligned}$$

Next, multiplying (1.4) by $-u/\gamma$ and integrating over $[-\eta, \eta]$, we have

$$\int_{-\infty}^0 \left(uv - \frac{u^2}{\gamma} \right) dx = \int_{-\infty}^0 \frac{1}{\gamma} uv'' dx = \frac{1}{\gamma} uv' \Big|_{-\infty}^0 - \int_{-\infty}^0 \frac{1}{\gamma} u'v' dx. \tag{3.4}$$

Since $\gamma = 9(2\beta^2 - 5\beta + 2)^{-1}$, it follows from direct calculation that

$$F(u) + \frac{u^2}{2\gamma} = \frac{1}{4}(u - u_-)^2(u - u_+)^2,$$

Substituting (3.4) into (3.3) yields

$$\begin{aligned}
& \int_{-\infty}^0 \left\{ \frac{d}{2}(u')^2 + \frac{1}{2}uv + F(u) \right\} dx \\
&= \int_{-\infty}^0 \frac{d}{2}(u')^2 + \frac{1}{2}(v')^2 + \frac{1}{2}\gamma \left(v - \frac{u}{\gamma} \right)^2 - \frac{1}{\gamma}u'v' + \frac{1}{4}(u - u_-)^2(u - u_+)^2 dx \\
&\quad - \frac{1}{2}vv' \Big|_{-\infty}^0 + \frac{1}{\gamma}uv' \Big|_{-\infty}^0 \\
&= \int_{-\infty}^0 \frac{1}{2} \left(d - \frac{1}{\gamma^2} \right) (u')^2 + \frac{1}{2} \left(v' - \frac{u'}{\gamma} \right)^2 + \frac{1}{2}\gamma \left(v - \frac{u}{\gamma} \right)^2 dx \\
&\quad + \int_{-\infty}^0 \frac{1}{4}(u - u_-)^2(u - u_+)^2 dx - \frac{1}{2}vv' \Big|_{-\infty}^0 + \frac{1}{\gamma}uv' \Big|_{-\infty}^0.
\end{aligned}$$

Then (3.1) easily follows in view of the boundary conditions on u and v .

Next we prove the existence of such a solution. Let \hat{v} be a C^∞ -function such that $\hat{v}(0) = \frac{2b}{\gamma}$, $-\hat{v}_{xx}(0) + \gamma\hat{v}(0) = b$ and $\hat{v}(x) = 0$ if $x \leq -1$. For convenience in notation, we define $\mathbf{H}_w = w + H_0^1(-\infty, 0)$. Set $\hat{u} = \gamma\hat{v} - \hat{v}_{xx}$. For $w \in \mathbf{H}_{\hat{u}}$, let

$$\hat{J}(w) = \int_{-\infty}^0 \frac{1}{2} [d(w')^2 + w\hat{\mathcal{L}}w] + F(w) dx, \tag{3.5}$$

where $\hat{\mathcal{L}}w$ denotes the unique solution of

$$-v'' + \gamma v = w, \quad v \in \mathbf{H}_{\hat{u}}. \quad (3.6)$$

As in the proof of Theorem 1.1 of [18], we pick a minimizing sequence $u_m \in \mathbf{H}_{\hat{u}}$ which converges to u in H_{loc}^1 . Moreover $u \in \mathbf{H}_{\hat{u}}$ and

$$\hat{J}(u) = \inf_{w \in \mathbf{H}_{\hat{u}}} \hat{J}(w).$$

Letting $v = \hat{\mathcal{L}}u$, we see that (u, v) satisfies (1.3)-(1.4), $u(0) = \frac{\gamma}{2}v(0) = b$ and $\lim_{x \rightarrow -\infty} (u(x), v(x)) = (u_-, v_-)$. This proves (i).

The above argument also shows that either $\lim_{x \rightarrow -\infty} (u(x), v(x)) = (u_-, v_-)$ or $\lim_{x \rightarrow -\infty} (u(x), v(x)) = (u_+, v_+)$. In the latter case, we replace $(u(x), v(x))$ by $(u(-x), v(-x))$ to establish (ii), since the proof of (3.2) is not different from that of (3.1). \blacksquare

Proof of Lemma 3.2 and Proposition 3.1. We actually just need to prove Lemma 3.2, since Proposition 3.1 then immediately follows.

Let u_c be a non-constant critical point of J and $\lim_{z \rightarrow -\infty} u_c(z) = u_{s_-}$, $\lim_{z \rightarrow +\infty} u_c(z) = u_{s_+}$, $s_-, s_+ \in \{-, +\}$. Note that $\mathbf{z}_{s_-} = (u_{s_-}, v_{s_-}, 0, 0)$ is a hyperbolic equilibrium of a first order Hamiltonian system (HS) with Hamiltonian function $H(u, v, p, q)$, which is associated to the second order Lagrangian system (1.3)-(1.4). Here, p denotes the momentum conjugate to u and q the momentum conjugate to v . With the Hamiltonian H being real analytic, the local unstable manifold $W_{loc}^u(\mathbf{z}_{s_-})$ of \mathbf{z}_{s_-} is a real analytic submanifold of \mathbb{R}^4 . Moreover the unstable space of the linearization of (HS) at \mathbf{z}_{s_-} is the graph of a linear map from \mathbb{R}^2 to itself, hence $W_{loc}^u(\mathbf{z}_{s_-})$ is the graph of a real analytic map φ from a small neighborhood \mathcal{U}_{s_-} of (u_{s_-}, v_{s_-}) into \mathbb{R}^2 , and $\varphi(u_{s_-}, v_{s_-}) = (0, 0)$. Similarly, $\mathbf{z}_{s_+} = (u_{s_+}, v_{s_+}, 0, 0)$ is hyperbolic and its local stable manifold $W_{loc}^s(\mathbf{z}_{s_+})$ is the graph of a real analytic map $\psi = (\psi_p, \psi_q)$ from a small neighborhood \mathcal{U}_{s_+} of (u_{s_+}, v_{s_+}) into \mathbb{R}^2 , and $\psi(u_{s_+}, v_{s_+}) = (0, 0)$.

Recall that we use the same ρ_1 as introduced in Proposition 2.7. Also, in Lemma 2.9, we choose $\delta = 1/2$ with the associated $\bar{r} > 0$. To employ Proposition 2.7, we pick $0 < \rho \leq \min\{\rho_1/2, \bar{r}\}$ so that $|P_{s_{\pm}}(u - u_{s_{\pm}}, v - v_{s_{\pm}})| \leq \rho$ implies $(u, v) \in \mathcal{U}_{s_{\pm}}$. Given $\theta \in \mathbb{R}$, we denote $(\mathbf{u}, \mathbf{v})_{\theta}^T := P_{s_-}^{-1}(\rho \cos \theta, \rho \sin \theta)^T + (u_{s_-}, v_{s_-})^T$ and $\mathbf{z}_{\theta} := ((\mathbf{u}, \mathbf{v})_{\theta}, \varphi((u, v)_{\theta}))$.

Let $v_c = \mathcal{L}u_c$. Remembering that $\rho_1 < \rho_0$, we may take x_1, x_2 associated to u_c as in Proposition 2.6(ii). Let $\Phi = (U, V, \Pi_U, \Pi_V)$ be the flow of (HS) at time $T = x_2 - x_1 + 1$ (both η and T depend on the critical point u_c under consideration). Then $P_{s_-}(u_c(x_1) - u_{s_-}, v_c(x_1) - v_{s_-})^T = (\rho \cos \theta_c, \rho \sin \theta_c)^T$ for some angle θ_c , and $P_{s_+}(U(z_{\theta_c}), V(z_{\theta_c}))^T$ lies in the open disk D_+ of center $P_{s_+}(u_{s_+}, v_{s_+})^T$ with radius ρ .

From now on in this proof, *let us assume that u_c is not isolated up to translations*, for the H^1 metric in the set of critical points of J . Let u be a critical point of J sufficiently close to u_c in the $H^1(\mathbf{R})$ topology but not equal to a translate of u_c . Then the trajectory parametrized by $(u, \mathcal{L}u, du', -\mathcal{L}u')$ in the phase space must contain a point \mathbf{z}_θ with θ arbitrarily close, but not equal, to the angle θ_c . Moreover, since $P_{s_+}(U(z_{\theta_c}), V(z_{\theta_c}))^T$ lies in the open disk D_+ of center $P_{s_+}(u_{s_+}, v_{s_+})^T$ with radius ρ , the same is true for $P_{s_+}(U(z_\theta), V(z_\theta))^T$ if u is close enough to u_c in $H^1(\mathbf{R})$ topology. As a consequence, θ is a zero of each of the functions $\chi_1(\theta) = \Pi_U(\mathbf{z}_\theta) - \psi_p(U(\mathbf{z}_\theta), V(\mathbf{z}_\theta))$ and $\chi_2(\theta) = \Pi_V(\mathbf{z}_\theta) - \psi_q(U(\mathbf{z}_\theta), V(\mathbf{z}_\theta))$.

The above argument shows that θ_c is not an isolated zero of the real-analytic functions χ_1 and χ_2 , which are defined in a small open interval \mathcal{I} containing θ_c . So these functions are identically zero on \mathcal{I} , which means that the flow η sends all the points \mathbf{z}_θ near \mathbf{z}_{θ_c} to points of $W_{loc}^s(\mathbf{z}_{s_+})$.

To each θ in \mathcal{I} , we associate the solution Z_θ of the Hamiltonian system with initial value $Z_\theta(0) = \mathbf{z}_\theta$. Let $u_\theta(x)$ be the first component of the vector $Z_\theta(x)$ and let $j(\theta) := J(u_\theta)$. Then it is not hard to see that the function $\theta \in \mathcal{I} \rightarrow u_\theta$ is continuous, and even of class C^1 , for the $H^1(\mathbb{R})$ metric on the target space. Thus, by the chain rule, $\frac{d}{d\theta}j(\theta) = 0$, and $j(\theta) = J(\theta_c)$. Moreover, all the angles in \mathcal{I} correspond to critical points u_θ that are not isolated, up to translation, in $H^1(\mathbf{R})$ topology.

Denote by Θ the set of angles θ associated to all non-isolated critical points J which converge to u_{s_-} as $x \rightarrow -\infty$. We have shown that Θ is an open subset of \mathbb{R} , and by assumption it contains θ_c . Clearly Θ is 2π -periodic, and the above argument shows that j is a 2π -periodic and locally constant function on Θ . Let \mathcal{J} be the maximal open interval in Θ containing θ_c and let $\hat{\theta} := \sup \mathcal{J}$. If $\hat{\theta} = +\infty$, then $\mathcal{J} = \Theta = \mathbb{R}$ and j is constant on \mathbb{R} , so Corollary 2.5 gives an L^∞ estimate on *all* the solutions Z_θ . Moreover these solutions converge to z_{s_\pm} as $x \rightarrow \pm\infty$, and $W^u(z_{s_-}) = W^s(z_{s_+}) = \bigcup_{(\theta, x) \in [0, 2\pi) \times \mathbb{R}} Z_\theta(x)$ is bounded. So, to complete the proof of Lemma 3.2, we just need to study the remaining case when $\hat{\theta}$ is a finite number.

Then, as $\theta \rightarrow \hat{\theta}$, $\theta \in \mathcal{J}$, Z_θ converges in the C_{loc}^1 topology to $Z_{\hat{\theta}}$, by continuous dependence of the solutions of the Hamiltonian system with respect to initial data. Since $\hat{\theta} \notin \mathcal{J}$, the convergence of u_θ to $u_{\hat{\theta}}$ does not hold for the $H^1(\mathbb{R})$ metric. As a consequence, from Proposition 2.7(iii), $u_{\hat{\theta}}$ is a critical point of j satisfying the estimate $\ell \leq j(u_{\hat{\theta}}) \leq j(\theta_c) - \ell$. From now on, set $(u_{s_1}, v_{s_1}) = \lim_{x \rightarrow +\infty} (u_{\hat{\theta}}, \mathcal{L}u_{\hat{\theta}})(x)$, $s_1 \in \{-, +\}$. Let $\underline{t} \in \mathbf{R}$ be such that for all $x > \underline{t}$, $P_{s_1}(u_{\hat{\theta}}, \mathcal{L}u_{\hat{\theta}})^T(x)$ lies in the open disk D_1 of center $P_{s_1}(u_{s_1}, v_{s_1})^T$ with radius ρ , while $P_{s_1}(u_{\hat{\theta}}, \mathcal{L}u_{\hat{\theta}})^T(\underline{t})$ sits on the boundary of D_1 . Then, for θ close enough to $\hat{\theta}$, there are two numbers $\underline{t}_\theta < \bar{t}_\theta$ such that for all $x \in (\underline{t}_\theta, \bar{t}_\theta)$, $P_{s_1}(u_\theta, \mathcal{L}u_\theta)^T(x)$ lies in the open disk D_1 , while $P_{s_1}(u_\theta, \mathcal{L}u_\theta)^T(\underline{t}_\theta)$

and $P_{s_1}(u_\theta, \mathcal{L}u_\theta)^\top(\bar{t}_\theta)$ both sit on the boundary of D_1 . Moreover $\lim_{\theta \rightarrow \hat{\theta}} \underline{t}_\theta = \underline{t}$ and $\lim_{\theta \rightarrow \hat{\theta}} \bar{t}_\theta = +\infty$. The existence of \underline{t}_θ is due to the convergence of Z_θ in C_{loc}^1 , and the existence of \bar{t}_θ is due to the lack of convergence in the $H^1(\mathbb{R})$ metric.

Now, it follows from the Harman-Grobman theorem that the distance between $Z_\theta(\underline{t}_\theta)$ and $W_{\text{loc}}^s(\mathbf{z}_{s_1})$ tends to zero as $\theta \rightarrow \hat{\theta}$, since $(\bar{t}_\theta - \underline{t}_\theta) \rightarrow \infty$. As a consequence, remembering that (2.8) holds on $W_{\text{loc}}^s(\mathbf{z}_{s_1})$, we obtain a similar estimate when θ is sufficiently close to $\hat{\theta}$:

$$\frac{d}{dx} |P_{s_1}(u_\theta(\underline{t}_\theta) - u_{s_1}, \mathcal{L}u_\theta(\underline{t}_\theta) - v_{s_1})^\top| \leq -\frac{\lambda}{4} |P_{s_1}(u_\theta(\underline{t}_\theta) - u_{s_1}, \mathcal{L}u_\theta(\underline{t}_\theta) - v_{s_1})^\top|.$$

Such an inequality, together with the implicit function theorem gives the continuity, and even the real-analyticity, of \underline{t}_θ as a function of θ on a small interval $(\hat{\theta} - \varepsilon, \hat{\theta}) \subset \mathcal{J}$. The continuity of \bar{t}_θ can be proved in the same manner, using an inequality analogous to (2.7).

So we can define two continuous functions $\underline{\alpha}, \bar{\alpha} : (\hat{\theta} - \varepsilon, \hat{\theta}) \rightarrow \mathbb{R}$ which satisfy

$$\begin{aligned} P_{s_1}(u_\theta - u_{s_1}, \mathcal{L}u_\theta - v_{s_1})^\top(\underline{t}_\theta) &= \rho(\cos \underline{\alpha}(\theta), \sin \underline{\alpha}(\theta))^\top, \\ P_{s_1}(u_\theta - u_{s_1}, \mathcal{L}u_\theta - v_{s_1})^\top(\bar{t}_\theta) &= \rho(\cos \bar{\alpha}(\theta), \sin \bar{\alpha}(\theta))^\top. \end{aligned}$$

Since $\lim_{\theta \rightarrow \hat{\theta}} (u_\theta, \mathcal{L}u_\theta)(\underline{t}_\theta) = (u_{\hat{\theta}}, \mathcal{L}u_{\hat{\theta}})(\underline{t})$, $\underline{\alpha}(\theta)$ has a finite limit $\underline{\alpha}(\hat{\theta})$ as $\theta \rightarrow \hat{\theta}$. In order to study the limit of $\bar{\alpha}$, we are going to use Lemma 2.9, remembering that we have fixed $\delta = 1/2$ and chosen $\rho \leq \bar{r}$. Since u_θ is a critical point of J , its energy is zero, so $E_{u_\theta(\underline{t}_\theta), \mathcal{L}u_\theta(\underline{t}_\theta), u_\theta(\bar{t}_\theta), \mathcal{L}u_\theta(\bar{t}_\theta)}(\bar{t}_\theta - \underline{t}_\theta) = 0$. Then, from (ii) in Lemma 2.9, we find a phase φ_1 independent of the parameters, such that, for θ close enough to $\hat{\theta}$,

$$\cos(\bar{\alpha}(\theta) - \underline{\alpha}(\theta) - (\bar{t}_\theta - \underline{t}_\theta)\omega + \varphi_1) \in [-1/2, 1/2].$$

But $(\bar{t}_\theta - \underline{t}_\theta) \rightarrow \infty$ and $\underline{\alpha}(\theta) \rightarrow \underline{\alpha}(\hat{\theta})$ as $\theta \rightarrow \hat{\theta}$. Moreover $\bar{\alpha}$ depends continuously on θ . So we must have $\lim_{\theta \rightarrow \hat{\theta}} \bar{\alpha}(\theta) = \infty$. As a consequence, given any $\alpha \in [0, 2\pi)$, there is a sequence $\{\theta_\alpha^{(n)}\}$ such that $\bar{\alpha}(\theta_\alpha^{(n)}) = \alpha + 2n\pi$ for all n large enough, and $\theta_\alpha^{(n)} \rightarrow \hat{\theta}$ as $n \rightarrow \infty$. Passing to a subsequence if necessary, it follows from Proposition 2.7 that $\{u_{\theta_\alpha^{(n)}}(\cdot - \bar{t}_{\theta_\alpha^{(n)}})\}$ converges for the local H^1 topology to a non-constant critical point of J , denoted by $u_\alpha^{(2)}$, with $P_{s_1}(u_\alpha^{(2)} - u_{s_1}, \mathcal{L}u_\alpha^{(2)} - v_{s_1})^\top(0) = \rho(\cos \alpha, \sin \alpha)^\top$ and $\ell \leq J(u_\alpha^{(2)}) \leq J(u_c) - \ell$. Then Corollary 2.5 gives an L^∞ estimate on $Z_\alpha^{(2)} = (u_\alpha^{(2)}, \mathcal{L}u_\alpha^{(2)}, d(u_\alpha^{(2)})', -\mathcal{L}(u_\alpha^{(2)})')$, which is independent of α . So $W^u(z_{s_1}) = \bigcup_{(\alpha, x) \in [0, 2\pi) \times \mathbb{R}} Z_\alpha^{(2)}(x)$ is bounded. This completes the proof of Lemma 3.2, so Proposition 3.1 is true. \blacksquare

Recall that the Lagrangian $L(u', v', u, v)$ is autonomous and u_* is a non-constant solution. By taking a small translation in x if necessary, from now on

we always assume that $u'_*(0) \neq 0$. This condition also holds for u^* , since it is the reverse orbit of u_* .

As a consequence of Proposition 3.1, we obtain the following result.

Corollary 3.4 *There exist $h_0, \sigma_0 > 0$ and, for any $0 < h < h_0$, a radius $\bar{\sigma}(h) > 0$ with $\lim_{h \rightarrow 0} \bar{\sigma}(h) = 0$, such that the local sublevel set*

$$\mathcal{V}_h = \{u \in H_{u^*} : u(0) = u^*(0), \|u - u^*\|_{H^1(\mathbb{R})} \leq \sigma_0 \text{ and } J(u) \leq J(u^*) + h\}$$

satisfies the following property:

$$u \in \mathcal{V}_h \Rightarrow \|u - u^*\|_{H^1(\mathbb{R})} < \bar{\sigma}(h).$$

Proof. From Corollary 2.8, for σ_1 small enough, the functional J satisfies the Palais-Smale condition on the closed ball of center u_* with radius σ_1 of $H^1(\mathbb{R})$ -norm. Since $u'_*(0) \neq 0$, by Proposition 3.1 there exists $\sigma_0 \leq \sigma_1$ such that u_* is the unique critical point of J on the closed ball of center u_* with radius σ_0 , so does u^* . Consequently in this closed ball, Theorem 1.1 tells that u^* is the unique minimizer of J , which completes the proof. \blacksquare

Consider a sufficiently large number z and define

$$\mathcal{V}_{h,z} := \{u \in H^1(-z, z) : u \equiv \hat{u} \text{ on } [-z, z] \text{ for some } \hat{u} \in \mathcal{V}_h\}. \quad (3.7)$$

Now, for $u \in \mathcal{V}_{h,z}$ with h small and z large, the functional J is C^2 and strictly convex on

$$\mathcal{C}_u := \{\tilde{u} \in H_{u^*} : \tilde{u} \equiv u \text{ on } [-z, z] \text{ and } \|\tilde{u} - u^*\|_{H^1(\mathbb{R})} \leq \bar{\sigma}(h)\},$$

which is a closed, bounded and convex subset of H_{u^*} . Indeed, if $\tilde{u} \in \mathcal{C}_u$, any other element of \mathcal{C}_u near \tilde{u} is of the form $\tilde{u} + w$ with $\|w\|_{H^1(\mathbb{R})}$ small and $w \equiv 0$ on $[-z, z]$. Thus a direct calculation gives

$$D^2 J(\tilde{u}) \cdot w \cdot w = \int_{\mathbb{R}} \{d(w')^2 - f'(\tilde{u})w^2 + (\mathcal{L}w')^2 + \gamma(\mathcal{L}w)^2\} \geq \hat{k}\|w\|_{H^1(\mathbb{R})}$$

for some $\hat{k} > 0$. Moreover, if a function $\tilde{u} \in \mathcal{C}_u$ satisfying $\|\tilde{u} - u^*\|_{H^1(\mathbb{R})} = \bar{\sigma}(h)$, then $J(\tilde{u}) > J(u^*) + h \geq \min_{\mathcal{C}_u} J$. So J has a minimizer, denoted by $b(u)$, which does not saturate the constraint $\|\tilde{u} - u^*\|_{H^1(\mathbb{R})} \leq \bar{\sigma}(h)$; that is, $(b(u), \mathcal{L}b(u))$ solves the system (1.3)-(1.4) outside the interval $[-z, z]$, and by the implicit function theorem, b is well-defined as a smooth function of u in H^1 topology. This provides a Lyapunov-Schmidt reduction $J_z = J \circ b$ of J defined on $\mathcal{V}_{h,z}$, and the following corollary holds.

Corollary 3.5 *For h_0 small enough, there is $z_0 > 0$ such that if $h \in (0, h_0)$ and $z > z_0$, then*

$$\rho(h) := \inf\{\|J'_z(u)\|_{(H^1(-z,z))^*} : u \in \mathcal{V}_{z,h} \text{ and } J_z(u) = J(u^*) + h\} > 0.$$

Proof. Suppose that the assertion of the corollary is false. Then $\rho(h) = 0$ for small h and Corollary 2.8 implies that a Palais-Smale sequence converges to a critical point of J in a small ball of center u^* at the critical level $J(u^*) + h$. Hence there would exist critical points of J in any small neighborhood of u^* . This is contrary to Proposition 3.1. \blacksquare

4 Construction of multi-front waves

With $u'_*(0) \neq 0$, we now get into details about how to construct the multi-front solutions. Let $h > 0$ be small and $D > 0$ large (to be determined later as depending on h). Pick an arbitrary finite interval of integers $[1, N]$ and an arbitrary finite sequence of positive integers $\mathbf{n} = (n_i)_{1 \leq i \leq N}$ such that $n_i \geq D$ for all i . Take $z > 0$ large enough so that $(u^*(-z) - u_-)^2 + (v^*(-z) - v_-)^2 \leq \bar{r}^2$ and $(u^*(z) - u_+)^2 + (v^*(z) - v_+)^2 \leq \bar{r}^2$, where \bar{r} is the small radius considered in Lemma 2.9.

Recall $\mathcal{V}_{h,z}$ from (3.7) and introduce a smooth map $b_{\mathbf{n}}$ from $(\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N$ into $H^1_{loc}(\mathbb{R}, \mathbb{R})$, defined as follows. For $(\mathbf{u}, \mathbf{x}) = ((u_i)_{0 \leq i \leq N}, (x_i)_{1 \leq i \leq N}) \in (\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N$, we associate a unique function $u = b_{\mathbf{n}}(\mathbf{u}, \mathbf{x})$, which satisfies the following conditions:

- (S₁) $\forall i \in [0, N] \cap 2\mathbb{Z}$, $u \equiv u_i(\cdot - C_i)$ on $(C_i - z, C_i + z)$,
- (S₂) $\forall i \in [0, N] \cap (2\mathbb{Z} + 1)$, $u \equiv u_i(C_i - \cdot)$ on $(C_i - z, C_i + z)$,
- (S₃) $\|u - u_-\|_{H^1(-\infty, -z)} \leq K\bar{r}$,
- (S₄) $\forall i \in [0, N - 1] \cap 2\mathbb{Z}$, $\|u - u_+\|_{H^1(C_i+z, C_{i+1}-z)} \leq K\bar{r}$,
- (S₅) $\forall i \in [0, N - 1] \cap (2\mathbb{Z} + 1)$, $\|u - u_-\|_{H^1(C_i+z, C_{i+1}-z)} \leq K\bar{r}$,
- (S₆) $\|u - u_{\pm}\|_{H^1(C_N+z, \infty)} \leq K\bar{r}$, where $u_{\pm} = u_+$ for N even, $u_{\pm} = u_-$ for N odd,
- (S₇) $C_0 = 0$, $C_{i+1} = C_i + X_i$ ($0 \leq i \leq N - 1$),
- (S₈) $X_{2j} = x_{2j} + \kappa_+ + \frac{2\pi n_{2j}}{\omega}$,
- (S₉) $X_{2j+1} = x_{2j+1} + \kappa_- + \frac{2\pi n_{2j+1}}{\omega}$,
- (S₁₀) $(u, \mathcal{L}u)$ satisfies (1.3)-(1.4) on each of the intervals $(-\infty, -z]$, $[C_i + z, C_{i+1} - z]$, $[C_N + z, +\infty)$.

Choosing \bar{r} small enough and a large K not depending on \bar{r} , we claim that conditions (S_1) - (S_{10}) determine u in a unique way, and explain why the corresponding function $b_{\mathbf{n}}(\mathbf{u}, \mathbf{x})$ is smooth. Observe that one can define the set $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$ consisting of all functions u satisfying conditions (S_1) - (S_6) . This set is convex, bounded, closed in the H^1 topology. Moreover, the controls (S_3) - (S_6) on u imply the strict convexity on J restricted to $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$. Indeed, if $u \in \mathcal{U}_{(\mathbf{u}, \mathbf{x})}$, any other element of $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$ near u is of the form $u + w$ with $\|w\|_{H^1(\mathbb{R})}$ small and $w \equiv 0$ on $\bigcup_{1 \leq i \leq N} [C_i - z, C_i + z]$, and then a direct calculation gives

$$D^2 J(u) \cdot w \cdot w = \int_{\mathbb{R}} \{d(w')^2 - f'(u)w^2 + (\mathcal{L}w')^2 + \gamma(\mathcal{L}w)^2\} \geq \bar{k}\|w\|_{H^1(\mathbb{R})}$$

for some $\bar{k} > 0$, exactly as in the proof of Corollary 3.4.

So J has a unique minimizer in $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$. Moreover for K large enough, if a function u belongs to $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$, and saturates at least one of the constraints (S_3) - (S_6) then $J(u) > \min_{\mathcal{U}_{(\mathbf{u}, \mathbf{x})}} J$. In conclusion, the minimizer does not saturate any of the constraints, so it is the only solution of (S_{10}) in $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$ and the implicit function theorem gives a smooth function $b_{\mathbf{n}}$ of (\mathbf{u}, \mathbf{x}) in the H^1 topology.

Up to this stage, a Lyapunov-Schmidt reduction has been performed, and the next task is to minimize the reduced functional $\mathbf{J} = J \circ b_{\mathbf{n}}$. The existence of a minimizer is easily established. Indeed, the set \mathcal{V}_h is a bounded, closed sublevel set of the weakly lower semicontinuous functional J , thus it is weakly compact in $H^1(-z, z)$. By the weak lower semicontinuity of \mathbf{J} , there exists a minimizer $(\bar{\mathbf{u}}, \bar{\mathbf{x}})$ in the weakly compact set $(\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N$.

Lemma 4.1 *Given z large, h small and choose D large enough, if $n_i \geq D$ for every i . then $\hat{u}_{\mathbf{n}} := b_{\mathbf{n}}(\bar{\mathbf{u}}, \bar{\mathbf{x}})$ is a local minimizer of J .*

To prove Lemma 4.1, we introduce the set $\mathcal{O} = \bigcup_{(\mathbf{u}, \mathbf{x}) \in (\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N} \mathcal{U}_{(\mathbf{u}, \mathbf{x})}$ consisting of functions u satisfying (S_1) - (S_6) for some $(\mathbf{u}, \mathbf{x}) \in (\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N$. The next lemma shows that \mathcal{O} contains a small ball in $H^1(\mathbb{R})$ with center at $\hat{u}_{\mathbf{n}}$. Clearly $\hat{u}_{\mathbf{n}}$ minimizes J on \mathcal{O} , by virtue of the construction used in the variational argument, and thus Lemma 4.1 is an immediate consequence.

Lemma 4.2 *Given z large and h small. Suppose that D is chosen large enough and $n_i \geq D$ for all i . If $(\bar{\mathbf{u}}, \bar{\mathbf{x}}) = ((\bar{u}_i)_{0 \leq i \leq N}, (\bar{x}_i)_{1 \leq i \leq N})$ is a minimizer of \mathbf{J} in $(\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N$ then*

- (i) $J_z(\bar{u}_i) < J(u^*) + h$ for all $0 \leq i \leq N$,
- (ii) $-\nu < \bar{x}_i < \nu$ for all $1 \leq i \leq N$.

The following lemma will be used to prove Lemma 4.2.

Lemma 4.3 *Let z and h be given as above, both are not depending on N . For any $\alpha > 0$ there exists $\bar{D}(\alpha)$, not depending on N , such that if $n_i \geq \bar{D}(\alpha) \forall 1 \leq i \leq N$ then*

$$\|J'_z(u_i) - \partial_{u_i} \mathbf{J}\|_{(H^1(-z,z))^*} < \alpha, \quad \forall (\mathbf{u}, \mathbf{x}) \in (\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N, \quad 0 \leq i \leq N.$$

The proof of Lemma 4.3 is standard (see e.g. [6]). We omit it.

Proof of Lemma 4.2. We argue indirectly. Suppose that $J_z(\bar{u}_l) = J(u^*) + h$ for some $l \in (0, N)$, applying Lemma 4.3 yields

$$\langle \partial_{u_l} \mathbf{J}(\bar{\mathbf{u}}, \bar{\mathbf{x}}), \nabla_{H^1(-z,z)} J_z(\bar{u}_l) \rangle \geq \frac{\rho(h)}{2},$$

with $\rho(h)$ given by Corollary 3.5. Then moving u_l slightly in the direction of $-\nabla_{H^1(-z,z)} J_z(\bar{u}_l)$ would decrease $\mathbf{J}(\mathbf{u}, \mathbf{x})$, which contradicts the minimality of $\mathbf{J}(\bar{\mathbf{u}}, \bar{\mathbf{x}})$. The proof of (i) is complete.

We next apply Lemma 2.9 to prove (ii). Fix z large and h small enough so that

$$\begin{aligned} r &:= |(u^*(z), v^*(z)) - (u_+, v_+)| \leq \bar{r}/2, \\ |(\bar{u}_i(\pm z), \mathcal{L}\bar{u}_i(\pm z)) - (u^*(\pm z), v^*(\pm z))| &< \epsilon \end{aligned}$$

with ϵ as in Lemma 2.9 and $n_i \geq 1/\epsilon$ being imposed. Suppose $\bar{x}_l = -\nu$ for some $l \in (1, N-1)$, it follows from Lemma 2.9 that

$$\partial_{x_l} \mathbf{J}(\bar{\mathbf{u}}, \bar{\mathbf{x}}) = -E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(\kappa_+ - 2z + 2\pi n/\omega - \nu) < 0,$$

where $\eta_1 = \bar{u}_l(C_l + z)$, $\eta_2 = \mathcal{L}\bar{u}_l(C_l + z)$, $\zeta_1 = \bar{u}_{l+1}(C_{l+1} - z)$, $\zeta_2 = \mathcal{L}\bar{u}_{l+1}(C_{l+1} - z)$. Then increasing x_l slightly would make \mathbf{J} small, which again contradicts the minimality of $\mathbf{J}(\bar{\mathbf{u}}, \bar{\mathbf{x}})$. Likewise, if $\bar{x}_l = \nu$ we could decrease \mathbf{J} by slightly decreasing x_l . Now the proof of Lemma 4.2 is complete.

We are now ready to prove the existence result of multi-front solutions, stated in Theorem 1.2. The stability of such solutions will be investigated in the next section.

Proof of Theorem 1.2. Take \bar{r} small enough so that $K\bar{r} < \sigma$, the small radius required as in the statement. Pick z large and h small enough so that the small number $\bar{\sigma}(h)$, defined in Corollary 3.4, is less than σ , and this then enables us to apply Lemma 4.1. To complete the existence proof, simply assign D_σ to be the number D stated in Lemma 4.1.

5 Stability

In this section a Lyapunov functional will be introduced to prove Theorem 1.4. This Lyapunov functional is well-defined for $u \in a_u + H^1(\mathbb{R})$ and $v \in a_v + H^1(\mathbb{R})$. From in the proof of Theorem 1.2, we know that $\hat{u}_{\mathbf{n}}$ is a local minimizer of J and $\hat{v}_{\mathbf{n}} = \mathcal{L}\hat{u}_{\mathbf{n}}$. To prove Theorem 1.4, we define

$$\mathcal{E}(u, v) := J(u) + \frac{\gamma}{2(1 + \hat{\delta})} \|v - \mathcal{L}(u)\|^2 \quad (5.1)$$

for $u \in \hat{u}_{\mathbf{n}} + H^1(\mathbb{R})$ and $v \in \hat{v}_{\mathbf{n}} + H^1(\mathbb{R})$. This together with Proposition 3.1 shows that $(\hat{u}_{\mathbf{n}}, \hat{v}_{\mathbf{n}})$ is a local minimizer of \mathcal{E} for the natural topology of the affine space $(\hat{u}_{\mathbf{n}} + H^1(\mathbb{R})) \times (\hat{v}_{\mathbf{n}} + L^2(\mathbb{R}))$, and it is an isolated critical point of \mathcal{E} up to translation in spatial variable. Also, \mathcal{E} satisfies the Palais-Smale condition in a small neighborhood of $(\hat{u}_{\mathbf{n}}, \hat{v}_{\mathbf{n}})$.

Consider the Cauchy problem:

$$u_t - du_{xx} = f(u) - v, \quad (5.2)$$

$$\tau v_t - v_{xx} = u - \gamma v, \quad (5.3)$$

with the initial data in the function space $Y = Y_u \times Y_v$. Here $Y_u = (\hat{u}_{\mathbf{n}} + H^1(\mathbb{R})) \cap C_b(\mathbb{R})$, $Y_v = \hat{v}_{\mathbf{n}} + H^1(\mathbb{R}) \cap C_b(\mathbb{R})$ and $C_b(\mathbb{R})$ is the set of bounded uniformly continuous functions on \mathbb{R} . For $(u_1, v_1), (u_2, v_2) \in Y$, define $\|(u_1, v_1) - (u_2, v_2)\|_Y = \|u_1 - u_2\|_{H^1(\mathbb{R})} + \|u_1 - u_2\|_{L^\infty(\mathbb{R})} + \|v_1 - v_2\|_{H^1(\mathbb{R})} + \|v_1 - v_2\|_{L^\infty(\mathbb{R})}$. Then with slightly modification, Theorem 2.1 of [41] (see also [46], Theorem 14.2) shows that, for given initial data in Y , (5.2)-(5.3) has a unique solution $(u(\cdot, t), v(\cdot, t))$. This is the result for the local existence of solutions. Indeed this solution exists globally in time and $(u(\cdot, t), v(\cdot, t)) \in C([0, \infty), Y)$. For the proofs, we refer to [41, 46] for the detail, including using the method of contracting rectangles. Similar results hold if we work on different function space; for instance, take $Y_u = \hat{u}_{\mathbf{n}} + C_0^2(\mathbb{R})$ and $Y_v = \hat{v}_{\mathbf{n}} + C_0^2(\mathbb{R})$ with the natural topology inherited from $C_0^2(\mathbb{R})$. Here $C_0^2(\mathbb{R})$ is the set of twice continuously differentiable functions with compact support.

The above results will be used in the proofs of Theorem 1.4 and Theorem 1.5. The next proposition shows that $\mathcal{E}(u, v)$ is a Lyapunov functional for the evolution flow generated by (1.1)-(1.2).

Proposition 5.1 *Assume that $0 < \tau < \gamma^2$. Let $\hat{\delta} > 0$ and satisfy $1 + \hat{\delta}/2 < \gamma^2/\tau$. Then for any smooth solution $(u(x, t), v(x, t))$ of (1.1)-(1.2),*

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(u(\cdot, t), v(\cdot, t)) &\leq -\frac{\hat{\delta}}{2(1 + \hat{\delta})} \|u_t\|^2 \\ &\quad - \frac{1}{1 + \hat{\delta}} \left(\frac{\gamma^2}{\tau} - 1 - \frac{\hat{\delta}}{2} \right) \|v - \mathcal{L}(u)\|^2 - \frac{\gamma}{(1 + \hat{\delta})\tau} \left\| \frac{\partial}{\partial x} (v - \mathcal{L}(u)) \right\|^2. \end{aligned}$$

Here $\|\cdot\|$ is the norm of $L^2(\mathbb{R})$.

Proof. Let $w = v - \mathcal{L}(u)$. If $u \in Y_u$ and $v \in Y_v$ then $v - \mathcal{L}(u) \in H^1(\mathbb{R})$. By Theorem 2.3 of [41] (see also [46], Theorem 14.3) and density argument, it suffices to treat the case $Y_u = \hat{u}_{\mathbf{n}} + C_0^2(\mathbb{R})$ and $Y_v = \hat{v}_{\mathbf{n}} + C_0^k(\mathbb{R})$. Clearly (1.1)-(1.2) is equivalent to

$$u_t = du_{xx} + u(u - \beta)(1 - u) - \mathcal{L}(u) - w, \quad (5.4)$$

$$\tau(w_t + \mathcal{L}(u_t)) = w_{xx} - \gamma w. \quad (5.5)$$

In terms of (u, w) , we rewrite (5.1) as

$$\mathcal{E}_1(u, w) := J(u) + \frac{\gamma}{2(1 + \hat{\delta})} \|w\|^2. \quad (5.6)$$

Let $(u(x, t), w(x, t))$ be a solution of (5.4)-(5.5). Multiplying (5.5) by w and integrating by parts, we obtain

$$\tau[(w, w_t)_{L^2} + (w, \mathcal{L}(u_t))_{L^2}] + \|w_x\|^2 + \gamma \|w\|^2 = 0. \quad (5.7)$$

Recall that \mathcal{L} is a self-adjoint operator from $L^2(\mathbb{R})$ to itself. Hence

$$(w, \mathcal{L}(u_t))_{L^2} = (\mathcal{L}(w), u_t)_{L^2}. \quad (5.8)$$

By making use of (5.7)-(5.8), a direct calculation gives

$$\begin{aligned} & \frac{d}{dt} \mathcal{E}_1(u(\cdot, t), w(\cdot, t)) \\ &= - \int_{-\infty}^{\infty} (du_{xx} + u(u - \beta)(1 - u) - \mathcal{L}(u)) u_t dx + \frac{\gamma}{1 + \hat{\delta}} (w, w_t)_{L^2} \\ &= - \|u_t\|^2 - (w, u_t)_{L^2} - \frac{\gamma}{1 + \hat{\delta}} \left((w, \mathcal{L}(u_t))_{L^2} + \frac{1}{\tau} (\|w_x\|^2 + \gamma \|w\|^2) \right) \\ &\leq -\frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|w\|^2 - \frac{\gamma}{(1 + \hat{\delta})} (\mathcal{L}(w), u_t)_{L^2} \\ &\quad - \frac{\gamma^2}{(1 + \hat{\delta})\tau} \|w\|^2 - \frac{\gamma}{(1 + \hat{\delta})\tau} \|w_x\|^2 \\ &\leq -\frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|w\|^2 + \frac{1}{(1 + \hat{\delta})} \|w\| \|u_t\| - \frac{\gamma^2}{(1 + \hat{\delta})\tau} \|w\|^2 - \frac{\gamma}{(1 + \hat{\delta})\tau} \|w_x\|^2 \\ &\leq -\frac{1}{2} \left(1 - \frac{1}{1 + \hat{\delta}} \right) \|u_t\|^2 + \left(\frac{1}{2} + \frac{1}{2(1 + \hat{\delta})} - \frac{\gamma^2}{(1 + \hat{\delta})\tau} \right) \|w\|^2 \\ &\quad - \frac{\gamma}{(1 + \hat{\delta})\tau} \|w_x\|^2 \\ &\leq -\frac{\hat{\delta}}{2(1 + \hat{\delta})} \|u_t\|^2 - \frac{1}{1 + \hat{\delta}} \left(\frac{\gamma^2}{\tau} - 1 - \frac{\hat{\delta}}{2} \right) \|w\|^2 - \frac{\gamma}{(1 + \hat{\delta})\tau} \|w_x\|^2 \leq 0. \end{aligned}$$

Thus $\mathcal{E}(u, v)$ and $\mathcal{E}_1(u, w)$ are non-increasing functions of t along the trajectory of a solution of (1.1)-(1.2). \blacksquare

Proof of Theorem 1.4. First by Proposition 3.1 and a slightly modified version of the proof of Corollary 3.4, the following assertion holds:

There exist $h_2, \sigma_2 > 0$ and, for any $0 < h < h_2$, a radius $\hat{\sigma}(h) > 0$ with $\lim_{h \rightarrow 0} \hat{\sigma}(h) = 0$, such that the local sublevel set

$$\mathcal{Y}_h = \{u \in \hat{u}_{\mathbf{n}} + H^1(\mathbb{R}) : u(0) = \hat{u}_{\mathbf{n}}(0), \|u - \hat{u}_{\mathbf{n}}\|_{H^1(\mathbb{R})} \leq \sigma_2 \text{ and } J(u) \leq J(\hat{u}_{\mathbf{n}}) + h\}$$

satisfies the property that if $u \in \mathcal{Y}_h$ then $\|u - \hat{u}_{\mathbf{n}}\|_{H^1(\mathbb{R})} < \hat{\sigma}(h)$. Moreover in \mathcal{Y}_h , $\hat{u}_{\mathbf{n}}$ is the unique critical point of J .

Let $\bar{h} < h_2$

$$\mathcal{O} = \{u \in \hat{u}_{\mathbf{n}} + H^1(\mathbb{R}) : u(0) = \hat{u}_{\mathbf{n}}(0) \text{ and } \|u - \hat{u}_{\mathbf{n}}\|_{H^1(\mathbb{R})} \leq \hat{\sigma}(\bar{h})\}.$$

and

$$\partial\mathcal{O} = \{u \in \hat{u}_{\mathbf{n}} + H^1(\mathbb{R}) : u(0) = \hat{u}_{\mathbf{n}}(0) \text{ and } \|u - \hat{u}_{\mathbf{n}}\|_{H^1(\mathbb{R})} = \hat{\sigma}(\bar{h})\}.$$

Define

$$\hat{h} = \inf_{u \in \partial\mathcal{O}} J(u) - J(\hat{u}_{\mathbf{n}}). \quad (5.9)$$

If \bar{h} is sufficiently small then $\hat{h} \in (0, \frac{h_2}{2})$. Set $\mathcal{O}^* = \cup_{y \in \mathbb{R}} \{u : u(\cdot + y) \in \mathcal{O}\}$ and $\partial\mathcal{O}^* = \cup_{y \in \mathbb{R}} \{u : u(\cdot + y) \in \partial\mathcal{O}\}$. Then any critical point of J in \mathcal{O}^* must be $\hat{u}_{\mathbf{n}}(\cdot + y)$ for some $y \in \mathbb{R}$. Consider the initial data which satisfy

$$\|u(\cdot, 0) - \hat{u}_{\mathbf{n}}\|_{H^1(\mathbb{R})} + \|v(\cdot, 0) - \hat{v}_{\mathbf{n}}\|_{H^1(\mathbb{R})} < \rho_{\mathbf{n}}. \quad (5.10)$$

Recall that there is a $C_0 > 0$ such that

$$\|\mathcal{L}(u(\cdot, t)) - \hat{v}_{\mathbf{n}}\|_{H^1(\mathbb{R})} \leq C_0 \|u(\cdot, t) - \hat{u}_{\mathbf{n}}\|_{L^2(\mathbb{R})}. \quad (5.11)$$

Let $\rho_{\mathbf{n}} < \min(\hat{\sigma}(\bar{h}), (C_0 + 1)^{-1} \sqrt{\frac{\hat{h}(1+\delta)}{2\gamma}})$. If we take $\rho_{\mathbf{n}}$ sufficiently small then $J(u(\cdot, 0)) < J(\hat{u}_{\mathbf{n}}) + \frac{\hat{h}}{4}$. Invoking (5.10)-(5.11) yields

$$\begin{aligned} & \|v(\cdot, 0) - \mathcal{L}(u(\cdot, 0))\|_{H^1(\mathbb{R})} \\ & \leq \|v(\cdot, 0) - \hat{v}_{\mathbf{n}}\|_{H^1(\mathbb{R})} + \|\mathcal{L}(u(\cdot, 0)) - \hat{v}_{\mathbf{n}}\|_{H^1(\mathbb{R})} \\ & \leq C_0 \|u(\cdot, 0) - \hat{u}_{\mathbf{n}}\|_{L^2(\mathbb{R})} + \|\mathcal{L}(u(\cdot, 0)) - \hat{v}_{\mathbf{n}}\|_{H^1(\mathbb{R})} \\ & < (C_0 + 1)\rho_{\mathbf{n}}. \end{aligned} \quad (5.12)$$

Hence

$$\frac{\gamma}{2(1+\hat{\delta})} \|v(\cdot, 0) - \mathcal{L}(u(\cdot, 0))\|_{H^1(\mathbb{R})}^2 < \frac{\gamma(C_0 + 1)^2 \rho_{\mathbf{n}}^2}{2(1+\hat{\delta})} \leq \frac{\hat{h}}{4}$$

and consequently

$$\mathcal{E}(u(\cdot, 0), v(\cdot, 0)) < J(\hat{u}_{\mathbf{n}}) + \frac{\hat{h}}{4} + \frac{\hat{h}}{4} < J(\hat{u}_{\mathbf{n}}) + \frac{\hat{h}}{2}. \quad (5.13)$$

It follows from Proposition 5.1 that, for all $t > 0$,

$$J(u(\cdot, t)) \leq \mathcal{E}(u(\cdot, t), v(\cdot, t)) < J(\hat{u}_{\mathbf{n}}) + \frac{\hat{h}}{2}. \quad (5.14)$$

Recall that $(u(\cdot, t), v(\cdot, t)) \in C([0, \infty), Y)$. Invoking (5.9) and (5.14) yields $(u(\cdot, t) \in \mathcal{O}^*$ for all $t > 0$.

Since $\mathcal{E}(u(\cdot, t), v(\cdot, t))$ is bounded from below, it follows from Proposition 5.1 that

$$\int_0^\infty \|u_t(\cdot, t)\|_{L^2(\mathbb{R})}^2 dt < \infty, \quad (5.15)$$

$$\int_0^\infty \|v(\cdot, t) - \mathcal{L}(u(\cdot, t))\|_{L^2(\mathbb{R})}^2 dt < \infty \quad (5.16)$$

and

$$\int_0^\infty \left\| \frac{\partial}{\partial x} (v(\cdot, t) - \mathcal{L}(u(\cdot, t))) \right\|_{L^2(\mathbb{R})}^2 dt < \infty. \quad (5.17)$$

Hence there exists a sequence $\{t_j\}$ such that as $t_j \rightarrow \infty$, $\|u_t(\cdot, t_j)\|_{L^2(\mathbb{R})} \rightarrow 0$, $\|v(\cdot, t_j) - \mathcal{L}(u(\cdot, t_j))\|_{H^1(\mathbb{R})} \rightarrow 0$ and $u(\cdot, t_j)$ converges to a critical point of J in \mathcal{O}^* . Since any critical point of J in \mathcal{O}^* must be $\hat{u}_{\mathbf{n}}(\cdot + y)$ for some $y \in \mathbb{R}$ and $(u(\cdot, t), v(\cdot, t)) \in C([0, \infty), Y)$, it follows that as $t \rightarrow \infty$, $J(u(\cdot, t))$ converges to $J(\hat{u}_{\mathbf{n}})$ and $u(\cdot, t)$ converges to $\hat{u}_{\mathbf{n}}(\cdot + y)$ for some $y \in \mathbb{R}$. This implies $\|\mathcal{L}(u(\cdot, t)) - \hat{v}_{\mathbf{n}}(\cdot + y)\|_{H^1(\mathbb{R})} \rightarrow 0$. Together with $\|v(\cdot, t) - \mathcal{L}(u(\cdot, t))\|_{H^1(\mathbb{R})} \rightarrow 0$ yields $\|v(\cdot, t) - \hat{v}_{\mathbf{n}}(\cdot + y)\|_{H^1(\mathbb{R})} \rightarrow 0$. The fact $\|u(\cdot, t) - \hat{u}_{\mathbf{n}}(\cdot + y)\|_{H^1(\mathbb{R})} \rightarrow 0$ follows from $J(u(\cdot, t)) - J(\hat{u}_{\mathbf{n}}) \rightarrow 0$. \blacksquare

6 Unstable waves

In this section the same notation as in the proof of Theorem 1.2 will be used to prove Theorem 1.5. We employ the mountain-pass principle to obtain such

critical points. Starting with the set $\mathcal{V}_{h,z}$ defined by (3.7), we construct a smooth map \check{b}_n from $(\mathcal{V}_{h,z})^2 \times [-\nu, \nu]$ to $H^1(\mathbb{R})$; here to each (u_0, u_1, x) in $\mathcal{V}_{h,z}^2 \times [-\nu, \nu]$, we associate the unique function $u \in H^1(\mathbb{R})$ satisfying the following conditions:

$$\begin{aligned}
(S'_1) \quad & u \equiv u_1(X - \cdot) \text{ on } (X - z, X + z), \\
(S'_2) \quad & u \equiv u_0 \text{ on } (-z, z), \\
(S'_3) \quad & \|u - u_-\|_{H^1(-\infty, -z)} \leq K\bar{r}, \\
(S'_4) \quad & \|u - u_-\|_{H^1(X+z, \infty)} \leq K\bar{r}, \\
(S'_5) \quad & X = x + \kappa_+ + \frac{\pi(2n+1)}{\omega}, \\
(S'_6) \quad & (u, \mathcal{L}u) \text{ satisfies (1.3)-(1.4) on each of the intervals } (-\infty, -z], \\
& [z, X - z], [X + z, +\infty).
\end{aligned}$$

With this definition of \check{b}_n , we define $\check{J} := J \circ \check{b}_n$. Then $\check{b}_n(u_0, u_1, x)$ is a critical point of J if (u_0, u_1, x) is a critical point of \check{J} in $\mathcal{V}_{h,z}^2 \times [-\nu, \nu]$.

Note that Lemma 4.3 still holds in the present situation. Hence for n large enough, there exists $\check{\rho}_h > 0$ such that if $(u_0, u_1, x) \in \mathcal{V}_{h,z}^2 \times [-\nu, \nu]$ and $J_z(\bar{u}_l) \in [J(u^*) + h/2, J(u^*) + h]$ with $l = 0$ or 1 then

$$\langle \partial_{u_l} \check{J}(u_0, u_1, x), \nabla_{H^1(-z, z)} J_z(u_l) \rangle \geq \check{\rho}_h. \quad (6.1)$$

Moreover, adapting Lemma 2.9 to the present situation, we see that, for each n large enough, there is a small μ_n such that if $x \in [-\nu, -\nu/2]$ then

$$\partial_x \check{J}(u_0, u_1, x) \geq \mu_n, \quad (6.2)$$

while for $x \in [\nu/2, \nu]$,

$$\partial_x \check{J}(u_0, u_1, x) \leq -\mu_n. \quad (6.3)$$

Pick $\mathbf{a} \in (\mathcal{V}_{h,z})^2 \times \{-\nu\}$ be such that $\check{J}(\mathbf{a}) \leq \inf_{(\mathcal{V}_{h,z})^2 \times \{-\nu\}} \check{J} + \mu_n \nu / 4$ and $\mathbf{b} \in (\mathcal{V}_{h,z})^2 \times \{\nu\}$ be such that $\check{J}(\mathbf{b}) \leq \inf_{(\mathcal{V}_{h,z})^2 \times \{\nu\}} \check{J} + \mu_n \nu / 4$. Set

$$\Gamma := \{\bar{\gamma} \in C([0, 1], (\mathcal{V}_{h,z})^2 \times [-\nu, \nu]) : \bar{\gamma}(0) = \mathbf{a}, \bar{\gamma}(1) = \mathbf{b}\}$$

and define

$$c_n := \inf_{\bar{\gamma} \in \Gamma} \max_{[0, 1]} \check{J} \circ \bar{\gamma}.$$

It follows from (6.2) and (6.3) that $c_n \geq \max(\check{J}(\mathbf{a}), \check{J}(\mathbf{b})) + \mu_n \nu / 4$.

For any $\bar{\gamma} \in \Gamma$, (6.1) together with the standard deformation theory gives a $\tilde{\gamma} \in \Gamma$ such that $\max_{[0, 1]} \check{J} \circ \tilde{\gamma} \leq \max_{[0, 1]} \check{J} \circ \bar{\gamma}$ and the image of $\tilde{\gamma}$ stays in the set $\mathcal{V}_{h/2, z}^2 \times [-\nu, \nu]$. Moreover, it follows from Corollary 2.8 that \check{J} satisfies the Palais-Smale condition. Since the critical points of \check{J} are isolated, we may apply

a result of Hofer [29] to find a “mountain-pass” type critical point $(u_0^\sharp, u_1^\sharp, x^\sharp)$ of \check{J} . This tells that for any neighborhood Ω of $(u_0^\sharp, u_1^\sharp, x^\sharp)$ there exists $(u'_0, u'_1, x') \in \Omega$ such that $\check{J}(u'_0, u'_1, x') < c_n$. Setting $(\check{u}_n, \check{v}_n) := (\check{b}_n(u_0^\sharp, u_1^\sharp, x^\sharp), \mathcal{L}\check{b}_n(u_0^\sharp, u_1^\sharp, x^\sharp))$ gives a two-bump solution of (1.3)-(1.4). Moreover $(\check{u}_n, \check{v}_n)$ satisfies Theorem 1.5(i),(ii) and $J(\check{u}_n) = c_n$.

Let us recall the Lyapunov functional \mathcal{E} defined in Section 5 for showing the instability of $(\check{u}_n, \check{v}_n)$. With $H^1 \times H^1$ topology, we can find a neighborhood of $(\check{u}_n, \check{v}_n)$ which possesses a single critical point only. Inside this set, any smaller neighborhood of $(\check{u}_n, \check{v}_n)$ contains a point $(\tilde{u}, \tilde{v}) := (\tilde{b}_n(u'_0, u'_1, x'), \mathcal{L}\tilde{b}_n(u'_0, u'_1, x'))$ with the property $J(\tilde{u}) < \mathcal{E}(\tilde{u}, \tilde{v}) < c_n$. If $(u(x, t), v(x, t))$ is a solution of (1.1)-(1.2) with the initial datum $(u(x, 0), v(x, 0)) = (\tilde{u}(x), \tilde{v}(x))$, it is clear that $J(u(x, t)) \leq \mathcal{E}(u(x, t), v(x, t)) < \mathcal{E}(\tilde{u}, \tilde{v}) < \mathcal{E}(\check{u}_n, \check{v}_n) < c_n$. Hence there exist $\tau_* > 0$ and a small neighborhood \mathcal{N} of $(\check{u}_n, \check{v}_n)$ such that $(u(x, t), v(x, t)) \notin \mathcal{N}$ if $t \geq \tau_*$. This completes the proof of (iii), so does Theorem 1.5.

7 Appendix

As a byproduct of [18], the proof of Theorem 1.1 shows that there exists a heteroclinic solution for the system

$$-du'' = k(u - u^3) - v, \quad (7.1)$$

$$-v'' = u - \gamma v. \quad (7.2)$$

Here (u_+, v_+) is replaced by $(\sqrt{1 - 1/(k\gamma)}, \sqrt{1 - 1/(k\gamma)}/\gamma)$ and $(u_-, v_-) = (-u_+, -v_+)$. The nonlinearity in (7.1) is an odd function, which gives the same type of potential as in the Allen-Cahn equation. The following observation indicates that system (1.3)-(1.4) can be converted into (7.1)-(7.2).

Suppose that (\bar{u}, \bar{v}) is a heteroclinic solution of (7.1)-(7.2). By setting $k = \frac{1}{3}(\beta^2 - \beta + 1)$ and

$$\begin{cases} u^* = (\beta + 1)/3 + \sqrt{k}\bar{u}, \\ v^* = (\beta + 1)/3\gamma + \sqrt{k}\bar{v}, \end{cases} \quad (7.3)$$

a simple calculation easily verifies that (u^*, v^*) is a heteroclinic solution of (1.3)-(1.4).

In this Appendix, we clarify the conditions on the parameters such that both (u_-, v_-) and (u_+, v_+) are saddle-focus equilibria. First from the assumptions of Theorem 1.1, $\gamma = 9/(2\beta^2 - 5\beta + 2)$ and $d\gamma^2 > 1$. In view of (7.1)-(7.2), since $u_+^2 = 1 - \frac{1}{k\gamma}$, it follows that $k\gamma > 1$. Therefore

$$k(1 - 3u_+^2) - \gamma d = \frac{1}{\gamma}(3 - 2k\gamma - \gamma^2 d) < 0. \quad (7.4)$$

Consider the linearization of (7.1)-(7.2) at $(\sqrt{1 - 1/(k\gamma)}, \sqrt{1 - 1/(k\gamma)}/\gamma)$. If

$$[k(1 - 3u_+^2) - \gamma d]^2 - 4d < 0, \quad (7.5)$$

then all the eigenvalues are complex numbers, which is case of saddle-focus. Combining (7.4) with (7.5) yields

$$\frac{1}{\gamma}(3 - 2k\gamma - \gamma^2 d) > -2\sqrt{d},$$

which can be rewritten as

$$\gamma d - 2\sqrt{d} + (2k - \frac{3}{\gamma}) < 0. \quad (7.6)$$

Solving (7.6) gives

$$\frac{1 - \sqrt{4 - 2\gamma k}}{\gamma} < \sqrt{d} < \frac{1 + \sqrt{4 - 2\gamma k}}{\gamma}, \quad (7.7)$$

provided that $k\gamma < 2$. Note that $k\gamma = \frac{3\beta^2 - 3\beta + 3}{2\beta^2 - 5\beta + 2} = \frac{3}{2} + \frac{9\beta}{2(2\beta^2 - 5\beta + 2)} > \frac{3}{2}$ for $\beta \in (0, \frac{1}{2})$. This together with $k\gamma < 2$ implies

$$\frac{9\beta}{2\beta^2 - 5\beta + 2} < 1$$

and consequently $9\beta < 2\beta^2 - 5\beta + 2$. Solving $\beta^2 - 7\beta + 1 > 0$ yields

$$\beta < \frac{7 - \sqrt{45}}{2}, \quad (7.8)$$

as $\beta \in (0, \frac{1}{2})$ rules out the possibility of $\beta > \frac{7 + \sqrt{45}}{2}$. Since $k\gamma > \frac{3}{2}$, it follows from (7.7) and $d\gamma^2 > 1$ that

$$\frac{1}{\gamma} < \sqrt{d} < \frac{2}{\gamma}. \quad (7.9)$$

In summary, the equilibria (u_-, v_-) and (u_+, v_+) are saddle-focus if and only if

$$\beta \in (0, \frac{7 - \sqrt{45}}{2}), \quad \gamma = 9/(2\beta^2 - 5\beta + 2) \text{ and } \frac{1}{\gamma} < \sqrt{d} < \frac{2}{\gamma},$$

as stated in (1.12).

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