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Multiple front standing waves in the FitzHugh-Nagumo equations

Chao-Nien Chen ^{*} Éric Séré [†]

Abstract: There have been several existence results for the standing waves of FitzHugh-Nagumo equations. Such waves are the connecting orbits of an autonomous second-order Lagrangian system and the corresponding kinetic energy is an indefinite quadratic form in the velocity terms. When the system has two stable hyperbolic equilibria, there exist two stable standing fronts, which will be used in this paper as building blocks, to construct stable standing waves with multiple fronts in case the equilibria are of saddle-focus type. The idea to prove existence is somewhat close in spirit to [6]. However several differences are required in the argument: facing a strongly indefinite functional, we need to perform a nonlocal Lyapunov-Schmidt reduction; in order to justify the stability of multiple front standing waves, we rely on a more precise variational characterization of such critical points. Based on this approach, both stable and unstable standing waves are established.

Key words: reaction-diffusion system, FitzHugh-Nagumo equations, standing wave, stability, Hamiltonian system, connecting orbit.

AMS subject classification: 34C37, 35J50, 35K57.

1 Introduction

Following a fascinating idea of Turing [36], reaction-diffusion systems [1, 7, 10, 21, 29] serve as models for studying pattern formation and wave propagation. Significant progress [7, 13, 12, 14, 15, 18, 30, 32, 33, 37] on the self-organized

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patterns has been made for the system of FitzHugh-Nagumo equations

$$u_t - du_{xx} = f(u) - v, \quad (1.1)$$

$$\tau v_t - v_{xx} = u - \gamma v. \quad (1.2)$$

Here $f(\xi) = \xi(\xi - \beta)(1 - \xi)$, $\beta \in (0, 1/2)$ and $d, \tau, \gamma \in (0, \infty)$. Historically the original model [21, 29] was derived as a simplification of the Hodgkin-Huxley equations [22] for nerve impulse propagation. In recent years (1.1)-(1.2) has been extensively studied as a paradigmatic activator-inhibitor system. Such systems are of great interest to the scientific community as breeding grounds for studying the generation of localized structures.

The standing wave solutions of (1.1)-(1.2) are the connecting orbits of a second order Hamiltonian system

$$-du'' = f(u) - v, \quad (1.3)$$

$$-v'' = u - \gamma v. \quad (1.4)$$

Associated with (1.3)-(1.4), the Lagrangian is

$$L(u_x, v_x, u, v) = \frac{d}{2}u_x^2 - \frac{1}{2}v_x^2 + uv - \frac{\gamma}{2}v^2 - \int_0^u f(\xi)d\xi. \quad (1.5)$$

As (1.4) is a linear equation, v can be solved from u , for instance, by making use of Green function. This leads to a variational formulation $J(u)$, as to be defined in (2.2) with a nonlocal term involved. In fact the action functional J has been employed [13], through a minimization argument, to obtain a basic type standing front solution of (1.1)-(1.2) as follows.

Theorem 1.1 *Let $(u_-, v_-) = (0, 0)$ and $(u_+, v_+) = (2(\beta + 1)/3, 2(\beta + 1)/3\gamma)$. If $\gamma = 9(2\beta^2 - 5\beta + 2)^{-1}$ and $d > \gamma^{-2}$, there exists a heteroclinic orbit (u^*, v^*) of (1.3)-(1.4) with asymptotic behavior $(u^*, v^*) \rightarrow (u_-, v_-)$ as $x \rightarrow -\infty$ and $(u^*, v^*) \rightarrow (u_+, v_+)$ as $x \rightarrow \infty$.*

Clearly (u_*, v_*) is also a heteroclinic orbit of (1.3)-(1.4) if we define $(u_*(x), v_*(x)) = (u^*(-x), v^*(-x))$. Note that two constant solutions (u_-, v_-) and (u_+, v_+) are in the same energy level only if $\gamma = 9(2\beta^2 - 5\beta + 2)^{-1}$. Like what is known in the Allen-Cahn equation, standing fronts appear in reaction-diffusion system with a balanced double-well potential. As a remark, such existence results can be extended to more general nonlinearities; that is, f is not necessary to be a cubic polynomial.

The goal of this paper is to construct multiple front solutions using (u^*, v^*) together with the reverse orbit (u_*, v_*) . We only deal with the case when the equilibria are of saddle-focus type; that is, the linearization of the Hamiltonian

system associated with (1.3)-(1.4) at (u_-, v_-) , as well as (u_+, v_+) , has eigenvalues $\pm\lambda \pm i\omega$. As to be seen in the Appendix, the values of the parameters have to satisfy

$$\beta \in (0, \frac{7 - \sqrt{45}}{2}), \quad \gamma = 9/(2\beta^2 - 5\beta + 2) \text{ and } \frac{1}{\gamma} < \sqrt{d} < \frac{2}{\gamma}. \quad (1.6)$$

Without loss of generality, we may assume that $u_x^*(0) \neq 0$. With prime denoting differentiation with respect to x , we occasionally use subscript for differentiation in notation. We now state a main existence result of the paper.

Theorem 1.2 *Assume that (1.6) is satisfied. Then there are two real numbers κ_+ , κ_- , and, for each sufficiently small $\sigma > 0$, a large constant $D_\sigma > 0$, such that for any positive integer N and any sequence of positive integers $\mathbf{n} = (n_i)_{1 \leq i \leq N}$ with $n_i \geq D_\sigma$ for every i , there exist positive numbers X_1, \dots, X_{N-1} and a solution $(\hat{u}_\mathbf{n}, \hat{v}_\mathbf{n})$ of (1.3)-(1.4) satisfying the following properties:*

(a) $\|(\hat{u}_\mathbf{n}, \hat{v}_\mathbf{n}) - (u^*, v^*)\|_{H^1(-\infty, A_1)} \leq \sigma.$

(b) For i odd in $[1, N]$,

$$\|(\hat{u}_\mathbf{n}, \hat{v}_\mathbf{n})(\cdot + C_i) - (u_*, v_*)\|_{H^1(-A_i, A_{i+1})} \leq \sigma, \quad |X_i - 2\pi n_i/\omega - \kappa_+| < \sigma.$$

(c) For i even in $[2, N]$,

$$\|(\hat{u}_\mathbf{n}, \hat{v}_\mathbf{n})(\cdot + C_i) - (u^*, v^*)\|_{H^1(-A_i, A_{i+1})} \leq \sigma, \quad |X_i - 2\pi n_i/\omega - \kappa_-| < \sigma.$$

Here, $C_1 = X_1$, $C_i = C_{i-1} + X_i$, $A_i = X_i/2$ for $1 \leq i \leq N$, and $A_{N+1} = +\infty$.

Let us remark that if N is odd, $(\hat{u}_\mathbf{n}, \hat{v}_\mathbf{n})$ is homoclinic to (u_-, v_-) while for N even, it is a heteroclinic connection between (u_-, v_-) and (u_+, v_+) . Such orbits are the standing waves of (1.1)-(1.2) with multiple fronts; for the Hamiltonian system they are often called multi-bump solutions.

As already mentioned, the range of parameters under consideration is such that the basic heteroclinics (u_*, v_*) and (u^*, v^*) connect two equilibria of saddle-focus type. In this situation, multi-bump solutions are known to exist provided the stable and unstable manifolds intersect transversally, as was proved by Devaney [19] by constructing a Smale horseshoe. Transversality condition in general is difficult to check for a given Hamiltonian although it is generically true. Instead to verify transversality, we follow a strategy, as introduced in [6], to prove that the basic heteroclinic is isolated up to translation invariance in the spatial variable, by solving an auxiliary boundary value problem. Then invoking this fact to show the existence of multi-bump solutions by variational argument.

The variational construction for multi-bump and chaotic solutions has a long history and the comments below are not exhaustive. To our knowledge, the earliest results were established by Bolotin [2, 3, 4] in the context of nonautonomous second order Lagrangian systems, the connecting orbits being minimizers of the action. In the case of twist maps on the annulus (also corresponding to nonautonomous Lagrangian systems), Mather [28] constructed chaotic connecting orbits by a minimization method in the region between two invariant circles. For non-autonomous first order Hamiltonian systems, multi-bump solutions were found by min-max methods [34, 35] under the assumption that critical points are isolated. This kind of approaches have been extended to second order systems and elliptic PDEs in [16, 17]. We refer to [31] and references therein for more recent development and related results in this direction. For autonomous problems of saddle-focus type a class of multi-bump solutions were obtained, for the special case of a fourth order equation related to water wave theory, by Buffoni [5] using a shooting argument. Subsequently a larger set of multi-bump solutions was constructed [6] by variational and degree arguments. This method was then adapted for studying the extended Fisher-Kolmogorov equations (of fourth order) [24]. In subsequent works [25, 26], the authors introduced a refined but more specific argument to obtain more precise results on the F-K model. As already mentioned, the present work is close in spirit to [6]. Note, however, that our system of autonomous second order Lagrangian equations is associated with a strongly indefinite variational problem and it cannot be reduced to a fourth order equation. Instead, we use a nonlocal Lyapunov-Schmidt reduction. Moreover our approach is purely variational, contrary to [6] where degree theory was employed; indeed such a variational construction is needed for the sake of stability analysis, as always an important issue in considering pattern formation as well as wave propagation.

For the stationary solutions of (1.1)-(1.2), stability questions have been studied in [11, 12, 13, 15, 30, 38] by various methods. In conjunction with strongly indefinite variational structure, the Maslov index [10, 12] and relative Morse index [11] provide useful information to determine the stability of such solutions, obtained as the critical points of the action functional. Let $\mathbb{C}^- = \{\zeta \in \mathbb{C} \text{ and } \operatorname{Re}\zeta < 0\}$, where $\operatorname{Re}\zeta$ denotes the real part of ζ . Denoted by Λ the linearization of (1.1)-(1.2) at a standing wave solution (u, v) . A standing wave (u, v) is said to be non-degenerate if zero is a simple eigenvalue of Λ .

Definition A non-degenerate standing wave (u, v) of (1.1)-(1.2) is spectrally stable if all the non-zero eigenvalues of Λ are in \mathbb{C}^- .

First, we state a stability result obtained from an index method developed in [9].

Theorem 1.3 Let (u, v) be a non-degenerate standing wave of (1.1)-(1.2). Suppose u is a local minimizer of J then (u, v) is spectrally stable, provided that $\tau < \gamma^2$.

In addition, the Lyapunov functional [15] reveals more dynamical aspects in the process of generating stable patterns. We give an extension of such a Lyapunov functional, which can be applied to the standing waves of (1.1)-(1.2).

Theorem 1.4 Let $\tau < \gamma^2$. Under the flow generated by (1.1)-(1.2) in the $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ topology, the standing wave (\hat{u}_n, \hat{v}_n) is asymptotically stable up to a phase shift in spatial variable. More precisely, for each $\epsilon > 0$ there is $\rho_n > 0$ such that if $(u(x, t), v(x, t))$ is a solution of (1.1)-(1.2) and

$$\|u(\cdot, 0) - \hat{u}_n\|_{H^1(\mathbb{R})} + \|v(\cdot, 0) - \hat{v}_n\|_{L^2(\mathbb{R})} < \rho_n,$$

then for all $t > 0$,

$$\inf_{Y \in \mathbb{R}} \{ \|u(\cdot, t) - \hat{u}_n(\cdot - Y)\|_{H^1(\mathbb{R})} + \|v(\cdot, t) - \hat{v}_n(\cdot - Y)\|_{L^2(\mathbb{R})} \} < \epsilon$$

and

$$\inf_{Y \in \mathbb{R}} \{ \|u(\cdot, t) - \hat{u}_n(\cdot - Y)\|_{H^1(\mathbb{R})} + \|v(\cdot, t) - \hat{v}_n(\cdot - Y)\|_{L^2(\mathbb{R})} \} \xrightarrow{t \rightarrow +\infty} 0.$$

As a final remark, there are plenty of unstable standing waves; however we do not attempt to describe all of them here but just state one result in the two-bump case.

Theorem 1.5 As in Theorem 1.2, assume (1.6) and take sufficiently small σ and large D_σ . For any positive integer $n \geq D_\sigma$ there exists a solution $(\check{u}_n, \check{v}_n)$ of (1.3)-(1.4). Moreover, for some $X \in \mathbb{R}$ with $|X - \pi(2n + 1)/\omega - \kappa_+| < \sigma$ and κ_+ as in Theorem 1.2, the following properties hold.

- (i) $\|(\check{u}_n, \check{v}_n) - (u^*, v^*)\|_{H^1(-\infty, X/2)} \leq \sigma$.
- (ii) $\|(\check{u}_n, \check{v}_n) - (u_*, v_*)(\cdot - X)\|_{H^1(X/2, +\infty)} \leq \sigma$.
- (iii) $(\check{u}_n, \check{v}_n)$ is unstable in the following sense: for any $\rho_0 > 0$, there exist $\epsilon_0 > 0$, $T_0 > 0$ and a solution $(u(x, t), v(x, t))$ of (1.1)-(1.2) such that

$$\|u(\cdot, 0) - \check{u}_n\|_{H^1(\mathbb{R})} + \|v(\cdot, 0) - \check{v}_n\|_{L^2(\mathbb{R})} < \rho_0,$$

while if $t \geq T_0$ then

$$\inf_{Y \in \mathbb{R}} \{ \|u(\cdot, t) - \check{u}_n(\cdot - Y)\|_{H^1(\mathbb{R})} + \|v(\cdot, t) - \check{v}_n(\cdot - Y)\|_{L^2(\mathbb{R})} \} \geq \epsilon_0.$$

The solution $(\check{u}_n, \check{v}_n)$ follows from mountain-pass argument, as to be shown in Section 6.

2 Variational setting

In this section we recall the variational setting [13] used to study (u^*, v^*) and discuss related properties, including a reduced functional J which is bounded from below. Recall from (1.5) that the Lagrangian associated with (1.3)-(1.4) is $L(u_x, v_x, u, v)$, which will be simply written as $L(u, v)$ frequently. Note that the main difference with [6] is that the present system is not reducible to a simple, almost linear, fourth-order system. So one has to deal with an indefinite Lagrangian (1.5). Fortunately, this Lagrangian is concave in v . We exploit this property as follows:

For a given $\phi \in H^1(\mathbb{R})$, let $\mathcal{L}\phi$ be the unique solution, in $H^1(\mathbb{R})$, of the equation

$$-g'' + \gamma g = \phi.$$

In the sequel, we work with affine functional spaces of the form $H_w = w + H^1(\mathbb{R})$, with $w = 0, u_+, v_+, u_*, u^*, v_*$ or v^* . For $a_u = 0, u_+, u_*, u^*$ respectively, and $u \in a_u + H^1(\mathbb{R})$, we also denote $\mathcal{L}u := a_v + \mathcal{L}(u - a_u)$, with $a_v = 0, v_+, v_*, v^*$ respectively.

Note that $\mathcal{L}u$ is the unique solution, in $a_v + H^1(\mathbb{R})$, of the equation

$$-v'' + \gamma v = u.$$

Clearly

$$\begin{aligned} \left\| \frac{d}{dx}(v - a_v) \right\|_{L^2(\mathbb{R})}^2 + \gamma \|v - a_v\|_{L^2(\mathbb{R})}^2 &= \int_{-\infty}^{\infty} (u - a_u)(v - a_v) dx \\ &\leq \|u - a_u\|_{L^2(\mathbb{R})} \|v - a_v\|_{L^2(\mathbb{R})}. \end{aligned}$$

Hence there is a $C_0 > 0$ such that

$$\|v - a_v\|_{H^1(\mathbb{R})} \leq C_0 \|u - a_u\|_{L^2(\mathbb{R})}. \quad (2.1)$$

Given $\phi \in H^1(\mathbb{R})$, define, for all $\psi \in H^1(\mathbb{R})$,

$$I(\psi) = \int_{-\infty}^{\infty} \left(\frac{1}{2} |\psi'|^2 + \frac{\gamma}{2} \psi^2 - \phi \psi \right) dx$$

Lemma 2.1 *Let $\phi \in H^1(\mathbb{R})$. Then*

$$I(\psi) - I(\mathcal{L}\phi) = \int_{-\infty}^{\infty} \frac{1}{2} (\psi' - (\mathcal{L}\phi)')^2 + \frac{\gamma}{2} (\psi - (\mathcal{L}\phi))^2 dx$$

for all $\psi \in H^1(\mathbb{R})$.

Proof. It follows from straightforward calculation, by making use of

$$\int_{-\infty}^{\infty} ((\mathcal{L}\phi')^2 + \gamma(\mathcal{L}\phi)^2)dx = \int_{-\infty}^{\infty} \phi\mathcal{L}\phi dx$$

and

$$\int_{-\infty}^{\infty} \psi'\mathcal{L}\phi' + \gamma\psi\mathcal{L}\phi dx = \int_{-\infty}^{\infty} \phi\psi dx.$$

■

For $w \in \{0, u_+, u^*, u_*\}$ and all $u \in H_w$, define

$$J(u) = \int_{-\infty}^{\infty} L(u, \mathcal{L}u) dx. \quad (2.2)$$

The next lemma is an immediate consequence of Lemma 2.1.

Lemma 2.2 *Taking $a_u = 0, u_+, u^*, u_*$ respectively and $a_v = 0, v_+, v^*, v_*$ respectively, if $u \in H_{a_u}$ then:*

$$J(u) = \max_{v \in H_{a_v}} \int_{-\infty}^{\infty} L(u, v) dx. \quad (2.3)$$

The operator \mathcal{L} has a good control in terms of local estimates:

Lemma 2.3 *There is a constant M such that, if $B - A \geq 1$ and $\phi \in H^1(\mathbb{R})$, then*

$$\|\mathcal{L}\phi\|_{H^1(A,B)} \leq M(\|\phi\|_{L^2(A,B)} + |\mathcal{L}\phi(A)| + |\mathcal{L}\phi(B)|).$$

Proof. Let $\tilde{\theta}$ be a smooth real-valued function defined on $[0, 1/2]$ such that $\tilde{\theta} \equiv 1$ in a neighborhood of zero and $\tilde{\theta} \equiv 0$ in a neighborhood of $1/2$. A test function $\psi_{A,B}$ is introduced as follows:

$$\begin{aligned} \psi_{A,B} &\equiv \mathcal{L}\phi \text{ away from } [A, B], \\ \psi_{A,B} &\equiv 0 \text{ on } [A + 1/2, B - 1/2], \\ \psi_{A,B} &\equiv \tilde{\theta}(\cdot - A)\mathcal{L}\phi \text{ on } [A, A + 1/2], \\ \psi_{A,B} &\equiv \tilde{\theta}(B - \cdot)\mathcal{L}\phi \text{ on } [B - 1/2, B]. \end{aligned}$$

By direct calculation

$$\begin{aligned} &\int_A^B ((\mathcal{L}\phi' - \psi'_{A,B})^2 + \gamma(\mathcal{L}\phi - \psi_{A,B})^2) dx \\ &= \int_A^B (-\psi''_{A,B} + \gamma\psi_{A,B} - \phi)(\psi_{A,B} - \mathcal{L}\phi) dx \\ &\leq (\|-\psi''_{A,B} + \gamma\psi_{A,B}\|_{L^2(A,B)} + \|\phi\|_{L^2(A,B)}) \|\psi_{A,B} - \mathcal{L}\phi\|_{L^2(A,B)}. \end{aligned}$$

Denoted by $O(s)$ a number bounded by Cs with C being a constant not depending on A, B, ϕ . Then

$$\|\psi_{A,B} - \mathcal{L}\phi\|_{H^1(A,B)} = O(\|-\psi''_{A,B} + \gamma\psi_{A,B}\|_{L^2(A,B)} + \|\phi\|_{L^2(A,B)}).$$

Applying the triangular inequality yields

$$\|\mathcal{L}\phi\|_{H^1(A,B)} = O(\|\psi_{A,B}\|_{H^1(A,B)} + \|-\psi''_{A,B} + \gamma\psi_{A,B}\|_{L^2(A,B)} + \|\phi\|_{L^2(A,B)}).$$

By the definition of $\psi_{A,B}$

$$\|\psi_{A,B}\|_{H^1(A,B)} + \|-\psi''_{A,B} + \gamma\psi_{A,B}\|_{L^2(A,B)} = O(|\mathcal{L}\phi(A)| + |\mathcal{L}\phi(B)| + \|\phi\|_{L^2(A,B)}).$$

Now the proof is complete. \blacksquare

3 Isolated critical points

The aim in this section is to show that any critical point of J is an isolated critical point, up to translations in x .

Proposition 3.1 *Suppose that u_c is a critical point of J in H_{u^*} and satisfies $u_c(0) = u^*(0)$. Then in the set of critical points of J having the same constraint at $x = 0$, u_c is an isolated point in the H^1 topology. The same assertion holds if H_{u^*} is replaced by H_{u_*} or H_{u_-} or H_{u_+} .*

In order to prove this Proposition, we use an alternative as in [6]: as the system is real analytic, either all trajectories of the unstable manifold of $(u_-, v_-, 0, 0)$ converge to $(u_+, v_+, 0, 0)$ and have the same action, or every heteroclinic is isolated. In the next proposition we will find a trajectory in the unstable manifold of $(u_-, v_-, 0, 0)$ which does not converge to $(u_+, v_+, 0, 0)$ as $x \rightarrow +\infty$ or, if it does, has an action larger than the action of (u^*, v^*) . This will allow us to prove Proposition 3.1.

Proposition 3.2 *For any $b \in \mathbf{R}$, there is a solution (u, v) of (1.3)-(1.4) which satisfies $u(0) = \frac{\gamma}{2}v(0) = b$ and one of the following conditions:*

(i) $\lim_{x \rightarrow -\infty} (u(x), v(x)) = (u_-, v_-)$ and

$$\int_{-\infty}^0 L(u, v) dx \geq \int_{-\infty}^0 \frac{1}{2} \left(d - \frac{1}{\gamma^2} \right) (u')^2 + \frac{1}{4} (u - u_-)^2 (u - u_+)^2 dx. \quad (3.1)$$

(ii) $\lim_{x \rightarrow \infty} (u(x), v(x)) = (u_+, v_+)$ and

$$\int_0^{\infty} L(u, v) dx \geq \int_0^{\infty} \frac{1}{2} \left(d - \frac{1}{\gamma^2} \right) (u')^2 + \frac{1}{4} (u - u_-)^2 (u - u_+)^2 dx. \quad (3.2)$$

Proof. Suppose that (u, v) satisfies (1.3)-(1.4), $u(0) = \frac{2}{\gamma}v(0) = b$ and $\lim_{x \rightarrow -\infty}(u(x), v(x)) = (u_-, v_-)$. Multiplying (1.4) by v and integrating over $(-\infty, 0)$, we get

$$-vv'|_{-\infty}^0 + \int_{-\infty}^0 (v')^2 + \gamma v^2 dx = \int_{-\infty}^0 uv dx.$$

By direct calculation

$$\begin{aligned} & \int_{-\infty}^0 \left\{ \frac{d}{2}(u')^2 + \frac{1}{2}uv + F(u) \right\} dx \\ &= \int_{-\infty}^0 \left\{ \frac{d}{2}(u')^2 + \frac{1}{2}(v')^2 + \frac{1}{2}\gamma v^2 + F(u) \right\} dx - \frac{1}{2}vv'|_{-\infty}^0 \\ &= \int_{-\infty}^0 \left\{ \frac{d}{2}(u')^2 + \frac{1}{2}(v')^2 + \frac{1}{2}\gamma \left(v - \frac{u}{\gamma} \right)^2 + \left(uv - \frac{u^2}{\gamma} \right) \right\} dx \\ & \quad + \int_{-\infty}^0 \left(F(u) + \frac{1}{2\gamma}u^2 \right) dx - \frac{1}{2}vv'|_{-\infty}^0. \end{aligned} \quad (3.3)$$

Next, multiplying (1.4) by $-u/\gamma$ and integrating over $[-\eta, \eta]$, we have

$$\int_{-\infty}^0 \left(uv - \frac{u^2}{\gamma} \right) dx = \int_{-\infty}^0 \frac{1}{\gamma}uv'' dx = \frac{1}{\gamma}uv'|_{-\infty}^0 - \int_{-\infty}^0 \frac{1}{\gamma}u'v' dx. \quad (3.4)$$

Since

$$F(u) + \frac{u^2}{2\gamma} = \frac{1}{4}(u - u_-)^2(u - u_+)^2,$$

substituting (3.4) into (3.3) yields

$$\begin{aligned} & \int_{-\infty}^0 \left\{ \frac{d}{2}(u')^2 + \frac{1}{2}uv + F(u) \right\} dx \\ &= \int_{-\infty}^0 \frac{d}{2}(u')^2 + \frac{1}{2}(v')^2 + \frac{1}{2}\gamma \left(v - \frac{u}{\gamma} \right)^2 - \frac{1}{\gamma}u'v' + \frac{1}{4}(u - u_-)^2(u - u_+)^2 dx \\ & \quad - \frac{1}{2}vv'|_{-\infty}^0 + \frac{1}{\gamma}uv'|_{-\infty}^0 \\ &= \int_{-\infty}^0 \frac{1}{2} \left(d - \frac{1}{\gamma^2} \right) (u')^2 + \frac{1}{2} \left(v' - \frac{u'}{\gamma} \right)^2 + \frac{1}{2}\gamma \left(v - \frac{u}{\gamma} \right)^2 dx \\ & \quad + \int_{-\infty}^0 \frac{1}{4}(u - u_-)^2(u - u_+)^2 dx - \frac{1}{2}vv'|_{-\infty}^0 + \frac{1}{\gamma}uv'|_{-\infty}^0. \end{aligned}$$

Then (3.1) easily follows in view of the boundary conditions on u and v .

Next we prove the existence of such a solution. Let \hat{v} be a C^∞ -function such that $\hat{v}(0) = \frac{2b}{\gamma}$, $-\hat{v}_{xx}(0) + \gamma\hat{v}(0) = b$ and $\hat{v}(x) = 0$ if $x \leq -1$. Define $\mathbf{H}_w = w + H_0^1(-\infty, 0)$ and set $\hat{u} = \gamma\hat{v} - \hat{v}_{xx}$. For $w \in \mathbf{H}_{\hat{u}}$, we define

$$\hat{J}(w) = \int_{-\infty}^0 \frac{1}{2} [d(w')^2 + w\hat{\mathcal{L}}w] + F(w)dx, \quad (3.5)$$

where $\hat{\mathcal{L}}w$ denotes the unique solution of

$$-v'' + \gamma v = w, \quad v \in \mathbf{H}_{\hat{v}}. \quad (3.6)$$

As in the proof of Theorem 1.1 of [13], we pick a minimizing sequence $u_m \in \mathbf{H}_{\hat{u}}$ which converges to u in H_{loc}^1 . Moreover $u \in \mathbf{H}_{\hat{u}}$ and

$$\hat{J}(u) = \inf_{w \in \mathbf{H}_{\hat{u}}} \hat{J}(w).$$

Letting $v = \hat{\mathcal{L}}u$, we see that (u, v) satisfies (1.3)-(1.4), $u(0) = \frac{\gamma}{2}v(0) = b$ and $\lim_{x \rightarrow -\infty} (u(x), v(x)) = (u_-, v_-)$. This proves (i).

The above argument also shows that either $\lim_{x \rightarrow -\infty} (u(x), v(x)) = (u_-, v_-)$ or $\lim_{x \rightarrow -\infty} (u(x), v(x)) = (u_+, v_+)$. In the latter case, we replace $(u(x), v(x))$ by $(u(-x), v(-x))$ to establish (ii), since the proof of (3.2) is not different from that of (3.1). ■

Proof of Proposition 3.1. We only treat the case of H_{u^*} , the others are analogue.

Note that $\mathbf{z}_- = (u_-, v_-, 0, 0)$ is a hyperbolic equilibrium of a first order Hamiltonian system (HS) with Hamiltonian function $H(u, v, p, q)$, which is associated to the second order Lagrangian system (1.3)-(1.4). With the Hamiltonian H being real analytic, the local unstable manifold $W_{loc}^u(\mathbf{z}_-)$ of \mathbf{z}_- is a real analytic submanifold of \mathbb{R}^4 . Moreover the unstable space of the linearization of (HS) at \mathbf{z}_- is the graph of a linear map from \mathbb{R}^2 to itself, hence $W_{loc}^u(\mathbf{z}_-)$ is the graph of a real analytic map φ from a ball of center (u_-, v_-) with radius 2ρ into \mathbb{R}^2 , and $\varphi(u_-, v_-) = (0, 0)$. Similarly, $\mathbf{z}_+ = (u_+, v_+, 0, 0)$ is hyperbolic and its local stable manifold $W_{loc}^s(\mathbf{z}_+)$ is the graph of a real analytic map $\psi = (\psi_p, \psi_q)$ from a ball of center (u_+, v_+) with radius 2ρ into \mathbb{R}^2 , and $\psi(u_+, v_+) = (0, 0)$.

Let u_c be a critical point of J in H_{u^*} such that $u_c(0) = u^*(0)$ and $v_c = \mathcal{L}u_c$. Take x_1 to be the smallest value of x such that $(u_- - u_c(x))^2 + (v_- - v_c(x))^2 = \rho^2$ and x_2 the largest value of x such that $(u_+ - u_c(x))^2 + (v_+ - v_c(x))^2 = \rho^2$. Let $\eta = (U, V, P, Q)$ be the flow of (HS) at time $T = x_2 - x_1$. For any $\theta \in \mathbb{R}$, $\mathbf{z}_\theta = (\rho \cos \theta, \rho \sin \theta, \varphi(\rho \cos \theta, \rho \sin \theta))$ is in $W_{loc}^u(\mathbf{z}_-)$. If u is a critical point of J satisfying the constraint $u(0) = u^*(0)$ and sufficiently close to u_c in the H^1 topology, then the heteroclinic trajectory parametrized by $(u, \mathcal{L}u, du_x, -\mathcal{L}u_x)$ in the phase space must contain a point \mathbf{z}_θ with θ arbitrarily close, but not

equal, to the angle θ_c satisfying $(\rho \cos \theta_c, \rho \sin \theta_c) = (u_c, v_c)(x_1)$. This implies that θ is a zero of each of the functions $\chi_1(\theta) = P(\mathbf{z}_\theta) - \psi_p(U(\mathbf{z}_\theta), V(\mathbf{z}_\theta))$ and $\chi_2(\theta) = Q(\mathbf{z}_\theta) - \psi_q(U(\mathbf{z}_\theta), V(\mathbf{z}_\theta))$.

If u_c were not isolated as critical point of J satisfying $u(0) = u^*(0)$, then θ_c would not be an isolated zero of the real-analytic functions χ_1 and χ_2 . Then these functions would be identically zero near θ_c , which would mean that the flow η sends all the points \mathbf{z}_θ near \mathbf{z}_{θ_c} to points of $W_{loc}^s(\mathbf{z}_+)$, and using again analyticity, this property would hold to all real values of θ . As a consequence, $W^u(\mathbf{z}_-)$ would coincide with $W^s(\mathbf{z}_+)$ and all the trajectories in $W^u(\mathbf{z}_-)$ would be heteroclinic connections, and the corresponding functions u_θ would form a real-analytic curve in H^1 . Being critical points of J , they would all be in the same critical level of the functional. This leads to a contradiction, since Proposition 3.2 gives a trajectory in $W^u(\mathbf{z}_-)$ with action larger than this critical level. Thus at least one of χ_i is non-constant and being real analytic it has isolated zeroes. Hence u^* is an isolated critical point, as expected; the proof is complete. ■

Corollary 3.3 *There exist $h_0, \sigma_0 > 0$ and, for any $0 < h < h_0$, a radius $\bar{\sigma}(h) > 0$ with $\lim_{h \rightarrow 0} \bar{\sigma}(h) = 0$, such that the local sublevel set*

$$\mathcal{V}_h = \{u \in H_{u^*} : u(0) = u^*(0), \|u - u^*\|_{H^1(\mathbb{R})} \leq \sigma_0 \text{ and } J(u) \leq J(u^*) + h\}$$

satisfies the following property:

$$u \in \mathcal{V}_h \Rightarrow \|u - u^*\|_{H^1(\mathbb{R})} < \bar{\sigma}(h) .$$

Proof. For σ_1 small enough, the functional J satisfies the Palais-Smale condition on the closed ball of center u_* with radius σ_1 (in H^1 -norm). By Proposition 3.1, there exists $\sigma_0 \leq \sigma_1$ such that u_* is the unique minimizer of J on the closed ball of center u_* with radius σ_0 . Thus the proof is complete. ■

Consider a sufficiently large number z and define

$$\mathcal{V}_{h,z} := \{u \in H^1(-z, z) : u \equiv \hat{u} \text{ on } [-z, z] \text{ for some } \hat{u} \in \mathcal{V}_h\} . \quad (3.7)$$

Now, for $u \in \mathcal{V}_{h,z}$ with h small and z large, the functional J is C^2 and strictly convex on

$$\mathcal{C}_u := \{\tilde{u} \in H_{u^*} : \tilde{u} \equiv u \text{ on } [-z, z] \text{ and } \|\tilde{u} - u^*\|_{H^1(\mathbb{R})} \leq \bar{\sigma}(h)\} ,$$

which is a closed, bounded and convex subset of H_{u^*} . Indeed, if $\tilde{u} \in \mathcal{C}_u$, any other element of \mathcal{C}_u near \tilde{u} is of the form $\tilde{u} + w$ with $\|w\|_{H^1(\mathbb{R})}$ small and $w \equiv 0$ on $[-z, z]$, and thus direction calculation gives

$$D^2 J(\tilde{u}) \cdot w \cdot w = \int_{\mathbb{R}} \{d(w')^2 - f'(\tilde{u})w^2 + (\mathcal{L}w')^2 + \gamma(\mathcal{L}w)^2\} \geq \hat{k} \|w\|_{H^1(\mathbb{R})}$$

for some $\hat{k} > 0$.

Moreover, if a function $\tilde{u} \in \mathcal{C}_u$ satisfying $\|\tilde{u} - u^*\|_{H^1(\mathbb{R})} = \bar{\sigma}(h)$, then $J(\tilde{u}) > J(u^*) + h \geq \min_{\mathcal{C}_u} J$. So J has a minimizer, denoted by $b(u)$, which does not saturate the constraint $\|\tilde{u} - u^*\|_{H^1(\mathbb{R})} \leq \bar{\sigma}(h)$; that is, $(b(u), \mathcal{L}b(u))$ solves the system (1.3)-(1.4) outside the interval $[-z, z]$, and by the implicit function theorem, b is well-defined as a smooth function of u in H^1 topology. This provides a Lyapunov-Schmidt reduction $J_z = J \circ b$ of J defined on $\mathcal{V}_{h,z}$, and the following corollary holds.

Corollary 3.4 *For h_0 small enough, there is $z_0 > 0$ such that if $h \in (0, h_0)$ and $z > z_0$, then*

$$\rho(h) := \inf\{\|J'_z(u)\|_{(H^1(-z,z))^*} : u \in \mathcal{V}_{z,h} \text{ and } J_z(u) = J(u^*) + h\} > 0.$$

This corollary is easily proved by an indirect argument. If $\rho(h) = 0$, a Palais-Smale sequence converges to a critical point of J in a small ball of center u^* at the critical level $J(u^*) + h$. Suppose the assertion of the corollary is false, there would exist critical points of J in any small neighborhood of u^* , which would violate Proposition 3.1.

4 Construction of multi-front waves

In the construction of multi-front solutions, the trajectories between two fronts need to be in good control. Such trajectories are very close to one of the two stable equilibria with asymptotical behavior being dominated by the linearized equations. Note that for any solution of the autonomous Lagrangian system (1.3)-(1.4), its energy

$$E(u_x, v_x, u, v) = \frac{d}{2}u_x^2 - \frac{1}{2}v_x^2 - uv + \frac{\gamma}{2}v^2 + \int_0^u \xi(\xi - \beta)(1 - \xi)d\xi \quad (4.1)$$

is conserved; that is, being constant along the trajectory. We now state a lemma in the same spirit of Lemma 3.1 of [6].

Lemma 4.1 *Take any $0 < \nu < \frac{\pi}{2\omega}$. There exists a small radius $\bar{r} > 0$ such that for any given points (η_1, η_2) and $(\zeta_1, \zeta_2) \in \mathbb{R}^2$ within a distance less than \bar{r} from (u_+, v_+) , the boundary value problem*

$$\begin{aligned} -du'' &= f(u) - v, \\ -v'' &= u - \gamma v, \\ (u(0), v(0)) &= (\eta_1, \eta_2), \quad (u(T), v(T)) = (\zeta_1, \zeta_2), \end{aligned}$$

has a solution, denoted by $(\bar{U}, \bar{V})_{T, \eta_1, \eta_2, \zeta_1, \zeta_2}(\cdot)$, staying in a small neighborhood of (u_+, v_+) and it is the only one having this property. Moreover, if

$\pm\lambda \pm i\omega$ are the eigenvalues of the linearization of (HS) at $(u_+, v_+, 0, 0)$ and $E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(T)$ denotes the associated energy for the solution $(\bar{U}, \bar{V})_{T, \eta_1, \eta_2, \zeta_1, \zeta_2}$, then the sign of the function $E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(\cdot)$ has the following property: there is a real number κ_+ and, for each $r \leq \bar{r}/2$, a smaller radius $\epsilon(r)$ proportional to r , such that, if $|(u^*(z), v^*(z)) - (u_+, v_+)| = r$, $|(\eta_1, \eta_2) - (u^*(z), v^*(z))| < \epsilon$, $|(\zeta_1, \zeta_2) - (u^*(z), v^*(z))| < \epsilon$ and $\tilde{n} \geq 1/\epsilon$ with \tilde{n} an integer, then

$$E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(\kappa_+ - 2z + 2\pi\tilde{n}/\omega - \nu) > 0,$$

$$E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(\kappa_+ - 2z + 2\pi\tilde{n}/\omega + \nu) < 0.$$

A similar assertion holds when replacing (u_+, v_+) by (u_-, v_-) and (u^*, v^*) by (u_*, v_*) , but here κ_+ should be replaced by a possibly different phase κ_- .

We refer to [6] for a proof; there the existence and local uniqueness of (\bar{U}, \bar{V}) follow from Lemma A.3, the sign property of the energy is a consequence of Lemma A.2, and see Lemma 3.1 for the detail.

We now get into details about how to construct the multi-front solutions. Let $h > 0$ be small and $D > 0$ large (to be determined later as depending on h). Pick an arbitrary finite interval of integers $[1, N]$ and an arbitrary finite sequence of positive integers $\mathbf{n} = (n_i)_{1 \leq i \leq N}$ such that $n_i \geq D$ for all i . Take $z > 0$ large enough so that $(u^*(-z) - u_-)^2 + (v^*(-z) - v_-)^2 \leq \bar{r}^2$ and $(u^*(z) - u_+)^2 + (v^*(z) - v_+)^2 \leq \bar{r}^2$, where \bar{r} is the small radius considered in Lemma 4.1.

Recall $\mathcal{V}_{h,z}$ from (3.7) and introduce a smooth map $b_{\mathbf{n}}$ from $(\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N$ into $H_{loc}^1(\mathbb{R}, \mathbb{R})$, defined as follows:

For $(\mathbf{u}, \mathbf{x}) = ((u_i)_{0 \leq i \leq N}, (x_i)_{1 \leq i \leq N}) \in (\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N$, we associate a unique function $u = b_{\mathbf{n}}(\mathbf{u}, \mathbf{x})$, which satisfies the following conditions:

- (S₁) $\forall i \in [0, N] \cap 2\mathbb{Z}$, $u \equiv u_i(\cdot - C_i)$ on $(C_i - z, C_i + z)$,
- (S₂) $\forall i \in [0, N] \cap (2\mathbb{Z} + 1)$, $u \equiv u_i(C_i - \cdot)$ on $(C_i - z, C_i + z)$,
- (S₃) $\|u - u_-\|_{H^1(-\infty, -z)} \leq K\bar{r}$,
- (S₄) $\forall i \in [0, N - 1] \cap 2\mathbb{Z}$, $\|u - u_+\|_{H^1(C_i+z, C_{i+1}-z)} \leq K\bar{r}$,
- (S₅) $\forall i \in [0, N - 1] \cap (2\mathbb{Z} + 1)$, $\|u - u_-\|_{H^1(C_i+z, C_{i+1}-z)} \leq K\bar{r}$,
- (S₆) $\|u - u_{\pm}\|_{H^1(C_N+z, \infty)} \leq K\bar{r}$, where $u_{\pm} = u_+$ for N even, $u_{\pm} = u_-$ for N odd,
- (S₇) $C_0 = 0$, $C_{i+1} = C_i + X_i$ ($0 \leq i \leq N - 1$),
- (S₈) $X_{2j} = x_{2j} + \kappa_+ + \frac{2\pi n_{2j}}{\omega}$,
- (S₉) $X_{2j+1} = x_{2j+1} + \kappa_- + \frac{2\pi n_{2j+1}}{\omega}$,
- (S₁₀) $(u, \mathcal{L}u)$ satisfies (1.3)-(1.4) on each of the intervals $(-\infty, -z]$, $[C_i + z, C_{i+1} - z]$, $[C_N + z, +\infty)$.

Choosing \bar{r} small enough and a large K not depending on \bar{r} , we claim that conditions (S_1) - (S_{10}) determine u in a unique way, and explain why the corresponding function $b_{\mathbf{n}}(\mathbf{u}, \mathbf{x})$ is smooth. Observe that one can define the set $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$ consisting of all functions u satisfying conditions (S_1) - (S_6) . This set is convex, bounded, closed in the H^1 topology. Moreover, the controls (S_3) - (S_6) on u imply the strict convexity on J restricted to $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$. Indeed, if $u \in \mathcal{U}_{(\mathbf{u}, \mathbf{x})}$, any other element of $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$ near u is of the form $u + w$ with $\|w\|_{H^1(\mathbb{R})}$ small and $w \equiv 0$ on $\bigcup_{1 \leq i \leq N} [C_i - z, C_i + z]$, and then direction calculation gives

$$D^2 J(u) \cdot w \cdot w = \int_{\mathbb{R}} \{d(w')^2 - f'(u)w^2 + (\mathcal{L}w')^2 + \gamma(\mathcal{L}w)^2\} \geq \bar{k}\|w\|_{H^1(\mathbb{R})}$$

for some $\bar{k} > 0$, exactly as in the proof of Corollary 3.3.

So J has a unique minimizer in $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$. Moreover for K large enough, if a function u belongs to $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$, and saturates at least one of the constraints (S_3) - (S_6) then $J(u) > \min_{\mathcal{U}_{(\mathbf{u}, \mathbf{x})}} J$. In conclusion, the minimizer does not saturate any of the constraints, so it is the only solution of (S_{10}) in $\mathcal{U}_{(\mathbf{u}, \mathbf{x})}$ and the implicit function theorem gives a smooth function $b_{\mathbf{n}}$ of (\mathbf{u}, \mathbf{x}) in the H^1 topology.

Up to this stage, a Lyapunov-Schmidt reduction has been performed, and the next task is to minimize the reduced functional $\mathbf{J} = J \circ b_{\mathbf{n}}$. The existence of a minimizer is easily established. Indeed, the set \mathcal{V}_h is a bounded, closed sublevel set of the weakly lower semicontinuous functional J , thus it is weakly compact in $H^1(-z, z)$. By the weak lower semicontinuity of \mathbf{J} , there exists a minimizer $(\bar{\mathbf{u}}, \bar{\mathbf{x}})$ in the weakly compact set $(\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N$.

Lemma 4.2 *Given z large, h small and choose D large enough I if $n_i \geq D$ for every i . then $\hat{u}_{\mathbf{n}} := b_{\mathbf{n}}(\bar{\mathbf{u}}, \bar{\mathbf{x}})$ is a local minimizer of J .*

To prove Lemma 4.2, we introduce the set $\mathcal{O} = \bigcup_{(\mathbf{u}, \mathbf{x}) \in (\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N} \mathcal{U}_{(\mathbf{u}, \mathbf{x})}$ consisting of functions u satisfying (S_1) - (S_6) for some $(\mathbf{u}, \mathbf{x}) \in (\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N$. The next lemma shows that \mathcal{O} contains a small ball in $H^1(\mathbb{R})$ with center at $\hat{u}_{\mathbf{n}}$. Clearly $\hat{u}_{\mathbf{n}}$ minimizes J on \mathcal{O} , by virtue of the construction used in the variational argument, and thus Lemma 4.2 is an immediate consequence.

Lemma 4.3 *Given z large and h small. Suppose that D is chosen large enough and $n_i \geq D$ for all i . If $(\bar{\mathbf{u}}, \bar{\mathbf{x}}) = ((\bar{u}_i)_{0 \leq i \leq N}, (\bar{x}_i)_{1 \leq i \leq N})$ is a minimizer of \mathbf{J} in $(\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N$ then*

- (i) $J_z(\bar{u}_i) < J(u^*) + h$ for all $0 \leq i \leq N$,
- (ii) $-\nu < \bar{x}_i < \nu$ for all $1 \leq i \leq N$.

The following lemma will be used to prove Lemma 4.3.

Lemma 4.4 *Let z and h be given as above, both are not depending on N . For any $\alpha > 0$ there exists $\bar{D}(\alpha)$, not depending on N , such that if $n_i \geq \bar{D}(\alpha) \forall 1 \leq i \leq N$ then*

$$\|J'_z(u_i) - \partial_{u_i} \mathbf{J}\|_{(H^1(-z,z))^*} < \alpha, \quad \forall (\mathbf{u}, \mathbf{x}) \in (\mathcal{V}_{h,z})^{N+1} \times [-\nu, \nu]^N, \quad 0 \leq i \leq N.$$

The proof of Lemma 4.4 is standard (see e.g. [6]). We omit it.

Proof of Lemma 4.3. We argue indirectly. Suppose that $J_z(\bar{u}_l) = J(u^*) + h$ for some $l \in (0, N)$, applying Lemma 4.4 yields

$$\langle \partial_{u_l} \mathbf{J}(\bar{\mathbf{u}}, \bar{\mathbf{x}}), \nabla_{H^1(-z,z)} J_z(\bar{u}_l) \rangle \geq \frac{\rho(h)}{2},$$

with $\rho(h)$ given by Corollary 3.4. Then moving u_l slightly in the direction of $-\nabla_{H^1(-z,z)} J_z(\bar{u}_l)$ would decrease $\mathbf{J}(\mathbf{u}, \mathbf{x})$, which contradicts the minimality of $\mathbf{J}(\bar{\mathbf{u}}, \bar{\mathbf{x}})$. The proof of (i) is complete.

We next apply Lemma 4.1 to prove (ii). Fix z large and h small enough so that

$$\begin{aligned} r &:= |(u^*(z), v^*(z)) - (u_+, v_+)| \leq \bar{r}/2, \\ |(\bar{u}_i(\pm z), \mathcal{L}\bar{u}_i(\pm z)) - (u^*(\pm z), v^*(\pm z))| &< \epsilon \end{aligned}$$

with ϵ as in Lemma 4.1 and $n_i \geq 1/\epsilon$ being imposed. Suppose $\bar{x}_l = -\nu$ for some $l \in (1, N-1)$, it follows from Lemma 4.1 that

$$\partial_{x_l} \mathbf{J}(\bar{\mathbf{u}}, \bar{\mathbf{x}}) = -E_{\eta_1, \eta_2, \zeta_1, \zeta_2}(\kappa_+ - 2z + 2\pi n/\omega - \nu) < 0,$$

where $\eta_1 = \bar{u}_l(C_l + z)$, $\eta_2 = \mathcal{L}\bar{u}_l(C_l + z)$, $\zeta_1 = \bar{u}_{l+1}(C_{l+1} - z)$, $\zeta_2 = \mathcal{L}\bar{u}_{l+1}(C_{l+1} - z)$. Then increasing x_l slightly would make \mathbf{J} small, which again contradicts the minimality of $\mathbf{J}(\bar{\mathbf{u}}, \bar{\mathbf{x}})$. Likewise, if $\bar{x}_l = \nu$ we could decrease \mathbf{J} by slightly decreasing x_l . Now the proof of Lemma 4.3 is complete.

We are now ready to prove the existence result of multi-front solutions, stated in Theorem 1.2. The stability of such solutions will be investigated in the next section.

Proof of Theorem 1.2. Take \bar{r} small enough so that $K\bar{r} < \sigma$, the small radius required as in the statement. Pick z large and h small enough so that the small number $\bar{\sigma}(h)$, defined in Corollary 3.3, is less than σ , and this then enables us to apply Lemma 4.2. To complete the existence proof, simply assign D_σ to be the number D stated in Lemma 4.2.

5 Stability

In this section a Lyapunov functional will be introduced to prove Theorem 1.4. For $u \in a_u + H$, $v \in a_v + H$, define

$$\mathcal{E}(u, v) := J(u) + \frac{\gamma}{2(1 + \hat{\delta})} \|v - \mathcal{L}(u)\|^2. \quad (5.1)$$

As in the proof of Theorem 1.2, $\hat{u}_{\mathbf{n}}$ is a local minimizer of J and $\hat{v}_{\mathbf{n}} = \mathcal{L}\hat{u}_{\mathbf{n}}$. This together with Proposition 3.1 shows that $(\hat{u}_{\mathbf{n}}, \hat{v}_{\mathbf{n}})$ is a local minimizer of \mathcal{E} in the $H^1(\mathbb{R}) \times L^2(\mathbb{R})$ topology, and it is an isolated critical point of \mathcal{E} up to translation in spatial variable. Moreover \mathcal{E} satisfies the Palais-Smale condition in a small neighborhood of $(\hat{u}_{\mathbf{n}}, \hat{v}_{\mathbf{n}})$.

The next proposition shows that $\mathcal{E}(u, v)$ is a Lyapunov functional for the evolution flow generated by (1.1)-(1.2), from which Theorem 1.4 immediately follows.

Proposition 5.1 *Assume that $0 < \tau < \gamma^2$. Let $\hat{\delta} > 0$ and satisfy $1 + \hat{\delta}/2 < \gamma^2/\tau$. Then for any smooth solution $(u(x, t), v(x, t))$ of (1.1)-(1.2),*

$$\begin{aligned} \frac{d}{dt} \mathcal{E}(u(\cdot, t), v(\cdot, t)) &\leq -\frac{\hat{\delta}}{2(1 + \hat{\delta})} \|u_t\|^2 \\ &\quad - \frac{1}{1 + \hat{\delta}} \left(\frac{\gamma^2}{\tau} - 1 - \frac{\hat{\delta}}{2} \right) \|v - \mathcal{L}(u)\|^2 - \frac{\gamma}{(1 + \hat{\delta})\tau} \|\nabla(v - \mathcal{L}(u))\|^2. \end{aligned}$$

Proof. Let $w = v - \mathcal{L}(u)$. It is easy to verify that (1.1)-(1.2) is equivalent to

$$u_t = d\Delta u + u(u - \beta)(1 - u) - \mathcal{L}(u) - w, \quad (5.2)$$

$$\tau(w_t + \mathcal{L}(u_t)) = \Delta w - \gamma w. \quad (5.3)$$

In terms of (u, w) , we rewrite (5.1) as

$$\mathcal{E}_1(u, w) := J(u) + \frac{\gamma}{2(1 + \hat{\delta})} \|w\|^2. \quad (5.4)$$

Let $(u(x, t), w(x, t))$ be a smooth solution of (5.2)-(5.3). Since $(w, \mathcal{L}(u_t))_{L^2} = (\mathcal{L}(w), u_t)_{L^2}$, a direct calculation gives

$$\begin{aligned} &\frac{d}{dt} \mathcal{E}_1(u(\cdot, t), w(\cdot, t)) \\ &= - \int_{\Omega} (d\Delta u + u(u - \beta)(1 - u) - \mathcal{L}(u)) u_t dx + \frac{\gamma}{1 + \hat{\delta}} (w, w_t)_{L^2} \\ &= - \|u_t\|^2 - (w, u_t)_{L^2} - \frac{\gamma}{1 + \hat{\delta}} \left((w, \mathcal{L}(u_t))_{L^2} + \frac{1}{\tau} (\|\nabla w\|^2 + \gamma \|w\|^2) \right) \end{aligned}$$

$$\begin{aligned}
&\leq -\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|w\|^2 - \frac{\gamma}{(1+\hat{\delta})}(\mathcal{L}(w), u_t)_{L^2} \\
&\quad - \frac{\gamma^2}{(1+\hat{\delta})\tau}\|w\|^2 - \frac{\gamma}{(1+\hat{\delta})\tau}\|\nabla w\|^2 \\
&\leq -\frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|w\|^2 + \frac{1}{(1+\hat{\delta})}\|w\|\|u_t\| - \frac{\gamma^2}{(1+\hat{\delta})\tau}\|w\|^2 - \frac{\gamma}{(1+\hat{\delta})\tau}\|\nabla w\|^2 \\
&\leq -\frac{1}{2}\left(1 - \frac{1}{1+\hat{\delta}}\right)\|u_t\|^2 + \left(\frac{1}{2} + \frac{1}{2(1+\hat{\delta})} - \frac{\gamma^2}{(1+\hat{\delta})\tau}\right)\|w\|^2 \\
&\quad - \frac{\gamma}{(1+\hat{\delta})\tau}\|\nabla w\|^2 \\
&\leq -\frac{\delta}{2(1+\hat{\delta})}\|u_t\|^2 - \frac{1}{1+\hat{\delta}}\left(\frac{\gamma^2}{\tau} - 1 - \frac{\hat{\delta}}{2}\right)\|w\|^2 - \frac{\gamma}{(1+\hat{\delta})\tau}\|\nabla w\|^2 \leq 0.
\end{aligned}$$

Thus $\mathcal{E}(u, v)$ and $\mathcal{E}_1(u, w)$ are non-increasing functions of t along the trajectory of a solution of (1.1)-(1.2). \blacksquare

6 Unstable waves

In this section the same notation as in the proof of Theorem 1.2 will be used to prove Theorem 1.5. We employ the mountain-pass principle to seek such critical points. Starting with the set $\mathcal{V}_{h,z}$ defined by (3.7), we construct a smooth map \check{b}_n from $(\mathcal{V}_{h,z})^2 \times [-\nu, \nu]$ to $H^1(\mathbb{R})$; here to each (u_0, u_1, x) in $\mathcal{V}_{h,z}^2 \times [-\nu, \nu]$, we associate the unique function $u \in H^1(\mathbb{R})$ satisfying the following conditions:

$$\begin{aligned}
(S'_1) \quad &u \equiv u_1(X - \cdot) \text{ on } (X - z, X + z), \\
(S'_2) \quad &u \equiv u_0 \text{ on } (-z, z), \\
(S'_3) \quad &\|u - u_-\|_{H^1(-\infty, -z)} \leq K\bar{r}, \\
(S'_4) \quad &\|u - u_-\|_{H^1(X+z, \infty)} \leq K\bar{r}, \\
(S'_5) \quad &X = x + \kappa_+ + \frac{\pi(2n+1)}{\omega}, \\
(S'_6) \quad &(u, \mathcal{L}u) \text{ satisfies (1.3)-(1.4) on each of the intervals } (-\infty, -z], \\
&\quad [z, X - z], [X + z, +\infty).
\end{aligned}$$

With this definition of \check{b}_n , we define $\check{J} := J \circ \check{b}_n$. Then $\check{b}_n(u_0, u_1, x)$ is a critical point of J if (u_0, u_1, x) is a critical point of \check{J} in $\mathcal{V}_{h,z}^2 \times [-\nu, \nu]$.

Note that Lemma 4.4 still holds in the present situation. Hence for n large enough, there exists $\check{\rho}_h > 0$ such that if $(u_0, u_1, x) \in \mathcal{V}_{h,z}^2 \times [-\nu, \nu]$ and $J_z(\bar{u}_l) \in$

$[J(u^*) + h/2, J(u^*) + h]$ with $l = 0$ or 1 then

$$\langle \partial_{u_l} \check{J}(u_0, u_1, x), \nabla_{H^1(-z, z)} J_z(u_l) \rangle \geq \check{\rho}_h. \quad (6.1)$$

Moreover, adapting Lemma 4.1 to the present situation, we see that, for each n large enough, there is a small μ_n such that if $x \in [-\nu, -\nu/2]$ then

$$\partial_x \check{J}(u_0, u_1, x) \geq \mu_n, \quad (6.2)$$

while for $x \in [\nu/2, \nu]$,

$$\partial_x \check{J}(u_0, u_1, x) \leq -\mu_n. \quad (6.3)$$

Pick $\mathbf{a} \in (\mathcal{V}_{h,z})^2 \times \{-\nu\}$ be such that $\check{J}(\mathbf{a}) \leq \inf_{(\mathcal{V}_{h,z})^2 \times \{-\nu\}} \check{J} + \mu_n \nu / 4$ and $\mathbf{b} \in (\mathcal{V}_{h,z})^2 \times \{\nu\}$ be such that $\check{J}(\mathbf{b}) \leq \inf_{(\mathcal{V}_{h,z})^2 \times \{\nu\}} \check{J} + \mu_n \nu / 4$. Set

$$\Gamma := \{\bar{\gamma} \in C^0([0, 1], (\mathcal{V}_{h,z})^2 \times [-\nu, \nu]) : \bar{\gamma}(0) = \mathbf{a}, \bar{\gamma}(1) = \mathbf{b}\}$$

and define

$$c_n := \inf_{\bar{\gamma} \in \Gamma} \max_{[0, 1]} \check{J} \circ \bar{\gamma}.$$

It follows from (6.2) and (6.3) that $c_n \geq \max(\check{J}(\mathbf{a}), \check{J}(\mathbf{b})) + \mu_n \nu / 4$.

For any $\bar{\gamma} \in \Gamma$, (6.1) together with the standard deformation theory gives a $\tilde{\gamma} \in \Gamma$ such that $\max_{[0, 1]} \check{J} \circ \tilde{\gamma} \leq \max_{[0, 1]} \check{J} \circ \bar{\gamma}$ and the image of $\tilde{\gamma}$ stays in the set $\mathcal{V}_{h/2, z}^2 \times [-\nu, \nu]$. Moreover it easily checked that \check{J} satisfies the Palais-Smale condition. Since the critical points of \check{J} are isolated, we may apply a result of Hofer [23] to find a ‘‘mountain-pass’’ type critical point $(u_0^\sharp, u_1^\sharp, x^\sharp)$ of \check{J} . This tells that for any neighborhood \mathcal{O} of $(u_0^\sharp, u_1^\sharp, x^\sharp)$ there exists $(u'_0, u'_1, x') \in \mathcal{O}$ such that $\check{J}(u'_0, u'_1, x') < c_n$. Setting $(\check{u}_n, \check{v}_n) := (\check{b}_n(u_0^\sharp, u_1^\sharp, x^\sharp), \mathcal{L}\check{b}_n(u_0^\sharp, u_1^\sharp, x^\sharp))$ gives a two-bump solution of (1.3)-(1.4) which satisfies Theorem 1.5(i),(ii), and $J(\check{u}_n) = c_n$.

Let us recall the Lyapunov functional \mathcal{E} defined in Section 5 for showing the instability of $(\check{u}_n, \check{v}_n)$. With $H^1 \times L^2$ topology, we can find a neighborhood of $(\check{u}_n, \check{v}_n)$ which possesses a single critical point only. Inside this set, any smaller neighborhood of $(\check{u}_n, \check{v}_n)$ contains a point $(\tilde{u}, \tilde{v}) := (\check{b}_n(u'_0, u'_1, x'), \mathcal{L}\check{b}_n(u'_0, u'_1, x'))$ with the property $J(\tilde{u}) < c_n$. If $(u(x, t), v(x, t))$ is a solution of (1.1)-(1.2) with the initial datum $(u(x, 0), v(x, 0)) = (\tilde{u}(x), \tilde{v}(x))$, it is clear that $\mathcal{E}(u(x, t), v(x, t)) < \mathcal{E}(\tilde{u}, \tilde{v}) < \mathcal{E}(\check{u}_n, \check{v}_n)$. Hence there exist $T_0 > 0$ and a small neighborhood \mathcal{N} of $(\check{u}_n, \check{v}_n)$ such that $(u(x, t), v(x, t)) \notin \mathcal{N}$ if $t \geq T_0$. This completes the proof of (iii), so does Theorem 1.5.

7 Appendix

As a byproduct of [13], the proof of Theorem 1.1 shows that there exists a heteroclinic solution for the system

$$-du'' = k(u - u^3) - v, \quad (7.1)$$

$$-v'' = u - \gamma v. \quad (7.2)$$

Here (u_+, v_+) is replaced by $(\sqrt{1 - 1/(k\gamma)}, \sqrt{1 - 1/(k\gamma)}/\gamma)$ and $(u_-, v_-) = (-u_+, -v_+)$. The nonlinearity in (7.1) is an odd function, which gives the same type of potential as in the Allen-Cahn equation. The following observation indicates that system (1.3)-(1.4) can be converted into (7.1)-(7.2).

Suppose that (\bar{u}, \bar{v}) is a heteroclinic solution of (7.1)-(7.2). By setting $k = \frac{1}{3}(\beta^2 - \beta + 1)$ and

$$\begin{cases} u^* = (\beta + 1)/3 + \sqrt{k}\bar{u}, \\ v^* = (\beta + 1)/3\gamma + \sqrt{k}\bar{v}, \end{cases} \quad (7.3)$$

a simple calculation easily verifies that (u^*, v^*) is a heteroclinic solution of (1.3)-(1.4).

In this Appendix, we clarify the conditions on the parameters such that both (u_-, v_-) and (u_+, v_+) are saddle-focus equilibria. First from the assumptions of Theorem 1.1, $\gamma = 9/(2\beta^2 - 5\beta + 2)$ and $d\gamma^2 > 1$. In view of (7.1)-(7.2), since $u_+^2 = 1 - \frac{1}{\gamma k}$, it follows that $k\gamma > 1$. Therefore

$$k(1 - 3u_+^2) - \gamma d = \frac{1}{\gamma}(3 - 2k\gamma - \gamma^2 d) < 0. \quad (7.4)$$

Consider the linearization of (7.1)-(7.2) at $(\sqrt{1 - 1/(k\gamma)}, \sqrt{1 - 1/(k\gamma)}/\gamma)$. If

$$[k(1 - 3u_+^2) - \gamma d]^2 - 4d < 0, \quad (7.5)$$

then all the eigenvalues are complex numbers, which is case of saddle-focus. Combining (7.4) with (7.5) yields

$$\frac{1}{\gamma}(3 - 2k\gamma - \gamma^2 d) > -2\sqrt{d},$$

which can be rewritten as

$$\gamma d - 2\sqrt{d} + (2k - \frac{3}{\gamma}) < 0. \quad (7.6)$$

Solving (7.6) gives

$$\frac{1 - \sqrt{4 - 2\gamma k}}{\gamma} < \sqrt{d} < \frac{1 + \sqrt{4 - 2\gamma k}}{\gamma}, \quad (7.7)$$

provided that $k\gamma < 2$. Note that $k\gamma = \frac{3\beta^2-3\beta+3}{2\beta^2-5\beta+2} = \frac{3}{2} + \frac{9\beta}{2(2\beta^2-5\beta+2)} > \frac{3}{2}$ for $\beta \in (0, \frac{1}{2})$. This together with $k\gamma < 2$ implies

$$\frac{9\beta}{2\beta^2 - 5\beta + 2} < 1$$

and consequently $9\beta < 2\beta^2 - 5\beta + 2$. Solving $\beta^2 - 7\beta + 1 > 0$ yields

$$\beta < \frac{7 - \sqrt{45}}{2}, \quad (7.8)$$

as $\beta \in (0, \frac{1}{2})$ rules out the possibility of $\beta > \frac{7+\sqrt{45}}{2}$. Since $k\gamma > \frac{3}{2}$, it follows from (7.7) and $d\gamma^2 > 1$ that

$$\frac{1}{\gamma} < \sqrt{d} < \frac{2}{\gamma}. \quad (7.9)$$

In summary, the equilibria (u_-, v_-) and (u_+, v_+) are saddle-focus if and only if

$$\beta \in (0, \frac{7 - \sqrt{45}}{2}), \quad \gamma = 9/(2\beta^2 - 5\beta + 2) \text{ and } \frac{1}{\gamma} < \sqrt{d} < \frac{2}{\gamma},$$

as stated in (1.6).

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