



**HAL**  
open science

## On the regularity of abnormal minimizers for rank 2 sub-Riemannian structures

Davide Barilari, Yacine Chitour, Frédéric Jean, Dario Prandi, Mario Sigalotti

► **To cite this version:**

Davide Barilari, Yacine Chitour, Frédéric Jean, Dario Prandi, Mario Sigalotti. On the regularity of abnormal minimizers for rank 2 sub-Riemannian structures. *Journal de Mathématiques Pures et Appliquées*, 2020, 133, pp.118-138. 10.1016/j.matpur.2019.04.008 . hal-01757343v2

**HAL Id: hal-01757343**

**<https://hal.science/hal-01757343v2>**

Submitted on 24 Oct 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON THE REGULARITY OF ABNORMAL MINIMIZERS FOR RANK 2 SUB-RIEMANNIAN STRUCTURES

D. BARILARI, Y. CHITOUR, F. JEAN, D. PRANDI, AND M. SIGALOTTI

ABSTRACT. We prove the  $C^1$  regularity for a class of abnormal length-minimizers in rank 2 sub-Riemannian structures. As a consequence of our result, all length-minimizers for rank 2 sub-Riemannian structures of step up to 4 are of class  $C^1$ .

## CONTENTS

1. Introduction	1
2. Notations and preliminary notions	3
3. Desingularisation and nilpotentization	5
4. Dynamics of abnormal extremals: preliminary results	7
5. Dynamics of abnormal extremals in a special case: proof of Proposition 3	9
6. Dynamics of abnormal extremals: the general case	10
7. Proof of Theorem 1	15
Appendix A. An elementary lemma	18
References	19

## 1. INTRODUCTION

The question of regularity of length-minimizers is one of the main open problems in sub-Riemannian geometry, cf. for instance [Mon02, Problem 10.1] or [Agr14, Problem II] and the survey [Mon14a].

Length-minimizers are solutions to a variational problem with constraints and satisfy a first-order necessary condition resulting from the Pontryagin Maximum Principle. With every length-minimizer  $\gamma : [0, T] \rightarrow M$  we can associate a lift  $\lambda : [0, T] \rightarrow T^*M$  in the cotangent space, satisfying a Hamiltonian equation. This lift can be either *normal* or *abnormal*, although a length-minimizer  $\gamma$  can actually admit several lifts, each of them being either normal or abnormal.

If a length-minimizer admits a normal lift, then it is smooth, i.e.,  $C^\infty$ , since normal lifts are solutions of smooth autonomous Hamiltonian systems in  $T^*M$ . Note that we assume length-minimizers to be parametrized by arclength and their regularity is meant with respect to this time parametrization. The question of regularity is then reduced to length-minimizers that are strictly abnormal, i.e., those which do not admit normal lifts. For such length-minimizers, from the first order necessary condition (and actually from the second order one as well) it is a priori not possible to deduce any regularity other than Lipschitz continuity.

In this paper we investigate the following.

**Open Problem.** *Are all length-minimizers in a sub-Riemannian manifold of class  $C^1$ ?*

If the sub-Riemannian structure has step 2, there are no strictly abnormal length-minimizers, see e.g. [AS95, ABB17], thus every length-minimizer admits a normal lift,

and is hence smooth. For step 3 structures, the situation is already more complicated and a positive answer to the above problem is known only for Carnot groups (where, actually, length-minimizers are proved to be  $C^\infty$ ), see [LDLMV13, TY13]. When the sub-Riemannian structure is analytic, more is known on the size of the set of points where a length-minimizer can lose regularity [Sus14], regardless of the rank and of the step of the distribution.

To state our main result, we introduce some notations. We refer the reader to Section 2 for precise definitions. Recall that a sub-Riemannian structure  $(D, g)$  on  $M$  is defined by a bracket generating distribution  $D$  endowed with a metric  $g$ . Hence  $D$  defines a flag of subspaces at every point  $x \in M$

$$D_x = D_x^1 \subset D_x^2 \subset D_x^3 \subset \dots \subset D_x^r = T_x M,$$

where  $D_x^i$  is the subspace of the tangent space spanned by Lie brackets of length at most  $i$  between horizontal vector fields. This induces a dual decreasing sequence of subspaces of  $T_x^* M$

$$0 = (D_x^r)^\perp \subset \dots \subset (D_x^4)^\perp \subset (D_x^3)^\perp \subset (D_x^2)^\perp \subset (D_x^1)^\perp \subset T_x^* M,$$

where perpendicularity is considered with respect to the duality product. By construction, any abnormal lift satisfies  $\lambda(t) \in (D^1)^\perp$  for every  $t$ . If the lift is strictly abnormal, then by Goh conditions  $\lambda(t) \in (D^2)^\perp$  for every  $t$ .

When the distribution has rank 2, it is known that if  $\lambda(t)$  does not cross  $(D^3)^\perp$ , then the length-minimizer is  $C^\infty$  [LS95, Sect. 6.2, Cor. 4]. Our main result pushes this analysis further and establishes that the answer to the Open Problem is positive for length-minimizers whose abnormal lift does not enter  $(D^4)^\perp$ .

**Theorem 1.** *Let  $(D, g)$  be a rank 2 sub-Riemannian structure on  $M$ . Assume that  $\gamma : [0, T] \rightarrow M$  is an abnormal minimizer parametrized by arclength. If  $\gamma$  admits a lift satisfying  $\lambda(t) \notin (D^4)^\perp$  for every  $t \in [0, T]$ , then  $\gamma$  is of class  $C^1$ .*

If the sub-Riemannian manifold has rank 2 and step at most 4, the assumption in Theorem 1 is trivially satisfied by every abnormal minimizer  $\gamma$  and we immediately obtain the following corollary.

**Corollary 2.** *Assume that the sub-Riemannian structure has rank 2 and step at most 4. Then all length-minimizers are of class  $C^1$ .*

It is legitimate to ask whether the  $C^1$  regularity in the Open Problem can be further improved. Indeed, the argument behind our proof permits to obtain  $C^\infty$  regularity of length-minimizers under an additional nilpotency condition on the Lie algebra generated by horizontal vector fields.

**Proposition 3.** *Assume that  $D$  is generated by two vector fields  $X_1, X_2$  such that the Lie algebra  $\text{Lie}\{X_1, X_2\}$  is nilpotent of step at most 4. Then for every sub-Riemannian structure  $(D, g)$  on  $M$ , the corresponding length-minimizers are of class  $C^\infty$ .*

The above proposition applies in particular to Carnot groups of rank 2 and step at most 4. In this case we recover the results obtained in [LM08, Example 4.6].

The strategy of proof of Theorem 1 is to show that, at points where they are not of class  $C^1$ , length-minimizers can admit only corner-like singularities. This is done by a careful asymptotic analysis of the differential equations satisfied by the abnormal lift, which exploits their Hamiltonian structure. We can then conclude thanks to the following result.

**Theorem 4** ([HL16]). *Let  $M$  be a sub-Riemannian manifold. Let  $T > 0$  and let  $\gamma : [-T, T] \rightarrow M$  be a horizontal curve parametrized by arclength. Assume that, in local*

coordinates, there exist

$$\dot{\gamma}^+(0) := \lim_{t \downarrow 0} \frac{\gamma(t) - \gamma(0)}{t}, \quad \dot{\gamma}^-(0) := \lim_{t \uparrow 0} \frac{\gamma(t) - \gamma(0)}{t}.$$

If  $\dot{\gamma}^+(0) \neq \dot{\gamma}^-(0)$ , then  $\gamma$  is not a length-minimizer.

We observe that the proof contained in [HL16] requires a previous result stated in [LM08]. A complete argument for the latter, addressing some issues raised in [Rif17, p. 1113-15], is provided in [MPV17]. For sub-Riemannian structures of rank 2 and step at most 4 (and indeed also for higher step, under an additional condition on the Lie algebra generated by horizontal vector fields), the fact that corners are not length-minimizers is already contained in [LM08].

We notice that the answer to the Open Problem is known to be positive also in a class of rank 2 Carnot groups (with no restriction on the step, but satisfying other additional conditions). For this class of structures in [Mon14b], it is proved the  $C^{1,\alpha}$  regularity for some suitable  $\alpha > 0$  (depending on the step).

We also refer to [MPV18, HL18] for recent results regarding these issues.

**1.1. Structure of the paper.** In Section 2 we recall some notations and preliminary notions. Section 3 is devoted to a desingularization and nilpotentization argument. Section 4 contains a preliminary analysis on the dynamics of abnormal extremals. To illustrate our approach in a simpler case, we discuss in Section 5 the proof of the main result for a nilpotent structure of step up to 4. Then in Sections 6 and 7 we complete our analysis to prove the general result. Appendix A contains a technical lemma.

**Acknowledgments.** We thank Ugo Boscain, Paolo Mason, Ludovic Rifford, and Luca Rizzi for many stimulating discussions. This work was supported by the Grant ANR-15-CE40-0018 SRGI ‘‘Sub-Riemannian geometry and interactions’’ and by a public grant as part of the Investissement d’avenir project, reference ANR-11-LABX-0056-LMH, LabEx LMH, in a joint call with Programme Gaspard Monge en Optimisation et Recherche Op erati onnelle.

## 2. NOTATIONS AND PRELIMINARY NOTIONS

Let  $M$  be a smooth  $n$ -dimensional manifold. A *sub-Riemannian structure* of rank  $m$  on  $M$  is a triplet  $(E, g^E, f)$  where  $E$  is a vector bundle of rank  $m$  over  $M$ ,  $g^E$  is an Euclidean metric on  $E$ , and  $f : E \rightarrow TM$  is a morphism of vector bundles such that  $f(E_x) \subseteq T_x M$  for every  $x \in M$ . Fix such a structure and define a family of subspaces of the tangent spaces by

$$D_x = \{X(x) \mid X \in D\} \subseteq T_x M, \quad \forall x \in M,$$

where  $D = \{f \circ Y \mid Y \text{ smooth section of } E\}$  is a submodule of the set of vector fields on  $M$ . We assume that the structure is *bracket generating*, i.e., the tangent space  $T_x M$  is spanned by the vector fields in  $D$  and their iterated Lie brackets evaluated at  $x$ .

The sub-Riemannian structure induces a quadratic form  $g_x$  on  $D_x$  by

$$g_x(v, v) = \inf\{g_x^E(u, u) \mid f(u) = v, u \in E_x\}, \quad v \in D_x.$$

In analogy with the classic sub-Riemannian case and to simplify notations, in the sequel we will refer to the sub-Riemannian structure as the pair  $(D, g)$  rather than  $(E, g^E, f)$ . This is justified since all the constructions and definitions below rely only on  $D$  and  $g$ . The triplet  $(M, D, g)$  is called a *sub-Riemannian manifold*.

*Remark 5.* Usually, a sub-Riemannian manifold denotes a triplet  $(M, D, g)$ , where  $M$  is a smooth manifold,  $D$  is a subbundle of  $TM$ , and  $g$  is a Riemannian metric on  $D$  (see, e.g., [Bel96]). This corresponds to the case where  $f(E_x)$  is of constant rank. The definition given above follows, for instance, [ABB17].

A *horizontal curve*  $\gamma : [0, T] \rightarrow M$  is an absolutely continuous path such that  $\dot{\gamma}(t) \in D_{\gamma(t)}$  for almost every (a.e. for short)  $t \in [0, T]$ . The *length* of a horizontal curve is defined by

$$\ell(\gamma) = \int_0^T \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

The *sub-Riemannian distance* between two arbitrary points  $x, y$  in  $M$  is then

$$d(x, y) = \inf\{\ell(\gamma) \mid \gamma(0) = x, \gamma(T) = y, \gamma \text{ horizontal}\}.$$

A *length-minimizer* is a horizontal curve  $\gamma$  which realizes the distance between its extremities, that is,  $\ell(\gamma) = d(\gamma(0), \gamma(T))$ . Note that any time-reparametrization of a length-minimizer is a length-minimizer as well.

A *generating frame* of the sub-Riemannian structure is a family of smooth vector fields  $X_1, \dots, X_k$  such that  $D$  is generated by  $X_1, \dots, X_k$  as a module and

$$g_x(v, v) = \inf \left\{ \sum_{i=1}^k u_i^2 \mid \sum_{i=1}^k u_i X_i(x) = v \right\}, \quad x \in U, v \in D_x.$$

There always exists a global generating frame (see [ABB17, Corollary 3.26]), with, in general, a number  $k$  of elements greater than the rank  $m$  of the structure. However, every point  $x \in M$  admits a neighborhood on which there exists a (local) generating frame with exactly  $k = m$  elements, e.g., by taking the image via  $f$  of a local orthonormal frame of  $(E, g^E)$ .

Fix now a (local or global) generating frame  $X_1, \dots, X_k$  of  $(D, g)$ . For any horizontal curve  $\gamma$  of finite length, there exists  $u \in L^\infty([0, T], \mathbb{R}^k)$  satisfying

$$(1) \quad \dot{\gamma}(t) = \sum_{i=1}^k u_i(t) X_i(\gamma(t)), \quad \text{for a.e. } t \in [0, T].$$

The curve is said to be *parametrized by arclength* if  $g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) = 1$  for a.e.  $t \in [0, T]$ , i.e., if there exists  $u \in L^\infty([0, T], \mathbb{S}^{k-1})$  satisfying (1). In that case  $\ell(\gamma) = T$ .

To state the first order necessary conditions, let us first introduce some notations. For  $\lambda \in T^*M$  and  $x = \pi(\lambda)$ , where  $\pi : T^*M \rightarrow M$  is the canonical projection, we set  $h_i(\lambda) = \langle \lambda, X_i(x) \rangle$ , for  $i = 1, \dots, k$  (here  $\langle \lambda, \cdot \rangle$  denotes the dual action of covectors on vectors). Recall also that, for a function  $H : T^*M \rightarrow \mathbb{R}$ , the corresponding Hamiltonian vector field  $\vec{H}$  is the unique vector field such that  $\sigma(\cdot, \vec{H}) = dH$ , where  $\sigma$  is the canonical symplectic form on the cotangent bundle.

Applying the Pontryagin Maximum Principle to the sub-Riemannian length minimization problem yields the following theorem.

**Theorem 6.** *Let  $(M, D, g)$  be a sub-Riemannian manifold with generating frame  $X_1, \dots, X_k$  and  $\gamma : [0, T] \rightarrow M$  be a length-minimizer. Then there exists a nontrivial absolutely continuous curve  $t \mapsto \lambda(t) \in T_{\gamma(t)}^*M$  such that one of the following conditions is satisfied:*

- (N)  $\dot{\lambda}(t) = \vec{H}(\lambda(t))$  for all  $t \in [0, T]$ , where  $H(\lambda) = \frac{1}{2} \sum_{i=1}^k h_i^2$ ,
- (A)  $\dot{\lambda}(t) = \sum_{i=1}^k u_i(t) \vec{h}_i(\lambda(t))$  for almost every  $t \in [0, T]$ , with  $u_1, \dots, u_k \in L^1([0, T])$ .  
Moreover,  $\lambda(t) \in (D_{\gamma(t)})^\perp$  for all  $t$ , i.e.,  $h_i(\lambda(t)) \equiv 0$  for  $i = 1, \dots, k$ .

In case (N) (respectively, case (A)),  $\lambda$  is called a *normal* (respectively, *abnormal*) *extremal*. Normal extremals are integral curves of  $\vec{H}$ . As such, they are smooth. A length-minimizer is normal (respectively, abnormal) if it admits a normal (respectively, abnormal) extremal lift. We stress that both conditions can be satisfied for the same curve  $\gamma$ , with different lifts  $\lambda_1$  and  $\lambda_2$ .

## 3. DESINGULARISATION AND NILPOTENTIZATION

**3.1. Desingularisation.** Let  $(M, D, g)$  be a sub-Riemannian manifold. We define recursively the following sequence of submodules of the set of vector fields,

$$D^1 = D, \quad D^{i+1} = D^i + [D, D^i].$$

At every point  $x \in M$ , the evaluation at  $x$  of these modules induces a flag of subspaces of the tangent space,

$$D_x^1 \subset D_x^2 \subset \dots \subset D_x^r = T_x M.$$

The smallest integer  $r = r(x)$  satisfying  $D_x^r = T_x M$  is called the *step of  $D$  at  $x$* . A point is said to be *regular* if the dimensions of the subspaces of the flag are locally constant in an open neighborhood of the point. When every point in  $M$  is regular, the sub-Riemannian manifold is said to be *equiregular*.

In general a sub-Riemannian manifold may admit non-regular points. However, for our purposes, we can restrict ourselves with no loss of generality to equiregular manifolds thanks to a desingularisation procedure.

**Lemma 7.** *Fix an integer  $m \geq 2$ . Assume that for every rank  $m$  equiregular sub-Riemannian structure the following property holds: every arclength parametrized abnormal minimizer admitting a lift  $\lambda(t) \notin (D^4)^\perp$  is of class  $C^1$ . Then the same property holds true for every rank  $m$  sub-Riemannian structure.*

*Proof.* Let  $(M, D, g)$  be a non-equiregular sub-Riemannian manifold of rank  $m$  and  $\gamma$  be an abnormal length-minimizer of  $(M, D, g)$  which admits an abnormal extremal lift such that  $\lambda(t) \notin (D^4)^\perp$  for every  $t \in [0, T]$ . Assume moreover that  $\gamma$  is parametrized by arclength. We have to prove that  $\gamma$  is of class  $C^1$ .

Fix  $t_0 \in [0, T]$  and a generating frame  $X_1, \dots, X_m$  on a neighborhood of  $\gamma(t_0)$ . By [Jea14, Lemma 2.5], there exists an equiregular sub-Riemannian manifold  $(\widetilde{M}, \widetilde{D}, \widetilde{g})$  of rank  $m$  with a generating frame  $\xi_1, \dots, \xi_m$  and a map  $\varpi : \widetilde{M} \rightarrow M$  onto a neighborhood  $U \subset M$  of  $\gamma(t_0)$  such that  $\varpi_* \xi_i = X_i$ . Up to reducing the interval  $[0, T]$  we assume that  $\gamma(t) \in U$  for all  $t \in [0, T]$ . Let  $u \in L^\infty([0, T], \mathbb{S}^{m-1})$  be such that

$$\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) X_i(\gamma(t)), \quad \text{a.e. } t.$$

By construction, since  $\gamma$  is a length-minimizer, there exists a length-minimizer  $\widetilde{\gamma}$  in  $\widetilde{M}$  with  $\varpi(\widetilde{\gamma}) = \gamma$  associated with the same  $u$ , that is,

$$\dot{\widetilde{\gamma}}(t) = \sum_{i=1}^m u_i(t) \xi_i(\widetilde{\gamma}(t)), \quad \text{a.e. } t,$$

which is parametrized by arclength as well. Hence the trajectory  $\gamma$  has at least the same regularity as  $\widetilde{\gamma}$ .

Moreover, if  $\lambda$  is an abnormal lift of  $\gamma$  in  $T^*M$ , then  $\widetilde{\lambda}$  admits an abnormal lift  $\widetilde{\lambda}$  in  $T^*\widetilde{M}$  such that  $\widetilde{\lambda}(t) = \varpi^* \lambda(t)$  for every  $t$ . Since  $\varpi^*(\widetilde{D}^k)^\perp = (D^k)^\perp$  for any positive integer  $k$ , the property  $\lambda(t) \notin (D^4)^\perp$  implies  $\widetilde{\lambda}(t) \notin (\widetilde{D}^4)^\perp$ .

It results from the hypothesis that  $\widetilde{\gamma}$  is  $C^1$ , so  $\gamma$  is of class  $C^1$  in an open neighborhood of  $t_0 \in [0, T]$ , which ends the proof.  $\square$

As a consequence of Lemma 7, we can assume in the rest of the paper that the sub-Riemannian manifold is equiregular.

**3.2. Nilpotentization.** Let us recall the construction of the nilpotent approximation (see for instance [Bel96] for details).

Let  $(M, D, g)$  be an equiregular sub-Riemannian manifold. We fix a point  $x \in M$  and a local generating frame  $X_1, \dots, X_m$  in a neighborhood of  $x$ .

For  $i = 1, \dots, m$ , let  $w_i$  be the smallest integer  $j$  such that  $\dim D_x^j \geq i$ . We define the dilations  $\delta_\nu : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for  $\nu \in \mathbb{R}$  as  $\delta_\nu(z) = (\nu^{w_1} z_1, \dots, \nu^{w_m} z_m)$ . Let  $z^x$  be a system of privileged coordinates at  $x$  and set  $\delta_\nu^x = \delta_\nu \circ z^x$ . Then, for  $i = 1, \dots, m$ , the vector field  $\varepsilon(\delta_{1/\varepsilon}^x)_* X_i$  converges locally uniformly as  $\varepsilon \rightarrow 0$  to a vector field  $\widehat{X}_i^x$  on  $\mathbb{R}^n$ . The space  $\mathbb{R}^n$  endowed with the sub-Riemannian structure having  $\widehat{X}_1^x, \dots, \widehat{X}_m^x$  as generating frame is called the *nilpotent approximation of  $(M, D, g)$  at  $x$*  and is denoted by  $\widehat{M}_x$ . This nilpotent approximation  $\widehat{M}_x$  is a Carnot group equipped with a left-invariant sub-Riemannian structure.

Since  $(M, D, g)$  is equiregular, we can locally choose systems of privileged coordinates  $z^x$  depending continuously on  $x$  [Jea14, Sect. 2.2.2]. Note that the  $w_i$ 's and  $\delta_\nu$  are independent of  $x$ . Thus an easy adaptation of the proof of [AGM15, Prop. 3.4] (see also [ABB17, Sect. 10.4.1]) shows that, for  $i = 1, \dots, m$ , the vector field  $\varepsilon(\delta_{1/\varepsilon}^x)_* X_i$  converges locally uniformly to  $\widehat{X}_i^{x_0}$  as  $\varepsilon \rightarrow 0$  and  $x \rightarrow x_0$ .

**Lemma 8.** *Let  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \subset [0, T]$ ,  $\bar{a} \in [0, T]$ , be such that  $a_n, b_n \rightarrow \bar{a}$  and  $a_n < b_n$  for any  $n \in \mathbb{N}$ . Given  $u \in L^\infty([0, T], \mathbb{S}^{m-1})$  and  $n \in \mathbb{N}$ , define  $u_n \in L^\infty([0, 1], \mathbb{S}^{m-1})$  by*

$$u_n(\tau) = u(a_n + \tau(b_n - a_n)).$$

*Assume that the sequence  $(u_n)_{n \in \mathbb{N}}$  converges to  $u_\star \in L^\infty([0, 1], \mathbb{R}^m)$  for the weak- $\star$  topology of  $L^\infty([0, 1], \mathbb{R}^m)$  and, moreover, that the trajectory  $\gamma : [0, T] \rightarrow M$  associated with  $u$  is a length-minimizer. If  $x = \gamma(\bar{a})$ , then the trajectory  $\gamma_\star : [0, 1] \rightarrow \widehat{M}_x$  satisfying*

$$\dot{\gamma}_\star(s) = \sum_{i=1}^m u_{\star, i}(s) \widehat{X}_i^x(\gamma_\star(s)), \quad \gamma_\star(0) = 0,$$

*is also a length-minimizer. In particular,  $u_\star(t) \in \mathbb{S}^{m-1}$  for almost every  $t \in [0, 1]$ .*

*Proof.* We consider a continuously varying family of privileged coordinates  $z^{\gamma(t)}$ ,  $t \in [0, T]$ , and the corresponding 1-parameter family of dilations  $\delta_\nu^t := \delta_\nu^{\gamma(t)}$ . It is not restrictive to assume that  $\delta_{\frac{1}{b_n - a_n}}^{a_n} \gamma(t)$  is well-defined for every  $n \in \mathbb{N}$  and  $t \in [a_n, b_n]$ .

Let  $\gamma_n$  be defined by  $\gamma_n(\tau) = \delta_{\frac{1}{b_n - a_n}}^{a_n}(\gamma(a_n + \tau(b_n - a_n)))$ . Then,  $\gamma_n$  is a length-minimizing curve for the sub-Riemannian structure on  $\mathbb{R}^n$  with orthonormal frame

$$(b_n - a_n) \left( \delta_{\frac{1}{b_n - a_n}}^{a_n} \right)_* X_1, \dots, (b_n - a_n) \left( \delta_{\frac{1}{b_n - a_n}}^{a_n} \right)_* X_m.$$

The corresponding control is  $u_n$ .

Since the sequence  $\left( (b_n - a_n) \left( \delta_{\frac{1}{b_n - a_n}}^{a_n} \right)_* X_i \right)_{n \in \mathbb{N}}$  converges locally uniformly to  $\widehat{X}_i^x$ , it follows by standard ODE theory that  $(\gamma_n)_{n \in \mathbb{N}}$  converges uniformly to  $\gamma_\star$ .

We claim that  $\widehat{d}(\gamma_\star(0), \gamma_\star(1)) = 1$ . Indeed,  $\ell(\gamma_\star) \geq \widehat{d}(\gamma_\star(0), \gamma_\star(1))$  and, by [Bel96, Theorem 7.32], we have

$$\begin{aligned} \widehat{d}(\gamma_\star(0), \gamma_\star(1)) &= \lim_{n \rightarrow \infty} \frac{1}{b_n - a_n} d(\gamma(a_n), \gamma(b_n)) \\ &= \lim_{n \rightarrow \infty} \int_0^1 |u(a_n + \tau(b_n - a_n))| d\tau = 1, \end{aligned}$$

where  $|\cdot|$  denotes the norm in  $\mathbb{R}^m$ . On the other hand, by weak- $\star$  convergence we have

$$\ell(\gamma_\star) = \|u_\star\|_{L^1([0, 1], \mathbb{R}^m)} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{L^1([0, 1], \mathbb{R}^m)} = 1,$$



proving the claim.

To conclude the proof, it suffices now to observe that the above implies that  $\gamma_\star$  is minimizing. In particular, since  $|u_\star(t)| \leq 1$  a.e. on  $[0, 1]$  by the properties of weak- $\star$  convergence, this shows that  $|u_\star(t)| = 1$  a.e. on  $[0, 1]$ .  $\square$

**Corollary 9.** *Let  $\gamma$ ,  $u$ ,  $\bar{a}$ ,  $(a_n)_{n \in \mathbb{N}}$ ,  $(b_n)_{n \in \mathbb{N}}$ , and  $u_\star$  be as in Lemma 8. Assume that there exist  $u_+, u_- \in \mathbb{S}^{m-1}$  such that  $u_\star = u_-$  almost everywhere on  $[0, 1/2]$  and  $u_\star = u_+$  almost everywhere on  $[1/2, 1]$ . Then  $u_- = u_+$ .*

*Proof.* If  $u_- \neq u_+$ , then  $\gamma_\star$  is not length-minimizing by Theorem 4, which contradicts Lemma 8.  $\square$

#### 4. DYNAMICS OF ABNORMAL EXTREMALS: PRELIMINARY RESULTS

In this section we present the dynamical system associated with the abnormal extremal, whose analysis is the basis for the proof of Theorem 1, and we derive a first result on its structure.

**4.1. Introduction to the dynamical system.** Let  $(M, D, g)$  be an equiregular sub-Riemannian manifold of rank 2. Since the arguments are local, in what follows we fix a local generating frame  $\{X_1, X_2\}$  of  $(D, g)$ .

Consider an abnormal length-minimizer  $\gamma : [0, T] \rightarrow M$  parametrized by arclength. Then  $T = d(\gamma(0), \gamma(T))$  and there exists  $u \in L^\infty([0, T], \mathbb{S}^1)$  such that

$$\dot{\gamma}(t) = u_1(t)X_1(\gamma(t)) + u_2(t)X_2(\gamma(t)), \quad \text{a.e. } t \in [0, T].$$

Moreover from Theorem 6,  $\gamma$  admits a lift  $\lambda : [0, T] \rightarrow T^*M$  which satisfies

$$\dot{\lambda}(t) = u_1 \vec{h}_1(\lambda(t)) + u_2 \vec{h}_2(\lambda(t)) \quad \text{and} \quad h_1(\lambda(t)) \equiv h_2(\lambda(t)) \equiv 0.$$

By a slight abuse of notation, set  $h_i(t) = \langle \lambda(t), X_i(\gamma(t)) \rangle$ ,  $i = 1, 2$ , and for every  $i_1, \dots, i_m \in \{1, 2\}$ ,

$$h_{i_1 \dots i_m}(t) = \langle \lambda(t), [X_{i_1}, \dots, [X_{i_{m-1}}, X_{i_m}]](\gamma(t)) \rangle.$$

Such a function  $h_{i_1 \dots i_m}$  is absolutely continuous and satisfies

$$(2) \quad \dot{h}_{i_1 \dots i_m}(t) = u_1(t)h_{1i_1 \dots i_m}(t) + u_2(t)h_{2i_1 \dots i_m}(t) \quad \text{for a.e. } t \in [0, T].$$

Differentiating the equalities  $h_1 \equiv h_2 \equiv 0$  and using (2) we obtain  $h_{12} \equiv 0$ . Differentiating again we get

$$(3) \quad 0 = \dot{h}_{12} = u_1 h_{112} + u_2 h_{212} \quad \text{a.e. on } [0, T].$$

*Remark 10.* The identities  $h_1(t) = h_2(t) = h_{12}(t) = 0$  imply that  $\lambda(t) \in (D^2)^\perp$  for every  $t$ . The latter is known as *Goh condition* and is in general (i.e., for sub-Riemannian structures of any rank) a necessary condition for the associated curve to be length-minimizing [AS99]. It is known that a generic sub-Riemannian structure of rank larger than 2 does not have non-constant abnormal extremals satisfying the Goh condition [CJT06].

Let  $h = (-h_{212}, h_{112})$  and  $(t_0, t_1) \subset (0, T)$  be a maximal (i.e., non-extendable) open interval on which  $h \neq 0$ . Equation (3) then implies that  $u = \pm h/|h|$  almost everywhere on  $(t_0, t_1)$ .

Moreover, by length-minimality of  $\gamma$  we can assume without loss of generality that  $u = h/|h|$  on  $(t_0, t_1)$  (see Lemma 22 in the appendix). Thus  $\gamma$  may be non-differentiable only at a time  $t$  such that  $h(t) = 0$ . In particular, if the step of the sub-Riemannian structure is not greater than 3, then  $\gamma$  is differentiable everywhere. We assume from now on that the step is at least 4.

Observe that from (2) and using  $u = h/|h|$  one obtains

$$(4) \quad \dot{h} = A \frac{h}{|h|}, \quad A = \begin{pmatrix} -h_{2112} & -h_{2212} \\ h_{1112} & h_{2112} \end{pmatrix}, \quad \text{on } (t_0, t_1).$$



Here, we used the relation  $h_{1212} = h_{2112}$ , which follows from the Jacobi identity

$$[X_1, [X_2, [X_1, X_2]]] = -[[X_1, X_2], [X_1, X_2]] - [X_2, [[X_1, X_2], X_1]] = [X_2, [X_1, [X_1, X_2]]].$$

Observe that the matrix  $A$  has zero trace and is absolutely continuous on the whole interval  $[0, T]$ .

**Lemma 11.** *Assume that  $\lambda(t) \notin (D_{\gamma(t)}^4)^\perp$  for every  $t \in [0, T]$ . If  $h(t_0) = 0$  for some  $t_0 \in [0, T]$ , then  $A(t_0) \neq 0$ .*

*Proof.* The fact that  $\gamma$  is abnormal implies that the non-zero covector  $\lambda(t)$  annihilates  $D_{\gamma(t)}$  for every  $t \in [0, T]$ . The Goh condition  $h_{12} \equiv 0$  guarantees that it also annihilates  $D_{\gamma(t)}^2$ . The fact that  $h(t_0) = 0$  says, moreover, that  $\lambda(t_0)$  annihilates  $D_{\gamma(t_0)}^3$ . If  $A(t_0)$  is equal to zero, then  $\lambda(t_0)$  annihilates  $D_{\gamma(t_0)}^4$ , which contradicts the assumption.  $\square$

**4.2. The sign of  $\det A$  is non-negative where  $h$  vanishes.** A key step in the proof of Theorem 1 is the following result.

**Proposition 12.** *Let  $(t_0, t_1)$  be a maximal open interval of  $[0, T]$  on which  $h \neq 0$  and assume that  $t_1 < T$ . Then  $\det A(t_1) \leq 0$ .*

*Proof.* Assume by contradiction that  $\det A(t_1) > 0$ . Since  $\text{trace } A(t_1) = 0$ , there exists  $P \in \text{GL}(2, \mathbb{R})$  such that

$$(5) \quad P^{-1}A(t_1)P = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, \quad a > 0.$$

Define the scalar functions  $\alpha$ ,  $\beta$  and  $\zeta$  through the relation

$$P^{-1}A(t)P = \begin{pmatrix} -\alpha(t) & \beta(t) \\ \zeta(t) & \alpha(t) \end{pmatrix},$$

and notice that  $\alpha, \beta, \zeta$  are absolutely continuous with bounded derivatives on  $(t_0, t_1)$ , since they are linear combinations of  $h_{2112}, h_{2212}, h_{1112}$ , according to (2). Clearly, (5) implies that  $\alpha(t) \rightarrow 0$ ,  $\beta(t) \rightarrow -a$ , and  $\zeta(t) \rightarrow a$  as  $t \rightarrow t_1$ .

Consider a time rescaling and a polar coordinates representation so that  $P^{-1}h(t) = \rho(s(t))e^{i\vartheta(s(t))}$ , where

$$s(t) := \int_{t_0}^t \frac{|P^{-1}h(\tau)|}{|h(\tau)|} d\tau.$$

It is useful to introduce  $\mu := (\zeta + \beta)/2$  and  $\eta := (\zeta - \beta)/2$ . Then, denoting by  $\rho'$  and  $\vartheta'$  the derivatives of  $\rho$  and  $\vartheta$  with respect to the parameter  $s$ , (4) can be rewritten as

$$\begin{cases} \rho' = (-\alpha \cos 2\vartheta + \mu \sin 2\vartheta), \\ \vartheta' = \frac{1}{\rho}(\alpha \sin 2\vartheta + \mu \cos 2\vartheta + \eta). \end{cases}$$

Let  $w = \alpha \sin 2\vartheta + \mu \cos 2\vartheta + \eta$  and notice that  $2a > w > a/2$  in a left-neighborhood of  $s(t_1)$ . Therefore,

$$\begin{aligned} (\rho^2 w)' &= 2\rho(-\alpha \cos 2\vartheta + \mu \sin 2\vartheta)w + \rho^2(\alpha' \sin 2\vartheta + \mu' \cos 2\vartheta + \eta') \\ &\quad + \rho^2(\alpha \cos 2\vartheta - \mu \sin 2\vartheta)2\vartheta' \\ &= \rho^2 w \frac{\alpha' \sin 2\vartheta + \mu' \cos 2\vartheta + \eta'}{w} \geq -M\rho^2 w, \end{aligned}$$

for some constant  $M > 0$ . This implies at once that  $t \mapsto e^{Mt} \rho^2(t)w(t)$  is increasing, and hence that it is impossible for  $\rho^2 w$  to tend to zero as  $s \rightarrow s(t_1)$ . This contradicts the assumption that  $\rho(t) \rightarrow 0$  as  $t \rightarrow t_1$ , completing the proof of the statement.  $\square$

## 5. DYNAMICS OF ABNORMAL EXTREMALS IN A SPECIAL CASE: PROOF OF PROPOSITION 3

In this section we prove Proposition 3. We present it here to illustrate in a simpler context the general procedure used later to complete the proof of Theorem 1.

Assume that  $D$  is generated by two vector fields  $X_1, X_2$  such that the Lie algebra  $\text{Lie}\{X_1, X_2\}$  is nilpotent of step at most 4. This means that all Lie brackets of  $X_1, X_2$  of length 5 vanish. In particular  $\dim M \leq 8$ .

*Proof of Proposition 3.* Without loss of generality, we assume that the step is equal to 4. Recall that for an abnormal minimizer on an interval  $I$  we have

$$h_1 \equiv h_2 \equiv h_{12} \equiv 0, \quad 0 = \dot{h}_{12} = u_1 h_{112} + u_2 h_{212} \quad \text{a.e. on } I.$$

The vector  $h = (-h_{212}, h_{112})$  satisfies the differential equation

$$(6) \quad \dot{h} = Au, \quad A = \begin{pmatrix} -h_{2112} & -h_{2212} \\ h_{1112} & h_{2112} \end{pmatrix}, \quad \text{a.e. on } I.$$

Notice that  $A$  is a constant matrix (with zero trace), as follows from (2) and the nilpotency assumption.

As we have already seen in the general case, on every interval where  $h(t) \neq 0$  we have that  $u$  is smooth and equal to either  $\frac{h(t)}{|h(t)|}$  or  $-\frac{h(t)}{|h(t)|}$ .

We are then reduced to the case where  $h$  vanishes at some point  $\bar{t} \in I$ . In this case the matrix  $A$  cannot be zero, as it follows from Lemma 11.

We consider the following alternative:

- (a)  $h(t) = 0$  for all  $t \in I$ ;
- (b)  $h$  does not vanishes identically on  $I$ .

Case (a). From (6) it follows that  $u(t)$  is in the kernel of  $A$  for a.e.  $t \in I$ . Since  $u$  is nonzero for a.e.  $t \in I$ , then necessarily  $A$  has one-dimensional kernel  $\ker A = \text{span}\{\bar{u}\}$ , where  $\bar{u}$  has norm one. Then  $u(t) = \sigma(t)\bar{u}$  for a.e.  $t \in I$ , with  $\sigma(t) \in \{-1, 1\}$  and

$$\dot{\gamma}(t) = \sigma(t)X_{\bar{u}}(\gamma(t)), \quad \text{a.e. } t \in I,$$

with  $X_{\bar{u}}$  a constant vector field. Since  $\gamma$  is a length-minimizer then  $\sigma$  is constant, and  $u$  is smooth, thanks to Lemma 22 in the appendix.

Case (b). Consider a maximal interval  $J = (t_0, t_1)$  on which  $h$  is never vanishing. Since  $J \subsetneq I$ , then either  $h(t_0) = 0$  or  $h(t_1) = 0$ .

The trajectories of (4) are time reparametrizations of those of the linear system  $\dot{z} = Az$ . Hence  $h$  stays in the stable or in the unstable manifold of  $A$ . Recall that  $\det A \leq 0$  by Proposition 12 and notice that if  $\det A = 0$  then  $A$  is conjugate to the nilpotent matrix  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Hence stable and unstable manifolds reduce to zero. We deduce that  $\det A < 0$ .

Denote by  $\lambda_{\pm}$  the eigenvalues of  $A$  and by  $v_{\pm}$  the corresponding unit eigenvectors. Since  $h$  belongs to the stable (respectively, unstable) manifold of  $A$  then  $\frac{h(t)}{|h(t)|}$  is constantly equal to  $v_-$  or  $-v_-$  on  $J$  (respectively,  $v_+$  or  $-v_+$ ). Fix  $t_* \in J$ . Then integrating (6) we get

$$h(t) = h(t_*) \pm (t - t_*)\lambda_- v_-, \quad t \in J,$$

or

$$h(t) = h(t_*) \pm (t - t_*)\lambda_+ v_+, \quad t \in J.$$

If  $h(t_1) = 0$ , then  $\lim_{t \downarrow t_0} h(t) \neq 0$  and  $J = I \cap (-\infty, t_1)$ . Similarly, if  $h(t_0) = 0$  then  $\lim_{t \uparrow t_1} h(t) \neq 0$  and  $J = I \cap (t_0, +\infty)$ .

If there exist two distinct maximal intervals of  $I$  on which  $h$  is never vanishing, then necessarily there exist  $\tau_1 \leq \tau_2$  in  $I$  such that these maximal intervals are of the form  $J_1 = I \cap (-\infty, \tau_1)$  and  $J_2 = I \cap (\tau_2, +\infty)$ . Notice that  $h$  vanishes on  $[\tau_1, \tau_2]$ .

If  $\tau_1 < \tau_2$ , we can apply case (a) on the interval  $(\tau_1, \tau_2)$ , which leads to a contradiction since  $A$  should have nontrivial kernel. We are thus left to consider the case where  $\tau_1 =$

$\tau_2 = \bar{t}$ , that is, when  $h(t) \neq 0$  for  $t \in I \setminus \{\bar{t}\}$ . In this case  $u$  is piecewise constant on  $I \setminus \{\bar{t}\}$  and satisfies

$$\lim_{t \downarrow \bar{t}} u(t) \in \{v_-, -v_-, v_+, -v_+\}, \quad \lim_{t \uparrow \bar{t}} u(t) \in \{v_-, -v_-, v_+, -v_+\}.$$

Theorem 4 and the length-minimizing assumption on  $\gamma$  imply that the two limits must be equal. Hence,  $u$  is constant on  $I$ , and in particular it is smooth.  $\square$

*Remark 13.* The technical ingredients of the above proof open the way to an alternative approach to the Sard conjecture for minimizers [Agr14] which is known in the free case [LDMO<sup>+</sup>16]. Indeed, assume that the hypotheses of Proposition 3 hold true and fix a point  $x \in M$ . We have proved that given any initial covector in  $(D_x^2)^\perp$  there exist at most four length minimizing curves whose extremal lift starts with this covector. Hence, such curves are parametrized by at most  $n - 3$  parameters. By taking into account the time parametrization, the set of final points of abnormal minimizers starting from  $x$  has codimension at most 2.

For recent results on the Sard conjecture for rank 2 structures in 3-dimensional manifolds, see [BdSRar], which extends the analysis in [ZZ95].

## 6. DYNAMICS OF ABNORMAL EXTREMALS: THE GENERAL CASE

The goal of this section is to prove the following result.

**Proposition 14.** *Let  $(t_0, t_1)$  be a maximal interval on which  $h \neq 0$ . Assume that  $t_1 < T$  and  $A(t_1) \neq 0$ . Then  $u(t)$  has a limit as  $t \uparrow t_1$ , which is an eigenvector of  $A(t_1)$ .*

We split the analysis in two steps. The first one, which is a rather straightforward adaptation of the proof of Proposition 3, corresponds to the case where  $\det A(t_1) < 0$ . We will then turn to the case where  $\det A(t_1) = 0$  (recall that, according to Proposition 12,  $\det A(t_1)$  cannot be positive).

For this purpose, we start by proving a preliminary result.

**6.1. A time-rescaling lemma.** The result below highlights the fact that equation (4) is “almost invariant” with respect to similarity of  $A$ .

**Lemma 15.** *For  $P \in \text{GL}(2, \mathbb{R})$  and  $t_* \in (t_0, t_1)$ , we consider the time reparameterization given by*

$$\varphi : [t_*, t_1) \ni t \mapsto s := \int_{t_*}^t \frac{d\tau}{|h(\tau)|}.$$

Let  $\mathfrak{h} = P^{-1}h \circ \varphi^{-1}$  and  $\mathfrak{A} = P^{-1}(A \circ \varphi^{-1})P$ . Then,

- (i)  $\varphi(t) \rightarrow +\infty$  as  $t \rightarrow t_1$ ;
- (ii) for any  $p \in [1, +\infty]$  we have  $\mathfrak{h} \in L^p((0, +\infty), \mathbb{R}^2)$ ;
- (iii) for every  $s \in (0, +\infty)$  we have

$$(7) \quad \mathfrak{h}'(s) = \mathfrak{A}(s)\mathfrak{h}(s).$$

*Proof.* We start by proving point (iii). Observe that  $\dot{\varphi} = 1/|h|$ . Then, simple computations yield

$$\mathfrak{h}' = \frac{P^{-1}\dot{h} \circ \varphi^{-1}}{\dot{\varphi} \circ \varphi^{-1}} = \mathfrak{A}\mathfrak{h}.$$

Assume now that  $\lim_{t \rightarrow t_1} \varphi(t) = s_* < +\infty$ . Then, since  $\mathfrak{h}(s_*) = h(t_1) = 0$ , we have that  $\mathfrak{h}$  is the solution to the (backward) Cauchy problem

$$\begin{cases} \mathfrak{h}' = \mathfrak{A}\mathfrak{h} & \text{on } (0, s_*), \\ \mathfrak{h}(s_*) = 0. \end{cases}$$

This implies that  $\mathfrak{h} \equiv 0$  on  $(0, s_*)$  and thus  $h \equiv 0$  on  $(t_*, t_1)$ , which contradicts the definition of the interval  $(t_0, t_1)$ .

To complete the proof of the statement, observe that  $t \mapsto h(t)$  is bounded on  $[t_*, t_1]$  and thus belongs to  $L^\infty((t_*, t_1), \mathbb{R}^2)$ . Then, for every  $p \geq 1$ ,

$$\begin{aligned} \int_0^{+\infty} |\mathfrak{h}(s)|^p ds &= \int_{t_*}^{t_1} |P^{-1}h|^p |h|^{-1} dt \leq \|P^{-1}\|^p \int_{t_*}^{t_1} |h|^{p-1} dt \\ &\leq \|P^{-1}\|^p \|h\|_{L^\infty}^{p-1} (t_1 - t_*) < +\infty. \end{aligned} \quad \square$$

**6.2. Proof of Proposition 14 in the case  $\det A(t_1) < 0$ .** Since  $\text{trace}(A) = 0$  and  $\det A(t_1) < 0$ , there exists  $P \in \text{GL}(2, \mathbb{R})$  such that

$$PA(t_1)P^{-1} = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}, \quad a > 0.$$

Up to applying the change of coordinates associated with  $P$  and defining the time-rescaled curves  $\mathfrak{h}$  and  $\mathfrak{A}$  as in Lemma 15, we have

$$(8) \quad \mathfrak{A}(s) = \begin{pmatrix} -\alpha(s) & \beta(s) \\ \zeta(s) & \alpha(s) \end{pmatrix},$$

where

$$(9) \quad \lim_{s \rightarrow \infty} \alpha(s) = a, \quad \lim_{s \rightarrow \infty} \zeta(s) = \lim_{s \rightarrow \infty} \beta(s) = 0.$$

Let  $\mathfrak{h} = \rho e^{i\vartheta}$  for  $\rho > 0$  and  $\vartheta \in [0, 2\pi)$ . We will prove that  $\vartheta(s) \rightarrow 0 \pmod{\pi}$  as  $s \rightarrow \infty$ .

Observe that, letting  $\mathfrak{h} = (x_1, x_2)$  with  $x_1, x_2 \in \mathbb{R}$ , we have

$$(10) \quad \frac{1}{2} \tan 2\vartheta = \frac{\sin \vartheta \cos \vartheta}{\cos^2 \vartheta - \sin^2 \vartheta} = \frac{x_1 x_2}{x_1^2 - x_2^2}.$$

By (7) and simple computations we obtain

$$(11) \quad (x_1 x_2)' = \zeta x_1^2 + \beta x_2^2, \quad \frac{(x_1^2 - x_2^2)'}{2} = -\alpha(x_1^2 + x_2^2) + (\beta - \zeta)x_1 x_2.$$

Upon integration and exploiting (9), we get

$$(12) \quad x_1 x_2 = o(R), \quad x_1^2 - x_2^2 = 2aR(1 + o(1)), \quad \text{where } R(s) := \int_s^{+\infty} |\mathfrak{h}(\sigma)|^2 d\sigma.$$

Observe that, by Lemma 15,  $\mathfrak{h} \in L^2((0, +\infty), \mathbb{R}^2)$  and, in particular,  $R \rightarrow 0$  as  $s \rightarrow +\infty$ . Finally, substituting the above in (10) shows that  $\tan 2\vartheta \rightarrow 0$ . From the second equation in (12), the sign of  $x_1^2 - x_2^2$  is positive as  $t \uparrow t_1$ , which implies that  $\vartheta \rightarrow 0 \pmod{\pi}$ . This completes the proof of Proposition 14 in the case  $\det A(t_1) < 0$ .

*Remark 16.* Recall that in the analysis above we suppose that  $u = \frac{h}{|h|}$ , and we actually prove that in this case  $u(t)$  converges to a unit eigenvector of  $A(t_1)$  associated with the negative eigenvalue  $-a$ . In the case where  $u = -\frac{h}{|h|}$ , an analogous argument yields that  $u(t)$  converges to a unit eigenvector of  $A(t_1)$  associated with the positive eigenvalue  $a$ .

**6.3. Proof of Proposition 14 in the case  $\det A(t_1) = 0$ .** Assume that  $\det A(t_1) = 0$  and recall that  $\text{trace} A(t_1) = 0$ . Since, moreover,  $A(t_1) \neq 0$ , there exists  $P \in \text{GL}(2, \mathbb{R})$  such that

$$(13) \quad PA(t_1)P^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

As before, using the change of variables of Lemma 15, we let

$$\mathfrak{A}(s) = \begin{pmatrix} -\alpha(s) & \beta(s) \\ \zeta(s) & \alpha(s) \end{pmatrix},$$

where  $\alpha, \beta, \zeta$  are linear combinations of  $h_{2112} \circ \varphi^{-1}$ ,  $h_{2212} \circ \varphi^{-1}$ , and  $h_{1112} \circ \varphi^{-1}$ , and hence absolutely continuous with bounded derivatives on  $(0, +\infty)$ , according to (2). Equality (13) implies that  $\alpha \rightarrow 0$ ,  $\beta \rightarrow 1$ , and  $\zeta \rightarrow 0$  as  $s \rightarrow +\infty$ .

We also introduce  $\mu := \zeta + \beta$  and we notice that  $\mu \rightarrow 1$  as  $s \rightarrow +\infty$ . (Beware that the same letters are used for different parameters in the proof of Proposition 12.) Then, (7) reads

$$\frac{\rho'}{\rho} = \mu \sin \vartheta \cos \vartheta - \alpha \cos 2\vartheta, \quad \vartheta' = -\mu \sin^2 \vartheta + \alpha \sin 2\vartheta + \zeta,$$

and can be written as

$$(14) \quad \frac{\rho'}{\rho} = \sin \vartheta \cos \vartheta + f, \quad \vartheta' = -\sin^2 \vartheta + g,$$

where the functions

$$f = -\alpha \cos 2\vartheta + (\mu - 1) \sin \vartheta \cos \vartheta, \quad g = \alpha \sin 2\vartheta + \zeta + (1 - \mu) \sin^2 \vartheta,$$

tend to zero as  $s \rightarrow +\infty$ .

Establishing Proposition 14, finally amounts to proving that  $\vartheta \rightarrow 0 \pmod{\pi}$ , as  $s \rightarrow +\infty$ .

**Lemma 17.** *We have the following dichotomy:*

- (i)  $\vartheta \rightarrow 0 \pmod{\pi}$ , as  $s \rightarrow +\infty$ ;
- (ii)  $\vartheta \rightarrow -\infty$  as  $s \rightarrow +\infty$ . Moreover, in this case, for any  $0 < \varepsilon < \pi/2$  there exists an increasing sequence of positive real numbers  $(s_n)_{n \in \mathbb{N}}$  tending to infinity such that

$$\begin{aligned} \vartheta(s_{2n}) &= \pi - \varepsilon \pmod{2\pi}, & \vartheta(s_{2n+1}) &= \varepsilon \pmod{2\pi}, \\ \vartheta'(s) &< 0 \quad \forall s \in [s_{2n}, s_{2n+1}]. \end{aligned}$$

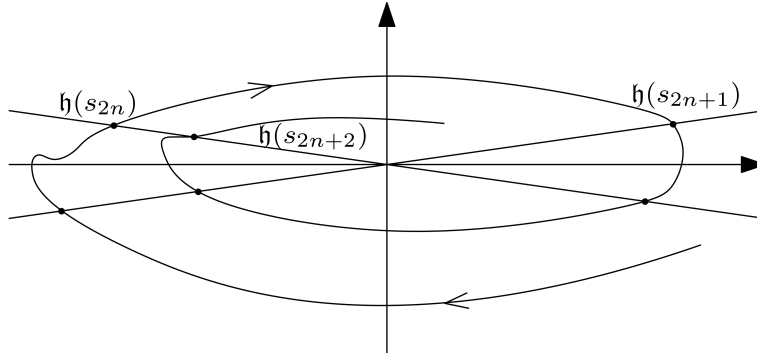


FIGURE 1. The sequence  $(s_n)_{n \in \mathbb{N}}$  in Lemma 17

*Proof.* Notice that the dynamics of  $\vartheta$  is a perturbation via  $g$  of

$$\vartheta' = -\sin^2 \vartheta.$$

The phase portrait of the latter on  $\mathbb{S}^1$  is made of two equilibria in 0 and  $\pi$  joined by two clock-wise oriented heteroclinic trajectories.

Assume that (i) does not hold. Therefore, there exists  $c > 0$  such that

$$(15) \quad \limsup_{s \rightarrow +\infty} |\sin \vartheta(s)| > c.$$

Let  $\varepsilon > 0$  be such that  $\sin \varepsilon \in (0, c)$  and  $s^* > 0$  be such that, for  $s > s^*$ ,  $\vartheta'(s) < -\varepsilon^2/2$  as soon as  $|\sin \vartheta(s)| > \varepsilon$ .

Pick  $q_1 > s^*$  such that  $|\sin \vartheta(q_1)| > c > \sin \varepsilon$ . Since  $\vartheta'$  is bounded from zero as long as  $|\sin \vartheta|$  stays larger than  $\sin \varepsilon$ , there exists  $r_1 > q_1$  such that  $|\sin \vartheta(r_1)| = \sin \varepsilon$ . By definition of  $c$ , there exists  $q_2 > r_1$  such that  $|\sin \vartheta(q_2)| > c$ . Moreover,  $q_1$  and  $q_2$  can be taken so that  $\vartheta(q_2) = \vartheta(q_1) - \pi$  and (15) holds with  $c$  arbitrarily close to 1. By iterating

the procedure leading from  $q_1$  to  $q_2$ , we prove that  $\vartheta \rightarrow -\infty$ . The construction also shows how to define the sequence  $(s_n)_{n \in \mathbb{N}}$  as in (ii) (cf. Figure 1).  $\square$

The rest of the argument consists in showing that case (ii) in Lemma 17 cannot hold true. For that purpose, we argue by contradiction.

**Lemma 18.** *Assume that property (ii) in Lemma 17 holds true. Then there exists  $0 < \varepsilon_0 < \pi/2$  such that for any  $0 < \varepsilon < \varepsilon_0$  there exists  $N_\varepsilon$  for which, given any  $n \geq N_\varepsilon$ ,*

$$(16) \quad \frac{2}{\varepsilon} (1 - \varepsilon^2) \leq s_{2n+1} - s_{2n} \leq \frac{2}{\varepsilon} (1 + \varepsilon^2),$$

and

$$(17) \quad (1 - \varepsilon)\varepsilon\rho(s_{2n}) \leq \rho(s) \sin \vartheta(s) \leq (1 + \varepsilon)\varepsilon\rho(s_{2n}), \text{ for } s \in [s_{2n}, s_{2n+1}].$$

As a consequence, for every  $n \geq N_\varepsilon$ , one has the following estimates

$$(18) \quad 2(1 - 2\varepsilon)\rho(s_{2n}) \leq \int_{s_{2n}}^{s_{2n+1}} \sin \vartheta(s) \rho(s) ds \leq 2(1 + 2\varepsilon)\rho(s_{2n}),$$

$$(19) \quad (1 - 2\varepsilon) \frac{\rho(s_{2n})}{\varepsilon} \leq \int_{s_{2n}}^{s_{2n+1}} \rho(s) ds \leq (1 + 2\varepsilon) \frac{\rho(s_{2n})}{\varepsilon},$$

$$(20) \quad \left| \int_{s_{2n}}^{s_{2n+1}} \cos \vartheta(s) \rho(s) ds \right| \leq \rho(s_{2n}).$$

*Proof.* Set  $M_f(s) = \sup_{\tau \geq s} |f(\tau)|$  and  $M_g(s) = \sup_{\tau \geq s} |g(\tau)|$ . Observe that these two functions tend to zero as  $s$  tends to infinity.

By Lemma 17, for  $n$  large enough and  $s \in [s_{2n}, s_{2n+1}]$ , equation (14) becomes

$$(21) \quad (\cot \vartheta)' = 1 - \frac{g}{\sin^2 \vartheta}.$$

For  $n$  large enough, for every  $s \in [s_{2n}, s_{2n+1}]$  we have

$$\left| \frac{g(s)}{\sin^2 \vartheta(s)} \right| \leq \frac{M_g(s_{2n})}{\sin^2 \varepsilon} \leq \frac{\varepsilon^2}{2}.$$

Equation (16) follows by integrating (21) on the interval  $[s_{2n}, s_{2n+1}]$ .

On the interval  $[s_{2n}, s_{2n+1}]$ , one has

$$\frac{\rho'(s)}{\rho(s)} + \frac{\vartheta'(s) \cos \vartheta(s)}{\sin \vartheta(s)} = f(s) + \frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)}.$$

For  $n$  large enough, for every  $s \in [s_{2n}, s_{2n+1}]$  we have

$$(22) \quad |f(s)| + \left| \frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)} \right| \leq M_f(s_{2n}) + \frac{M_g(s_{2n})}{\sin \varepsilon} \leq \frac{\varepsilon^2}{4}.$$

By integrating between  $s_{2n}$  and any  $s \in [s_{2n}, s_{2n+1}]$ , one gets

$$\left| \ln \left( \frac{\rho(s) \sin \vartheta(s)}{\rho(s_{2n}) \sin \varepsilon} \right) \right| = \left| \int_{s_{2n}}^s \left( f(s) + \frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)} \right) ds \right| \leq \frac{(s_{2n+1} - s_{2n})\varepsilon^2}{4} \leq \frac{\varepsilon}{2}(1 + \varepsilon^2),$$

yielding (17) for  $\varepsilon$  small enough.

We now turn to the proof of the three estimates (18)–(20). The first one simply follows by integrating (17) on  $[s_{2n}, s_{2n+1}]$  and using (16). Estimate (19) is obtained by first dividing (17) by  $\sin \vartheta(s)$  and then integrating the resulting inequalities on  $[s_{2n}, s_{2n+1}]$ . One gets that

$$(1 - \varepsilon)\varepsilon\rho(s_{2n}) \int_{s_{2n}}^{s_{2n+1}} \frac{ds}{\sin \vartheta(s)} \leq \int_{s_{2n}}^{s_{2n+1}} \rho(s) ds \leq (1 + \varepsilon)\varepsilon\rho(s_{2n}) \int_{s_{2n}}^{s_{2n+1}} \frac{ds}{\sin \vartheta(s)}.$$

On the other hand, the following holds true,

$$\int_{s_{2n}}^{s_{2n+1}} \frac{ds}{\sin \vartheta(s)} = \int_{s_{2n}}^{s_{2n+1}} \frac{\vartheta'(s)}{-\sin^3 \vartheta(s) \left(1 - \frac{g(s)}{\sin^2 \vartheta(s)}\right)} ds,$$

which implies that

$$(1 - \varepsilon^2) \int_{\varepsilon}^{\pi - \varepsilon} \frac{d\vartheta}{\sin^3 \vartheta} \leq \int_{s_{2n}}^{s_{2n+1}} \frac{ds}{\sin \vartheta(s)} \leq (1 + \varepsilon^2) \int_{\varepsilon}^{\pi - \varepsilon} \frac{d\vartheta}{\sin^3 \vartheta}.$$

A direct computation shows that  $\int_{\varepsilon}^{\pi - \varepsilon} \frac{d\vartheta}{\sin^3 \vartheta} = \frac{1}{\varepsilon^2(1+o(\varepsilon))}$  as  $\varepsilon$  tends to zero. One finally deduces estimate (19).

To derive estimate (20), one notices that

$$\begin{aligned} \int_{s_{2n}}^{s_{2n+1}} \cos \vartheta(s) \rho(s) ds &= \int_{s_{2n}}^{s_{2n+1}} \frac{\sin \vartheta(s) \cos \vartheta(s) \rho(s)}{\sin \vartheta(s)} ds \\ &= \int_{s_{2n}}^{s_{2n+1}} \frac{\rho'(s) - f(s) \rho(s)}{\sin \vartheta(s)} ds \\ &= - \int_{s_{2n}}^{s_{2n+1}} \frac{f(s) \rho(s)}{\sin \vartheta(s)} ds + \frac{\rho(s_{2n+1}) - \rho(s_{2n})}{\sin \varepsilon} \\ &\quad + \int_{s_{2n}}^{s_{2n+1}} \frac{\rho(s) \cos \vartheta(s) \vartheta'(s)}{\sin^2 \vartheta(s)} ds. \end{aligned}$$

By using the expression of  $\vartheta'$  in the last integral, one deduces that

$$2 \int_{s_{2n}}^{s_{2n+1}} \cos \vartheta(s) \rho(s) ds = \frac{\rho(s_{2n+1}) - \rho(s_{2n})}{\sin \varepsilon} - \int_{s_{2n}}^{s_{2n+1}} \rho(s) \frac{f(s) - \frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)}}{\sin \vartheta(s)} ds.$$

By using (17) for  $s = s_{2n}$  and  $s = s_{2n+1}$  and then (22), one deduces (20).  $\square$

Fix a sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$ , strictly decreasing to 0. For each  $k \in \mathbb{N}$ , we use  $(s_{k,n})_{n \in \mathbb{N}}$  to denote the sequence  $(s_n)_{n \in \mathbb{N}}$  given by Lemma 17 and corresponding to  $\varepsilon = \varepsilon_k$ . For all  $k \in \mathbb{N}$  let  $n_k \geq N_{\varepsilon_k}$  be an integer to be fixed later, where  $N_{\varepsilon_k}$  is as in Lemma 18. We use  $(\xi_\ell)_{\ell \in \mathbb{N}}$  to denote the sequence defined by

$$\xi_{2k} = s_{k,2n_k}, \quad \xi_{2k+1} = s_{k,2n_k+1}, \quad \forall k \in \mathbb{N}.$$

We choose  $k \mapsto n_k$  so that the sequence  $(\xi_\ell)_{\ell \in \mathbb{N}}$  is strictly increasing and tends to infinity as  $\ell \rightarrow +\infty$ .

Let  $t_\ell = \varphi^{-1}(\xi_\ell)$ , where  $\varphi$  is the change of variables introduced in Lemma 15. For every  $\ell \geq 0$  consider the function  $u_\ell \in L^\infty([0, 1], \mathbb{S}^1)$  defined by  $u_\ell(\tau) = u(t_{2\ell} + \tau(t_{2\ell+1} - t_{2\ell}))$ . By the weak- $\star$  compactness of all bounded subsets of  $L^\infty([0, 1], \mathbb{R}^2)$ , we can assume without loss of generality that  $u_\ell \rightharpoonup u_\star$  in the weak- $\star$  topology. Applying Lemma 8 with  $a_\ell = t_{2\ell}$  and  $b_\ell = t_{2\ell+1}$ , we deduce that  $u_\star$  is minimizing and  $|u_\star| \equiv 1$  almost everywhere in  $[0, 1]$ .

For every subinterval  $[a, b]$  of  $[0, 1]$ , by the properties of weak- $\star$  convergence, we have that

$$\int_a^b v^T u_\ell(\tau) d\tau \rightarrow \int_a^b v^T u_\star(\tau) d\tau, \quad \forall v \in \mathbb{R}^2.$$

Moreover, one has

$$\begin{aligned} \int_a^b v^T u_\ell(\tau) d\tau &= \frac{1}{t_{2\ell+1} - t_{2\ell}} \int_{(1-a)t_{2\ell} + at_{2\ell+1}}^{(1-b)t_{2\ell} + bt_{2\ell+1}} \frac{v^T h}{|h|} dt \\ (23) \quad &= \frac{1}{t_{2\ell+1} - t_{2\ell}} \int_{\varphi((1-a)t_{2\ell} + at_{2\ell+1})}^{\varphi((1-b)t_{2\ell} + bt_{2\ell+1})} v^T P h(s) ds, \end{aligned}$$

where  $P$  has been introduced in (13).



In addition

$$(24) \quad t_{2\ell+1} - t_{2\ell} = \int_{\xi_{2\ell}}^{\xi_{2\ell+1}} |P\mathfrak{h}(s)| ds.$$

**Lemma 19.** *Under the above assumptions, there exists a unit vector  $v_\star \in \mathbb{R}^2$  such that  $u_\star(t) = v_\star$  for a.e.  $t \in [0, 1]$ . Moreover,  $v_\star$  is parallel to  $P(1, 0)$ .*

*Proof.* Let  $v_\star, w_\star \in \mathbb{R}^2$  be two orthogonal unit vectors such that  $v_\star$  is parallel to  $P(1, 0)$ . Notice that  $P^T w_\star$  is orthogonal to  $(1, 0)$ , that is, it is parallel to  $(0, 1)$ . We start by showing that  $w_\star^T u_\star(t) = 0$  for a.e.  $t \in [0, 1]$ . This amounts to showing that for all  $0 \leq a < b \leq 1$  it holds

$$\frac{1}{t_{2\ell+1} - t_{2\ell}} \int_{\varphi((1-a)t_{2\ell} + at_{2\ell+1})}^{\varphi((1-b)t_{2\ell} + bt_{2\ell+1})} \mathfrak{h}_2(s) ds \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty.$$

Since  $\mathfrak{h}_2 = \rho \sin \vartheta$  is positive on  $[\xi_{2\ell}, \xi_{2\ell+1}]$  by construction, using (24) it is enough to show that

$$\frac{\int_{\xi_{2\ell}}^{\xi_{2\ell+1}} \rho(s) \sin \vartheta(s) ds}{\int_{\xi_{2\ell}}^{\xi_{2\ell+1}} |P\mathfrak{h}(s)| ds} \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty.$$

Since  $|P\mathfrak{h}(s)| \geq \|P^{-1}\|^{-1} \rho(s)$  for all  $s$ , the latter limit holds true according to (18) and (19) in Lemma 18, applied to  $\varepsilon = \varepsilon_\ell$  for  $\ell \geq 0$ .

Recall that the control  $u_\star$  is minimizing and  $|u_\star(t)| = 1$  for a.e.  $t \in [0, 1]$ . From what precedes, one deduces that  $u_\star$  is almost everywhere perpendicular to  $w_\star$ , hence equal to  $v_\star$  or  $-v_\star$ . It then follows from Lemma 22 in the appendix that, up to replacing  $v_\star$  by  $-v_\star$ , the equality  $u_\star(t) = v_\star$  holds for a.e.  $t \in [0, 1]$ .  $\square$

Let  $\bar{v} \in \mathbb{R}^2$  be such that  $P^T \bar{v} = (1, 0)$ . We have, according to Lemma 19,

$$\lim_{\ell \rightarrow \infty} \int_0^1 \bar{v}^T u_\ell(\tau) d\tau = \int_0^1 \bar{v}^T u_\star(\tau) d\tau = \bar{v}^T v_\star \neq 0.$$

We conclude the proof by contradiction by showing that the limit in the left-hand side is zero. Indeed, according to (23), we have

$$\left| \int_0^1 \bar{v}^T u_\ell(\tau) d\tau \right| = \frac{\left| \int_{\xi_{2\ell}}^{\xi_{2\ell+1}} \rho(s) \cos \vartheta(s) ds \right|}{\int_{\xi_{2\ell}}^{\xi_{2\ell+1}} |P\mathfrak{h}(s)| ds} \leq \|P^{-1}\| \frac{\left| \int_{\xi_{2\ell}}^{\xi_{2\ell+1}} \rho(s) \cos \vartheta(s) ds \right|}{\int_{\xi_{2\ell}}^{\xi_{2\ell+1}} \rho(s) ds}.$$

The right-hand side of the above equation tends to zero thanks to (19) and (20) in Lemma 18 applied to  $\varepsilon = \varepsilon_\ell$  for  $\ell \geq 0$ .

We have therefore proved that (ii) in Lemma 17 cannot hold true, which completes the proof of Proposition 14.

## 7. PROOF OF THEOREM 1

Let  $M$  be as in the statement of Theorem 1. Denote, as in the previous sections, by  $\gamma : [0, T] \rightarrow M$  a length-minimizing trajectory parametrized by arclength and by  $\lambda : [0, T] \rightarrow T^*M$  an abnormal extremal lift of  $\gamma$ .

Proposition 14, together with Theorem 4, proves the  $C^1$  regularity of  $\gamma$  provided that  $h$  vanishes only at isolated points.

We consider in this section the case where  $t_0 \in (0, T)$  is a density point of  $\{t \in [0, T] \mid h(t) = 0\}$ . We want to prove that  $u(t)$  (up to modification on a set of measure zero) has a limit as  $t \uparrow t_0$  and as  $t \downarrow t_0$ . By symmetry, we restrict our attention to the existence of the limit of  $u(t)$  as  $t \uparrow t_0$ .

We are going to consider separately the situations where  $h \equiv 0$  on a left neighborhood of  $t_0$  and where there exists a sequence of maximal open intervals  $(t_0^n, t_1^n)$  with  $h|_{(t_0^n, t_1^n)} \neq 0$  and such that  $t_1^n \rightarrow t_0$ .

Assume for now on that  $h \equiv 0$  on a left neighborhood  $(t_0 - \eta, t_0]$  of  $t_0$ . Then, since  $\dot{h} = Au$  almost everywhere on  $(t_0 - \eta, t_0]$ , we have that  $u(t)$  belongs to  $\ker A(t)$  for almost every  $t$  in  $(t_0 - \eta, t_0]$ . By Lemma 11, moreover,  $\ker A(t)$  is one-dimensional for every  $t \in (t_0 - \eta, t_0]$ .

Fix an open neighborhood  $V_0$  of  $\lambda(t_0)$  in  $T^*M$  such that there exists a smooth map  $V_0 \ni \lambda \mapsto v(\lambda) \in \mathbb{S}^1$  such that  $v(\lambda(t)) \in \ker A(t)$  if  $\lambda(t) \in V_0$  and  $t \in (t_0 - \eta, t_0]$ . Up to reducing  $\eta$ , we assume that  $\lambda(t) \in V_0$  for every  $t \in (t_0 - \eta, t_0]$ . Notice that  $\lambda|_{(t_0 - \eta, t_0]}$  is a solution of the time-varying system

$$\dot{\lambda} = \sigma(t)\vec{X}_{v(\lambda)}(\lambda),$$

where  $\sigma : (t_0 - \eta, t_0] \rightarrow \{-1, 1\}$  is measurable. Hence, by length-minimality of  $\gamma$  and by Lemma 22 in the appendix, either  $u = v$  almost everywhere on  $(t_0 - \eta, t_0]$  or  $u = -v$  almost everywhere on  $(t_0 - \eta, t_0]$ . We conclude that  $u$  is continuous on  $(t_0 - \eta, t_0]$  and the proof in this case is concluded.

We are left to consider the case where every left neighborhood of  $t_0$  contains a maximal interval  $(\tau_0, \tau_1)$  such that  $h \neq 0$  on  $(\tau_0, \tau_1)$ .

Notice that, by Proposition 12 and by continuity of  $t \mapsto A(t)$ , we have that  $\det A(t_0) \leq 0$ .

The case  $\det A(t_0) < 0$  can be ruled out thanks to the following lemma.

**Lemma 20.** *Let  $\det A(t_0) < 0$ . There exists  $\eta \in (0, t_0)$  such that, for any maximal interval  $(\tau_0, \tau_1) \subset (0, t_0)$  on which  $h(t) \neq 0$ , then  $\tau_0 < t_0 - \eta$ .*

*Proof.* As we have already seen in Section 4.1, on every interval where  $h(t) \neq 0$  we have that  $u$  is smooth and equal to either  $\frac{h(t)}{|h(t)|}$  or  $-\frac{h(t)}{|h(t)|}$ . Thus the function  $h$  on  $(\tau_0, \tau_1)$  is either a maximal solution to  $\dot{x} = A(t)\frac{x}{|x|}$  or a maximal solution to  $\dot{x} = -A(t)\frac{x}{|x|}$ . Let us assume that it is a maximal solution of  $\dot{x} = A(t)\frac{x}{|x|}$ , the proof being identical in the second case.

For every  $v \in \mathbb{R}^2 \setminus \{0\}$  and every  $\vartheta > 0$  denote by  $C_\vartheta(v)$  the cone of all vectors in  $\mathbb{R}^2 \setminus \{0\}$  making an (unoriented) angle smaller than  $\vartheta$  with  $v$  or  $-v$ .

Let  $\eta_0 \in (0, t_0)$  be such that  $\det(A(t)) < 0$  for every  $t \in [t_0 - \eta_0, t_0]$ . For  $t \in [t_0 - \eta_0, t_0]$ , denote by  $v_-(t)$  and  $v_+(t)$  two unit eigenvectors of  $A(t)$ , the first corresponding to a negative and the second to a positive eigenvalue.

Let  $\eta \in (0, \eta_0)$  and  $\vartheta_0 > 0$  be such that  $C_{\vartheta_0}(v_+(t_0)) \cap C_{\vartheta_0}(v_-(t_0)) = \emptyset$  and  $v_\pm(t) \in C_{\vartheta_0}(v_\pm(t_0))$  for every  $t \in [t_0 - \eta, t_0]$ . Notice that, for every fixed  $\bar{t} \in [t_0 - \eta, t_0]$ , the vector field  $x \mapsto A(\bar{t})x$  points inward  $C_{\vartheta_0}(v_+(\bar{t}))$  at every nonzero point of its boundary (see Figure 2). Hence  $C_{\vartheta_0}(v_+(\bar{t}))$  is positively invariant for the dynamics of  $\dot{x} = A(\bar{t})\frac{x}{|x|}$  on  $[t_0 - \eta, t_0]$ .

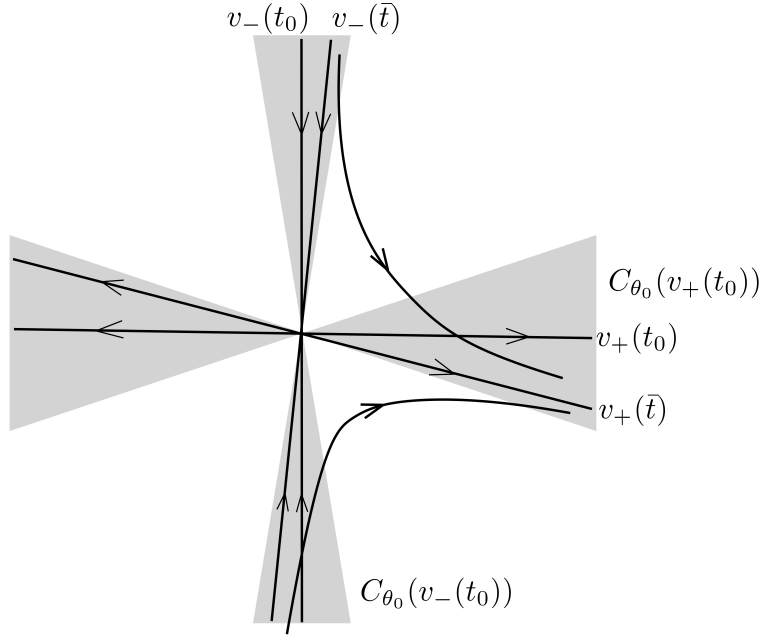
In order to prove the statement, we argue by contradiction. Assume that  $h : (\tau_0, \tau_1) \rightarrow \mathbb{R}^2 \setminus \{0\}$  is a maximal solution of  $\dot{x} = A(t)\frac{x}{|x|}$  with  $(\tau_0, \tau_1) \subset (t_0 - \eta, t_0)$ . Then  $h(\tau)$  tends to 0 as  $\tau$  tends to  $\tau_0$  or  $\tau_1$  and it follows from Proposition 14 that  $\frac{h(\tau)}{|h(\tau)|}$  converges to an eigenvector of  $A(\tau_0)$  as  $\tau \downarrow \tau_0$  and to an eigenvector of  $A(\tau_1)$  as  $\tau \uparrow \tau_1$ . More precisely, from Remark 16 there holds

$$\lim_{\tau \downarrow \tau_0} \frac{h(\tau)}{|h(\tau)|} \rightarrow \pm v_+(\tau_0), \quad \lim_{\tau \uparrow \tau_1} \frac{h(\tau)}{|h(\tau)|} \rightarrow \pm v_-(\tau_1).$$

This contradicts the positive invariance of  $C_{\vartheta_0}(v_+(\tau_0))$  for the equation  $\dot{x} = A(t)\frac{x}{|x|}$  on  $(\tau_0, \tau_1)$ .  $\square$

In the case  $\det A(t_0) = 0$  the proof follows the steps of the construction of Section 6.3. In particular, let  $P \in \text{GL}(2, \mathbb{R})$  be such that

$$P^{-1}A(t)P = \begin{pmatrix} -a(t) & b(t) \\ c(t) & a(t) \end{pmatrix},$$


 FIGURE 2. Phase portrait of  $\dot{x} = A(\bar{t})x$  for  $\bar{t} \in [t_0 - \eta, t_0]$ 

where  $a, b, c$  are affine combinations of  $h_{2112}, h_{2212}$ , and  $h_{1112}$  with  $a \rightarrow 0$ ,  $b \rightarrow 1$ , and  $c \rightarrow 0$  as  $t \rightarrow t_0$ . Let

$$(25) \quad P^{-1}h(t) = r(t)e^{i\omega(t)},$$

with  $\omega(t)$  uniquely defined modulus  $2\pi$  only when  $h(t) \neq 0$ .

The crucial point is the following counterpart to Lemma 17, whose proof can be obtained using exactly the same arguments.

**Lemma 21.** *We have the following dichotomy:*

- (i) for any  $0 < \varepsilon < \pi/2$ , for  $\eta$  small enough,  $|\sin(\omega(t))| < \varepsilon$  for all  $t \in (t_0 - \eta, t_0)$  such that  $h(t) \neq 0$ ;
- (ii) for any  $0 < \varepsilon < \pi/2$  there exists an increasing sequence  $(t_n)_{n \in \mathbb{N}}$  in  $(0, t_0)$  tending to  $t_0$  and such that

$$\begin{aligned} \omega(t_{2n}) &= \pi - \varepsilon \pmod{2\pi}, & \omega(t_{2n+1}) &= \varepsilon \pmod{2\pi}, \\ h(t) &\neq 0, \sin(\omega(t)) > 0, \dot{\omega}(t) < 0 & \forall t \in [t_{2n}, t_{2n+1}] \end{aligned}$$

or

$$\begin{aligned} \omega(t_{2n}) &= -\varepsilon \pmod{2\pi}, & \omega(t_{2n+1}) &= \varepsilon - \pi \pmod{2\pi}, \\ h(t) &\neq 0, \sin(\omega(t)) < 0, \dot{\omega}(t) < 0 & \forall t \in [t_{2n}, t_{2n+1}] \end{aligned}$$

holds true.

Case (ii) can be excluded by similar computations as in Section 6.3, since it contradicts the optimality of  $\gamma$ .

Consider now case (i). Let  $v_*, w_* \in \mathbb{R}^2$  be two orthogonal unit vectors such that  $v_*$  is parallel to  $P(1, 0)$ . According to (25), if  $\sin(\omega(t)) = 0$  and  $r(t) \neq 0$ , then  $u(t) = h(t)/|h(t)|$  is equal to  $v_*$  or  $-v_*$ . For every  $\eta \in (0, t_0)$  we set

$$I_\eta^+ = \{t \in (t_0 - \eta, t_0) \mid v_*^T u(t) > 0\}, \quad I_\eta^- = \{t \in (t_0 - \eta, t_0) \mid v_*^T u(t) < 0\}.$$

Property (i) implies that, for  $\eta$  small,  $I_\eta^+ \cup I_\eta^-$  contains  $\{t \in (t_0 - \eta, t_0) \mid h(t) \neq 0\}$ . Moreover, if  $t_0$  is a density point for  $I = I_\eta^+ \cap \{t \in (t_0 - \eta, t_0) \mid h(t) \neq 0\}$  (respectively,

$I = I_\eta^- \cap \{t \in (t_0 - \eta, t_0) \mid h(t) \neq 0\}$ , then,

$$\lim_{t \in I, t \rightarrow t_0} u(t) = v_\star \quad (\text{respectively, } \lim_{t \in I, t \rightarrow t_0} u(t) = -v_\star).$$

Let  $\Phi_\eta = \{t \in (t_0 - \eta, t_0) \mid h(t) = 0\}$ . For almost every  $t \in \Phi_\eta$ ,  $u(t)$  is in the kernel of  $A(t)$  and  $|u(t)| = 1$ . Notice that, if  $t_0$  is a density point for  $J = \{t \in (0, t_0) \mid \ker A(t) \neq (0)\}$ , then the kernel of  $A(t)$  converges to the kernel of  $A(t_0)$  as  $t \in J$ ,  $t \rightarrow t_0$ . By construction of  $P$ , moreover,  $\ker(A(t_0)) = \text{span}(P(1, 0)) = \text{span}(v_\star)$ . Hence, for  $\eta$  small enough, almost every  $t \in \Phi_\eta$  is in  $I_\eta^+ \cup I_\eta^-$ .

To summarize, for  $\eta$  small enough,  $I_\eta^+ \cup I_\eta^-$  has full measure in  $(t_0 - \eta, t_0)$ . Moreover,

$$(26) \quad \lim_{t \in I_\eta^+, t \rightarrow t_0} u(t) = v_\star, \quad \lim_{t \in I_\eta^-, t \rightarrow t_0} u(t) = -v_\star.$$

We next prove that  $u$  converges either to  $v_\star$  or to  $-v_\star$  as  $t \rightarrow t_0$  by showing that, for  $\eta$  small enough, either  $I_\eta^+$  or  $I_\eta^-$  has measure zero.

Suppose by contradiction that there exists a sequence of intervals  $(\tau_0^n, \tau_1^n)$  in  $(0, t_0)$  such that  $\tau_0^n, \tau_1^n \rightarrow t_0$  as  $n \rightarrow \infty$  and both  $|(\tau_0^n, \tau_1^n) \cap I^+|$  and  $|(\tau_0^n, \tau_1^n) \cap I^-|$  are positive, where  $|\cdot|$  denotes the Lebesgue measure and  $I^\pm = \{t \in (0, t_0) \mid \pm v_\star^T u(t) > 0\}$ . Moreover, up to restricting  $(\tau_0^n, \tau_1^n)$ , we can assume that

$$(27) \quad |(\tau_0^n, \tau_1^n) \cap I^+| = |(\tau_0^n, \tau_1^n) \cap I^-| > 0.$$

This can be seen, for instance, by considering a continuous deformation of an interval around a Lebesgue point of  $(\tau_0^n, \tau_1^n) \cap I^+$  towards an interval around a Lebesgue point of  $(\tau_0^n, \tau_1^n) \cap I^-$ .

For every  $n \in \mathbb{N}$ , let  $u_n \in L^\infty([0, 1], \mathbb{R}^2)$  be defined by  $u_n(\tau) = u(\tau_0^n + \tau(\tau_1^n - \tau_0^n))$ . Up to extracting a subsequence,  $u_n$  weakly- $\star$  converges to some  $u_\star$ . Condition (27) and the limits in (26) imply that

$$(28) \quad \int_0^1 u_\star(t) dt = 0.$$

Thanks to (26) we also have that  $w_\star^T u_n$   $L^\infty$ -converges to zero as  $n \rightarrow \infty$ . In particular,  $w_\star^T u_\star \equiv 0$ . By Lemma 8,  $u_\star$  is optimal and  $v_\star^T u_\star$  has values in  $\{-1, 1\}$ . Hence, by Lemma 22 in the appendix,  $v_\star^T u_\star$  is constantly equal to  $+1$  or  $-1$ . This contradicts (28) and the proof is concluded.  $\square$

#### APPENDIX A. AN ELEMENTARY LEMMA

**Lemma 22.** *Let  $(M, D, g)$  be a sub-Riemannian manifold. Let  $V$  be a Lipschitz continuous vector field on  $T^*M$  such that  $\pi_* V(\lambda) \in D_{\pi(\lambda)} \setminus \{0\}$  for every  $\lambda \in T^*M$ . Let  $\lambda : [0, T] \rightarrow T^*M$  satisfy  $\dot{\lambda}(t) = \sigma(t)V(\lambda(t))$  with  $\sigma \in L^\infty([0, T], [-1, 1])$ . Assume that  $\gamma = \pi \circ \lambda : [0, T] \rightarrow M$  is a length-minimizer. Then  $\sigma$  has constant sign, i.e., either  $\sigma \geq 0$  a.e. on  $[0, T]$  or  $\sigma \leq 0$  a.e. on  $[0, T]$ .*

*Proof.* Set  $\kappa = \int_0^T \sigma(t) dt$  and notice that  $\lambda(T) = e^{\kappa V}(\lambda(0))$ . If  $\sigma$  does not have constant sign, then  $[0, 1] \ni t \mapsto \pi \circ e^{t\kappa V}(\lambda(0))$  is a curve connecting  $\gamma(0)$  to  $\gamma(T)$  and having length smaller than  $\gamma$ .  $\square$

A particular case of the lemma occurs when  $V = \vec{H}$  is the Hamiltonian vector field on  $T^*M$  associated with the Hamiltonian  $\lambda \mapsto \langle \lambda, X(\pi(\lambda)) \rangle$ , where  $X$  is a smooth horizontal never-vanishing vector field on  $M$ . This means that if a solution of  $\dot{\gamma}(t) = \sigma(t)X(\gamma(t))$  is a length-minimizer then  $\sigma$  has constant sign.

## REFERENCES

- [ABB17] A. Agrachev, D. Barilari, and U. Boscain. Introduction to Riemannian and sub-Riemannian geometry. Lecture notes available at <http://webusers.imj-prg.fr/~davide.barilari/Notes.php>, preprint SISSA 09/2012/M. Version Nov 17, 2017.
- [AGM15] L. Ambrosio, R. Ghezzi, and V. Magnani. BV functions and sets of finite perimeter in sub-Riemannian manifolds. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(3):489–517, 2015.
- [Agr14] A. A. Agrachev. Some open problems. In *Geometric control theory and sub-Riemannian geometry*, volume 5 of *Springer INdAM Ser.*, pages 1–13. Springer, Cham, 2014.
- [AS95] A. A. Agrachev and A. V. Sarychev. Strong minimality of abnormal geodesics for 2-distributions. *J. Dynam. Control Systems*, 1(2):139–176, 1995.
- [AS99] A. A. Agrachev and A. V. Sarychev. Sub-Riemannian metrics: Minimality of abnormal geodesics versus subanalyticity. *ESAIM Control Optim. Calc. Var.*, 4, 1999.
- [BdSRar] A. Belotto da Silva and L. Rifford. The Sard conjecture on Martinet surfaces. *Duke Math. J.*, to appear.
- [Bel96] A. Bellaïche. The tangent space in sub-Riemannian geometry. In *Sub-Riemannian geometry*, volume 144 of *Progr. Math.*, pages 1–78. Birkhäuser, Basel, 1996.
- [CJT06] Y. Chitour, F. Jean, and E. Trélat. Genericity results for singular curves. *J. Differential Geom.*, 73(1):45–73, 2006.
- [HL16] E. Hakavuori and E. Le Donne. Non-minimality of corners in subriemannian geometry. *Invent. Math.*, pages 1–12, 2016.
- [HL18] E. Hakavuori and E. Le Donne. Blowups and blowdowns of geodesics in Carnot groups. *ArXiv e-prints*, June 2018.
- [Jea14] F. Jean. *Control of Nonholonomic Systems: from Sub-Riemannian Geometry to Motion Planning*. Springer International Publishing, SpringerBriefs in Mathematics, 2014.
- [LDLMV13] E. Le Donne, G. P. Leonardi, R. Monti, and D. Vittone. Extremal curves in nilpotent Lie groups. *Geom. Funct. Anal.*, 23(4):1371–1401, 2013.
- [LDMO<sup>+</sup>16] E. Le Donne, R. Montgomery, A. Ottazzi, P. Pansu, and D. Vittone. Sard property for the end-point map on some Carnot groups. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(6):1639–1666, 2016.
- [LM08] G. P. Leonardi and R. Monti. End-point equations and regularity of sub-Riemannian geodesics. *Geom. Funct. Anal.*, 18(2):552–582, 2008.
- [LS95] W. S. Liu and H. J. Sussmann. Shortest paths for sub-Riemannian metrics of rank two distributions. *Memoirs AMS*, 118(564), 1995.
- [Mon02] R. Montgomery. *A tour of subriemannian geometries, their geodesics and applications*, volume 91 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.
- [Mon14a] R. Monti. The regularity problem for sub-Riemannian geodesics. In *Geometric control theory and sub-Riemannian geometry*, volume 5 of *Springer INdAM Ser.*, pages 313–332. Springer, Cham, 2014.
- [Mon14b] R. Monti. Regularity results for sub-Riemannian geodesics. *Calc. Var. Partial Differential Equations*, 49(1-2):549–582, 2014.
- [MPV17] R. Monti, A. Pigati, and D. Vittone. On tangent cones to length minimizers in Carnot–Carathéodory spaces. *Preprint CVGMT*, 2017.
- [MPV18] R. Monti, A. Pigati, and D. Vittone. Existence of tangent lines to carnot–carathéodory geodesics. *Calculus of Variations and Partial Differential Equations*, 57(3):75, Apr 2018.
- [Rif17] L. Rifford. Singulières minimisantes en géométrie sous-Riemannienne. *Astérisque*, (390):Exp. No. 1113, 277–301, 2017. Séminaire Bourbaki. Vol. 2015/2016. Exposés 1104–1119.
- [Sus14] H. J. Sussmann. A regularity theorem for minimizers of real-analytic subriemannian metrics. *53rd IEEE Conference on Decision and Control*, pages 4801–4806, 2014.
- [TY13] K. Tan and X. Yang. Subriemannian geodesics of Carnot groups of step 3. *ESAIM Control Optim. Calc. Var.*, 19(1):274–287, 2013.
- [ZZ95] I. Zelenko and M. Zhitomirskii. Rigid paths of generic 2-distributions on 3-manifolds. *Duke Math. J.*, 79(2):281–307, 08 1995.

UNIV. PARIS DIDEROT, INSTITUT DE MATHÉMATIQUES DE JUSSIEU-PARIS RIVE GAUCHE, CNRS, SORBONNE UNIVERSITÉ. BÂTIMENT SOPHIE-GERMAIN, CASE 7012, 75205 PARIS CEDEX 13, FRANCE  
*E-mail address:* `davide.barilari@imj-prg.fr`

UNIVERSITÉ PARIS-SUD, L2S, CENTRALESUPÉLEC, UNIVERSITÉ PARIS-SACLAY, GIF-SUR-YVETTE, FRANCE  
*E-mail address:* `yacine.chitour@l2s.centralesupelec.fr`

UNITÉ DE MATHÉMATIQUES APPLIQUÉES, ENSTA PARISTECH, UNIVERSITÉ PARIS-SACLAY, 91120 PALAISEAU, FRANCE  
*E-mail address:* `frederic.jean@ensta-paristech.fr`

CRNS, L2S, CENTRALESUPÉLEC, UNIVERSITÉ PARIS-SACLAY, GIF-SUR-YVETTE, FRANCE  
*E-mail address:* `dario.prandi@l2s.centralesupelec.fr`

INRIA & LABORATOIRE JACQUES-LOUIS LIONS, SORBONNE UNIVERSITÉ, 75005 PARIS, FRANCE  
*E-mail address:* `mario.sigalotti@inria.fr`