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# ON THE REGULARITY OF ABNORMAL MINIMIZERS FOR RANK 2 SUB-RIEMANNIAN STRUCTURES

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ABSTRACT. We prove the  $C^1$  regularity for a class of abnormal length-minimizers in rank 2 sub-Riemannian structures. As a consequence of our result, all length-minimizers for rank 2 sub-Riemannian structures of step up to 4 are of class  $C^1$ .

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## 1. INTRODUCTION

The question of regularity of length-minimizers is one of the main open problems in sub-Riemannian geometry, cf. for instance [Mon02, Problem 10.1] or [Agr14, Problem II] and the survey [Mon14a].

Length-minimizers are solutions to a variational problem with constraints and satisfy a first-order necessary condition resulting from the Pontryagin Maximum Principle. With every length-minimizer  $\gamma : [0, T] \rightarrow M$  we can associate a lift  $\lambda : [0, T] \rightarrow T^*M$  in the cotangent space, satisfying a Hamiltonian equation. This lift can be either *normal* or *abnormal*, although a length-minimizer  $\gamma$  can actually admit several lifts, each of them being either normal or abnormal.

If a length-minimizer admits a normal lift, then it is smooth, i.e.,  $C^\infty$ , since normal lifts are solutions of smooth autonomous Hamiltonian systems in  $T^*M$ . Note that we assume length-minimizers to be parametrized by arclength and their regularity is meant with respect to this time parametrization. The question of regularity is then reduced to length-minimizers that are strictly abnormal, i.e., those which do not admit normal lifts. For such length-minimizers, from the first order necessary condition (and actually from the second order one as well) it is a priori not possible to deduce any regularity other than Lipschitz continuity.

In this paper we investigate the following.

**Open Problem.** *Are all length-minimizers in a sub-Riemannian manifold of class  $C^1$ ?*

If the sub-Riemannian structure has step 2, there are no strictly abnormal length-minimizers, see e.g. [AS95, ABB17], thus every length-minimizer admits a normal lift,

and is hence smooth. For step 3 structures, the situation is already more complicated and a positive answer to the above problem is known only for Carnot groups (where, actually, length-minimizers are proved to be  $C^\infty$ ), see [LDLMV13, TY13]. When the sub-Riemannian structure is analytic, more is known on the size of the set of points where a length-minimizer can lose regularity [Sus14], regardless of the rank and of the step of the distribution.

To state our main result, we introduce some notations. We refer the reader to Section 2 for precise definitions. Recall that a sub-Riemannian structure  $(D, g)$  on  $M$  is defined by a bracket generating distribution  $D$  endowed with a metric  $g$ . Hence  $D$  defines a flag of subspaces at every point  $x \in M$

$$D_x = D_x^1 \subset D_x^2 \subset D_x^3 \subset \dots \subset D_x^r = T_x M,$$

where  $D_x^i$  is the subspace of the tangent space spanned by Lie brackets of length at most  $i$  between horizontal vector fields. This induces a dual decreasing sequence of subspaces of  $T_x^* M$

$$0 = (D_x^r)^\perp \subset \dots \subset (D_x^4)^\perp \subset (D_x^3)^\perp \subset (D_x^2)^\perp \subset (D_x^1)^\perp \subset T_x^* M,$$

where perpendicularity is considered with respect to the duality product. By construction, any abnormal lift satisfies  $\lambda(t) \in (D^1)^\perp$  for every  $t$ . If the lift is strictly abnormal, then by Goh conditions  $\lambda(t) \in (D^2)^\perp$  for every  $t$ .

When the distribution has rank 2, it is known that if  $\lambda(t)$  does not cross  $(D^3)^\perp$ , then the length-minimizer is  $C^\infty$  [LS95, Sect. 6.2, Cor. 4]. Our main result pushes this analysis further and establishes that the answer to the Open Problem is positive for length-minimizers whose abnormal lift does not enter  $(D^4)^\perp$ .

**Theorem 1.** *Let  $(D, g)$  be a rank 2 sub-Riemannian structure on  $M$ . Assume that  $\gamma : [0, T] \rightarrow M$  is an abnormal minimizer parametrized by arclength. If  $\gamma$  admits a lift satisfying  $\lambda(t) \notin (D^4)^\perp$  for every  $t \in [0, T]$ , then  $\gamma$  is of class  $C^1$ .*

If the sub-Riemannian manifold has rank 2 and step at most 4, the assumption in Theorem 1 is trivially satisfied by every abnormal minimizer  $\gamma$  and we immediately obtain the following corollary.

**Corollary 2.** *Assume that the sub-Riemannian structure has rank 2 and step at most 4. Then all length-minimizers are of class  $C^1$ .*

It is legitimate to ask whether the  $C^1$  regularity in the Open Problem can be further improved. Indeed, the argument behind our proof permits to obtain  $C^\infty$  regularity of length-minimizers under an additional nilpotency condition on the Lie algebra generated by horizontal vector fields.

**Proposition 3.** *Assume that  $D$  is generated by two vector fields  $X_1, X_2$  such that the Lie algebra  $\text{Lie}\{X_1, X_2\}$  is nilpotent of step at most 4. Then for every sub-Riemannian structure  $(D, g)$  on  $M$ , the corresponding length-minimizers are of class  $C^\infty$ .*

The above proposition applies in particular to Carnot groups of rank 2 and step at most 4. In this case we recover the results obtained in [LM08, Example 4.6].

The strategy of proof of Theorem 1 is to show that, at points where they are not of class  $C^1$ , length-minimizers can admit only corner-like singularities. This is done by a careful asymptotic analysis of the differential equations satisfied by the abnormal lift, which exploits their Hamiltonian structure. We can then conclude thanks to the following result.

**Theorem 4** ([HL16]). *Let  $M$  be a sub-Riemannian manifold. Let  $T > 0$  and let  $\gamma : [-T, T] \rightarrow M$  be a horizontal curve parametrized by arclength. Assume that, in local*

coordinates, there exist

$$\dot{\gamma}^+(0) := \lim_{t \downarrow 0} \frac{\gamma(t) - \gamma(0)}{t}, \quad \dot{\gamma}^-(0) := \lim_{t \uparrow 0} \frac{\gamma(t) - \gamma(0)}{t}.$$

If  $\dot{\gamma}^+(0) \neq \dot{\gamma}^-(0)$ , then  $\gamma$  is not a length-minimizer.

We observe that the proof contained in [HL16] requires a previous result stated in [LM08]. A complete argument for the latter, addressing some issues raised in [Rif17, p. 1113-15], is provided in [MPV17]. For sub-Riemannian structures of rank 2 and step at most 4 (and indeed also for higher step, under an additional condition on the Lie algebra generated by horizontal vector fields), the fact that corners are not length-minimizers is already contained in [LM08].

We notice that the answer to the Open Problem is known to be positive also in a class of rank 2 Carnot groups (with no restriction on the step, but satisfying other additional conditions). For this class of structures in [Mon14b], it is proved the  $C^{1,\alpha}$  regularity for some suitable  $\alpha > 0$  (depending on the step).

We also refer to [MPV18, HL18] for recent results regarding these issues.

**1.1. Structure of the paper.** In Section 2 we recall some notations and preliminary notions. Section 3 is devoted to a desingularization and nilpotentization argument. Section 4 contains a preliminary analysis on the dynamics of abnormal extremals. To illustrate our approach in a simpler case, we discuss in Section 5 the proof of the main result for a nilpotent structure of step up to 4. Then in Sections 6 and 7 we complete our analysis to prove the general result. Appendix A contains a technical lemma.

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## 2. NOTATIONS AND PRELIMINARY NOTIONS

Let  $M$  be a smooth  $n$ -dimensional manifold. A *sub-Riemannian structure* of rank  $m$  on  $M$  is a triplet  $(E, g^E, f)$  where  $E$  is a vector bundle of rank  $m$  over  $M$ ,  $g^E$  is an Euclidean metric on  $E$ , and  $f : E \rightarrow TM$  is a morphism of vector bundles such that  $f(E_x) \subseteq T_x M$  for every  $x \in M$ . Fix such a structure and define a family of subspaces of the tangent spaces by

$$D_x = \{X(x) \mid X \in D\} \subseteq T_x M, \quad \forall x \in M,$$

where  $D = \{f \circ Y \mid Y \text{ smooth section of } E\}$  is a submodule of the set of vector fields on  $M$ . We assume that the structure is *bracket generating*, i.e., the tangent space  $T_x M$  is spanned by the vector fields in  $D$  and their iterated Lie brackets evaluated at  $x$ .

The sub-Riemannian structure induces a quadratic form  $g_x$  on  $D_x$  by

$$g_x(v, v) = \inf\{g_x^E(u, u) \mid f(u) = v, u \in E_x\}, \quad v \in D_x.$$

In analogy with the classic sub-Riemannian case and to simplify notations, in the sequel we will refer to the sub-Riemannian structure as the pair  $(D, g)$  rather than  $(E, g^E, f)$ . This is justified since all the constructions and definitions below rely only on  $D$  and  $g$ . The triplet  $(M, D, g)$  is called a *sub-Riemannian manifold*.

*Remark 5.* Usually, a sub-Riemannian manifold denotes a triplet  $(M, D, g)$ , where  $M$  is a smooth manifold,  $D$  is a subbundle of  $TM$ , and  $g$  is a Riemannian metric on  $D$  (see, e.g., [Bel96]). This corresponds to the case where  $f(E_x)$  is of constant rank. The definition given above follows, for instance, [ABB17].

A *horizontal curve*  $\gamma : [0, T] \rightarrow M$  is an absolutely continuous path such that  $\dot{\gamma}(t) \in D_{\gamma(t)}$  for almost every (a.e. for short)  $t \in [0, T]$ . The *length* of a horizontal curve is defined by

$$\ell(\gamma) = \int_0^T \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

The *sub-Riemannian distance* between two arbitrary points  $x, y$  in  $M$  is then

$$d(x, y) = \inf\{\ell(\gamma) \mid \gamma(0) = x, \gamma(T) = y, \gamma \text{ horizontal}\}.$$

A *length-minimizer* is a horizontal curve  $\gamma$  which realizes the distance between its extremities, that is,  $\ell(\gamma) = d(\gamma(0), \gamma(T))$ . Note that any time-reparametrization of a length-minimizer is a length-minimizer as well.

A *generating frame* of the sub-Riemannian structure is a family of smooth vector fields  $X_1, \dots, X_k$  such that  $D$  is generated by  $X_1, \dots, X_k$  as a module and

$$g_x(v, v) = \inf \left\{ \sum_{i=1}^k u_i^2 \mid \sum_{i=1}^k u_i X_i(x) = v \right\}, \quad x \in U, v \in D_x.$$

There always exists a global generating frame (see [ABB17, Corollary 3.26]), with, in general, a number  $k$  of elements greater than the rank  $m$  of the structure. However, every point  $x \in M$  admits a neighborhood on which there exists a (local) generating frame with exactly  $k = m$  elements, e.g., by taking the image via  $f$  of a local orthonormal frame of  $(E, g^E)$ .

Fix now a (local or global) generating frame  $X_1, \dots, X_k$  of  $(D, g)$ . For any horizontal curve  $\gamma$  of finite length, there exists  $u \in L^\infty([0, T], \mathbb{R}^k)$  satisfying

$$(1) \quad \dot{\gamma}(t) = \sum_{i=1}^k u_i(t) X_i(\gamma(t)), \quad \text{for a.e. } t \in [0, T].$$

The curve is said to be *parametrized by arclength* if  $g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t)) = 1$  for a.e.  $t \in [0, T]$ , i.e., if there exists  $u \in L^\infty([0, T], \mathbb{S}^{k-1})$  satisfying (1). In that case  $\ell(\gamma) = T$ .

To state the first order necessary conditions, let us first introduce some notations. For  $\lambda \in T^*M$  and  $x = \pi(\lambda)$ , where  $\pi : T^*M \rightarrow M$  is the canonical projection, we set  $h_i(\lambda) = \langle \lambda, X_i(x) \rangle$ , for  $i = 1, \dots, k$  (here  $\langle \lambda, \cdot \rangle$  denotes the dual action of covectors on vectors). Recall also that, for a function  $H : T^*M \rightarrow \mathbb{R}$ , the corresponding Hamiltonian vector field  $\vec{H}$  is the unique vector field such that  $\sigma(\cdot, \vec{H}) = dH$ , where  $\sigma$  is the canonical symplectic form on the cotangent bundle.

Applying the Pontryagin Maximum Principle to the sub-Riemannian length minimization problem yields the following theorem.

**Theorem 6.** *Let  $(M, D, g)$  be a sub-Riemannian manifold with generating frame  $X_1, \dots, X_k$  and  $\gamma : [0, T] \rightarrow M$  be a length-minimizer. Then there exists a nontrivial absolutely continuous curve  $t \mapsto \lambda(t) \in T_{\gamma(t)}^*M$  such that one of the following conditions is satisfied:*

- (N)  $\dot{\lambda}(t) = \vec{H}(\lambda(t))$  for all  $t \in [0, T]$ , where  $H(\lambda) = \frac{1}{2} \sum_{i=1}^k h_i^2$ ,
- (A)  $\dot{\lambda}(t) = \sum_{i=1}^k u_i(t) \vec{h}_i(\lambda(t))$  for almost every  $t \in [0, T]$ , with  $u_1, \dots, u_k \in L^1([0, T])$ .  
Moreover,  $\lambda(t) \in (D_{\gamma(t)})^\perp$  for all  $t$ , i.e.,  $h_i(\lambda(t)) \equiv 0$  for  $i = 1, \dots, k$ .

In case (N) (respectively, case (A)),  $\lambda$  is called a *normal* (respectively, *abnormal*) *extremal*. Normal extremals are integral curves of  $\vec{H}$ . As such, they are smooth. A length-minimizer is normal (respectively, abnormal) if it admits a normal (respectively, abnormal) extremal lift. We stress that both conditions can be satisfied for the same curve  $\gamma$ , with different lifts  $\lambda_1$  and  $\lambda_2$ .

## 3. DESINGULARISATION AND NILPOTENTIZATION

**3.1. Desingularisation.** Let  $(M, D, g)$  be a sub-Riemannian manifold. We define recursively the following sequence of submodules of the set of vector fields,

$$D^1 = D, \quad D^{i+1} = D^i + [D, D^i].$$

At every point  $x \in M$ , the evaluation at  $x$  of these modules induces a flag of subspaces of the tangent space,

$$D_x^1 \subset D_x^2 \subset \dots \subset D_x^r = T_x M.$$

The smallest integer  $r = r(x)$  satisfying  $D_x^r = T_x M$  is called the *step of  $D$  at  $x$* . A point is said to be *regular* if the dimensions of the subspaces of the flag are locally constant in an open neighborhood of the point. When every point in  $M$  is regular, the sub-Riemannian manifold is said to be *quiregular*.

In general a sub-Riemannian manifold may admit non-regular points. However, for our purposes, we can restrict ourselves with no loss of generality to equiregular manifolds thanks to a desingularisation procedure.

**Lemma 7.** *Fix an integer  $m \geq 2$ . Assume that for every rank  $m$  equiregular sub-Riemannian structure the following property holds: every arclength parametrized abnormal minimizer admitting a lift  $\lambda(t) \notin (D^4)^\perp$  is of class  $C^1$ . Then the same property holds true for every rank  $m$  sub-Riemannian structure.*

*Proof.* Let  $(M, D, g)$  be a non-equiregular sub-Riemannian manifold of rank  $m$  and  $\gamma$  be an abnormal length-minimizer of  $(M, D, g)$  which admits an abnormal extremal lift such that  $\lambda(t) \notin (D^4)^\perp$  for every  $t \in [0, T]$ . Assume moreover that  $\gamma$  is parametrized by arclength. We have to prove that  $\gamma$  is of class  $C^1$ .

Fix  $t_0 \in [0, T]$  and a generating frame  $X_1, \dots, X_m$  on a neighborhood of  $\gamma(t_0)$ . By [Jea14, Lemma 2.5], there exists an equiregular sub-Riemannian manifold  $(\widetilde{M}, \widetilde{D}, \widetilde{g})$  of rank  $m$  with a generating frame  $\xi_1, \dots, \xi_m$  and a map  $\varpi : \widetilde{M} \rightarrow M$  onto a neighborhood  $U \subset M$  of  $\gamma(t_0)$  such that  $\varpi_* \xi_i = X_i$ . Up to reducing the interval  $[0, T]$  we assume that  $\gamma(t) \in U$  for all  $t \in [0, T]$ . Let  $u \in L^\infty([0, T], \mathbb{S}^{m-1})$  be such that

$$\dot{\gamma}(t) = \sum_{i=1}^m u_i(t) X_i(\gamma(t)), \quad \text{a.e. } t.$$

By construction, since  $\gamma$  is a length-minimizer, there exists a length-minimizer  $\widetilde{\gamma}$  in  $\widetilde{M}$  with  $\varpi(\widetilde{\gamma}) = \gamma$  associated with the same  $u$ , that is,

$$\dot{\widetilde{\gamma}}(t) = \sum_{i=1}^m u_i(t) \xi_i(\widetilde{\gamma}(t)), \quad \text{a.e. } t,$$

which is parametrized by arclength as well. Hence the trajectory  $\gamma$  has at least the same regularity as  $\widetilde{\gamma}$ .

Moreover, if  $\lambda$  is an abnormal lift of  $\gamma$  in  $T^*M$ , then  $\widetilde{\lambda}$  admits an abnormal lift  $\widetilde{\lambda}$  in  $T^*\widetilde{M}$  such that  $\widetilde{\lambda}(t) = \varpi^* \lambda(t)$  for every  $t$ . Since  $\varpi^*(\widetilde{D}^k)^\perp = (D^k)^\perp$  for any positive integer  $k$ , the property  $\lambda(t) \notin (D^4)^\perp$  implies  $\widetilde{\lambda}(t) \notin (\widetilde{D}^4)^\perp$ .

It results from the hypothesis that  $\widetilde{\gamma}$  is  $C^1$ , so  $\gamma$  is of class  $C^1$  in an open neighborhood of  $t_0 \in [0, T]$ , which ends the proof.  $\square$

As a consequence of Lemma 7, we can assume in the rest of the paper that the sub-Riemannian manifold is equiregular.

**3.2. Nilpotentization.** Let us recall the construction of the nilpotent approximation (see for instance [Bel96] for details).

Let  $(M, D, g)$  be an equiregular sub-Riemannian manifold. We fix a point  $x \in M$  and a local generating frame  $X_1, \dots, X_m$  in a neighborhood of  $x$ .

For  $i = 1, \dots, m$ , let  $w_i$  be the smallest integer  $j$  such that  $\dim D_x^j \geq i$ . We define the dilations  $\delta_\nu : \mathbb{R}^n \rightarrow \mathbb{R}^n$  for  $\nu \in \mathbb{R}$  as  $\delta_\nu(z) = (\nu^{w_1} z_1, \dots, \nu^{w_m} z_m)$ . Let  $z^x$  be a system of privileged coordinates at  $x$  and set  $\delta_\nu^x = \delta_\nu \circ z^x$ . Then, for  $i = 1, \dots, m$ , the vector field  $\varepsilon(\delta_{1/\varepsilon}^x)_* X_i$  converges locally uniformly as  $\varepsilon \rightarrow 0$  to a vector field  $\widehat{X}_i^x$  on  $\mathbb{R}^n$ . The space  $\mathbb{R}^n$  endowed with the sub-Riemannian structure having  $\widehat{X}_1^x, \dots, \widehat{X}_m^x$  as generating frame is called the *nilpotent approximation of  $(M, D, g)$  at  $x$*  and is denoted by  $\widehat{M}_x$ . This nilpotent approximation  $\widehat{M}_x$  is a Carnot group equipped with a left-invariant sub-Riemannian structure.

Since  $(M, D, g)$  is equiregular, we can locally choose systems of privileged coordinates  $z^x$  depending continuously on  $x$  [Jea14, Sect. 2.2.2]. Note that the  $w_i$ 's and  $\delta_\nu$  are independent of  $x$ . Thus an easy adaptation of the proof of [AGM15, Prop. 3.4] (see also [ABB17, Sect. 10.4.1]) shows that, for  $i = 1, \dots, m$ , the vector field  $\varepsilon(\delta_{1/\varepsilon}^x)_* X_i$  converges locally uniformly to  $\widehat{X}_i^{x_0}$  as  $\varepsilon \rightarrow 0$  and  $x \rightarrow x_0$ .

**Lemma 8.** *Let  $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}} \subset [0, T]$ ,  $\bar{a} \in [0, T]$ , be such that  $a_n, b_n \rightarrow \bar{a}$  and  $a_n < b_n$  for any  $n \in \mathbb{N}$ . Given  $u \in L^\infty([0, T], \mathbb{S}^{m-1})$  and  $n \in \mathbb{N}$ , define  $u_n \in L^\infty([0, 1], \mathbb{S}^{m-1})$  by*

$$u_n(\tau) = u(a_n + \tau(b_n - a_n)).$$

*Assume that the sequence  $(u_n)_{n \in \mathbb{N}}$  converges to  $u_\star \in L^\infty([0, 1], \mathbb{R}^m)$  for the weak- $\star$  topology of  $L^\infty([0, 1], \mathbb{R}^m)$  and, moreover, that the trajectory  $\gamma : [0, T] \rightarrow M$  associated with  $u$  is a length-minimizer. If  $x = \gamma(\bar{a})$ , then the trajectory  $\gamma_\star : [0, 1] \rightarrow \widehat{M}_x$  satisfying*

$$\dot{\gamma}_\star(s) = \sum_{i=1}^m u_{\star,i}(s) \widehat{X}_i^x(\gamma_\star(s)), \quad \gamma_\star(0) = 0,$$

*is also a length-minimizer. In particular,  $u_\star(t) \in \mathbb{S}^{m-1}$  for almost every  $t \in [0, 1]$ .*

*Proof.* We consider a continuously varying family of privileged coordinates  $z^{\gamma(t)}$ ,  $t \in [0, T]$ , and the corresponding 1-parameter family of dilations  $\delta_\nu^t := \delta_\nu^{\gamma(t)}$ . It is not restrictive to assume that  $\delta_{\frac{1}{b_n - a_n}}^{a_n} \gamma(t)$  is well-defined for every  $n \in \mathbb{N}$  and  $t \in [a_n, b_n]$ .

Let  $\gamma_n$  be defined by  $\gamma_n(\tau) = \delta_{\frac{1}{b_n - a_n}}^{a_n}(\gamma(a_n + \tau(b_n - a_n)))$ . Then,  $\gamma_n$  is a length-minimizing curve for the sub-Riemannian structure on  $\mathbb{R}^n$  with orthonormal frame

$$(b_n - a_n) \left( \delta_{\frac{1}{b_n - a_n}}^{a_n} \right)_* X_1, \dots, (b_n - a_n) \left( \delta_{\frac{1}{b_n - a_n}}^{a_n} \right)_* X_m.$$

The corresponding control is  $u_n$ .

Since the sequence  $\left( (b_n - a_n) \left( \delta_{\frac{1}{b_n - a_n}}^{a_n} \right)_* X_i \right)_{n \in \mathbb{N}}$  converges locally uniformly to  $\widehat{X}_i^x$ , it follows by standard ODE theory that  $(\gamma_n)_{n \in \mathbb{N}}$  converges uniformly to  $\gamma_\star$ .

We claim that  $\widehat{d}(\gamma_\star(0), \gamma_\star(1)) = 1$ . Indeed,  $\ell(\gamma_\star) \geq \widehat{d}(\gamma_\star(0), \gamma_\star(1))$  and, by [Bel96, Theorem 7.32], we have

$$\begin{aligned} \widehat{d}(\gamma_\star(0), \gamma_\star(1)) &= \lim_{n \rightarrow \infty} \frac{1}{b_n - a_n} d(\gamma(a_n), \gamma(b_n)) \\ &= \lim_{n \rightarrow \infty} \int_0^1 |u(a_n + \tau(b_n - a_n))| d\tau = 1, \end{aligned}$$

where  $|\cdot|$  denotes the norm in  $\mathbb{R}^m$ . On the other hand, by weak- $\star$  convergence we have

$$\ell(\gamma_\star) = \|u_\star\|_{L^1([0, 1], \mathbb{R}^m)} \leq \liminf_{n \rightarrow \infty} \|u_n\|_{L^1([0, 1], \mathbb{R}^m)} = 1,$$

proving the claim.

To conclude the proof, it suffices now to observe that the above implies that  $\gamma_\star$  is minimizing. In particular, since  $|u_\star(t)| \leq 1$  a.e. on  $[0, 1]$  by the properties of weak- $\star$  convergence, this shows that  $|u_\star(t)| = 1$  a.e. on  $[0, 1]$ .  $\square$

**Corollary 9.** *Let  $\gamma$ ,  $u$ ,  $\bar{a}$ ,  $(a_n)_{n \in \mathbb{N}}$ ,  $(b_n)_{n \in \mathbb{N}}$ , and  $u_\star$  be as in Lemma 8. Assume that there exist  $u_+, u_- \in \mathbb{S}^{m-1}$  such that  $u_\star = u_-$  almost everywhere on  $[0, 1/2]$  and  $u_\star = u_+$  almost everywhere on  $[1/2, 1]$ . Then  $u_- = u_+$ .*

*Proof.* If  $u_- \neq u_+$ , then  $\gamma_\star$  is not length-minimizing by Theorem 4, which contradicts Lemma 8.  $\square$

#### 4. DYNAMICS OF ABNORMAL EXTREMALS: PRELIMINARY RESULTS

In this section we present the dynamical system associated with the abnormal extremal, whose analysis is the basis for the proof of Theorem 1, and we derive a first result on its structure.

**4.1. Introduction to the dynamical system.** Let  $(M, D, g)$  be an equiregular sub-Riemannian manifold of rank 2. Since the arguments are local, in what follows we fix a local generating frame  $\{X_1, X_2\}$  of  $(D, g)$ .

Consider an abnormal length-minimizer  $\gamma : [0, T] \rightarrow M$  parametrized by arclength. Then  $T = d(\gamma(0), \gamma(T))$  and there exists  $u \in L^\infty([0, T], \mathbb{S}^1)$  such that

$$\dot{\gamma}(t) = u_1(t)X_1(\gamma(t)) + u_2(t)X_2(\gamma(t)), \quad \text{a.e. } t \in [0, T].$$

Moreover from Theorem 6,  $\gamma$  admits a lift  $\lambda : [0, T] \rightarrow T^*M$  which satisfies

$$\dot{\lambda}(t) = u_1 \vec{h}_1(\lambda(t)) + u_2 \vec{h}_2(\lambda(t)) \quad \text{and} \quad h_1(\lambda(t)) \equiv h_2(\lambda(t)) \equiv 0.$$

By a slight abuse of notation, set  $h_i(t) = \langle \lambda(t), X_i(\gamma(t)) \rangle$ ,  $i = 1, 2$ , and for every  $i_1, \dots, i_m \in \{1, 2\}$ ,

$$h_{i_1 \dots i_m}(t) = \langle \lambda(t), [X_{i_1}, \dots, [X_{i_{m-1}}, X_{i_m}]](\gamma(t)) \rangle.$$

Such a function  $h_{i_1 \dots i_m}$  is absolutely continuous and satisfies

$$(2) \quad \dot{h}_{i_1 \dots i_m}(t) = u_1(t)h_{1i_1 \dots i_m}(t) + u_2(t)h_{2i_1 \dots i_m}(t) \quad \text{for a.e. } t \in [0, T].$$

Differentiating the equalities  $h_1 \equiv h_2 \equiv 0$  and using (2) we obtain  $h_{12} \equiv 0$ . Differentiating again we get

$$(3) \quad 0 = \dot{h}_{12} = u_1 h_{112} + u_2 h_{212} \quad \text{a.e. on } [0, T].$$

*Remark 10.* The identities  $h_1(t) = h_2(t) = h_{12}(t) = 0$  imply that  $\lambda(t) \in (D^2)^\perp$  for every  $t$ . The latter is known as *Goh condition* and is in general (i.e., for sub-Riemannian structures of any rank) a necessary condition for the associated curve to be length-minimizing [AS99]. It is known that a generic sub-Riemannian structure of rank larger than 2 does not have non-constant abnormal extremals satisfying the Goh condition [CJT06].

Let  $h = (-h_{212}, h_{112})$  and  $(t_0, t_1) \subset (0, T)$  be a maximal (i.e., non-extendable) open interval on which  $h \neq 0$ . Equation (3) then implies that  $u = \pm h/|h|$  almost everywhere on  $(t_0, t_1)$ .

Moreover, by length-minimality of  $\gamma$  we can assume without loss of generality that  $u = h/|h|$  on  $(t_0, t_1)$  (see Lemma 22 in the appendix). Thus  $\gamma$  may be non-differentiable only at a time  $t$  such that  $h(t) = 0$ . In particular, if the step of the sub-Riemannian structure is not greater than 3, then  $\gamma$  is differentiable everywhere. We assume from now on that the step is at least 4.

Observe that from (2) and using  $u = h/|h|$  one obtains

$$(4) \quad \dot{h} = A \frac{h}{|h|}, \quad A = \begin{pmatrix} -h_{2112} & -h_{2212} \\ h_{1112} & h_{2112} \end{pmatrix}, \quad \text{on } (t_0, t_1).$$



Here, we used the relation  $h_{1212} = h_{2112}$ , which follows from the Jacobi identity

$$[X_1, [X_2, [X_1, X_2]]] = -[[X_1, X_2], [X_1, X_2]] - [X_2, [[X_1, X_2], X_1]] = [X_2, [X_1, [X_1, X_2]]].$$

Observe that the matrix  $A$  has zero trace and is absolutely continuous on the whole interval  $[0, T]$ .

**Lemma 11.** *Assume that  $\lambda(t) \notin (D_{\gamma(t)}^4)^\perp$  for every  $t \in [0, T]$ . If  $h(t_0) = 0$  for some  $t_0 \in [0, T]$ , then  $A(t_0) \neq 0$ .*

*Proof.* The fact that  $\gamma$  is abnormal implies that the non-zero covector  $\lambda(t)$  annihilates  $D_{\gamma(t)}$  for every  $t \in [0, T]$ . The Goh condition  $h_{12} \equiv 0$  guarantees that it also annihilates  $D_{\gamma(t)}^2$ . The fact that  $h(t_0) = 0$  says, moreover, that  $\lambda(t_0)$  annihilates  $D_{\gamma(t_0)}^3$ . If  $A(t_0)$  is equal to zero, then  $\lambda(t_0)$  annihilates  $D_{\gamma(t_0)}^4$ , which contradicts the assumption.  $\square$

**4.2. The sign of  $\det A$  is non-negative where  $h$  vanishes.** A key step in the proof of Theorem 1 is the following result.

**Proposition 12.** *Let  $(t_0, t_1)$  be a maximal open interval of  $[0, T]$  on which  $h \neq 0$  and assume that  $t_1 < T$ . Then  $\det A(t_1) \leq 0$ .*

*Proof.* Assume by contradiction that  $\det A(t_1) > 0$ . Since  $\text{trace } A(t_1) = 0$ , there exists  $P \in \text{GL}(2, \mathbb{R})$  such that

$$(5) \quad P^{-1}A(t_1)P = \begin{pmatrix} 0 & -a \\ a & 0 \end{pmatrix}, \quad a > 0.$$

Define the scalar functions  $\alpha$ ,  $\beta$  and  $\zeta$  through the relation

$$P^{-1}A(t)P = \begin{pmatrix} -\alpha(t) & \beta(t) \\ \zeta(t) & \alpha(t) \end{pmatrix},$$

and notice that  $\alpha, \beta, \zeta$  are absolutely continuous with bounded derivatives on  $(t_0, t_1)$ , since they are linear combinations of  $h_{2112}, h_{2212}, h_{1112}$ , according to (2). Clearly, (5) implies that  $\alpha(t) \rightarrow 0$ ,  $\beta(t) \rightarrow -a$ , and  $\zeta(t) \rightarrow a$  as  $t \rightarrow t_1$ .

Consider a time rescaling and a polar coordinates representation so that  $P^{-1}h(t) = \rho(s(t))e^{i\vartheta(s(t))}$ , where

$$s(t) := \int_{t_0}^t \frac{|P^{-1}h(\tau)|}{|h(\tau)|} d\tau.$$

It is useful to introduce  $\mu := (\zeta + \beta)/2$  and  $\eta := (\zeta - \beta)/2$ . Then, denoting by  $\rho'$  and  $\vartheta'$  the derivatives of  $\rho$  and  $\vartheta$  with respect to the parameter  $s$ , (4) can be rewritten as

$$\begin{cases} \rho' = (-\alpha \cos 2\vartheta + \mu \sin 2\vartheta), \\ \vartheta' = \frac{1}{\rho}(\alpha \sin 2\vartheta + \mu \cos 2\vartheta + \eta). \end{cases}$$

Let  $w = \alpha \sin 2\vartheta + \mu \cos 2\vartheta + \eta$  and notice that  $2a > w > a/2$  in a left-neighborhood of  $s(t_1)$ . Therefore,

$$\begin{aligned} (\rho^2 w)' &= 2\rho(-\alpha \cos 2\vartheta + \mu \sin 2\vartheta)w + \rho^2(\alpha' \sin 2\vartheta + \mu' \cos 2\vartheta + \eta') \\ &\quad + \rho^2(\alpha \cos 2\vartheta - \mu \sin 2\vartheta)2\vartheta' \\ &= \rho^2 w \frac{\alpha' \sin 2\vartheta + \mu' \cos 2\vartheta + \eta'}{w} \geq -M\rho^2 w, \end{aligned}$$

for some constant  $M > 0$ . This implies at once that  $t \mapsto e^{Mt} \rho^2(t)w(t)$  is increasing, and hence that it is impossible for  $\rho^2 w$  to tend to zero as  $s \rightarrow s(t_1)$ . This contradicts the assumption that  $\rho(t) \rightarrow 0$  as  $t \rightarrow t_1$ , completing the proof of the statement.  $\square$

## 5. DYNAMICS OF ABNORMAL EXTREMALS IN A SPECIAL CASE: PROOF OF PROPOSITION 3

In this section we prove Proposition 3. We present it here to illustrate in a simpler context the general procedure used later to complete the proof of Theorem 1.

Assume that  $D$  is generated by two vector fields  $X_1, X_2$  such that the Lie algebra  $\text{Lie}\{X_1, X_2\}$  is nilpotent of step at most 4. This means that all Lie brackets of  $X_1, X_2$  of length 5 vanish. In particular  $\dim M \leq 8$ .

*Proof of Proposition 3.* Without loss of generality, we assume that the step is equal to 4. Recall that for an abnormal minimizer on an interval  $I$  we have

$$h_1 \equiv h_2 \equiv h_{12} \equiv 0, \quad 0 = \dot{h}_{12} = u_1 h_{112} + u_2 h_{212} \quad \text{a.e. on } I.$$

The vector  $h = (-h_{212}, h_{112})$  satisfies the differential equation

$$(6) \quad \dot{h} = Au, \quad A = \begin{pmatrix} -h_{2112} & -h_{2212} \\ h_{1112} & h_{2112} \end{pmatrix}, \quad \text{a.e. on } I.$$

Notice that  $A$  is a constant matrix (with zero trace), as follows from (2) and the nilpotency assumption.

As we have already seen in the general case, on every interval where  $h(t) \neq 0$  we have that  $u$  is smooth and equal to either  $\frac{h(t)}{|h(t)|}$  or  $-\frac{h(t)}{|h(t)|}$ .

We are then reduced to the case where  $h$  vanishes at some point  $\bar{t} \in I$ . In this case the matrix  $A$  cannot be zero, as it follows from Lemma 11.

We consider the following alternative:

- (a)  $h(t) = 0$  for all  $t \in I$ ;
- (b)  $h$  does not vanishes identically on  $I$ .

Case (a). From (6) it follows that  $u(t)$  is in the kernel of  $A$  for a.e.  $t \in I$ . Since  $u$  is nonzero for a.e.  $t \in I$ , then necessarily  $A$  has one-dimensional kernel  $\ker A = \text{span}\{\bar{u}\}$ , where  $\bar{u}$  has norm one. Then  $u(t) = \sigma(t)\bar{u}$  for a.e.  $t \in I$ , with  $\sigma(t) \in \{-1, 1\}$  and

$$\dot{\gamma}(t) = \sigma(t)X_{\bar{u}}(\gamma(t)), \quad \text{a.e. } t \in I,$$

with  $X_{\bar{u}}$  a constant vector field. Since  $\gamma$  is a length-minimizer then  $\sigma$  is constant, and  $u$  is smooth, thanks to Lemma 22 in the appendix.

Case (b). Consider a maximal interval  $J = (t_0, t_1)$  on which  $h$  is never vanishing. Since  $J \subsetneq I$ , then either  $h(t_0) = 0$  or  $h(t_1) = 0$ .

The trajectories of (4) are time reparametrizations of those of the linear system  $\dot{z} = Az$ . Hence  $h$  stays in the stable or in the unstable manifold of  $A$ . Recall that  $\det A \leq 0$  by Proposition 12 and notice that if  $\det A = 0$  then  $A$  is conjugate to the nilpotent matrix  $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ . Hence stable and unstable manifolds reduce to zero. We deduce that  $\det A < 0$ .

Denote by  $\lambda_{\pm}$  the eigenvalues of  $A$  and by  $v_{\pm}$  the corresponding unit eigenvectors. Since  $h$  belongs to the stable (respectively, unstable) manifold of  $A$  then  $\frac{h(t)}{|h(t)|}$  is constantly equal to  $v_-$  or  $-v_-$  on  $J$  (respectively,  $v_+$  or  $-v_+$ ). Fix  $t_* \in J$ . Then integrating (6) we get

$$h(t) = h(t_*) \pm (t - t_*)\lambda_- v_-, \quad t \in J,$$

or

$$h(t) = h(t_*) \pm (t - t_*)\lambda_+ v_+, \quad t \in J.$$

If  $h(t_1) = 0$ , then  $\lim_{t \downarrow t_0} h(t) \neq 0$  and  $J = I \cap (-\infty, t_1)$ . Similarly, if  $h(t_0) = 0$  then  $\lim_{t \uparrow t_1} h(t) \neq 0$  and  $J = I \cap (t_0, +\infty)$ .

If there exist two distinct maximal intervals of  $I$  on which  $h$  is never vanishing, then necessarily there exist  $\tau_1 \leq \tau_2$  in  $I$  such that these maximal intervals are of the form  $J_1 = I \cap (-\infty, \tau_1)$  and  $J_2 = I \cap (\tau_2, +\infty)$ . Notice that  $h$  vanishes on  $[\tau_1, \tau_2]$ .

If  $\tau_1 < \tau_2$ , we can apply case (a) on the interval  $(\tau_1, \tau_2)$ , which leads to a contradiction since  $A$  should have nontrivial kernel. We are thus left to consider the case where  $\tau_1 =$

$\tau_2 = \bar{t}$ , that is, when  $h(t) \neq 0$  for  $t \in I \setminus \{\bar{t}\}$ . In this case  $u$  is piecewise constant on  $I \setminus \{\bar{t}\}$  and satisfies

$$\lim_{t \downarrow \bar{t}} u(t) \in \{v_-, -v_-, v_+, -v_+\}, \quad \lim_{t \uparrow \bar{t}} u(t) \in \{v_-, -v_-, v_+, -v_+\}.$$

Theorem 4 and the length-minimizing assumption on  $\gamma$  imply that the two limits must be equal. Hence,  $u$  is constant on  $I$ , and in particular it is smooth.  $\square$

*Remark 13.* The technical ingredients of the above proof open the way to an alternative approach to the Sard conjecture for minimizers [Agr14] which is known in the free case [LDMO<sup>+</sup>16]. Indeed, assume that the hypotheses of Proposition 3 hold true and fix a point  $x \in M$ . We have proved that given any initial covector in  $(D_x^2)^\perp$  there exist at most four length minimizing curves whose extremal lift starts with this covector. Hence, such curves are parametrized by at most  $n - 3$  parameters. By taking into account the time parametrization, the set of final points of abnormal minimizers starting from  $x$  has codimension at most 2.

For recent results on the Sard conjecture for rank 2 structures in 3-dimensional manifolds, see [BdSRar], which extends the analysis in [ZZ95].

## 6. DYNAMICS OF ABNORMAL EXTREMALS: THE GENERAL CASE

The goal of this section is to prove the following result.

**Proposition 14.** *Let  $(t_0, t_1)$  be a maximal interval on which  $h \neq 0$ . Assume that  $t_1 < T$  and  $A(t_1) \neq 0$ . Then  $u(t)$  has a limit as  $t \uparrow t_1$ , which is an eigenvector of  $A(t_1)$ .*

We split the analysis in two steps. The first one, which is a rather straightforward adaptation of the proof of Proposition 3, corresponds to the case where  $\det A(t_1) < 0$ . We will then turn to the case where  $\det A(t_1) = 0$  (recall that, according to Proposition 12,  $\det A(t_1)$  cannot be positive).

For this purpose, we start by proving a preliminary result.

**6.1. A time-rescaling lemma.** The result below highlights the fact that equation (4) is “almost invariant” with respect to similarity of  $A$ .

**Lemma 15.** *For  $P \in \text{GL}(2, \mathbb{R})$  and  $t_* \in (t_0, t_1)$ , we consider the time reparameterization given by*

$$\varphi : [t_*, t_1) \ni t \mapsto s := \int_{t_*}^t \frac{d\tau}{|h(\tau)|}.$$

Let  $\mathfrak{h} = P^{-1}h \circ \varphi^{-1}$  and  $\mathfrak{A} = P^{-1}(A \circ \varphi^{-1})P$ . Then,

- (i)  $\varphi(t) \rightarrow +\infty$  as  $t \rightarrow t_1$ ;
- (ii) for any  $p \in [1, +\infty]$  we have  $\mathfrak{h} \in L^p((0, +\infty), \mathbb{R}^2)$ ;
- (iii) for every  $s \in (0, +\infty)$  we have

$$(7) \quad \mathfrak{h}'(s) = \mathfrak{A}(s)\mathfrak{h}(s).$$

*Proof.* We start by proving point (iii). Observe that  $\dot{\varphi} = 1/|h|$ . Then, simple computations yield

$$\mathfrak{h}' = \frac{P^{-1}\dot{h} \circ \varphi^{-1}}{\dot{\varphi} \circ \varphi^{-1}} = \mathfrak{A}\mathfrak{h}.$$

Assume now that  $\lim_{t \rightarrow t_1} \varphi(t) = s_* < +\infty$ . Then, since  $\mathfrak{h}(s_*) = h(t_1) = 0$ , we have that  $\mathfrak{h}$  is the solution to the (backward) Cauchy problem

$$\begin{cases} \mathfrak{h}' = \mathfrak{A}\mathfrak{h} & \text{on } (0, s_*), \\ \mathfrak{h}(s_*) = 0. \end{cases}$$

This implies that  $\mathfrak{h} \equiv 0$  on  $(0, s_*)$  and thus  $h \equiv 0$  on  $(t_*, t_1)$ , which contradicts the definition of the interval  $(t_0, t_1)$ .

To complete the proof of the statement, observe that  $t \mapsto h(t)$  is bounded on  $[t_*, t_1]$  and thus belongs to  $L^\infty((t_*, t_1), \mathbb{R}^2)$ . Then, for every  $p \geq 1$ ,

$$\begin{aligned} \int_0^{+\infty} |\mathfrak{h}(s)|^p ds &= \int_{t_*}^{t_1} |P^{-1}h|^p |h|^{-1} dt \leq \|P^{-1}\|^p \int_{t_*}^{t_1} |h|^{p-1} dt \\ &\leq \|P^{-1}\|^p \|h\|_{L^\infty}^{p-1} (t_1 - t_*) < +\infty. \end{aligned} \quad \square$$

**6.2. Proof of Proposition 14 in the case  $\det A(t_1) < 0$ .** Since  $\text{trace}(A) = 0$  and  $\det A(t_1) < 0$ , there exists  $P \in \text{GL}(2, \mathbb{R})$  such that

$$PA(t_1)P^{-1} = \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}, \quad a > 0.$$

Up to applying the change of coordinates associated with  $P$  and defining the time-rescaled curves  $\mathfrak{h}$  and  $\mathfrak{A}$  as in Lemma 15, we have

$$(8) \quad \mathfrak{A}(s) = \begin{pmatrix} -\alpha(s) & \beta(s) \\ \zeta(s) & \alpha(s) \end{pmatrix},$$

where

$$(9) \quad \lim_{s \rightarrow \infty} \alpha(s) = a, \quad \lim_{s \rightarrow \infty} \zeta(s) = \lim_{s \rightarrow \infty} \beta(s) = 0.$$

Let  $\mathfrak{h} = \rho e^{i\vartheta}$  for  $\rho > 0$  and  $\vartheta \in [0, 2\pi)$ . We will prove that  $\vartheta(s) \rightarrow 0 \pmod{\pi}$  as  $s \rightarrow \infty$ .

Observe that, letting  $\mathfrak{h} = (x_1, x_2)$  with  $x_1, x_2 \in \mathbb{R}$ , we have

$$(10) \quad \frac{1}{2} \tan 2\vartheta = \frac{\sin \vartheta \cos \vartheta}{\cos^2 \vartheta - \sin^2 \vartheta} = \frac{x_1 x_2}{x_1^2 - x_2^2}.$$

By (7) and simple computations we obtain

$$(11) \quad (x_1 x_2)' = \zeta x_1^2 + \beta x_2^2, \quad \frac{(x_1^2 - x_2^2)'}{2} = -\alpha(x_1^2 + x_2^2) + (\beta - \zeta)x_1 x_2.$$

Upon integration and exploiting (9), we get

$$(12) \quad x_1 x_2 = o(R), \quad x_1^2 - x_2^2 = 2aR(1 + o(1)), \quad \text{where } R(s) := \int_s^{+\infty} |\mathfrak{h}(\sigma)|^2 d\sigma.$$

Observe that, by Lemma 15,  $\mathfrak{h} \in L^2((0, +\infty), \mathbb{R}^2)$  and, in particular,  $R \rightarrow 0$  as  $s \rightarrow +\infty$ . Finally, substituting the above in (10) shows that  $\tan 2\vartheta \rightarrow 0$ . From the second equation in (12), the sign of  $x_1^2 - x_2^2$  is positive as  $t \uparrow t_1$ , which implies that  $\vartheta \rightarrow 0 \pmod{\pi}$ . This completes the proof of Proposition 14 in the case  $\det A(t_1) < 0$ .

*Remark 16.* Recall that in the analysis above we suppose that  $u = \frac{h}{|h|}$ , and we actually prove that in this case  $u(t)$  converges to a unit eigenvector of  $A(t_1)$  associated with the negative eigenvalue  $-a$ . In the case where  $u = -\frac{h}{|h|}$ , an analogous argument yields that  $u(t)$  converges to a unit eigenvector of  $A(t_1)$  associated with the positive eigenvalue  $a$ .

**6.3. Proof of Proposition 14 in the case  $\det A(t_1) = 0$ .** Assume that  $\det A(t_1) = 0$  and recall that  $\text{trace} A(t_1) = 0$ . Since, moreover,  $A(t_1) \neq 0$ , there exists  $P \in \text{GL}(2, \mathbb{R})$  such that

$$(13) \quad PA(t_1)P^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

As before, using the change of variables of Lemma 15, we let

$$\mathfrak{A}(s) = \begin{pmatrix} -\alpha(s) & \beta(s) \\ \zeta(s) & \alpha(s) \end{pmatrix},$$

where  $\alpha, \beta, \zeta$  are linear combinations of  $h_{2112} \circ \varphi^{-1}$ ,  $h_{2212} \circ \varphi^{-1}$ , and  $h_{1112} \circ \varphi^{-1}$ , and hence absolutely continuous with bounded derivatives on  $(0, +\infty)$ , according to (2). Equality (13) implies that  $\alpha \rightarrow 0$ ,  $\beta \rightarrow 1$ , and  $\zeta \rightarrow 0$  as  $s \rightarrow +\infty$ .

We also introduce  $\mu := \zeta + \beta$  and we notice that  $\mu \rightarrow 1$  as  $s \rightarrow +\infty$ . (Beware that the same letters are used for different parameters in the proof of Proposition 12.) Then, (7) reads

$$\frac{\rho'}{\rho} = \mu \sin \vartheta \cos \vartheta - \alpha \cos 2\vartheta, \quad \vartheta' = -\mu \sin^2 \vartheta + \alpha \sin 2\vartheta + \zeta,$$

and can be written as

$$(14) \quad \frac{\rho'}{\rho} = \sin \vartheta \cos \vartheta + f, \quad \vartheta' = -\sin^2 \vartheta + g,$$

where the functions

$$f = -\alpha \cos 2\vartheta + (\mu - 1) \sin \vartheta \cos \vartheta, \quad g = \alpha \sin 2\vartheta + \zeta + (1 - \mu) \sin^2 \vartheta,$$

tend to zero as  $s \rightarrow +\infty$ .

Establishing Proposition 14, finally amounts to proving that  $\vartheta \rightarrow 0 \pmod{\pi}$ , as  $s \rightarrow +\infty$ .

**Lemma 17.** *We have the following dichotomy:*

- (i)  $\vartheta \rightarrow 0 \pmod{\pi}$ , as  $s \rightarrow +\infty$ ;
- (ii)  $\vartheta \rightarrow -\infty$  as  $s \rightarrow +\infty$ . Moreover, in this case, for any  $0 < \varepsilon < \pi/2$  there exists an increasing sequence of positive real numbers  $(s_n)_{n \in \mathbb{N}}$  tending to infinity such that

$$\begin{aligned} \vartheta(s_{2n}) &= \pi - \varepsilon \pmod{2\pi}, & \vartheta(s_{2n+1}) &= \varepsilon \pmod{2\pi}, \\ \vartheta'(s) &< 0 \quad \forall s \in [s_{2n}, s_{2n+1}]. \end{aligned}$$

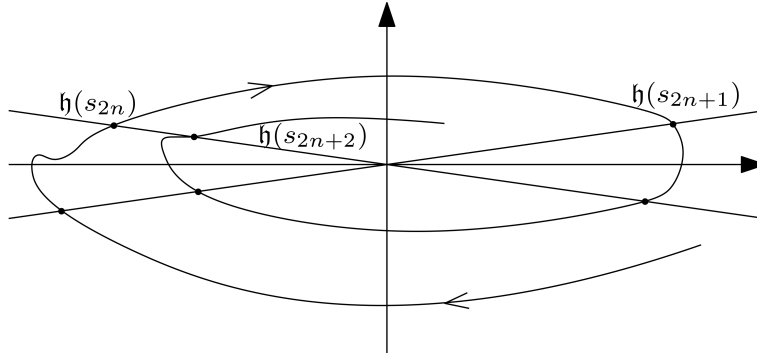


FIGURE 1. The sequence  $(s_n)_{n \in \mathbb{N}}$  in Lemma 17

*Proof.* Notice that the dynamics of  $\vartheta$  is a perturbation via  $g$  of

$$\vartheta' = -\sin^2 \vartheta.$$

The phase portrait of the latter on  $\mathbb{S}^1$  is made of two equilibria in 0 and  $\pi$  joined by two clock-wise oriented heteroclinic trajectories.

Assume that (i) does not hold. Therefore, there exists  $c > 0$  such that

$$(15) \quad \limsup_{s \rightarrow +\infty} |\sin \vartheta(s)| > c.$$

Let  $\varepsilon > 0$  be such that  $\sin \varepsilon \in (0, c)$  and  $s^* > 0$  be such that, for  $s > s^*$ ,  $\vartheta'(s) < -\varepsilon^2/2$  as soon as  $|\sin \vartheta(s)| > \varepsilon$ .

Pick  $q_1 > s^*$  such that  $|\sin \vartheta(q_1)| > c > \sin \varepsilon$ . Since  $\vartheta'$  is bounded from zero as long as  $|\sin \vartheta|$  stays larger than  $\sin \varepsilon$ , there exists  $r_1 > q_1$  such that  $|\sin \vartheta(r_1)| = \sin \varepsilon$ . By definition of  $c$ , there exists  $q_2 > r_1$  such that  $|\sin \vartheta(q_2)| > c$ . Moreover,  $q_1$  and  $q_2$  can be taken so that  $\vartheta(q_2) = \vartheta(q_1) - \pi$  and (15) holds with  $c$  arbitrarily close to 1. By iterating

the procedure leading from  $q_1$  to  $q_2$ , we prove that  $\vartheta \rightarrow -\infty$ . The construction also shows how to define the sequence  $(s_n)_{n \in \mathbb{N}}$  as in (ii) (cf. Figure 1).  $\square$

The rest of the argument consists in showing that case (ii) in Lemma 17 cannot hold true. For that purpose, we argue by contradiction.

**Lemma 18.** *Assume that property (ii) in Lemma 17 holds true. Then there exists  $0 < \varepsilon_0 < \pi/2$  such that for any  $0 < \varepsilon < \varepsilon_0$  there exists  $N_\varepsilon$  for which, given any  $n \geq N_\varepsilon$ ,*

$$(16) \quad \frac{2}{\varepsilon} (1 - \varepsilon^2) \leq s_{2n+1} - s_{2n} \leq \frac{2}{\varepsilon} (1 + \varepsilon^2),$$

and

$$(17) \quad (1 - \varepsilon)\varepsilon\rho(s_{2n}) \leq \rho(s) \sin \vartheta(s) \leq (1 + \varepsilon)\varepsilon\rho(s_{2n}), \text{ for } s \in [s_{2n}, s_{2n+1}].$$

As a consequence, for every  $n \geq N_\varepsilon$ , one has the following estimates

$$(18) \quad 2(1 - 2\varepsilon)\rho(s_{2n}) \leq \int_{s_{2n}}^{s_{2n+1}} \sin \vartheta(s) \rho(s) ds \leq 2(1 + 2\varepsilon)\rho(s_{2n}),$$

$$(19) \quad (1 - 2\varepsilon) \frac{\rho(s_{2n})}{\varepsilon} \leq \int_{s_{2n}}^{s_{2n+1}} \rho(s) ds \leq (1 + 2\varepsilon) \frac{\rho(s_{2n})}{\varepsilon},$$

$$(20) \quad \left| \int_{s_{2n}}^{s_{2n+1}} \cos \vartheta(s) \rho(s) ds \right| \leq \rho(s_{2n}).$$

*Proof.* Set  $M_f(s) = \sup_{\tau \geq s} |f(\tau)|$  and  $M_g(s) = \sup_{\tau \geq s} |g(\tau)|$ . Observe that these two functions tend to zero as  $s$  tends to infinity.

By Lemma 17, for  $n$  large enough and  $s \in [s_{2n}, s_{2n+1}]$ , equation (14) becomes

$$(21) \quad (\cot \vartheta)' = 1 - \frac{g}{\sin^2 \vartheta}.$$

For  $n$  large enough, for every  $s \in [s_{2n}, s_{2n+1}]$  we have

$$\left| \frac{g(s)}{\sin^2 \vartheta(s)} \right| \leq \frac{M_g(s_{2n})}{\sin^2 \varepsilon} \leq \frac{\varepsilon^2}{2}.$$

Equation (16) follows by integrating (21) on the interval  $[s_{2n}, s_{2n+1}]$ .

On the interval  $[s_{2n}, s_{2n+1}]$ , one has

$$\frac{\rho'(s)}{\rho(s)} + \frac{\vartheta'(s) \cos \vartheta(s)}{\sin \vartheta(s)} = f(s) + \frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)}.$$

For  $n$  large enough, for every  $s \in [s_{2n}, s_{2n+1}]$  we have

$$(22) \quad |f(s)| + \left| \frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)} \right| \leq M_f(s_{2n}) + \frac{M_g(s_{2n})}{\sin \varepsilon} \leq \frac{\varepsilon^2}{4}.$$

By integrating between  $s_{2n}$  and any  $s \in [s_{2n}, s_{2n+1}]$ , one gets

$$\left| \ln \left( \frac{\rho(s) \sin \vartheta(s)}{\rho(s_{2n}) \sin \varepsilon} \right) \right| = \left| \int_{s_{2n}}^s \left( f(s) + \frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)} \right) ds \right| \leq \frac{(s_{2n+1} - s_{2n})\varepsilon^2}{4} \leq \frac{\varepsilon}{2}(1 + \varepsilon^2),$$

yielding (17) for  $\varepsilon$  small enough.

We now turn to the proof of the three estimates (18)–(20). The first one simply follows by integrating (17) on  $[s_{2n}, s_{2n+1}]$  and using (16). Estimate (19) is obtained by first dividing (17) by  $\sin \vartheta(s)$  and then integrating the resulting inequalities on  $[s_{2n}, s_{2n+1}]$ . One gets that

$$(1 - \varepsilon)\varepsilon\rho(s_{2n}) \int_{s_{2n}}^{s_{2n+1}} \frac{ds}{\sin \vartheta(s)} \leq \int_{s_{2n}}^{s_{2n+1}} \rho(s) ds \leq (1 + \varepsilon)\varepsilon\rho(s_{2n}) \int_{s_{2n}}^{s_{2n+1}} \frac{ds}{\sin \vartheta(s)}.$$

On the other hand, the following holds true,

$$\int_{s_{2n}}^{s_{2n+1}} \frac{ds}{\sin \vartheta(s)} = \int_{s_{2n}}^{s_{2n+1}} \frac{\vartheta'(s)}{-\sin^3 \vartheta(s) \left(1 - \frac{g(s)}{\sin^2 \vartheta(s)}\right)} ds,$$

which implies that

$$(1 - \varepsilon^2) \int_{\varepsilon}^{\pi - \varepsilon} \frac{d\vartheta}{\sin^3 \vartheta} \leq \int_{s_{2n}}^{s_{2n+1}} \frac{ds}{\sin \vartheta(s)} \leq (1 + \varepsilon^2) \int_{\varepsilon}^{\pi - \varepsilon} \frac{d\vartheta}{\sin^3 \vartheta}.$$

A direct computation shows that  $\int_{\varepsilon}^{\pi - \varepsilon} \frac{d\vartheta}{\sin^3 \vartheta} = \frac{1}{\varepsilon^2(1+o(\varepsilon))}$  as  $\varepsilon$  tends to zero. One finally deduces estimate (19).

To derive estimate (20), one notices that

$$\begin{aligned} \int_{s_{2n}}^{s_{2n+1}} \cos \vartheta(s) \rho(s) ds &= \int_{s_{2n}}^{s_{2n+1}} \frac{\sin \vartheta(s) \cos \vartheta(s) \rho(s)}{\sin \vartheta(s)} ds \\ &= \int_{s_{2n}}^{s_{2n+1}} \frac{\rho'(s) - f(s) \rho(s)}{\sin \vartheta(s)} ds \\ &= - \int_{s_{2n}}^{s_{2n+1}} \frac{f(s) \rho(s)}{\sin \vartheta(s)} ds + \frac{\rho(s_{2n+1}) - \rho(s_{2n})}{\sin \varepsilon} \\ &\quad + \int_{s_{2n}}^{s_{2n+1}} \frac{\rho(s) \cos \vartheta(s) \vartheta'(s)}{\sin^2 \vartheta(s)} ds. \end{aligned}$$

By using the expression of  $\vartheta'$  in the last integral, one deduces that

$$2 \int_{s_{2n}}^{s_{2n+1}} \cos \vartheta(s) \rho(s) ds = \frac{\rho(s_{2n+1}) - \rho(s_{2n})}{\sin \varepsilon} - \int_{s_{2n}}^{s_{2n+1}} \rho(s) \frac{f(s) - \frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)}}{\sin \vartheta(s)} ds.$$

By using (17) for  $s = s_{2n}$  and  $s = s_{2n+1}$  and then (22), one deduces (20).  $\square$

Fix a sequence  $(\varepsilon_k)_{k \in \mathbb{N}}$ , strictly decreasing to 0. For each  $k \in \mathbb{N}$ , we use  $(s_{k,n})_{n \in \mathbb{N}}$  to denote the sequence  $(s_n)_{n \in \mathbb{N}}$  given by Lemma 17 and corresponding to  $\varepsilon = \varepsilon_k$ . For all  $k \in \mathbb{N}$  let  $n_k \geq N_{\varepsilon_k}$  be an integer to be fixed later, where  $N_{\varepsilon_k}$  is as in Lemma 18. We use  $(\xi_\ell)_{\ell \in \mathbb{N}}$  to denote the sequence defined by

$$\xi_{2k} = s_{k,2n_k}, \quad \xi_{2k+1} = s_{k,2n_k+1}, \quad \forall k \in \mathbb{N}.$$

We choose  $k \mapsto n_k$  so that the sequence  $(\xi_\ell)_{\ell \in \mathbb{N}}$  is strictly increasing and tends to infinity as  $\ell \rightarrow +\infty$ .

Let  $t_\ell = \varphi^{-1}(\xi_\ell)$ , where  $\varphi$  is the change of variables introduced in Lemma 15. For every  $\ell \geq 0$  consider the function  $u_\ell \in L^\infty([0, 1], \mathbb{S}^1)$  defined by  $u_\ell(\tau) = u(t_{2\ell} + \tau(t_{2\ell+1} - t_{2\ell}))$ . By the weak- $\star$  compactness of all bounded subsets of  $L^\infty([0, 1], \mathbb{R}^2)$ , we can assume without loss of generality that  $u_\ell \rightharpoonup u_\star$  in the weak- $\star$  topology. Applying Lemma 8 with  $a_\ell = t_{2\ell}$  and  $b_\ell = t_{2\ell+1}$ , we deduce that  $u_\star$  is minimizing and  $|u_\star| \equiv 1$  almost everywhere in  $[0, 1]$ .

For every subinterval  $[a, b]$  of  $[0, 1]$ , by the properties of weak- $\star$  convergence, we have that

$$\int_a^b v^T u_\ell(\tau) d\tau \rightarrow \int_a^b v^T u_\star(\tau) d\tau, \quad \forall v \in \mathbb{R}^2.$$

Moreover, one has

$$\begin{aligned} \int_a^b v^T u_\ell(\tau) d\tau &= \frac{1}{t_{2\ell+1} - t_{2\ell}} \int_{(1-a)t_{2\ell} + at_{2\ell+1}}^{(1-b)t_{2\ell} + bt_{2\ell+1}} \frac{v^T h}{|h|} dt \\ (23) \quad &= \frac{1}{t_{2\ell+1} - t_{2\ell}} \int_{\varphi((1-a)t_{2\ell} + at_{2\ell+1})}^{\varphi((1-b)t_{2\ell} + bt_{2\ell+1})} v^T P h(s) ds, \end{aligned}$$

where  $P$  has been introduced in (13).

In addition

$$(24) \quad t_{2\ell+1} - t_{2\ell} = \int_{\xi_{2\ell}}^{\xi_{2\ell+1}} |P\mathfrak{h}(s)| ds.$$

**Lemma 19.** *Under the above assumptions, there exists a unit vector  $v_\star \in \mathbb{R}^2$  such that  $u_\star(t) = v_\star$  for a.e.  $t \in [0, 1]$ . Moreover,  $v_\star$  is parallel to  $P(1, 0)$ .*

*Proof.* Let  $v_\star, w_\star \in \mathbb{R}^2$  be two orthogonal unit vectors such that  $v_\star$  is parallel to  $P(1, 0)$ . Notice that  $P^T w_\star$  is orthogonal to  $(1, 0)$ , that is, it is parallel to  $(0, 1)$ . We start by showing that  $w_\star^T u_\star(t) = 0$  for a.e.  $t \in [0, 1]$ . This amounts to showing that for all  $0 \leq a < b \leq 1$  it holds

$$\frac{1}{t_{2\ell+1} - t_{2\ell}} \int_{\varphi((1-a)t_{2\ell} + at_{2\ell+1})}^{\varphi((1-b)t_{2\ell} + bt_{2\ell+1})} \mathfrak{h}_2(s) ds \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty.$$

Since  $\mathfrak{h}_2 = \rho \sin \vartheta$  is positive on  $[\xi_{2\ell}, \xi_{2\ell+1}]$  by construction, using (24) it is enough to show that

$$\frac{\int_{\xi_{2\ell}}^{\xi_{2\ell+1}} \rho(s) \sin \vartheta(s) ds}{\int_{\xi_{2\ell}}^{\xi_{2\ell+1}} |P\mathfrak{h}(s)| ds} \rightarrow 0 \quad \text{as } \ell \rightarrow +\infty.$$

Since  $|P\mathfrak{h}(s)| \geq \|P^{-1}\|^{-1} \rho(s)$  for all  $s$ , the latter limit holds true according to (18) and (19) in Lemma 18, applied to  $\varepsilon = \varepsilon_\ell$  for  $\ell \geq 0$ .

Recall that the control  $u_\star$  is minimizing and  $|u_\star(t)| = 1$  for a.e.  $t \in [0, 1]$ . From what precedes, one deduces that  $u_\star$  is almost everywhere perpendicular to  $w_\star$ , hence equal to  $v_\star$  or  $-v_\star$ . It then follows from Lemma 22 in the appendix that, up to replacing  $v_\star$  by  $-v_\star$ , the equality  $u_\star(t) = v_\star$  holds for a.e.  $t \in [0, 1]$ .  $\square$

Let  $\bar{v} \in \mathbb{R}^2$  be such that  $P^T \bar{v} = (1, 0)$ . We have, according to Lemma 19,

$$\lim_{\ell \rightarrow \infty} \int_0^1 \bar{v}^T u_\ell(\tau) d\tau = \int_0^1 \bar{v}^T u_\star(\tau) d\tau = \bar{v}^T v_\star \neq 0.$$

We conclude the proof by contradiction by showing that the limit in the left-hand side is zero. Indeed, according to (23), we have

$$\left| \int_0^1 \bar{v}^T u_\ell(\tau) d\tau \right| = \frac{\left| \int_{\xi_{2\ell}}^{\xi_{2\ell+1}} \rho(s) \cos \vartheta(s) ds \right|}{\int_{\xi_{2\ell}}^{\xi_{2\ell+1}} |P\mathfrak{h}(s)| ds} \leq \|P^{-1}\| \frac{\left| \int_{\xi_{2\ell}}^{\xi_{2\ell+1}} \rho(s) \cos \vartheta(s) ds \right|}{\int_{\xi_{2\ell}}^{\xi_{2\ell+1}} \rho(s) ds}.$$

The right-hand side of the above equation tends to zero thanks to (19) and (20) in Lemma 18 applied to  $\varepsilon = \varepsilon_\ell$  for  $\ell \geq 0$ .

We have therefore proved that (ii) in Lemma 17 cannot hold true, which completes the proof of Proposition 14.

## 7. PROOF OF THEOREM 1

Let  $M$  be as in the statement of Theorem 1. Denote, as in the previous sections, by  $\gamma : [0, T] \rightarrow M$  a length-minimizing trajectory parametrized by arclength and by  $\lambda : [0, T] \rightarrow T^*M$  an abnormal extremal lift of  $\gamma$ .

Proposition 14, together with Theorem 4, proves the  $C^1$  regularity of  $\gamma$  provided that  $h$  vanishes only at isolated points.

We consider in this section the case where  $t_0 \in (0, T)$  is a density point of  $\{t \in [0, T] \mid h(t) = 0\}$ . We want to prove that  $u(t)$  (up to modification on a set of measure zero) has a limit as  $t \uparrow t_0$  and as  $t \downarrow t_0$ . By symmetry, we restrict our attention to the existence of the limit of  $u(t)$  as  $t \uparrow t_0$ .

We are going to consider separately the situations where  $h \equiv 0$  on a left neighborhood of  $t_0$  and where there exists a sequence of maximal open intervals  $(t_0^n, t_1^n)$  with  $h|_{(t_0^n, t_1^n)} \neq 0$  and such that  $t_1^n \rightarrow t_0$ .



Assume for now on that  $h \equiv 0$  on a left neighborhood  $(t_0 - \eta, t_0]$  of  $t_0$ . Then, since  $\dot{h} = Au$  almost everywhere on  $(t_0 - \eta, t_0]$ , we have that  $u(t)$  belongs to  $\ker A(t)$  for almost every  $t$  in  $(t_0 - \eta, t_0]$ . By Lemma 11, moreover,  $\ker A(t)$  is one-dimensional for every  $t \in (t_0 - \eta, t_0]$ .

Fix an open neighborhood  $V_0$  of  $\lambda(t_0)$  in  $T^*M$  such that there exists a smooth map  $V_0 \ni \lambda \mapsto v(\lambda) \in \mathbb{S}^1$  such that  $v(\lambda(t)) \in \ker A(t)$  if  $\lambda(t) \in V_0$  and  $t \in (t_0 - \eta, t_0]$ . Up to reducing  $\eta$ , we assume that  $\lambda(t) \in V_0$  for every  $t \in (t_0 - \eta, t_0]$ . Notice that  $\lambda|_{(t_0 - \eta, t_0]}$  is a solution of the time-varying system

$$\dot{\lambda} = \sigma(t)\vec{X}_{v(\lambda)}(\lambda),$$

where  $\sigma : (t_0 - \eta, t_0] \rightarrow \{-1, 1\}$  is measurable. Hence, by length-minimality of  $\gamma$  and by Lemma 22 in the appendix, either  $u = v$  almost everywhere on  $(t_0 - \eta, t_0]$  or  $u = -v$  almost everywhere on  $(t_0 - \eta, t_0]$ . We conclude that  $u$  is continuous on  $(t_0 - \eta, t_0]$  and the proof in this case is concluded.

We are left to consider the case where every left neighborhood of  $t_0$  contains a maximal interval  $(\tau_0, \tau_1)$  such that  $h \neq 0$  on  $(\tau_0, \tau_1)$ .

Notice that, by Proposition 12 and by continuity of  $t \mapsto A(t)$ , we have that  $\det A(t_0) \leq 0$ .

The case  $\det A(t_0) < 0$  can be ruled out thanks to the following lemma.

**Lemma 20.** *Let  $\det A(t_0) < 0$ . There exists  $\eta \in (0, t_0)$  such that, for any maximal interval  $(\tau_0, \tau_1) \subset (0, t_0)$  on which  $h(t) \neq 0$ , then  $\tau_0 < t_0 - \eta$ .*

*Proof.* As we have already seen in Section 4.1, on every interval where  $h(t) \neq 0$  we have that  $u$  is smooth and equal to either  $\frac{h(t)}{|h(t)|}$  or  $-\frac{h(t)}{|h(t)|}$ . Thus the function  $h$  on  $(\tau_0, \tau_1)$  is either a maximal solution to  $\dot{x} = A(t)\frac{x}{|x|}$  or a maximal solution to  $\dot{x} = -A(t)\frac{x}{|x|}$ . Let us assume that it is a maximal solution of  $\dot{x} = A(t)\frac{x}{|x|}$ , the proof being identical in the second case.

For every  $v \in \mathbb{R}^2 \setminus \{0\}$  and every  $\vartheta > 0$  denote by  $C_\vartheta(v)$  the cone of all vectors in  $\mathbb{R}^2 \setminus \{0\}$  making an (unoriented) angle smaller than  $\vartheta$  with  $v$  or  $-v$ .

Let  $\eta_0 \in (0, t_0)$  be such that  $\det(A(t)) < 0$  for every  $t \in [t_0 - \eta_0, t_0]$ . For  $t \in [t_0 - \eta_0, t_0]$ , denote by  $v_-(t)$  and  $v_+(t)$  two unit eigenvectors of  $A(t)$ , the first corresponding to a negative and the second to a positive eigenvalue.

Let  $\eta \in (0, \eta_0)$  and  $\vartheta_0 > 0$  be such that  $C_{\vartheta_0}(v_+(t_0)) \cap C_{\vartheta_0}(v_-(t_0)) = \emptyset$  and  $v_\pm(t) \in C_{\vartheta_0}(v_\pm(t_0))$  for every  $t \in [t_0 - \eta, t_0]$ . Notice that, for every fixed  $\bar{t} \in [t_0 - \eta, t_0]$ , the vector field  $x \mapsto A(\bar{t})x$  points inward  $C_{\vartheta_0}(v_+(t_0))$  at every nonzero point of its boundary (see Figure 2). Hence  $C_{\vartheta_0}(v_+(t_0))$  is positively invariant for the dynamics of  $\dot{x} = A(t)\frac{x}{|x|}$  on  $[t_0 - \eta, t_0]$ .

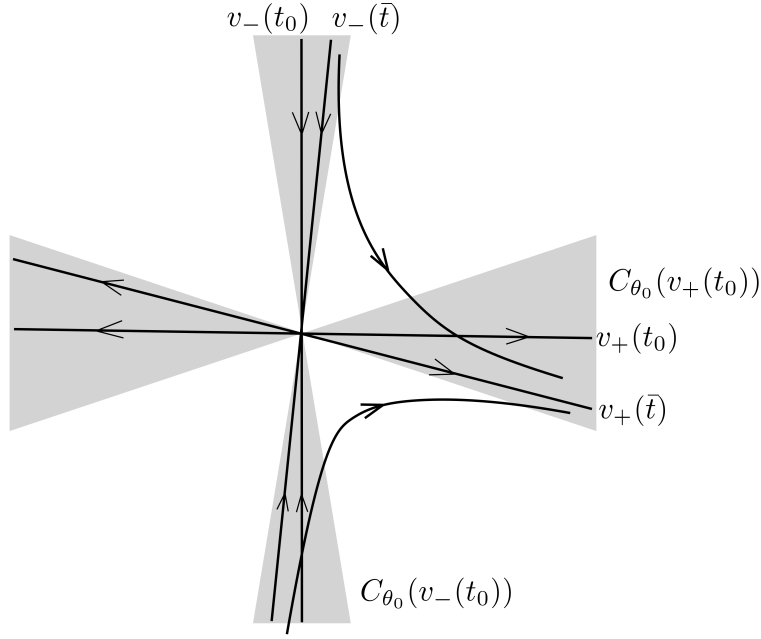
In order to prove the statement, we argue by contradiction. Assume that  $h : (\tau_0, \tau_1) \rightarrow \mathbb{R}^2 \setminus \{0\}$  is a maximal solution of  $\dot{x} = A(t)\frac{x}{|x|}$  with  $(\tau_0, \tau_1) \subset (t_0 - \eta, t_0)$ . Then  $h(\tau)$  tends to 0 as  $\tau$  tends to  $\tau_0$  or  $\tau_1$  and it follows from Proposition 14 that  $\frac{h(\tau)}{|h(\tau)|}$  converges to an eigenvector of  $A(\tau_0)$  as  $\tau \downarrow \tau_0$  and to an eigenvector of  $A(\tau_1)$  as  $\tau \uparrow \tau_1$ . More precisely, from Remark 16 there holds

$$\lim_{\tau \downarrow \tau_0} \frac{h(\tau)}{|h(\tau)|} \rightarrow \pm v_+(\tau_0), \quad \lim_{\tau \uparrow \tau_1} \frac{h(\tau)}{|h(\tau)|} \rightarrow \pm v_-(\tau_1).$$

This contradicts the positive invariance of  $C_{\vartheta_0}(v_+(t_0))$  for the equation  $\dot{x} = A(t)\frac{x}{|x|}$  on  $(\tau_0, \tau_1)$ .  $\square$

In the case  $\det A(t_0) = 0$  the proof follows the steps of the construction of Section 6.3. In particular, let  $P \in \text{GL}(2, \mathbb{R})$  be such that

$$P^{-1}A(t)P = \begin{pmatrix} -a(t) & b(t) \\ c(t) & a(t) \end{pmatrix},$$


 FIGURE 2. Phase portrait of  $\dot{x} = A(\bar{t})x$  for  $\bar{t} \in [t_0 - \eta, t_0]$ 

where  $a, b, c$  are affine combinations of  $h_{2112}, h_{2212}$ , and  $h_{1112}$  with  $a \rightarrow 0$ ,  $b \rightarrow 1$ , and  $c \rightarrow 0$  as  $t \rightarrow t_0$ . Let

$$(25) \quad P^{-1}h(t) = r(t)e^{i\omega(t)},$$

with  $\omega(t)$  uniquely defined modulus  $2\pi$  only when  $h(t) \neq 0$ .

The crucial point is the following counterpart to Lemma 17, whose proof can be obtained using exactly the same arguments.

**Lemma 21.** *We have the following dichotomy:*

- (i) for any  $0 < \varepsilon < \pi/2$ , for  $\eta$  small enough,  $|\sin(\omega(t))| < \varepsilon$  for all  $t \in (t_0 - \eta, t_0)$  such that  $h(t) \neq 0$ ;
- (ii) for any  $0 < \varepsilon < \pi/2$  there exists an increasing sequence  $(t_n)_{n \in \mathbb{N}}$  in  $(0, t_0)$  tending to  $t_0$  and such that

$$\begin{aligned} \omega(t_{2n}) &= \pi - \varepsilon \pmod{2\pi}, & \omega(t_{2n+1}) &= \varepsilon \pmod{2\pi}, \\ h(t) &\neq 0, \sin(\omega(t)) > 0, \dot{\omega}(t) < 0 & \forall t \in [t_{2n}, t_{2n+1}] \end{aligned}$$

or

$$\begin{aligned} \omega(t_{2n}) &= -\varepsilon \pmod{2\pi}, & \omega(t_{2n+1}) &= \varepsilon - \pi \pmod{2\pi}, \\ h(t) &\neq 0, \sin(\omega(t)) < 0, \dot{\omega}(t) < 0 & \forall t \in [t_{2n}, t_{2n+1}] \end{aligned}$$

holds true.

Case (ii) can be excluded by similar computations as in Section 6.3, since it contradicts the optimality of  $\gamma$ .

Consider now case (i). Let  $v_*, w_* \in \mathbb{R}^2$  be two orthogonal unit vectors such that  $v_*$  is parallel to  $P(1, 0)$ . According to (25), if  $\sin(\omega(t)) = 0$  and  $r(t) \neq 0$ , then  $u(t) = h(t)/|h(t)|$  is equal to  $v_*$  or  $-v_*$ . For every  $\eta \in (0, t_0)$  we set

$$I_\eta^+ = \{t \in (t_0 - \eta, t_0) \mid v_*^T u(t) > 0\}, \quad I_\eta^- = \{t \in (t_0 - \eta, t_0) \mid v_*^T u(t) < 0\}.$$

Property (i) implies that, for  $\eta$  small,  $I_\eta^+ \cup I_\eta^-$  contains  $\{t \in (t_0 - \eta, t_0) \mid h(t) \neq 0\}$ . Moreover, if  $t_0$  is a density point for  $I = I_\eta^+ \cap \{t \in (t_0 - \eta, t_0) \mid h(t) \neq 0\}$  (respectively,

$I = I_\eta^- \cap \{t \in (t_0 - \eta, t_0) \mid h(t) \neq 0\}$ , then,

$$\lim_{t \in I, t \rightarrow t_0} u(t) = v_\star \quad (\text{respectively, } \lim_{t \in I, t \rightarrow t_0} u(t) = -v_\star).$$

Let  $\Phi_\eta = \{t \in (t_0 - \eta, t_0) \mid h(t) = 0\}$ . For almost every  $t \in \Phi_\eta$ ,  $u(t)$  is in the kernel of  $A(t)$  and  $|u(t)| = 1$ . Notice that, if  $t_0$  is a density point for  $J = \{t \in (0, t_0) \mid \ker A(t) \neq (0)\}$ , then the kernel of  $A(t)$  converges to the kernel of  $A(t_0)$  as  $t \in J$ ,  $t \rightarrow t_0$ . By construction of  $P$ , moreover,  $\ker(A(t_0)) = \text{span}(P(1, 0)) = \text{span}(v_\star)$ . Hence, for  $\eta$  small enough, almost every  $t \in \Phi_\eta$  is in  $I_\eta^+ \cup I_\eta^-$ .

To summarize, for  $\eta$  small enough,  $I_\eta^+ \cup I_\eta^-$  has full measure in  $(t_0 - \eta, t_0)$ . Moreover,

$$(26) \quad \lim_{t \in I_\eta^+, t \rightarrow t_0} u(t) = v_\star, \quad \lim_{t \in I_\eta^-, t \rightarrow t_0} u(t) = -v_\star.$$

We next prove that  $u$  converges either to  $v_\star$  or to  $-v_\star$  as  $t \rightarrow t_0$  by showing that, for  $\eta$  small enough, either  $I_\eta^+$  or  $I_\eta^-$  has measure zero.

Suppose by contradiction that there exists a sequence of intervals  $(\tau_0^n, \tau_1^n)$  in  $(0, t_0)$  such that  $\tau_0^n, \tau_1^n \rightarrow t_0$  as  $n \rightarrow \infty$  and both  $|(\tau_0^n, \tau_1^n) \cap I^+|$  and  $|(\tau_0^n, \tau_1^n) \cap I^-|$  are positive, where  $|\cdot|$  denotes the Lebesgue measure and  $I^\pm = \{t \in (0, t_0) \mid \pm v_\star^T u(t) > 0\}$ . Moreover, up to restricting  $(\tau_0^n, \tau_1^n)$ , we can assume that

$$(27) \quad |(\tau_0^n, \tau_1^n) \cap I^+| = |(\tau_0^n, \tau_1^n) \cap I^-| > 0.$$

This can be seen, for instance, by considering a continuous deformation of an interval around a Lebesgue point of  $(\tau_0^n, \tau_1^n) \cap I^+$  towards an interval around a Lebesgue point of  $(\tau_0^n, \tau_1^n) \cap I^-$ .

For every  $n \in \mathbb{N}$ , let  $u_n \in L^\infty([0, 1], \mathbb{R}^2)$  be defined by  $u_n(\tau) = u(\tau_0^n + \tau(\tau_1^n - \tau_0^n))$ . Up to extracting a subsequence,  $u_n$  weakly- $\star$  converges to some  $u_\star$ . Condition (27) and the limits in (26) imply that

$$(28) \quad \int_0^1 u_\star(t) dt = 0.$$

Thanks to (26) we also have that  $w_\star^T u_n$   $L^\infty$ -converges to zero as  $n \rightarrow \infty$ . In particular,  $w_\star^T u_\star \equiv 0$ . By Lemma 8,  $u_\star$  is optimal and  $v_\star^T u_\star$  has values in  $\{-1, 1\}$ . Hence, by Lemma 22 in the appendix,  $v_\star^T u_\star$  is constantly equal to  $+1$  or  $-1$ . This contradicts (28) and the proof is concluded.  $\square$

#### APPENDIX A. AN ELEMENTARY LEMMA

**Lemma 22.** *Let  $(M, D, g)$  be a sub-Riemannian manifold. Let  $V$  be a Lipschitz continuous vector field on  $T^*M$  such that  $\pi_* V(\lambda) \in D_{\pi(\lambda)} \setminus \{0\}$  for every  $\lambda \in T^*M$ . Let  $\lambda : [0, T] \rightarrow T^*M$  satisfy  $\dot{\lambda}(t) = \sigma(t)V(\lambda(t))$  with  $\sigma \in L^\infty([0, T], [-1, 1])$ . Assume that  $\gamma = \pi \circ \lambda : [0, T] \rightarrow M$  is a length-minimizer. Then  $\sigma$  has constant sign, i.e., either  $\sigma \geq 0$  a.e. on  $[0, T]$  or  $\sigma \leq 0$  a.e. on  $[0, T]$ .*

*Proof.* Set  $\kappa = \int_0^T \sigma(t) dt$  and notice that  $\lambda(T) = e^{\kappa V}(\lambda(0))$ . If  $\sigma$  does not have constant sign, then  $[0, 1] \ni t \mapsto \pi \circ e^{t\kappa V}(\lambda(0))$  is a curve connecting  $\gamma(0)$  to  $\gamma(T)$  and having length smaller than  $\gamma$ .  $\square$

A particular case of the lemma occurs when  $V = \vec{H}$  is the Hamiltonian vector field on  $T^*M$  associated with the Hamiltonian  $\lambda \mapsto \langle \lambda, X(\pi(\lambda)) \rangle$ , where  $X$  is a smooth horizontal never-vanishing vector field on  $M$ . This means that if a solution of  $\dot{\gamma}(t) = \sigma(t)X(\gamma(t))$  is a length-minimizer then  $\sigma$  has constant sign.

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