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# ON THE REGULARITY OF ABNORMAL MINIMIZERS FOR RANK 2 SUB-RIEMANNIAN STRUCTURES 

D. BARILARI, Y. CHITOUR, F. JEAN, D. PRANDI, AND M. SIGALOTTI


#### Abstract

We prove the $C^{1}$ regularity for a class of abnormal length-minimizers in rank 2 sub-Riemannian structures. As a consequence of our result, all length-minimizers for rank 2 sub-Riemannian structures of step up to 4 are of class $C^{1}$.


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## 1. Introduction

The question of regularity of length-minimizers is one of the main open problems in sub-Riemannian geometry, cf. for instance [Mon02, Problem 10.1] or [Agr14, Problem II] and the survey [Mon14].

Length-minimizers are solutions to a variational problem with constraints and satisfy a first-order necessary condition resulting from the Pontryagin Maximum Principle. With every length-minimizer $\gamma:[0, T] \rightarrow M$ we can associate a lift $\lambda:[0, T] \rightarrow T^{*} M$ in the cotangent space, satisfying a Hamiltonian equation. This lift can be either normal or abnormal, although a length-minimizer $\gamma$ can actually admit several lifts, each of them being either normal or abnormal.
If a length-minimizer admits a normal lift, then it is smooth, i.e., $C^{\infty}$, since normal lifts are solutions of smooth autonomous Hamiltonian systems in $T^{*} M$. Note that we assume length-minimizers to be arclength parameterized and their regularity is meant with respect to this time parameterization. The question of regularity is then reduced to length-minimizers that are strictly abnormal, i.e., those which do not admit normal lifts. For such length-minimizers, from the first order necessary condition (and actually from the second order one as well) it is a priori not possible to deduce any regularity other than Lipschitz continuity.

In this paper we investigate the following.
Open Problem. Are all length-minimizers in a sub-Riemannian manifold of class $C^{1}$ ?
If the sub-Riemannian structure has step 2 , there are no strictly abnormal lengthminimizers, see e.g. [AS95, ABB17], thus every length-minimizer admits a normal lift, and
is hence smooth. For step 3 structures, the situation is already more complicated and a positive answer to the above problem is known only for Carnot groups (where, actually, length-minimizers are proved to be $C^{\infty}$ ), see [LDLMV13, TY13].

To state our main result, we introduce some notations. We refer the reader to Section 2 for precise definitions. Recall that a sub-Riemannian structure $(D, g)$ on $M$ is defined by a bracket generating distribution $D$ endowed with a metric $g$. Hence $D$ defines a flag of subspaces at every point $x \in M$

$$
D_{x}=D_{x}^{1} \subset D_{x}^{2} \subset D_{x}^{3} \subset \cdots \subset D_{x}^{r}=T_{x} M,
$$

where $D_{x}^{i}$ is the subspace of the tangent space spanned by Lie brackets of length at most $i$ between horizontal vector fields. This induces a dual decreasing sequence of subspaces of $T_{x}^{*} M$

$$
0=\left(D_{x}^{r}\right)^{\perp} \subset \cdots \subset\left(D_{x}^{4}\right)^{\perp} \subset\left(D_{x}^{3}\right)^{\perp} \subset\left(D_{x}^{2}\right)^{\perp} \subset\left(D_{x}^{1}\right)^{\perp} \subset T_{x}^{*} M,
$$

where perpendicularity is considered with respect to the duality product. By construction, any abnormal lift satisfies $\lambda(t) \in\left(D^{1}\right)^{\perp}$ for every $t$. If the lift is strictly abnormal, then by Goh conditions $\lambda(t) \in\left(D^{2}\right)^{\perp}$ for every $t$.

When the distribution has rank 2 , it is known that if $\lambda(t)$ does not cross $\left(D^{3}\right)^{\perp}$, then the length-minimizer is $C^{\infty}$ [LS95, Sect. 6.2, Cor. 4]. Our main result pushes this analysis further and establishes that the answer to the Open Problem is positive for length-minimizers whose abnormal lift does not enter in $\left(D^{4}\right)^{\perp}$.

Theorem 1. Let $(D, g)$ be a rank 2 sub-Riemannian structure on $M$. Assume that $\gamma$ : $[0, T] \rightarrow M$ is an arclength parameterized abnormal minimizer. If $\gamma$ admits a lift satisfying $\lambda(t) \notin\left(D^{4}\right)^{\perp}$ for every $t \in[0, T]$, then $\gamma$ is of class $C^{1}$.

If the sub-Riemannian manifold has rank 2 and step at most 4, the assumption in Theorem 1 is trivially satisfied by every abnormal minimizer $\gamma$ and we immediately obtain the following corollary.
Corollary 2. Assume that the sub-Riemannian structure has rank 2 and step at most 4. Then all length-minimizers are of class $C^{1}$.

It is legitimate to ask whether the $C^{1}$ regularity in the Open Problem can be further improved. Indeed, the argument behind our proof permits to obtain $C^{\infty}$ regularity of length-minimizers under an additional nilpotency condition on the Lie algebra generated by horizontal vector fields.

Proposition 3. Assume that $D$ is generated by two vector fields $X_{1}, X_{2}$ such that Lie $\left\{X_{1}, X_{2}\right\}$ is a nilpotent Lie algebra of step at most 4. Then for every sub-Riemannian structure $(D, g)$ on $M$, the corresponding length-minimizers are of class $C^{\infty}$.

Notice that, from a geometric viewpoint, the assumption of Proposition 3 implies that $M$ is a homogeneous space. Proposition 3 applies in particular to Carnot groups of rank 2 and step at most 4. In this case we recover the results obtained in [LM08, Example 4.6]. Moreover, under the hypotheses of the above proposition, we also deduce a Sard property for minimizers (Corollary 13).

The strategy of proof of Theorem 1 is to show that, at points where they are not of class $C^{1}$, length-minimizers can admit only corner-like singularities. This is done by a careful asymptotic analysis of the differential equations satisfied by the abnormal lift, which exploits their Hamiltonian structure. We can then conclude thanks to the following result.

Theorem 4 ([HL16]). Let $M$ be a sub-Riemannian manifold. Let $T>0$ and let $\gamma$ : $[-T, T] \rightarrow M$ be a horizontal curve such that, in local coordinates, there exist

$$
\dot{\gamma}^{+}(0):=\lim _{t \downarrow 0} \frac{\gamma(t)-\gamma(0)}{t}, \quad \dot{\gamma}^{-}(0):=\lim _{t \rightarrow 0} \frac{\gamma(t)-\gamma(0)}{t} .
$$

If $\dot{\gamma}^{+}(0) \neq \dot{\gamma}^{-}(0)$, then $\gamma$ is not a length-minimizer.
We oberve that the proof contained in [HL16] requires a previous result stated in [LM08]. A complete argument for the latter, addressing some issues raised in [Rif17, p. 1113-15], is provided in [MPV17]. For sub-Riemannian structures of rank 2 and step at most 4 (and indeed also for higher step, under an additional condition on the Lie algebra generated by horizontal vector fields), the fact that corners are not length-minimizers is already contained in [LM08].
1.1. Structure of the paper. In Section 2 we recall some notations and preliminary notions. Section 3 is devoted to a desingularization and nilpotentization argument. Section 4 contains a preliminary analysis on the dynamics of abnormal extremals. To illustrate our approach in a simpler case, we discuss in Section 5 the proof of the main result for a nilpotent structure of rank up to 4 . Then in Sections 6 and 7 we complete our analysis to prove the general result. Appendix A contains a technical lemma.

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## 2. Notations and preliminary notions

Let $M$ be a smooth $n$-dimensional manifold. A sub-Riemannian structure of rank $m$ on $M$ is a triplet $\left(E, g^{E}, f\right)$ where $E$ is a vector bundle of rank $m$ over $M, g^{E}$ is an Euclidean metric on $E$, and $f: E \rightarrow T M$ is a morphism of vector bundles such that $f\left(E_{x}\right) \subseteq T_{x} M$ for every $x \in M$. Fix such a structure and define a family of subspaces of the tangent spaces by

$$
D_{x}=\{X(x) \mid X \in D\} \subseteq T_{x} M, \quad \forall x \in M,
$$

where $D=\{f \circ Y \mid Y$ smooth section of $E\}$ is a submodule of the set of vector fields on $M$. We assume that the structure is bracket generating, i.e., the tangent space $T_{x} M$ is spanned by the vector fields in $D$ and their iterated Lie brackets evaluated at $x$.

The sub-Riemannian structure induces a quadratic form $g_{x}$ on $D_{x}$ by

$$
g_{x}(v, v)=\inf \left\{g_{x}^{E}(u, u) \mid f(u)=v, u \in E_{x}\right\}, \quad v \in D_{x}
$$

In analogy with the classic sub-Riemannian case and to simplify notations, in the sequel we will refer to the sub-Riemannian structure as the pair $(D, g)$ rather than $\left(E, g^{E}, f\right)$. This is justified since all the constructions and definitions below rely only on $D$ and $g$. The triplet ( $M, D, g$ ) is called a sub-Riemannian manifold.

Remark 5. Usually, a sub-Riemannian manifold denotes a triplet ( $M, D, g$ ), where $M$ is a smooth manifold, $D$ is a subbundle of $T M$, and $g$ is a Riemannian metric on $D$ (see, e.g., [Bel96]). This corresponds to the case where $f\left(E_{x}\right)$ is of constant rank. The definition given above follows, for instance, [ABB17].

A horizontal curve $\gamma:[0, T] \rightarrow M$ is an absolutely continuous path such that $\dot{\gamma}(t) \in$ $D_{\gamma(t)}$ for almost every (a.e. for short) $t \in[0, T]$. The length of a horizontal curve is defined by

$$
\ell(\gamma)=\int_{0}^{T} \sqrt{g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))} d t .
$$

The sub-Riemannian distance between two arbitrary points $x, y$ in $M$ is then

$$
d(x, y)=\inf \{\ell(\gamma) \mid \gamma(0)=x, \gamma(T)=y, \gamma \text { horizontal }\} .
$$

A length-minimizer is a horizontal curve $\gamma$ which realizes the distance between its extremities, that is, $\ell(\gamma)=d(\gamma(0), \gamma(T))$. Note that any time-reparameterization of a lengthminimizer is a length-minimizer as well.

A generating frame of the sub-Riemannian structure is a family of vector fields $X_{1}, \ldots, X_{k}$ such that $D$ is generated by $X_{1}, \ldots, X_{k}$ as a module and

$$
g_{x}(v, v)=\inf \left\{\sum_{i=1}^{k} u_{i}^{2} \mid \sum_{i=1}^{k} u_{i} X_{i}(x)=v\right\}, \quad x \in U, v \in D_{x} .
$$

There always exists a global generating frame (see [ABB17, Corollary 3.26]), with, in general, a number $k$ of elements greater than the rank $m$ of the structure. However, every point $x \in M$ admits a neighborhood on which there exists a (local) generating frame with exactly $k=m$ elements, e.g., by taking the image via $f$ of a local orthonormal frame of $\left(E, g^{E}\right)$.

Fix now a (local or global) generating frame $X_{1}, \ldots, X_{k}$ of $(D, g)$. For any horizontal curve $\gamma$ of finite length, there exists $u \in L^{\infty}\left([0, T], \mathbb{R}^{k}\right)$ satisfying

$$
\begin{equation*}
\dot{\gamma}(t)=\sum_{i=1}^{k} u_{i}(t) X_{i}(\gamma(t)), \quad \text { for a.e. } t \in[0, T] . \tag{1}
\end{equation*}
$$

The curve is said to be arclength parameterized if $g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))=1$ for a.e. $t \in[0, T]$, i.e., if there exists $u \in L^{\infty}\left([0, T], \mathbb{S}^{k-1}\right)$ satisfying (1). In that case $\ell(\gamma)=T$.

To state the first order necessary conditions, let us first introduce some notations. For $\lambda \in T^{*} M$ and $x=\pi(\lambda)$, where $\pi: T^{*} M \rightarrow M$ is the canonical projection, we set $h_{i}(\lambda)=\left\langle\lambda, X_{i}(x)\right\rangle$, for $i=1, \ldots, k$ (here $\langle\lambda, \cdot\rangle$ denotes the dual action of covectors on vectors). Recall also that, for a function $H: T^{*} M \rightarrow \mathbb{R}$, the corresponding Hamiltonian vector field $\vec{H}$ is the unique vector field such that $\sigma(\cdot, \vec{H})=d H$, where $\sigma$ is the canonical symplectic form on the cotangent bundle.
Applying the Pontryagin Maximum Principle to the sub-Riemannian length minimization problem yields the following theorem.
Theorem 6. Let $(M, D, g)$ be a sub-Riemannian manifold with generating frame $X_{1}, \ldots, X_{k}$ and $\gamma:[0, T] \rightarrow M$ be a length-minimizer. Then there exists a nontrivial absolutely continuous curve $t \mapsto \lambda(t) \in T_{\gamma(t)}^{*} M$ such that one of the following conditions is satisfied:
(N) $\dot{\lambda}(t)=\vec{H}(\lambda(t))$ for all $t \in[0, T]$, where $H(\lambda)=\frac{1}{2} \sum_{i=1}^{k} h_{i}^{2}$,
(A) $\dot{\lambda}(t)=\sum_{i=1}^{k} u_{i}(t) \vec{h}_{i}(\lambda(t))$ for almost every $t \in[0, T]$, with $u_{1}, \ldots, u_{k} \in L^{1}([0, T])$. Moreover, $\lambda(t) \in\left(D_{\gamma(t)}\right)^{\perp}$ for all $t$, i.e., $h_{i}(\lambda(t)) \equiv 0$ for $i=1, \ldots, k$.
In case (N) (respectively, case (A)), $\lambda$ is called a normal (respectively, abnormal) extremal. Normal extremals are integral curves of $\vec{H}$. As such, they are smooth. A lengthminimizer is normal (respectively, abnormal) if it admits a normal (respectively, abnormal) extremal lift. We stress that both conditions can be satisfied for the same curve $\gamma$, with different lifts $\lambda_{1}$ and $\lambda_{2}$.

## 3. Desingularisation and nilpotentization

3.1. Desingularisation. Let $(M, D, g)$ be a sub-Riemannian manifold. We define recursively the following sequence of submodules of the set of vector fields,

$$
D^{1}=D, \quad D^{i+1}=D^{i}+\left[D, D^{i}\right] .
$$

At every point $x \in M$, the evaluation at $x$ of these modules induces a flag of subspaces of the tangent space,

$$
D_{x}^{1} \subset D_{x}^{2} \subset \cdots \subset D_{x}^{r}=T_{x} M .
$$

The smallest integer $r=r(x)$ satisfying $D_{x}^{r}=T_{x} M$ is called the step of $D$ at $x$. A point is said to be regular if the dimensions of the subspaces of the flag are locally constant in an open neighborhood of the point. When every point in $M$ is regular, the sub-Riemannian manifold is said to be equiregular.
In general a sub-Riemannian manifold may admit non-regular points. However, for our purposes, we can restrict ourselves with no loss of generality to equiregular manifolds thanks to a desingularisation procedure.

Lemma 7. Fix an integer $m \geq 2$. Assume that for every rank $m$ equiregular subRiemannian structure the following property holds: every arclength parameterized abnormal minimizer admitting a lift $\lambda(t) \notin\left(D^{4}\right)^{\perp}$ is of class $C^{1}$. Then the same property holds true for every rank $m$ sub-Riemannian structure.

Proof. Let $(M, D, g)$ be a non-equiregular sub-Riemannian manifold of rank $m$ and $\gamma$ be an abnormal length-minimizer of ( $M, D, g$ ) which admits an abnormal extremal lift such that $\lambda(t) \notin\left(D^{4}\right)^{\perp}$ for every $t \in[0, T]$. Assume moreover that $\gamma$ is arclength parameterized. We have to prove that $\gamma$ is of class $C^{1}$.

Fix $t_{0} \in[0, T]$ and a generating frame $X_{1}, \ldots, X_{m}$ on a neighborhood of $\gamma\left(t_{0}\right)$. By [Jea14, Lemma 2.5], there exists an equiregular sub-Riemannian manifold ( $\widetilde{M}, \widetilde{D}, \widetilde{g}$ ) of rank $m$ with a generating frame $\xi_{1}, \ldots, \xi_{m}$ and a map $\varpi: \widetilde{M} \rightarrow M$ onto a neighborhood $U \subset M$ of $\gamma\left(t_{0}\right)$ such that $\varpi_{*} \xi_{i}=X_{i}$. Up to reducing the interval $[0, T]$ we assume that $\gamma(t) \in U$ for all $t \in[0, T]$. Let $u \in L^{\infty}\left([0, T], \mathbb{S}^{m-1}\right)$ be such that

$$
\dot{\gamma}(t)=\sum_{i=1}^{m} u_{i}(t) X_{i}(\gamma(t)), \quad \text { a.e. } t .
$$

By construction, since $\gamma$ is a length-minimizer, there exists a length-minimizer $\widetilde{\gamma}$ in $\widetilde{M}$ with $\varpi(\widetilde{\gamma})=\gamma$ associated with the same $u$, that is,

$$
\dot{\tilde{\gamma}}(t)=\sum_{i=1}^{m} u_{i}(t) \xi_{i}(\widetilde{\gamma}(t)), \quad \text { a.e. } t
$$

which is arclength parameterized as well. Hence the trajectory $\gamma$ has at least the same regularity as $\widetilde{\gamma}$.
Moreover, if $\lambda$ is an abnormal lift of $\gamma$ in $T^{*} M$, then $\widetilde{\gamma}$ admits an abnormal lift $\widetilde{\lambda}$ in $T^{*} \widetilde{M}$ such that $\widetilde{\lambda}(t)=\varpi^{*} \lambda(t)$ for every $t$. Since $\varpi^{*}\left(\widetilde{D}^{k}\right)^{\perp}=\left(D^{k}\right)^{\perp}$ for any positive integer $k$, the property $\lambda(t) \notin\left(D^{4}\right)^{\perp}$ implies $\tilde{\lambda}(t) \notin\left(\widetilde{D}^{4}\right)^{\perp}$.

It results from the hypothesis that $\widetilde{\gamma}$ is $C^{1}$, so $\gamma$ is of class $C^{1}$ in an open neighborhood of $t_{0} \in[0, T]$, which ends the proof.

As a consequence of Lemma 7, we can assume in the rest of the paper that the subRiemannian manifold is equiregular.
3.2. Nilpotentization. Let us recall the construction of the nilpotent approximation (see for instance [Bel96] for details).

Let $(M, D, g)$ be an equiregular sub-Riemannian manifold. We fix a point $x \in M$ and a local generating frame $X_{1}, \ldots, X_{m}$ in a neighborhood of $x$.

For $i=1, \ldots, n$, let $w_{i}$ be the smallest integer $j$ such that $\operatorname{dim} D_{x}^{j} \geq i$. We define the dilations $\delta_{\nu}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ for $\nu \in \mathbb{R}$ as $\delta_{\nu}(z)=\left(\nu^{w_{1}} z_{1}, \ldots, \nu^{w_{n}} z_{n}\right)$. Let $z^{x}$ be a system of privileged coordinates at $x$ and set $\delta_{\nu}^{x}=\delta_{\nu} \circ z^{x}$. Then, for $i=1, \ldots, m$, the vector field $\varepsilon\left(\delta_{1 / \varepsilon}^{x}\right)_{*} X_{i}$ converges locally uniformly as $\varepsilon \rightarrow 0$ to a vector field $\widehat{X}_{i}^{x}$ on $\mathbb{R}^{n}$. The space $\mathbb{R}^{n}$ endowed with the sub-Riemannian structure having $\widehat{X}_{1}^{x}, \ldots, \widehat{X}_{m}^{x}$ as
generating frame is called the nilpotent approximation of $(M, D, g)$ at $x$ and is denoted by $\widehat{M}_{x}$. This nilpotent approximation $\widehat{M}_{x}$ is a Carnot group equipped with a left-invariant sub-Riemannian structure.

Since ( $M, D, g$ ) is equiregular, we can locally choose systems of privileged coordinates $z^{x}$ depending continuously on $x$ [Jea14, Sect. 2.2.2]. Note that the $w_{i}$ 's and $\delta_{\nu}$ are independent of $x$. Thus an easy adaptation of the proof of [AGM15, Prop. 3.4] (see also [ABB17, Sect. 10.4.1]) shows that, for $i=1, \ldots, m$, the vector field $\varepsilon\left(\delta_{1 / \varepsilon}^{x}\right)_{*} X_{i}$ converges locally uniformly to $\widehat{X}_{i}^{x_{0}}$ as $\varepsilon \rightarrow 0$ and $x \rightarrow x_{0}$.
Lemma 8. Let $\left(a_{n}\right)_{n \in \mathbb{N}},\left(b_{n}\right)_{n \in \mathbb{N}} \subset[0, T], \bar{a} \in[0, T]$, be such that $a_{n}, b_{n} \rightarrow \bar{a}$ and $a_{n}<b_{n}$ for any $n \in \mathbb{N}$. Given $u \in L^{\infty}\left([0, T], \mathbb{S}^{m-1}\right)$ and $n \in \mathbb{N}$, define $u_{n} \in L^{\infty}\left([0,1], \mathbb{S}^{m-1}\right)$ by

$$
u_{n}(\tau)=u\left(a_{n}+\tau\left(b_{n}-a_{n}\right)\right) .
$$

Assume that $\left(u_{n}\right)_{n \in \mathbb{N}}$ converges to $u_{\star} \in L^{\infty}\left([0,1], \mathbb{R}^{m}\right)$ for the weak- $\star$ topology of $L^{\infty}\left([0,1], \mathbb{R}^{m}\right)$ and, moreover, that the trajectory $\gamma:[0, T] \rightarrow M$ associated with $u$ is a length-minimizer. If $x=\gamma(\bar{a})$, then the trajectory $\gamma_{\star}:[0,1] \rightarrow \widehat{M}_{x}$ satisfying

$$
\dot{\gamma}_{\star}(s)=\sum_{i=1}^{m} u_{\star, i}(s) \widehat{X}_{i}^{x}\left(\gamma_{\star}(s)\right), \quad \gamma_{\star}(0)=0
$$

is also a length-minimizer. In particular, $u_{\star}(t) \in \mathbb{S}^{m-1}$ for almost every $t \in[0,1]$.
Proof. We consider a continuously varying family of privileged coordinates $z^{\gamma(t)}, t \in[0, T]$, and the corresponding 1-parameter family of dilations $\delta_{\nu}^{t}:=\delta_{\nu}^{\gamma(t)}$. It is not restrictive to assume that $\delta_{\frac{a_{n}}{b_{n}-a_{n}}}^{a_{n}} \gamma(t)$ is well-defined for every $n \in \mathbb{N}$ and $t \in\left[a_{n}, b_{n}\right]$.

Let $\gamma_{n}$ be defined by $\gamma_{n}(\tau)=\delta_{\frac{b_{n}}{b_{n}-a_{n}}}^{a_{n}}\left(\gamma\left(a_{n}+\tau\left(b_{n}-a_{n}\right)\right)\right)$. Then, $\gamma_{n}$ is a lengthminimizing curve for the sub-Riemannian structure on $\mathbb{R}^{n}$ with orthonormal frame

$$
\varepsilon_{n}\left(\delta_{\frac{1}{a_{n}-a_{n}}}^{a_{n}}\right)_{*} X_{1}, \ldots, \varepsilon_{n}\left(\frac{\left.\delta_{\frac{1}{a_{n}-a_{n}}}^{a_{n}}\right)_{*} X_{m} . . . ~ . ~}{\text {. }}\right.
$$

The corresponding control is $u_{n}$.
Since the sequence $\left(\left(b_{n}-a_{n}\right)\left(\delta_{\frac{1}{b_{n}-a_{n}}}^{a_{n}}\right)_{*} X_{i}\right)_{n \in \mathbb{N}}$ converges locally uniformly to $\widehat{X}_{i}^{x}$, it follows by standard ODE theory that $\left(\gamma_{n}\right)_{n \in \mathbb{N}}$ converges uniformly to $\gamma_{\star}$.

We claim that $\widehat{d}\left(\gamma_{\star}(0), \gamma_{\star}(1)\right)=1$. Indeed, $\ell\left(\gamma_{\star}\right) \geq \widehat{d}\left(\gamma_{\star}(0), \gamma_{\star}(1)\right)$ and, by [Bel96, Theorem 7.32], we have

$$
\begin{aligned}
\widehat{d}\left(\gamma_{\star}(0), \gamma_{\star}(1)\right) & =\lim _{n \rightarrow \infty} \frac{1}{b_{n}-a_{n}} d\left(\gamma\left(a_{n}\right), \gamma\left(b_{n}\right)\right) \\
& =\lim _{n \rightarrow \infty} \int_{0}^{1}\left|u\left(a_{n}+\tau\left(b_{n}-a_{n}\right)\right)\right| d \tau=1,
\end{aligned}
$$

where $|\cdot|$ denotes the norm in $\mathbb{R}^{m}$. On the other hand, by weak- $\star$ convergence we have

$$
\ell\left(\gamma_{\star}\right)=\left\|u_{\star}\right\|_{L^{1}\left([0,1], \mathbb{R}^{m}\right)} \leq \liminf _{n \rightarrow \infty}\left\|u_{n}\right\|_{L^{1}\left([0,1], \mathbb{R}^{m}\right)}=1,
$$

proving the claim.
To conclude the proof, it suffices now to observe that the above implies that $\gamma_{\star}$ is minimizing. In particular, since $\left|u_{\star}(t)\right| \leq 1$ a.e. on $[0,1]$ by the properties of weak- $\star$ convergence, this shows that $\left|u_{\star}(t)\right|=1$ a.e. on $[0,1]$.
Corollary 9. Let $\gamma, u, \bar{a},\left(a_{n}\right)_{n \in \mathbb{N}},\left(b_{n}\right)_{n \in \mathbb{N}}$, and $u_{\star}$ be as in Lemma 8. Assume that there exist $u_{+}, u_{-} \in \mathbb{S}^{m-1}$ such that $u_{\star}=u_{-}$almost everywhere on $[0,1 / 2]$ and $u_{\star}=u_{+}$almost everywhere on $[1 / 2,1]$. Then $u_{-}=u_{+}$.
Proof. If $u_{-} \neq u_{+}$, then $\gamma_{\star}$ is not length-minimizing by Theorem 4, which contradicts Lemma 8.

## 4. Dynamics of abnormal extremals: preliminary results

In this section we present the dynamical system associated with the abnormal extremal, whose analysis is the basis for the proof of Theorem 1, and we derive a first result on its structure.
4.1. Introduction to the dynamical system. Let $(M, D, g)$ be an equiregular subRiemannian manifold of rank 2. Since the arguments are local, in what follows we fix a local generating frame $\left\{X_{1}, X_{2}\right\}$ of $(D, g)$.

Consider an abnormal length-minimizer $\gamma:[0, T] \rightarrow M$ parameterized by arclength. Then $T=d(\gamma(0), \gamma(T))$ and there exists $u \in L^{\infty}\left([0, T], \mathbb{S}^{1}\right)$ such that

$$
\dot{\gamma}(t)=u_{1}(t) X_{1}(\gamma(t))+u_{2}(t) X_{2}(\gamma(t)), \quad \text { a.e. } t \in[0, T] .
$$

Moreover from Theorem 6, $\gamma$ admits a lift $\lambda:[0, T] \rightarrow T^{*} M$ which satisfies

$$
\dot{\lambda}(t)=u_{1} \vec{h}_{1}(\lambda(t))+u_{2} \vec{h}_{2}(\lambda(t)) \quad \text { and } \quad h_{1}(\lambda(t)) \equiv h_{2}(\lambda(t)) \equiv 0 .
$$

By a slight abuse of notation, set $h_{i}(t)=\left\langle\lambda(t), X_{i}(\gamma(t))\right\rangle, i=1,2$, and for every $i_{1}, \ldots, i_{m} \in$ $\{1,2\}$,

$$
h_{i_{1} \cdots i_{m}}(t)=\left\langle\lambda(t),\left[X_{i_{1}}, \ldots,\left[X_{i_{m}-1}, X_{i_{m}}\right]\right](\gamma(t))\right\rangle .
$$

Such a function $h_{i_{1} \cdots i_{m}}$ is absolutely continuous and satisfies

$$
\begin{equation*}
\dot{h}_{i_{1} \cdots i_{m}}(t)=u_{1}(t) h_{1 i_{1} \cdots i_{m}}(t)+u_{2}(t) h_{2 i_{1} \cdots i_{m}}(t) \quad \text { for a.e. } t \in[0, T] . \tag{2}
\end{equation*}
$$

Differentiating the equalities $h_{1} \equiv h_{2} \equiv 0$ and using (2) we obtain $h_{12} \equiv 0$. Differentiating again we get

$$
\begin{equation*}
0=\dot{h}_{12}=u_{1} h_{112}+u_{2} h_{212} \quad \text { a.e. on }[0, T] . \tag{3}
\end{equation*}
$$

Remark 10. The identities $h_{1}(t)=h_{2}(t)=h_{12}(t)=0$ imply that $\lambda(t) \in\left(D^{2}\right)^{\perp}$ for every $t$. The latter is known as Goh condition and is in general (i.e., for sub-Riemannian structures of any rank) a necessary condition for the associated curve to be length-minimizing [AS99]. It is known that a generic sub-Riemannian structure of rank larger than 2 does not have non-constant abnormal extremals satisfying the Goh condition [CJT06].
Let $h=\left(-h_{212}, h_{112}\right)$ and $\left(t_{0}, t_{1}\right) \subset(0, T)$ be a maximal (i.e., non-extendable) open interval on which $h \neq 0$. Equation (3) then implies that $u= \pm h /|h|$ almost everywhere on ( $t_{0}, t_{1}$ ).

Moreover, by length-minimality of $\gamma$ we can assume without loss of generality that $u=h /|h|$ on ( $t_{0}, t_{1}$ ) (see Lemma 21 in the appendix). Thus $\gamma$ may be non-differentiable only at a time $t$ such that $h(t)=0$. In particular, if the step of the sub-Riemannian structure is not greater than 3 , then $\gamma$ is differentiable everywhere. We assume from now on that the step is at least 4 .

Observe that from (2) and using $u=h /|h|$ one obtains

$$
\dot{h}=A \frac{h}{|h|}, \quad A=\left(\begin{array}{cc}
-h_{2112} & -h_{2212}  \tag{4}\\
h_{1112} & h_{2112}
\end{array}\right), \quad \text { on }\left(t_{0}, t_{1}\right) .
$$

Here, we used the relation $h_{1212}=h_{2112}$, which follows from the Jacobi identity. Observe that the matrix $A$ has zero trace and is absolutely continuous on the whole interval $[0, T]$.
Lemma 11. Assume that $\lambda(t) \notin\left(D_{\gamma(t)}^{4}\right)^{\perp}$ for every $t \in[0, T]$. If $h\left(t_{0}\right)=0$ for some $t_{0} \in[0, T]$, then $A\left(t_{0}\right) \neq 0$.

Proof. The fact that $\gamma$ is abnormal implies that the non-zero covector $\lambda(t)$ annihilates $D_{\gamma(t)}$ for every $t \in[0, T]$. The Goh condition $h_{12} \equiv 0$ guarantees that it also annihilates $D_{\gamma(t)}^{2}$. The fact that $h\left(t_{0}\right)=0$ says, moreover, that $\lambda\left(t_{0}\right)$ annihilates $D_{\gamma\left(t_{0}\right)}^{3}$. If $A\left(t_{0}\right)$ is equal to zero, then $\lambda\left(t_{0}\right)$ annihilates $D_{\gamma\left(t_{0}\right)}^{4}$, which contradicts the assumption.
4.2. The sign of $\operatorname{det} A$ is non-negative where $h$ vanishes. A key step in the proof of Theorem 1 is the following result.

Proposition 12. Let $\left(t_{0}, t_{1}\right)$ be a maximal open interval of $[0, T]$ on which $h \neq 0$ and assume that $t_{1}<T$. Then $\operatorname{det} A\left(t_{1}\right) \leq 0$.

Proof. Assume by contradiction that $\operatorname{det} A\left(t_{1}\right)>0$. Since trace $A\left(t_{1}\right)=0$, there exists $P \in \mathrm{GL}(2, \mathbb{R})$ such that

$$
P^{-1} A\left(t_{1}\right) P=\left(\begin{array}{cc}
0 & -a  \tag{5}\\
a & 0
\end{array}\right), \quad a>0
$$

Define the scalar functions $\alpha, \beta$ and $\zeta$ through the relation

$$
P^{-1} A(t) P=\left(\begin{array}{cc}
-\alpha(t) & \beta(t) \\
\zeta(t) & \alpha(t)
\end{array}\right)
$$

and notice that $\alpha, \beta, \zeta$ are absolutely continuous with bounded derivatives on $\left(t_{0}, t_{1}\right)$, since they are linear combinations of $h_{2112}, h_{2212}, h_{1112}$, according to (2). Clearly, (5) implies that $\alpha(t) \rightarrow 0, \beta(t) \rightarrow-a$, and $\zeta(t) \rightarrow a$ as $t \rightarrow t_{1}$.

Consider a time rescaling and a polar coordinates representation so that $P^{-1} h(t)=$ $\rho(s(t)) e^{i \vartheta(s(t))}$, where

$$
s(t):=\int_{t_{0}}^{t} \frac{\left|P^{-1} h(\tau)\right|}{|h(\tau)|} d \tau
$$

It is useful to introduce $\mu:=(\zeta+\beta) / 2$ and $\eta:=(\zeta-\beta) / 2$. Then, denoting by $\rho^{\prime}$ and $\vartheta^{\prime}$ the derivatives of $\rho$ and $\vartheta$ with respect to the parameter $s$, (4) can be rewritten as

$$
\left\{\begin{array}{l}
\rho^{\prime}=(-\alpha \cos 2 \vartheta+\mu \sin 2 \vartheta) \\
\vartheta^{\prime}=\frac{1}{\rho}(\alpha \sin 2 \vartheta+\mu \cos 2 \vartheta+\eta)
\end{array}\right.
$$

Let $w=\alpha \sin 2 \vartheta+\mu \cos 2 \vartheta+\eta$ and notice that $2 a>w>a / 2$ in a left-neighborhood of $s\left(t_{1}\right)$. Therefore,

$$
\begin{aligned}
\left(\rho^{2} w\right)^{\prime}= & 2 \rho(-\alpha \cos 2 \vartheta+\mu \sin 2 \vartheta) w+\rho^{2}\left(\alpha^{\prime} \sin 2 \vartheta+\mu^{\prime} \cos 2 \vartheta+\dot{\eta}\right) \\
& +\rho^{2}(\alpha \cos 2 \vartheta-\mu \sin 2 \vartheta) 2 \vartheta^{\prime} \\
= & \rho^{2} w \frac{\alpha^{\prime} \sin 2 \vartheta+\mu^{\prime} \cos 2 \vartheta+\eta^{\prime}}{w} \geq-M \rho^{2} w
\end{aligned}
$$

for some constant $M>0$. This implies at once that $t \mapsto e^{M t} \rho^{2}(t) w(t)$ is increasing, and hence that it is impossible for $\rho^{2} w$ to tend to zero as $s \rightarrow s\left(t_{1}\right)$. This contradicts the assumption that $\rho(t) \rightarrow 0$ as $t \rightarrow t_{1}$, completing the proof of the statement.

## 5. Dynamics of abnormal extremals in a special case: proof of Proposition 3

In this section we prove Proposition 3. We present it here to illustrate in a simpler context the general procedure used later to complete the proof of Theorem 1.

Assume that $D$ is generated by two vector fields $X_{1}, X_{2}$ such that the Lie algebra Lie $\left\{X_{1}, X_{2}\right\}$ is nilpotent of step at most 4. This means that all Lie brackets of $X_{1}, X_{2}$ of length 5 vanish. Notice that this implies that $M$ is a homogeneous space (see [Bel96], [Jea14, Sect. 2.3.1] or [ABB17, Sect. 10.5]). In particular $\operatorname{dim} M \leq 8$.

Proof of Proposition 3. Without loss of generality, we assume that the step is equal to 4 . Recall that for an abnormal minimizer on an interval $I$ we have

$$
h_{1} \equiv h_{2} \equiv h_{12} \equiv 0, \quad 0=\dot{h}_{12}=u_{1} h_{112}+u_{2} h_{212} \quad \text { a.e. on } I .
$$

The vector $h=\left(-h_{212}, h_{112}\right)$ satisfies the differential equation

$$
\dot{h}=A u, \quad A=\left(\begin{array}{cc}
-h_{2112} & -h_{2212}  \tag{6}\\
h_{1112} & h_{2112}
\end{array}\right), \quad \text { a.e. on } I .
$$

Notice that $A$ is a constant matrix (with zero trace), as follows from (2) and the nilpotency assumption.

As we have already seen in the general case, on every interval where $h(t) \neq 0$ we have that $u$ is smooth and equal to either $\frac{h(t)}{|h(t)|}$ or $-\frac{h(t)}{|h(t)|}$.

We are then reduced to the case where $h$ vanishes at some point $\bar{t} \in I$. In this case the matrix $A$ cannot be zero, as it follows from Lemma 11.

We consider separately the following cases:
(a) $h(t)=0$ for all $t \in I$;
(b) there exist $\bar{t}, t_{*} \in I$ such that $h(\bar{t})=0$ and $h\left(t_{*}\right) \neq 0$.

Case (a). From (6) it follows that $u(t)$ is in the kernel of $A$ for a.e. $t \in I$. Since $u$ is nonzero for a.e. $t \in I$, then necessarily $A$ has one-dimensional kernel $\operatorname{ker} A=\operatorname{span}\{\bar{u}\}$, where $\bar{u}$ has norm one. Then $u(t)=\sigma(t) \bar{u}$ for a.e. $t \in I$, with $\sigma(t) \in\{-1,1\}$ and

$$
\dot{\gamma}(t)=\sigma(t) X_{\bar{u}}(\gamma(t)), \quad \text { a.e. } t \in I,
$$

with $X_{\bar{u}}$ a constant vector field. Since $\gamma$ is a length-minimizer then $\sigma$ is constant, and $u$ is smooth, thanks to Lemma 21 in the appendix.
Case (b). Consider the maximal neighborhood $J=\left(t_{0}, t_{1}\right)$ of $t_{*}$ on which $h$ is nonvanishing. In particular $h\left(t_{0}\right)=0$ or $h\left(t_{1}\right)=0$. Consider the case $h\left(t_{1}\right)=0$.

The trajectories of (4) are time reparameterizations of those of the linear system $\dot{z}=A z$. Hence $h$ stays in the stable or in the unstable manifold of $A$. Recall that $\operatorname{det} A \leq 0$ by Proposition 12 and notice that if $\operatorname{det} A=0$ then no nontrivial stable nor unstable manifold exists. We deduce that $\operatorname{det} A<0$.
Denote by $\lambda_{ \pm}$the eigenvalues of $A$ and by $v_{ \pm}$the corresponding unit eigenvectors. Since $h$ belongs to the stable (respectively, unstable) manifold of $A$ then $\frac{h(t)}{h(t) \mid}$ is constantly equal to $v_{-}$or $-v_{-}$on $J$ (respectively, $v_{+}$or $-v_{+}$). Then we can integrate (6) and get

$$
h(t)=h\left(t_{*}\right) \pm\left(t-t_{*}\right) \lambda_{-} v_{-}, \quad t \in J,
$$

or

$$
h(t)=h\left(t_{*}\right) \pm\left(t-t_{*}\right) \lambda_{+} v_{+}, \quad t \in J .
$$

In particular $\lim _{t \downarrow t_{0}} h(t) \neq 0$ and $J=I \cap\left(-\infty, t_{1}\right)$. Similarly, if $h\left(t_{0}\right)=0$ then one has $\lim _{t \uparrow t_{1}} h(t) \neq 0$ and $J=I \cap\left(t_{0},+\infty\right)$.

If there exist two distinct maximal intervals $J_{1}, J_{2} \subset I$ where $h$ does not vanish, then

$$
I \backslash\left(J_{1} \cup J_{2}\right)=\left[\tau_{0}, \tau_{1}\right],
$$

for some $\tau_{0} \leq \tau_{1}$ in $I$. If $\tau_{0}<\tau_{1}$, we can apply case (a) on the interval $\left(\tau_{0}, \tau_{1}\right)$, which leads to a contradiction since $A$ should have nontrivial kernel. We are thus left to consider the case where $\tau_{0}=\tau_{1}=\bar{t}$, that is, where $h(t) \neq 0$ for $t \in I \backslash\{\bar{t}\}$. In this case $u$ is piecewise constant on $I \backslash\{\bar{t}\}$ and satisfies

$$
\lim _{t \downarrow \bar{t}} u(t) \in\left\{v_{-},-v_{-}, v_{+},-v_{+}\right\}, \quad \lim _{t \uparrow \bar{t}} u(t) \in\left\{v_{-},-v_{-}, v_{+},-v_{+}\right\} .
$$

Theorem 4 and the length-minimizing assumption on $\gamma$ imply that the two limits must be equal. Hence, $u$ is constant on $I$, and in particular it is smooth.

As a byproduct of the previous proof we get that the Sard conjecture for minimizers holds [Agr14], which is known in the free case [LDMO $\left.{ }^{+} 16\right]$. More precisely we have the following.

Corollary 13. Let $(D, g)$ be a rank 2 sub-Riemannian structure on $M$. Assume that $D$ is generated by two vector fields $X_{1}, X_{2}$ such that Lie $\left\{X_{1}, X_{2}\right\}$ is a nilpotent Lie algebra of step at most 4. Then the set of points reached by abnormal minimizers has codimension at least 2.

Proof. We proved that given any initial covector in $\left(D^{2}\right)^{\perp}$, there exists at most four length-minimizing curves, whose extremal lifts start with this covector (one if $h \neq 0$, four if $h=0$ ). Hence the set of final points of abnormal minimizers has codimension 2 , that is, the codimension of $\left(D^{2}\right)^{\perp}$ (which is equal to three) minus one, taking into account the time parameterization.

For recent results on the Sard conjecture for rank 2 structures in 3-dimensional manifolds, see [BdSRar], which extends the analysis in [ZZ95].

## 6. Dynamics of abnormal extremals: The general case

The goal of this section is to prove the following result.
Proposition 14. Let $\left(t_{0}, t_{1}\right)$ be a maximal interval on which $h \neq 0$. Assume that $t_{1}<T$ and $A\left(t_{1}\right) \neq 0$. Then $u(t)$ has a limit as $t \uparrow t_{1}$.

We split the analysis in two steps. The first one, which is a rather straightforward adaptation of the proof of Proposition 3 , corresponds to the case where $\operatorname{det} A\left(t_{1}\right)<0$. We will then turn to the case where $\operatorname{det} A\left(t_{1}\right)=0$ (recall that, according to Proposition 12, $\operatorname{det} A\left(t_{1}\right)$ cannot be positive).

For this purpose, we start by proving a preliminary result.
6.1. A time-rescaling lemma. The result below highlights the fact that equation (4) is "almost invariant" with respect to similarity of $A$.

Lemma 15. For $P \in \operatorname{GL}(2, \mathbb{R})$ and $t_{*} \in\left(t_{0}, t_{1}\right)$, we consider the time reparameterization given by

$$
\varphi:\left[t_{*}, t_{1}\right) \ni t \mapsto s:=\int_{t_{*}}^{t} \frac{d \tau}{|h(\tau)|}
$$

Let $\mathfrak{h}=P^{-1} h \circ \varphi^{-1}$ and $\mathfrak{A}=P^{-1}\left(A \circ \varphi^{-1}\right) P$. Then,
(i) $\varphi(t) \rightarrow+\infty$ as $t \rightarrow t_{1}$;
(ii) for any $p \in[1,+\infty]$ we have $\mathfrak{h} \in L^{p}\left((0,+\infty), \mathbb{R}^{2}\right)$;
(iii) for every $s \in(0,+\infty)$ we have

$$
\begin{equation*}
\mathfrak{h}^{\prime}(s)=\mathfrak{A}(s) \mathfrak{h}(s) . \tag{7}
\end{equation*}
$$

Proof. We start by proving point (iii). Observe that $\dot{\varphi}=1 /|h|$. Then, simple computations yield

$$
\mathfrak{h}^{\prime}=\frac{P^{-1} \dot{h} \circ \varphi^{-1}}{\dot{\varphi} \circ \varphi^{-1}}=\mathfrak{A} \mathfrak{h} .
$$

Assume now that $\lim _{t \rightarrow t_{1}} \varphi(t)=s_{*}<+\infty$. Then, since $\mathfrak{h}\left(s_{*}\right)=h\left(t_{1}\right)=0$, we have that $\mathfrak{h}$ is the solution to the (backward) Cauchy problem

$$
\left\{\begin{array}{l}
\mathfrak{h}^{\prime}=\mathfrak{A h} \quad \text { on }\left(0, s_{*}\right), \\
\mathfrak{h}\left(s_{*}\right)=0 .
\end{array}\right.
$$

This implies that $\mathfrak{h} \equiv 0$ on $\left(0, s_{*}\right)$ and thus $h \equiv 0$ on $\left(t_{*}, t_{1}\right)$, which contradicts the definition of the interval $\left(t_{0}, t_{1}\right)$.

To complete the proof of the statement, observe that $t \mapsto h(t)$ is bounded on $\left[t_{*}, t_{1}\right]$ and thus belongs to $L^{\infty}\left(\left(t_{*}, t_{1}\right), \mathbb{R}^{2}\right)$. Then, for every $p \geq 1$,

$$
\begin{aligned}
\int_{0}^{+\infty}|\mathfrak{h}(s)|^{p} d s & =\int_{t_{*}}^{t_{1}}\left|P^{-1} h\right|^{p}|h|^{-1} d t \leq\left\|P^{-1}\right\|^{p} \int_{t_{*}}^{t_{1}}|h|^{p-1} d t \\
& \leq\left\|P^{-1}\right\|^{p}\|h\|_{L^{\infty}}^{p-1}\left(t_{1}-t_{*}\right)<+\infty .
\end{aligned}
$$

6.2. Proof of Proposition 14 in the case $\operatorname{det} A\left(t_{1}\right)<0$. Since $\operatorname{trace}(A)=0$ and $\operatorname{det} A\left(t_{1}\right)<0$, there exists $P \in \operatorname{GL}(2, \mathbb{R})$ such that

$$
P A\left(t_{1}\right) P^{-1}=\left(\begin{array}{cc}
-a & 0 \\
0 & a
\end{array}\right), \quad a>0
$$

Up to applying the change of coordinates associated with $P$ and defining the time-rescaled curves $\mathfrak{h}$ and $\mathfrak{A}$ as in Lemma 15, we have

$$
\mathfrak{A}(s)=\left(\begin{array}{cc}
-\alpha(s) & \beta(s)  \tag{8}\\
\zeta(s) & \alpha(s)
\end{array}\right)
$$

where

$$
\begin{equation*}
\lim _{s \rightarrow \infty} \alpha(s)=a, \quad \lim _{s \rightarrow \infty} \zeta(s)=\lim _{s \rightarrow \infty} \beta(s)=0 \tag{9}
\end{equation*}
$$

Let $\mathfrak{h}=\rho e^{i \vartheta}$ for $\rho>0$ and $\vartheta \in[0,2 \pi)$. To prove the statement it is enough to show that $\vartheta$ has a limit as $s \rightarrow \infty$. Let us show that $\lim _{s \rightarrow \infty} \tan 2 \vartheta=0$.

Observe that, letting $\mathfrak{h}=\left(x_{1}, x_{2}\right)$ with $x_{1}, x_{2} \in \mathbb{R}$, we have

$$
\begin{equation*}
\frac{1}{2} \tan 2 \vartheta=\frac{\sin \vartheta \cos \vartheta}{\cos ^{2} \vartheta-\sin ^{2} \vartheta}=\frac{x_{1} x_{2}}{x_{2}^{2}-x_{1}^{2}} . \tag{10}
\end{equation*}
$$

By (7) and simple computations we obtain

$$
\begin{equation*}
\left(x_{1} x_{2}\right)^{\prime}=\zeta x_{1}^{2}+\beta x_{2}^{2}, \quad \frac{\left(x_{1}^{2}-x_{2}^{2}\right)^{\prime}}{2}=-\alpha\left(x_{1}^{2}+x_{2}^{2}\right)+(\beta-\zeta) x_{1} x_{2} \tag{11}
\end{equation*}
$$

Upon integration and exploiting (9), we get

$$
x_{1} x_{2}=o(R), \quad x_{1}^{2}-x_{2}^{2}=2 a R(1+o(1)), \quad \text { where } \quad R(s):=\int_{s}^{+\infty}|\mathfrak{h}|^{2} d s .
$$

Observe that, by Lemma $15, \mathfrak{h} \in L^{2}\left((0,+\infty), \mathbb{R}^{2}\right)$ and, in particular, $R \rightarrow 0$ as $s \rightarrow+\infty$. Finally, substituting the above in (10) shows that $\tan 2 \vartheta \rightarrow 0$, completing the proof of Proposition 14 in the case $\operatorname{det} A\left(t_{1}\right)<0$.
6.3. Proof of Proposition 14 in the case $\operatorname{det} A\left(t_{1}\right)=0$. Assume that $\operatorname{det} A\left(t_{1}\right)=0$ and recall that trace $A\left(t_{1}\right)=0$. Since, moreover, $A\left(t_{1}\right) \neq 0$, there exists $P \in \operatorname{GL}(2, \mathbb{R})$ such that

$$
P A\left(t_{1}\right) P^{-1}=\left(\begin{array}{ll}
0 & 1  \tag{12}\\
0 & 0
\end{array}\right) .
$$

As before, using the change of variables of Lemma 15 , we let

$$
\mathfrak{A}(s)=\left(\begin{array}{cc}
-\alpha(s) & \beta(s) \\
\zeta(s) & \alpha(s)
\end{array}\right)
$$

where $\alpha, \beta, \zeta$ are linear combinations of $h_{2112} \circ \varphi^{-1}, h_{2212} \circ \varphi^{-1}$, and $h_{1112} \circ \varphi^{-1}$, and hence absolutely continuous with bounded derivatives on $(0,+\infty)$, according to (2). Equality (12) implies that $\alpha \rightarrow 0, \beta \rightarrow 1$, and $\zeta \rightarrow 0$ as $s \rightarrow+\infty$.

We also introduce $\mu:=\zeta+\beta$ and we notice that $\mu \rightarrow 1$ as $s \rightarrow+\infty$. (Beware that the same letters are used for different parameters in the proof of Proposition 12.) Then, (7) reads

$$
\frac{\rho^{\prime}}{\rho}=\mu \sin \vartheta \cos \vartheta-\alpha \cos 2 \vartheta, \quad \vartheta^{\prime}=-\mu \sin ^{2} \vartheta+\alpha \sin 2 \vartheta+\zeta
$$

and can be written as

$$
\begin{equation*}
\frac{\rho^{\prime}}{\rho}=\sin \vartheta \cos \vartheta+f, \quad \vartheta^{\prime}=-\sin ^{2} \vartheta+g \tag{13}
\end{equation*}
$$

where the functions

$$
f=-\alpha \cos 2 \vartheta+(1-\mu) \sin \vartheta \cos \vartheta, \quad g=\alpha \sin 2 \vartheta+\zeta+(1-\mu) \sin ^{2} \vartheta,
$$

tend to zero as $s \rightarrow+\infty$.

## Lemma 16. We have the following dichotomy:

(i) $\sin \vartheta \rightarrow 0$ as $s \rightarrow+\infty$. (That is, $\vartheta$ admits a limit as $s \rightarrow+\infty$, which is equal to 0 modulus $\pi$.)
(ii) $\vartheta \rightarrow-\infty$ as $s \rightarrow+\infty$. Moreover, in this case, for any $0<\varepsilon<\pi / 2$ there exist two sequences of positive real numbers $\left(s_{n}\right)_{n \in \mathbb{N}}$ and $\left(\bar{s}_{n}\right)_{n \in \mathbb{N}}$ tending to infinity and such that

\[

\]



Figure 1. The sequences $\left(s_{n}\right)_{n \in \mathbb{N}}$ and $\left(\bar{s}_{n}\right)_{n \in \mathbb{N}}$ in Lemma 16
Proof. Notice that the dynamics of $\vartheta$ is a perturbation via $g$ of

$$
\vartheta^{\prime}=-\sin ^{2} \vartheta
$$

The phase portrait of the latter on $\mathbb{S}^{1}$ is made of two equilibria in 0 and $\pi$ joined by two clock-wise oriented heteroclinic trajectories.

Assume that (i) does not hold. Therefore, there exists $c>0$ such that

$$
\begin{equation*}
\limsup _{s \rightarrow+\infty}|\sin \vartheta(s)|>c \tag{14}
\end{equation*}
$$

Let $\varepsilon>0$ be such that $\sin \varepsilon \in(0, c)$ and $s^{*}>0$ be such that, for $s>s^{*}, \vartheta^{\prime}(s)<-\varepsilon^{2} / 2$ as soon as $|\sin \vartheta(s)|>\varepsilon$.

Pick $q_{1}>s^{*}$ such that $\left|\sin \vartheta\left(q_{1}\right)\right|>c>\sin \varepsilon$. Since $\vartheta^{\prime}$ is bounded from zero as long as $|\sin \vartheta|$ stays larger than $\sin \varepsilon$, there exists $r_{1}>q_{1}$ such that $\left|\sin \vartheta\left(r_{1}\right)\right|=\sin \varepsilon$. By definition of $c$, there exists $q_{2}>r_{1}$ such that $\left|\sin \vartheta\left(q_{2}\right)\right|>c$. Moreover, $q_{1}$ and $q_{2}$ can be taken so that $\vartheta\left(q_{2}\right)=\vartheta\left(q_{1}\right)-\pi$ and (14) holds with $c$ arbitrarily close to 1 . By iterating the procedure leading from $q_{1}$ to $q_{2}$, we prove that $\vartheta \rightarrow-\infty$. The construction also shows how to construct the sequences $\left(s_{n}\right)_{n \in \mathbb{N}}$ and $\left(\bar{s}_{n}\right)_{n \in \mathbb{N}}$ as in (ii).

The rest of the argument consists in showing that case (ii) in Lemma 16 cannot hold true. For that purpose, we argue by contradiction.

Lemma 17. Assume that property (ii) in Lemma 16 holds true. Then there exists $0<$ $\varepsilon_{0}<\pi / 2$ such that for any $0<\varepsilon<\varepsilon_{0}$ there exists $N_{\varepsilon}$ for which, given any $n \geq N_{\varepsilon}$,

$$
\begin{equation*}
\frac{2}{\varepsilon}\left(1-\varepsilon^{2}\right) \leq s_{2 n+1}-s_{2 n} \leq \frac{2}{\varepsilon}\left(1+\varepsilon^{2}\right) \tag{15}
\end{equation*}
$$

and

$$
\begin{equation*}
(1-\varepsilon) \varepsilon \rho\left(s_{2 n}\right) \leq \rho(s) \sin \vartheta(s) \leq(1+\varepsilon) \varepsilon \rho\left(s_{2 n}\right), \text { for } s \in\left[s_{2 n}, s_{2 n+1}\right] \tag{16}
\end{equation*}
$$

As a consequence, for every $n \geq N_{\varepsilon}$, one has the following estimates

$$
\begin{align*}
& 2(1-2 \varepsilon) \rho\left(s_{2 n}\right) \leq \int_{s_{2 n}}^{s_{2 n+1}} \sin \vartheta(s) \rho(s) d s \leq 2(1+2 \varepsilon) \rho\left(s_{2 n}\right)  \tag{17}\\
&(1-4 \varepsilon) \frac{2 \rho\left(s_{2 n}\right)}{\varepsilon} \leq \int_{s_{2 n}}^{s_{2 n+1}} \rho(s) d s \leq(1+4 \varepsilon) \frac{2 \rho\left(s_{2 n}\right)}{\varepsilon}  \tag{18}\\
&\left|\int_{s_{2 n}}^{s_{2 n+1}} \cos \vartheta(s) \rho(s) d s\right| \leq \rho\left(s_{2 n}\right) \tag{19}
\end{align*}
$$

Proof. Set $M_{f}(s)=\sup _{\tau \geq s}|f(\tau)|$ and $M_{g}(s)=\sup _{\tau \geq s}|g(\tau)|$. Observe that these two functions tend to zero as $s$ tends to infinity.

By Lemma 16, for $n$ large enough and $s \in\left[s_{2 n}, s_{2 n+1}\right]$, equation (13) becomes

$$
\begin{equation*}
(\cot \vartheta)^{\prime}=1-\frac{g}{\sin ^{2} \vartheta} . \tag{20}
\end{equation*}
$$

For $n$ large enough, for every $s \in\left[s_{2 n}, s_{2 n+1}\right]$ we have

$$
\left|\frac{g(s)}{\sin ^{2} \vartheta(s)}\right| \leq \frac{M_{g}\left(s_{2 n}\right)}{\sin ^{2} \varepsilon} \leq \frac{\varepsilon^{2}}{2}
$$

Equation (15) follows by integrating (20) on the interval [ $\left.s_{2 n}, s_{2 n+1}\right]$.
On the interval $\left[s_{2 n}, s_{2 n+1}\right]$, one has

$$
\frac{\rho^{\prime}(s)}{\rho(s)}+\frac{\vartheta^{\prime}(s) \cos \vartheta(s)}{\sin \vartheta(s)}=-f(s)-\frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)} .
$$

For $n$ large enough, for every $s \in\left[s_{2 n}, s_{2 n+1}\right]$ we have

$$
\begin{equation*}
\left|f(s)+\frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)}\right| \leq M_{f}\left(s_{2 n}\right)+\frac{M_{g}\left(s_{2 n}\right)}{\sin \varepsilon} \leq \frac{\varepsilon^{2}}{4} \tag{21}
\end{equation*}
$$

By integrating between $s_{2 n}$ and any $s \in\left[s_{2 n}, s_{2 n+1}\right]$, one gets

$$
\left|\ln \left(\frac{\rho(s) \sin \vartheta(s)}{\rho\left(s_{2 n}\right) \sin \varepsilon}\right)\right|=\left|\int_{s_{2 n}}^{s}\left(f(s)+\frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)}\right) d s\right| \leq \frac{\left(s_{2 n+1}-s_{2 n}\right) \varepsilon^{2}}{4} \leq \frac{\varepsilon}{2}
$$

yielding (16) for $\varepsilon$ small enough.
We now turn to the proof of the three estimates (17)-(19). The first one simply follows by integrating (16) on $\left[s_{2 n}, s_{2 n+1}\right]$ and using (15). Estimate (18) is obtained by first dividing (16) by $\sin \vartheta(s)$ and then integrating the resulting inequalities on $\left[s_{2 n}, s_{2 n+1}\right]$. One gets that

$$
(1-\varepsilon) \varepsilon \rho\left(s_{2 n}\right) \int_{s_{2 n}}^{s_{2 n+1}} \frac{d s}{\sin \vartheta(s)} \leq \int_{s_{2 n}}^{s_{2 n+1}} \rho(s) d s \leq(1+\varepsilon) \varepsilon \rho\left(s_{2 n}\right) \int_{s_{2 n}}^{s_{2 n+1}} \frac{d s}{\sin \vartheta(s)}
$$

On the other hand, the following holds true,

$$
\int_{s_{2 n}}^{s_{2 n+1}} \frac{d s}{\sin \vartheta(s)}=\int_{s_{2 n}}^{s_{2 n+1}} \frac{\vartheta^{\prime}(s)}{-\sin ^{3} \vartheta(s)\left(1-\frac{g(s)}{\sin ^{2} \vartheta(s)}\right)} d s
$$

which implies that

$$
\left(1-\varepsilon^{2}\right) \int_{\varepsilon}^{\pi-\varepsilon} \frac{d \vartheta}{\sin ^{3} \vartheta} \leq \int_{s_{2 n}}^{s_{2 n+1}} \frac{d s}{\sin \vartheta(s)} \leq\left(1+\varepsilon^{2}\right) \int_{\varepsilon}^{\pi-\varepsilon} \frac{d \vartheta}{\sin ^{3} \vartheta} .
$$

A direct computation shows that $\int_{\varepsilon}^{\pi-\varepsilon} \frac{d \vartheta}{\sin ^{3} \vartheta}=\frac{1}{\varepsilon^{2}(1+o(\varepsilon))}$ as $\varepsilon$ tends to zero. One finally deduces estimate (18).

To derive estimate (19), one notices that

$$
\begin{aligned}
\int_{s_{2 n}}^{s_{2 n+1}} \cos \vartheta(s) \rho(s) d s= & \int_{s_{2 n}}^{s_{2 n+1}} \frac{\sin \vartheta(s) \cos \vartheta(s) \rho(s)}{\sin \vartheta(s)} d s \\
= & \int_{s_{2 n}}^{s_{2 n+1}} \frac{\rho^{\prime}(s)-f(s) \rho(s)}{\sin \vartheta(s)} d s \\
= & -\int_{s_{2 n}}^{s_{2 n+1}} \frac{f(s) \rho(s)}{\sin \vartheta(s)} d s+\frac{\rho\left(s_{2 n+1}\right)-\rho\left(s_{2 n}\right)}{\sin \varepsilon} \\
& +\int_{s_{2 n}}^{s_{2 n+1}} \frac{\rho(s) \cos \vartheta(s) \vartheta^{\prime}(s)}{\sin \vartheta(s)} d s .
\end{aligned}
$$

By using the expression of $\vartheta^{\prime}$ in the last integral, one deduces that

$$
2 \int_{s_{2 n}}^{s_{2 n+1}} \cos \vartheta(s) \rho(s) d s=\frac{\rho\left(s_{2 n+1}\right)-\rho\left(s_{2 n}\right)}{\sin \varepsilon}-\int_{s_{2 n}}^{s_{2 n+1}} \rho(s) \frac{f(s)+\frac{\cos \vartheta(s) g(s)}{\sin \vartheta(s)}}{\sin \vartheta(s)} d s
$$

By using (16) for $s=s_{2 n}$ and $s=s_{2 n+1}$ and then (21), one deduces (19).
Fix a sequence $\left(\varepsilon_{k}\right)_{k \in \mathbb{N}}$, strictly decreasing to 0 . For each $k \in \mathbb{N}$, we use $\left(s_{k, n}\right)_{n \in \mathbb{N}}$ to denote the sequence $\left(s_{n}\right)_{n \in \mathbb{N}}$ given by Lemma 16 and corresponding to $\varepsilon=\varepsilon_{k}$. For all $k \in \mathbb{N}$ let $n_{k} \geq N_{\varepsilon_{k}}$ be an integer to be fixed later, where $N_{\varepsilon_{k}}$ is as in Lemma 17. We use $\left(\xi_{\ell}\right)_{\ell \in \mathbb{N}}$ to denote the sequence defined by

$$
\xi_{2 k}=s_{k, 2 n_{k}}, \quad \xi_{2 k+1}=s_{k, 2 n_{k}+1}, \quad \forall k \in \mathbb{N}
$$

We choose $k \mapsto n_{k}$ so that the sequence $\left(\xi_{\ell}\right)_{\ell \in \mathbb{N}}$ is strictly increasing and tends to infinity as $\ell \rightarrow+\infty$.

Let $t_{\ell}=\varphi^{-1}\left(\xi_{\ell}\right)$, where $\varphi$ is the change of variables introduced in Lemma 15 . For every $\ell \geq 0$ consider the function $u_{\ell} \in L^{\infty}\left([0,1], \mathbb{S}^{1}\right)$ defined by $u_{\ell}(\tau)=u\left(t_{2 \ell}+\tau\left(t_{2 \ell+1}-t_{2 \ell}\right)\right)$. By the weak- $\star$ compactness of all bounded subsets of $L^{\infty}\left([0,1], \mathbb{R}^{2}\right)$, we can assume without loss of generality that $u_{\ell} \rightharpoonup u_{\star}$ in the weak- $\begin{gathered}\text { topology. Applying Lemma } 8 \text { with } a_{\ell}=t_{2 \ell}\end{gathered}$ and $b_{\ell}=t_{2 \ell+1}$, we deduce that $u_{\star}$ is minimizing and $\left|u_{\star}\right| \equiv 1$ almost everywhere in $[0,1]$.

For every subinterval $[a, b]$ of $[0,1]$, by the properties of weak-ぇ convergence, we have that

$$
\int_{a}^{b} v^{T} u_{\ell}(\tau) d \tau \rightarrow \int_{a}^{b} v^{T} u_{\star}(\tau) d \tau, \quad \forall v \in \mathbb{R}^{2}
$$

Moreover, one has

$$
\begin{align*}
\int_{a}^{b} v^{T} u_{\ell}(\tau) d \tau & =\frac{1}{t_{\ell+1}-t_{\ell}} \int_{(1-a) t_{2 \ell}+a t_{2 \ell+1}}^{(1-b) t_{2}+b t_{2 \ell+1}} \frac{v^{T} h}{|h|} d t \\
& =\frac{1}{t_{\ell+1}-t_{\ell}} \int_{\varphi\left((1-a) t_{2 \ell}+a t_{2 \ell+1}\right)}^{\varphi\left((1-b) t_{2 \ell}+b t_{2 \ell+1}\right)} v^{T} P \mathfrak{h}(s) d s \tag{22}
\end{align*}
$$

where $P$ has been introduced in (12).
In addition

$$
\begin{equation*}
t_{\ell+1}-t_{\ell}=\int_{\xi_{2 \ell}}^{\xi_{2 \ell+1}}|P \mathfrak{h}(s)| d s \tag{23}
\end{equation*}
$$

Lemma 18. Under the above assumptions, there exists a unit vector $v_{\star} \in \mathbb{R}^{2}$ such that $u_{\star}(t)=v_{\star}$ for a.e. $t \in[0,1]$. Moreover, $v_{\star}$ is parallel to $P(1,0)$.

Proof. Let $v_{\star}, w_{\star} \in \mathbb{R}^{2}$ be two orthogonal unit vectors such that $v_{\star}$ is parallel to $P(1,0)$. Notice that $P^{T} w_{\star}$ is orthogonal to $(1,0)$, that is, it is parallel to $(0,1)$. We start by showing that $w_{\star}^{T} u_{\star}(t)=0$ for a.e. $t \in[0,1]$. This amounts to show that for all $0 \leq a<b \leq 1$ it holds

$$
\frac{1}{t_{\ell+1}-t_{\ell}} \int_{\varphi\left((1-a) t_{2 \ell}+a t_{2 \ell+1}\right)}^{\varphi\left((1-b) t_{2}+b t_{2 \ell+1}\right)} \mathfrak{h}_{2}(s) d s \rightarrow 0 \quad \text { as } \ell \rightarrow+\infty .
$$

Since $\mathfrak{h}_{2}=\rho \sin \vartheta$ is positive on $\left[\xi_{2 \ell}, \xi_{2 \ell+1}\right]$ by construction, using (23) it is enough to show that

$$
\frac{\int_{\xi_{2}}^{\xi_{2 \ell+1}} \rho(s) \sin \vartheta(s) d s}{\int_{\xi_{2 \ell}}^{\xi_{22+1}}|P \mathfrak{h}(s)| d s} \longrightarrow 0 \quad \text { as } \ell \rightarrow+\infty .
$$

Since $|P \mathfrak{h}(s)| \geq\left\|P^{-1}\right\|^{-1} \rho(s)$ for all $s$, the latter limit holds true according to (17) and (18) in Lemma 17, applied to $\varepsilon=\varepsilon_{\ell}$ for $\ell \geq 0$.

Recall that the control $u_{\star}$ is minimizing and $\left|u_{\star}(t)\right|=1$ for a.e. $t \in[0,1]$. From what precedes, one deduces that $u_{\star}$ is almost everywhere perpendicular to $w_{\star}$, hence equal to $v_{\star}$ or $-v_{\star}$. It then follows from Lemma 21 in the appendix that, up to replacing $v_{\star}$ by $-v_{\star}$, the equality $u_{\star}(t)=v_{\star}$ holds for a.e. $t \in[0,1]$.
Let $\bar{v} \in \mathbb{R}^{2}$ be such that $P^{T} \bar{v}=(1,0)$. We have, according to Lemma 18,

$$
\lim _{\ell \rightarrow \infty} \int_{0}^{1} \bar{v}^{T} u_{\ell}(\tau) d \tau=\int_{0}^{1} \bar{v}^{T} u_{\star}(\tau) d \tau=\bar{v}^{T} v_{\star} \neq 0 .
$$

We conclude the proof by contradiction by showing that the limit in the left-hand side is zero. Indeed, according to (22), we have

$$
\left|\int_{0}^{1} \bar{v}^{T} u_{\ell}(\tau) d \tau\right|=\frac{\left|\int_{\xi_{2 \ell}}^{\xi_{22+1}} \rho(s) \cos \vartheta(s) d s\right|}{\int_{\xi_{2 \ell}}^{\xi_{2 \ell+1}}|P \mathfrak{h}(s)| d s} \leq\left\|P^{-1}\right\| \frac{\left|\int_{\xi_{2 \ell}}^{\xi_{2 \ell+1}} \rho(\tau) \cos \vartheta(\tau) d \tau\right|}{\int_{\xi_{2 \ell}}^{\xi_{2 \ell+1}} \rho(\tau) d s} .
$$

The right-hand side of the above equation tends to zero thanks to (18) and (19) in Lemma 17 applied to $\varepsilon=\varepsilon_{\ell}$ for $\ell \geq 0$.

We have therefore proved that (ii) in Lemma 16 cannot hold true, which completes the proof of Proposition 14.

## 7. Proof of Theorem 1

Let $M$ be as in the statement of Theorem 1. Denote, as in the previous sections, by $\gamma:[0, T] \rightarrow M$ a length-minimizing trajectory parameterized by arclength and by $\lambda:[0, T] \rightarrow T^{*} M$ an abnormal extremal lift of $\gamma$.

Proposition 14, together with Theorem 4, proves the $C^{1}$ regularity of $\gamma$ provided that $h$ vanishes only at isolated points.

We consider in this section the case where $t_{0} \in(0, T)$ is a density point of $\{t \in[0, T] \mid$ $h(t)=0\}$. We want to prove that $u(t)$ (up to modification on a set of measure zero) has a limit as $t \uparrow t_{0}$ and as $t \downarrow t_{0}$. By symmetry, we restrict our attention to the existence of the limit of $u(t)$ as $t \uparrow t_{0}$.

We are going to consider separately the situations where $h \equiv 0$ on a left neighborhood of $t_{0}$ and where there exists a sequence of maximal open intervals $\left(t_{0}^{n}, t_{1}^{n}\right)$ with $\left.h\right|_{\left(t_{0}^{n}, t_{1}^{n}\right)} \neq 0$ and such that $t_{1}^{n} \rightarrow t_{0}$.
Assume for now on that $h \equiv 0$ on a left neighborhood $\left(t_{0}-\eta, t_{0}\right]$ of $t_{0}$. Then, since $\dot{h}=A u$ almost everywhere on $\left(t_{0}-\eta, t_{0}\right.$ ], we have that $u(t)$ belongs to ker $A(t)$ for almost every $t$ in $\left(t_{0}-\eta, t_{0}\right]$. By Lemma 11, moreover, $\operatorname{ker} A(t)$ is one-dimensional for every $t \in\left(t_{0}-\eta, t_{0}\right]$.

Fix an open neighborhood $V_{0}$ of $\lambda\left(t_{0}\right)$ in $T^{*} M$ such that there exists a smooth map $V_{0} \ni \lambda \mapsto v(\lambda) \in \mathbb{S}^{1}$ such that $v(\lambda(t)) \in \operatorname{ker} A(t)$ if $\lambda(t) \in V_{0}$ and $t \in\left(t_{0}-\eta, t_{0}\right]$. Up to


Figure 2. Phase portrait of $\dot{x}(t)=A(\bar{t}) x(t)$ for $\bar{t} \in\left[t_{0}-\eta, t_{0}\right]$
reducing $\eta$, we assume that $\lambda(t) \in V_{0}$ for every $t \in\left(t_{0}-\eta, t_{0}\right]$. Notice that $\left.\lambda\right|_{\left(t_{0}-\eta, t_{0}\right]}$ is a solution of the time-varying system

$$
\dot{\lambda}=\sigma(t) \vec{X}_{v(\lambda)}(\lambda),
$$

where $\sigma:\left(t_{0}-\eta, t_{0}\right] \rightarrow\{-1,1\}$ is measurable. Hence, by length-minimality of $\gamma$ and by Lemma 21 in the appendix, either $u=v$ almost everywhere on $\left(t_{0}-\eta, t_{0}\right.$ ] or $u=-v$ almost everywhere on $\left(t_{0}-\eta, t_{0}\right.$ ]. We conclude that $u$ is continuous on $\left(t_{0}-\eta, t_{0}\right]$ and the proof in this case in concluded.

We are left to consider the case where every left neighborhood of $t_{0}$ contains a maximal interval ( $\tau_{0}, \tau_{1}$ ) such that $h \neq 0$ on ( $\tau_{0}, \tau_{1}$ ).

Notice that, by Proposition 12 and by continuity of $t \mapsto A(t)$, we have that $\operatorname{det} A\left(t_{0}\right) \leq 0$.
The case $\operatorname{det} A\left(t_{0}\right)<0$ can be ruled out thanks to the following lemma.
Lemma 19. Let $\operatorname{det} A\left(t_{0}\right)<0$. Then there exists $\eta \in\left(0, t_{0}\right)$ such that, if $0 \leq \tau_{0}<\tau_{1} \leq t_{1}$ and $x:\left(\tau_{0}, \tau_{1}\right) \rightarrow \mathbb{R}^{2} \backslash\{0\}$ is a maximal solution to $\dot{x}=A \frac{x}{|x|}$, then $\tau_{0}<t_{0}-\eta$.
Proof. For every $v \in \mathbb{R}^{2} \backslash\{0\}$ and every $\vartheta>0$ denote by $C_{\vartheta}(v)$ the cone of all vectors in $\mathbb{R}^{2} \backslash\{0\}$ making an (unoriented) angle smaller than $\vartheta$ with $v$ or $-v$.

Let $\eta_{0} \in\left(0, t_{0}\right)$ be such that $\operatorname{det}(A(t))<0$ for every $t \in\left[t_{0}-\eta_{0}, t_{0}\right]$. For $t \in\left[t_{0}-\eta_{0}, t_{0}\right]$, denote by $v_{-}(t)$ and $v_{+}(t)$ two eigenvectors of $A(t)$, the first corresponding to a negative and the second to a positive eigenvalue.

Let $\eta \in\left(0, \eta_{0}\right)$ and $\vartheta_{0}>0$ be such that $C_{\vartheta_{0}}\left(v_{+}\left(t_{0}\right)\right) \cap C_{\vartheta_{0}}\left(v_{-}\left(t_{0}\right)\right)=\emptyset$ and $v_{ \pm}(t) \in$ $C_{\vartheta_{0}}\left(v_{ \pm}\left(t_{0}\right)\right)$ for every $t \in\left[t_{0}-\eta, t_{0}\right]$.

Notice that $C_{\vartheta_{0}}\left(v_{+}\left(t_{0}\right)\right)$ is positively invariant for the dynamics of $\dot{x}(t)=A(\bar{t}) x(t)$ for every fixed $\bar{t} \in\left[t_{0}-\eta, t_{0}\right]$ (see Figure 2).

Assume by contradiction that $x:\left(\tau_{0}, \tau_{1}\right) \rightarrow \mathbb{R}^{2} \backslash\{0\}$ is a maximal solution of $\dot{x}=A \frac{x}{|x|}$ with $\left(\tau_{0}, \tau_{1}\right) \subset\left(t_{0}-\eta, t_{0}\right)$. It follows from Proposition 14 that

$$
\lim _{\tau \downarrow \tau_{0}} \frac{x(\tau)}{|x(\tau)|} \rightarrow \pm v_{+}\left(\tau_{1}\right), \quad \lim _{\tau \rightarrow \tau_{1}} \frac{x(\tau)}{|x(\tau)|} \rightarrow \pm v_{-}\left(\tau_{1}\right) .
$$

This contradicts the positive invariance of $C_{\vartheta_{0}}\left(v_{+}\left(t_{0}\right)\right)$ for the equation $\dot{x}=A_{\left\lvert\, \frac{x}{|x|}\right.}$ on $\left(\tau_{0}, \tau_{1}\right)$.

In the case $\operatorname{det} A\left(t_{0}\right)=0$ the proof follows the steps of the construction of Section 6.3. In particular, let $P \in \mathrm{GL}(2, \mathbb{R})$ be such that

$$
P^{-1} A(t) P=\left(\begin{array}{cc}
-a(t) & 1-c(t) \\
c(t) & a(t)
\end{array}\right)
$$

where $a, b, c$ are affine combinations of $h_{2112}, h_{2212}$, and $h_{1112}$ with $a \rightarrow 0$, and $c \rightarrow 0$ as $t \rightarrow t_{0}$. Let

$$
\begin{equation*}
P^{-1} h(t)=r(t) e^{i \omega(t)} \tag{24}
\end{equation*}
$$

with $\omega(t)$ uniquely defined modulus $2 \pi$ only when $h(t) \neq 0$.
The crucial point is the following counterpart to Lemma 16, whose proof can be obtained using exactly the same arguments.

Lemma 20. We have the following dichotomy:
(i) for any $0<\varepsilon<\pi / 2$, for $\eta$ small enough, $|\sin (\omega(t))|<\varepsilon$ for all $t \in\left(t_{0}-\eta, t_{0}\right)$ such that $h(t) \neq 0$;
(ii) for any $0<\varepsilon<\pi / 2$ there exists an increasing sequence $\left(t_{n}\right)_{n \in \mathbb{N}}$ in $\left(0, t_{0}\right)$ tending to $t_{0}$ and such that

$$
\begin{gathered}
\omega\left(t_{2 n}\right)=\pi-\varepsilon \quad \bmod 2 \pi, \quad \omega\left(t_{2 n+1}\right)=\varepsilon \quad \bmod 2 \pi \\
h(t) \neq 0, \sin (\omega(t))>0, \dot{\omega}(t)<0 \quad \forall t \in\left[t_{2 n}, t_{2 n+1}\right]
\end{gathered}
$$

or

$$
\begin{gathered}
\omega\left(t_{2 n}\right)=-\varepsilon \quad \bmod 2 \pi n, \quad \omega\left(t_{2 n+1}\right)=\varepsilon-\pi \quad \bmod 2 \pi n \\
h(t) \neq 0, \quad \sin (\omega(t))<0, \quad \dot{\omega}(t)<0 \quad \forall t \in\left[t_{2 n}, t_{2 n+1}\right]
\end{gathered}
$$

holds true.
Case (ii) can be excluded by similar computations as in Section 6.3, since it contradicts the optimality of $\gamma$.

Consider now case (i). Let $v_{\star}, w_{\star} \in \mathbb{R}^{2}$ be two orthogonal unit vectors such that $v_{\star}$ is parallel to $P(1,0)$. According to $(24)$, if $\sin (\omega(t))=0$ and $r(t) \neq 0$, then $u(t)=h(t) /|h(t)|$ is equal to $v_{\star}$ or $-v_{\star}$. For every $\eta \in\left(0, t_{0}\right)$ we set

$$
I_{\eta}^{+}=\left\{t \in\left(t_{0}-\eta, t_{0}\right) \mid v_{\star}^{T} u(t)>0\right\}, \quad I_{\eta}^{-}=\left\{t \in\left(t_{0}-\eta, t_{0}\right) \mid v_{\star}^{T} u(t)<0\right\} .
$$

Property (i) implies that, for $\eta$ small, $I_{\eta}^{+} \cup I_{\eta}^{-}$contains $\left\{t \in\left(t_{0}-\eta, t_{0}\right) \mid h(t) \neq 0\right\}$. Moreover, if $t_{0}$ is a density point for $I=I_{\eta}^{+} \cap\left\{t \in\left(t_{0}-\eta, t_{0}\right) \mid h(t) \neq 0\right\}$ (respectively, $\left.I=I_{\eta}^{-} \cap\left\{t \in\left(t_{0}-\eta, t_{0}\right) \mid h(t) \neq 0\right\}\right)$, then,

$$
\left.\lim _{t \in I, t \rightarrow t_{0}} u(t)=v_{\star} \quad \text { (respectively, } \lim _{t \in I, t \rightarrow t_{0}} u(t)=-v_{\star}\right) .
$$

Let $\Phi_{\eta}=\left\{t \in\left(t_{0}-\eta, t_{0}\right) \mid h(t)=0\right\}$. For almost every $t \in \Phi_{\eta}, u(t)$ is in the kernel of $A(t)$ and $|u(t)|=1$. Notice that, if $t_{0}$ is a density point for $J=\left\{t \in\left(0, t_{0}\right) \mid \operatorname{ker} A(t) \neq(0)\right\}$, then the kernel of $A(t)$ converges to the kernel of $A\left(t_{0}\right)$ as $t \in J, t \rightarrow t_{0}$. By construction of $P$, moreover, $\operatorname{ker}\left(A\left(t_{0}\right)\right)=\operatorname{span}(P(1,0))=\operatorname{span}\left(v_{\star}\right)$. Hence, for $\eta$ small enough, almost every $t \in \Phi_{\eta}$ is in $I_{\eta}^{+} \cup I_{\eta}^{-}$.

To summarize, for $\eta$ small enough, $I_{\eta}^{+} \cup I_{\eta}^{-}$has full measure in $\left(t_{0}-\eta, t_{0}\right)$. Moreover,

$$
\begin{equation*}
\lim _{t \in I_{\eta}^{+}, t \rightarrow t_{0}} u(t)=v_{\star}, \quad \lim _{t \in I_{\eta}^{-}, t \rightarrow t_{0}} u(t)=-v_{\star} . \tag{25}
\end{equation*}
$$

We next prove that $u$ converges either to $v_{\star}$ or to $-v_{\star}$ as $t \rightarrow t_{0}$ by showing that, for $\eta$ small enough, either $I_{\eta}^{+}$or $I_{\eta}^{-}$has measure zero.

Suppose by contradiction that there exists a sequence of intervals $\left(\tau_{0}^{n}, \tau_{1}^{n}\right)$ in $\left(0, t_{0}\right)$ such that $\tau_{0}^{n}, \tau_{1}^{n} \rightarrow t_{0}$ as $n \rightarrow \infty$ and both $\left|\left(\tau_{0}^{n}, \tau_{1}^{n}\right) \cap I^{+}\right|$and $\left|\left(\tau_{0}^{n}, \tau_{1}^{n}\right) \cap I^{-}\right|$are positive, where
$|\cdot|$ denotes the Lebesgue measure and $I^{ \pm}=\left\{t \in\left(0, t_{0}\right) \mid \pm v_{\star}^{T} u(t)>0\right\}$. Moreover, up to restricting $\left(\tau_{0}^{n}, \tau_{1}^{n}\right)$, we can assume that

$$
\begin{equation*}
\left|\left(\tau_{0}^{n}, \tau_{1}^{n}\right) \cap I^{+}\right|=\left|\left(\tau_{0}^{n}, \tau_{1}^{n}\right) \cap I^{-}\right|>0 . \tag{26}
\end{equation*}
$$

This can be seen, for instance, by considering a continuous deformation of an interval around a Lebesgue point of $\left(\tau_{0}^{n}, \tau_{1}^{n}\right) \cap I^{+}$towards an interval around a Lebesgue point of $\left(\tau_{0}^{n}, \tau_{1}^{n}\right) \cap I^{-}$.

For every $n \in \mathbb{N}$, let $u_{n} \in L^{\infty}\left([0,1], \mathbb{R}^{2}\right)$ be defined by $u_{n}(\tau)=u\left(\tau_{0}^{n}+\tau\left(\tau_{1}^{n}-\tau_{0}^{n}\right)\right)$. Up to extracting a subsequence, $u_{n}$ weakly- $\star$ converges to some $u_{\star}$. Condition (26) and the limits in (25) imply that

$$
\begin{equation*}
\int_{0}^{1} u_{\star}(t) d t=0 \tag{27}
\end{equation*}
$$

Thanks to (25) we also have that $w_{\star}^{T} u_{n} L^{\infty}$-converges to zero as $n \rightarrow \infty$. In particular, $w_{\star}^{T} u_{\star} \equiv 0$. By Lemma $8, u_{\star}$ is optimal and $v_{\star}^{T} u_{\star}$ has values in $\{-1,1\}$. Hence, by Lemma 21 in the appendix, $v_{\star}^{T} u_{\star}$ is constantly equal to +1 or -1 . This contradicts (27) and the proof is concluded.

## Appendix A. An elementary lemma

Lemma 21. Let $(M, D, g)$ be a sub-Riemannian manifold. Let $V$ be a Lipschitz continuous vector field on $T^{*} M$ such that $\pi_{*} V(\lambda) \in D_{\pi(\lambda)} \backslash\{0\}$ for every $\lambda \in T^{*} M$. Let $\lambda:[0, T] \rightarrow$ $T^{*} M$ satisfy $\dot{\lambda}(t)=\sigma(t) V(\lambda(t))$ with $\sigma \in L^{\infty}([0, T],[-1,1])$. Assume that $\gamma=\pi \circ \lambda$ : $[0, T] \rightarrow M$ is a length-minimizer. Then $\sigma$ has constant sign, i.e., either $\sigma \geq 0$ a.e. on $[0, T]$ or $\sigma \leq 0$ a.e. on $[0, T]$.

Proof. Set $\kappa=\int_{0}^{T} \sigma(t) d t$ and notice that $\lambda(T)=e^{\kappa V}(\lambda(0))$. If $\sigma$ does not have constant sign, then $[0,1] \ni t \mapsto \pi \circ e^{t \kappa V}(\lambda(0))$ is a curve connecting $\gamma(0)$ to $\gamma(T)$ and having length smaller than $\gamma$.

A particular case of the lemma occurs when $V=\vec{H}$ is the Hamiltonian vector field on $T^{*} M$ associated with the Hamiltonian $\lambda \mapsto\langle\lambda, X(\pi(\lambda))\rangle$, where $X$ is a smooth horizontal never-vanishing vector field on $M$. This means that if a solution of $\dot{\gamma}(t)=\sigma(t) X(\gamma(t))$ is a length-minimizer then $\sigma$ has constant sign.

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