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DYNAMICS OF ONE FOLD SYMMETRIC PATCHES FOR THE AGGREGATION EQUATION AND COLLAPSE TO SINGULAR MEASURE

TAOUFIK HMIDI AND DONG LI

Abstract. We are concerned with the dynamics of one fold symmetric patches for the two-dimensional aggregation equation associated to the Newtonian potential. We reformulate a suitable graph model and prove a local well-posedness result in sub-critical and critical spaces. The global existence is obtained only for small initial data using a weak damping property hidden in the velocity terms. This allows to analyze the concentration phenomenon of the aggregation patches near the blow up time. In particular, we prove that the patch collapses to a collection of disjoint segments and we provide a description of the singular measure through a careful study of the asymptotic behavior of the graph.

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1. Introduction

This paper is devoted to the study of the two-dimensional aggregation equation with the Newtonian potential:

\[
\begin{align*}
\partial_t \rho + \text{div}(v \rho) &= 0, \quad t \geq 0, \quad x \in \mathbb{R}^2, \\
v(t, x) &= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} \rho(t, y) dy, \\
\rho(0, x) &= \rho_0(x).
\end{align*}
\]

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This model with more general potential interactions, with or without dissipation, is used to explain some behavior in physics and population dynamics. As a matter of fact it appears in vortex densities in superconductors [1, 20, 25], material sciences [24, 31], cooperative controls and biological swarming [2, 11, 12, 23, 29, 30, 33], etc... During the last few decades, a lot of intensive research activity has been devoted to explore several mathematical and numerical aspects of this equation. It is known according to [4, 31] that classical solutions can be constructed for short time. They develop finite time singularity if and only if the initial data is strictly positive at some points and the blow up time is explicitly given by $T^* = \frac{1}{\max \rho_0}$. This follows from the equivalent form

$$\partial \rho + v \cdot \nabla \rho = \rho^2$$

which, written with Lagrangian coordinates, gives exactly a Riccati equation. Note that similarly to Yudovich result for Euler equations [35], weak unique solutions in $L^1 \cap L^\infty$ can be constructed following the same strategy, for more details see [4, 5, 6, 7, 8, 21, 22, 19, 26, 27]. Since $L^1$ norm is conserved at least at the formal level, then lot of efforts were done in order to extend the classical solutions beyond the first blowup time. In [32], Poupaud established the existence of global generalized solutions with defect measure when the initial data is a non negative bounded Radon measure. He also showed that when the second moment of the initial data is bounded then such solutions atomic part appears in finite time. This result is at some extent in contrast with what is established for Euler equations. Indeed, according to Delort’s result [18] global weak solutions without defect measure can be established when the initial vorticity is a non negative bounded Radon measure and the associated velocity has finite local energy. During the time those solutions do not develop atomic part contrary to the aggregation equation. This illustrates somehow the gap between both equations not only at the level of classical solutions but also for the weak solutions. The literature dealing with measure valued solutions for the aggregation equation with different potentials is very abundant and we we refer the reader to the papers [10, 13, 14, 15, 28] and the references therein.

Now we shall discuss another subject concerning the aggregation patches. Assume that the initial data takes the patch form

$$\rho_0 = 1_{D_0}$$

with $D_0$ a bounded domain, then solutions can be uniquely constructed up to the time $T^* = 1$ and one can check that

$$\rho(t) = \frac{1}{1-t} 1_{D_t} \quad \text{with} \quad (\partial_t + v \cdot \nabla) 1_{D_t} = 0.$$  

Note that $v$ is computed from $\rho$ through Biot-Savart law. To filter the time factor in the velocity field and find analogous equation to Euler equations it is more convenient to rescale the time as it was done in [4]. Indeed, set

$$\tau = -\ln(1-t), \quad u(\tau, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} 1_{D_\tau}(y) dy, \quad \tilde{D}_\tau = D_t$$

then we get

$$(\partial_\tau + u \cdot \nabla) 1_{\tilde{D}_\tau} = 0, \tilde{D}_0 = D_0.$$  

We observe that with this formulation, the blow up occurs at infinity and so the solutions do exist globally in time. To alleviate the notations we shall write this latter equation with the initial variables. Hence the vortex patch problem reduces to understand the evolution equation

$$
\begin{cases}
\partial_t \rho + v \cdot \nabla \rho = 0, \quad t \geq 0, \\
v(t, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{x - y}{|x - y|^2} dy, \\
\rho(0) = 1_{D_0}.
\end{cases}
$$

(1.2)
Let us point out that the area of the domain $D_t$ shrinks to zero exponentially, that is,

$$\forall t \geq 0, \| \rho(t) \|_{L^1} = e^{-t}|D_0|.$$  

The solution to this problem is global in time and takes the form $\rho(t) = 1_{D_t}, D_t = \psi(t,D_0)$ where $\psi$ denotes the flow associated to the velocity $v$. Similarly to Euler equations [3, 16], Bertozzi, Garnett, Laurent and Verdera proved in [9] the global in time persistence of the boundary regularity in Hölder spaces $C^{1+s}, s \in (0,1)$. However the asymptotic behavior of the patches for large time is still not well-understood despite some interesting numerical simulations giving some indications on the concentration dynamics. Notice first that the area of the patch shrinks to zero which entails that the associated domains will converge in Hausdorff distance to negligible sets. The geometric structure of such sets is not well explored and hereafter we will give two pedagogic and interesting simple examples illustrating the concentration, and one can find more details in [4]. The first example is the disc which shrinks to its center leading after normalization procedure to the convergence to Dirac mass. The second one is the ellipse patch which collapses to a segment along the big axis and the normalized patch converges weakly to Wigner’s semicircle law of density

$$x_1 \mapsto \frac{2\sqrt{x_0^2 - x_1^2}}{\pi x_0^2} 1_{[-x_0,x_0]}, x_0 = a - b.$$ 

It seems that the mechanisms governing the concentration are very complexe and related in part for some special class to the initial distribution of the local mass. Indeed, the numerical experiments implemented in [4] for some regular shapes indicate that generically the concentration is organized along a skeleton structure. The aim of this paper is to investigate this phenomenon and try to give a complete answer for special class of initial data where the concentration occurs along disjoint segments lying in the same line. More precisely, we will deal with a one-fold symmetric patch, and by rotation invariant we can suppose that it coincides with the real axis. We assume in addition that the boundary of the upper part is the graph of a slightly smooth function with small amplitude. Then we will show that we can track the dynamics of the graph globally in time and prove that the normalized solution converges weakly towards a probability measure supported in the union of disjoint segments lying in the real axis. The results will be formulated rigorously in Section 2. The paper is organized as follows. In next section we formulate the graph equation and state our main results. In Sections 3 and 4 we shall discuss basic tools that we use frequently throughout the paper. In Section 5 we prove the local well-posedness for the graph equation. The global existence with small initial data is proved in Section 6. The last section deals with the asymptotic behavior of the normalized density and its convergence towards a singular measure.

2. Graph reformulation and main results

The main purpose of this section is to describe the boundary motion of the patch associated to the equation (1.2) under suitable symmetry structure. One of the basic properties of the aggregation equation that we shall use in a crucial way concerns its group of symmetry which is much more rich than Euler equations. Actually and in addition to rotation and translation invariance, the aggregation equation is in fact invariant by reflexion. To check this property and without loss of generality we can look for the invariance with respect to the real axis. Set

$$X = (x, y) \in \mathbb{R}^2 \quad \text{and} \quad \bar{X} = (x, -y)$$

and introduce

$$\tilde{\rho}(t, X) = \rho(t, \bar{X}), \quad \tilde{v}(t, X) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{X - Y}{|X - Y|^2} \tilde{\rho}(t, Y) dY.$$ 

Using straightforward change of variables, it is quite easy to get

$$v(t, X) = \tilde{v}(t, \bar{X}), \quad [v \cdot \nabla \rho](t, X) = [\tilde{v} \cdot \nabla \tilde{\rho}](t, \bar{X}).$$
Therefore we find that \( \hat{\rho} \) satisfies also the aggregation equation
\[
\partial_t \hat{\rho} + \hat{v} \cdot \nabla \hat{\rho} = 0.
\]
Combining this property with the uniqueness of Yudovich solutions, it follows that if the initial data belongs to \( L^1 \cap L^\infty \) and admits an axis of symmetry then the solution remains invariant with respect to the same axis. In the framework of the vortex patches this result means that if the initial data is given by \( \rho_0 = 1_{D_0} \) and the domain \( D_0 \) is symmetric with respect to the real axis, then we can simply track its evolution by knowing the dynamics of its extended domain: we just remove the extra lines located on the real axis.

Along the current study, we shall focus on the domains \( D_0 \) such that the boundary part lying in the upper half-plane is described by the graph of a \( C^1 \) positive function \( f_0 : \mathbb{R} \to \mathbb{R}_+ \) with compact support. This is equivalent to say
\[
D_0 = \{ (x, y) \in \mathbb{R}^2; x \in \text{supp } f_0, -f_0(x) \leq y \leq f_0(x) \}.
\]
We point out that concretely we shall consider the evolution not of \( D_0 \) but of its extended set defined by
\[
\hat{D}_0 = \{ (x, y) \in \mathbb{R}^2; x \in \mathbb{R}, -f_0(x) \leq y \leq f_0(x) \}.
\]
This does not matter since the domain \( \hat{D}_t \) remains symmetric with the respect to the real axis and then we can simply track its evolution by knowing the dynamics of its extended domain: we just remove the extra lines located on the real axis.

One of the main objective of this paper is to follow the dynamics of the graph and investigate local and global well-posedness issues in different function spaces. In the next lines, we shall derive the evolution equation governing the motion of the initial graph \( f_0 \). Assume that in a short time interval \([0, T]\) the part of the boundary in the upper half-plane is described by the graph of a \( C^1 \)-function \( f_t : \mathbb{R} \to \mathbb{R}_+ \). This forces the points of the boundary of \( \partial D_0 \) located on the real axis to be cusp singularities. As a material point located at the boundary remains on the boundary then any parametrization \( s \mapsto \gamma_t(s) \) of the boundary should satisfy
\[
(\partial_t \gamma_t(s) - v(t, \gamma_t(s))) \cdot \hat{n}(\gamma_t(s)) = 0,
\]
with \( \hat{n}(\gamma_t) \) being a normal unit vector to the boundary at the point \( \gamma_t(s) \). Now take the parametrization in the graph form \( \gamma_t : x \mapsto (x, f(t, x)) \), then the preceding equation reduces to the nonlinear transport equation
\[
\begin{cases}
\partial_t f(t, x) + u_1(t, x) \partial_x f(t, x) = u_2(t, x), & t \geq 0, x \in \mathbb{R} \\
f(0, x) = f_0(x),
\end{cases}
\]
where \( (u_1, u_2)(t, x) \) is the velocity \( (v_1, v_2)(t, X) \) computed at the point \( X = (x, f(t, x)) \). Sometimes and along this paper we use the following notations
\[
f_t(x) = f(t, x) \quad \text{and} \quad f'(t, x) = \partial_x f(t, x).
\]
To reformulate the equation (2.1) in a closed form we shall recover the velocity components with respect to the graph parametrization. We start with the computation of \( v_1(X) \). Here and for the sake of simplicity we drop the time parameter from the graph and the domain of the patch. One
writes according to Fubini’s theorem

\[-2\pi v_1(X) = \int_{D} \frac{x - y_1}{|X - Y|^2} dY, \quad Y = (y_1, y_2)\]

\[-2\pi v_1(X) = \int_{R} (x - y_1) \int_{-f(y_1)}^{f(y_1)} \frac{dy_2}{(x - y_1)^2 + (f(x) - y_2)^2} dy_1.\]

Using the change of variables \(y_2 - f(x) = (x - y_1)Z\) we find

\[2\pi v_1(X) = \int_{R} \left\{ \arctan \left( \frac{f(y) - f(x)}{y - x} \right) + \arctan \left( \frac{f(y) + f(x)}{y - x} \right) \right\} dy = \int_{R} \left\{ \arctan \left( \frac{f(x + y) - f(x)}{y} \right) + \arctan \left( \frac{f(x + y) + f(x)}{y} \right) \right\} dy.\]

To compute \(v_2\) in terms of \(f\) we proceed as before and we find

\[-2\pi v_2(X) = \int_{D} \frac{f(x) - y_2}{|X - Y|^2} dA(Y) = \int_{R} \int_{-f(y_1)}^{f(y_1)} \frac{f(x) - y_2}{(x - y_1)^2 + (f(x) - y_2)^2} dy_2 dy_1.\]

Therefore we obtain the following expression

\[4\pi v_2(x, f(x)) = \int_{R} \log \left( \frac{y^2 + (f(x + y) - f(x))^2}{y^2 + (f(x + y) + f(x))^2} \right) dy.\]

With the notation adopted before for \((u_1, u_2)\) we finally get the formulas

\[u_1(t, x) = \frac{1}{2\pi} \int_{R} \left\{ \arctan \left( \frac{f_t(x + y) - f_t(x)}{y} \right) + \arctan \left( \frac{f_t(x + y) + f_t(x)}{y} \right) \right\} dy\]

\[u_2(t, x) = \frac{1}{4\pi} \int_{R} \log \left( \frac{y^2 + (f_t(x + y) - f_t(x))^2}{y^2 + (f_t(x + y) + f_t(x))^2} \right) dy.\] (2.2)

We emphasize that for the coherence of the model the graph equation (2.1) is supplemented with the initial condition \(f_0(x) \geq 0\). According to Proposition 6.2, the positivity is preserved for enough smooth solutions. Furthermore, and once again according to this proposition we have a maximum principle estimate:

\[\forall t \geq 0, \forall x \in R, \quad 0 \leq f(t, x) \leq \|f_0\|_{L^\infty}.\]

Notice that the model remains meaningful even though the function \(f_t\) changes the sign. In this case the geometric domain of the patch is simply obtained by looking to the region delimited by the curve of \(f_t\) and its symmetric with respect to the real axis. This is also equivalent to deal with positive function \(f_t\) but its graph will be less regular and belongs only to the Lipschitz class. Another essential element that will be analyzed later in Proposition 6.2 concerns the support of the
solutions which remains confined through the time. More precisely, if \( \text{supp} f_0 \subset [a, b] \) with \( a < b \) then provided that the graph exists for \( t \in [0, T] \) one has

\[
\text{supp} f(t) \subset [a, b].
\]

This follows from the fact that the flow associated to the horizontal velocity \( u_1 \) is contractive on the boundary. It is not clear whether global weak solutions satisfying the maximum principle can be constructed. However, to deal with classical solutions one should control higher regularity of the graph and it seems from the transport structure of the equation that the optimal scaling for local well-posedness theory is Lipschitz class. Denote by \( g(t, x) = \partial_x f(t, x) \) the slope of the graph then it is quite obvious from (2.1) that

\[
(2.3) \quad \partial_t g + u_1 \partial_x g = -\partial_x u_1 g + \partial_x u_2.
\]

For the computation of the source term we proceed in a classical way using the differentiation under the integral sign and we get successively,

\[
2\pi \partial_x u_1(x) = \text{p.v.} \int_{\mathbb{R}} \frac{f'(x + y) - f'(x)}{y^2 + (f(x + y) - f(x))^2} y dy
\]

\[
+ \text{p.v.} \int_{\mathbb{R}} \frac{f'(x + y) + f'(x)}{y^2 + (f(x + y) + f(x))^2} y dy
\]

(2.4)

and

\[
2\pi \partial_x u_2(x) = \text{p.v.} \int_{\mathbb{R}} \frac{(f(x + y) - f(x))(f'(x + y) - f'(x))}{y^2 + (f(x + y) - f(x))^2} dy
\]

\[- \text{p.v.} \int_{\mathbb{R}} \frac{(f(x + y) + f(x))(f'(x + y) + f'(x))}{y^2 + (f(x + y) + f(x))^2} dy,
\]

(2.5)

where the notation p.v. is the Cauchy principal value. It is worthy to point out that the first two integrals appearing in the right hand side of the expressions of \( \partial_x u_1 \) and \( \partial_x u_2 \) are in fact connected to Cauchy operator associated to the curve \( f \) defined in (5.1). This operator is well-studied in the literature and some details will be given later in the Section 5. Next, we shall check that the integrals appearing in the right-hand side of the preceding formulas can actually be restricted over a compact set related to the support of \( f \). Let \([-M, M]\) be a symmetric segment containing the set \( K_0 - K_0 \), with \( K_0 \) being the convex hull of the support of \( f_0 \) denoted by \( \text{supp} f_0 \). It is clear that the support of \( \partial_x u_1 f' \) is contained in \( K_0 \) and thus for \( x \in K_0 \) one has

\[
\text{p.v.} \int_{\mathbb{R}} \frac{f'(x + y) - f'(x)}{y^2 + (f(x + y) - f(x))^2} y dy = \text{p.v.} \int_{-M}^{M} \frac{f'(x + y) - f'(x)}{y^2 + (f(x + y) - f(x))^2} y dy.
\]
Consequently, we obtain for \( x \in \mathbb{R} \),

\[
2\pi f'(x)\partial_x u_1(x) = \text{p.v.} \int_{-M}^{M} \frac{f'(x+y) - f'(x)}{y^2 + (f(x+y) - f(x))^2} y dy
\]

\[
- \text{p.v.} \int_{-M}^{M} \frac{f'(x+y) + f'(x)}{y^2 + (f(x+y) + f(x))^2} y dy.
\]

Coming back to the integral representation defining \( \partial_x u_2 \) one can see, using a cancellation between both integrals, that the support of \( \partial_x u_2 \) is contained in \( K_0 \). Furthermore, for \( x \in K_0 \) one may write,

\[
2\pi \partial_x u_2(x) = \text{p.v.} \int_{-M}^{M} \frac{(f(x+y) - f(x))(f'(x+y) - f'(x))}{y^2 + (f(x+y) - f(x))^2} dy
\]

\[
- \text{p.v.} \int_{-M}^{M} \frac{(f(x+y) + f(x))(f'(x+y) + f'(x))}{y^2 + (f(x+y) + f(x))^2} dy.
\]

Gathering the preceding identities we deduce that

\[
(2.6) \quad 2\pi\left(-\partial_x u_1 f'(x) + \partial_x u_2\right) = F(x) - G(x)
\]

with

\[
F(x) \equiv \text{p.v.} \int_{-M}^{M} \left[\frac{f(x+y) - f(x) - y f'(x)}{y^2 + (f(x+y) - f(x))^2} \right] \left( f'(x+y) - f'(x) \right) dy
\]

and

\[
G(x) \equiv \text{p.v.} \int_{-M}^{M} \left[\frac{f(x+y) + f(x) + y f'(x)}{y^2 + (f(x+y) + f(x))^2} \right] \left( f'(x+y) + f'(x) \right) dy.
\]

Keeping in mind, and this will be useful at some points, that the foregoing integrals can be also extended to the full real axis. Sometimes and in order to reduce the size of the integral representation, we use the notations

\[
(2.7) \quad \Delta_y^\pm f(x) = f(x+y) \pm f(x).
\]

Thus \( F \) and \( G \) take the form

\[
(2.8) \quad F(x) = \text{p.v.} \int_{-M}^{M} \left[\frac{\Delta_y^- f(x) - y f'(x)}{y^2 + (\Delta_y^- f(x))^2} \right] \Delta_y^- f'(x) dy
\]

and

\[
(2.9) \quad G(x) = \text{p.v.} \int_{-M}^{M} \left[\frac{\Delta_y^+ f(x) + y f'(x)}{y^2 + (\Delta_y^+ f(x))^2} \right] \Delta_y^+ f'(x) dy.
\]

The first main first result of this paper is devoted to the local well-posedness issue. We shall discuss two results related to sub-critical and critical regularities. Denote by \( X \) one of the following spaces: Hölder spaces \( C^s(\mathbb{R}) \) with \( s \in (0, 1) \) or Dini space \( C^r(\mathbb{R}) \). For more details about classical properties of these spaces we refer the reader to the Section 4.
Theorem 2.1. Let $f_0$ be a positive compactly supported function such that $f_0' \in X$. Then, the following results hold true.

1. The equation (2.1) admits a unique local solution such that $f' \in L^\infty([0,T],X)$, where the time existence $T$ is related to the norm $\|f_0'\|_X$ and the size of the support of $f_0$. In addition, the solution satisfies the maximum principle
   \[ \forall t \in [0,T], \quad \|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty}. \]

2. There exists a constant $\varepsilon > 0$ depending only on the size of the support of $f_0$ such that if
   \[ \|f_0'\|_{C^\alpha} < \varepsilon \]
   then the equation (2.1) admits a unique global solution $f' \in L^\infty(\mathbb{R}_+;C^\alpha(\mathbb{R}))$. Moreover
   \[ \forall t \geq 0, \quad \|\partial_x f(t)\|_{L^\infty} \leq C_0 e^{-t} \]
   with $C_0$ a constant depending only on $\|f_0'\|_{C^\alpha}$.

Before outlining the strategy of the proofs some comments are in order.

Remarks.  (1) The global existence result is only proved for the sub-critical case. The critical case is more delicate to handle due to the lack of strong damping which is only proved in the sub-critical case.

(2) From Sobolev embeedings we deduce according to the assumption on $f_0$ listed in Theorem 2.1 that $f_0$ belongs to the space $\in C^1_c(\mathbb{R})$ of compactly supported $C^1$ functions.

(3) The maximum principle holds true globally in time, however it is note clear whether some suitable weak global solutions could be constructed in this setting.

Now we shall give some details about the proofs. First we establish local-in-time a priori estimates based on the transport structure of the equation combined with some refined studies on modified curved Cauchy operators implemented in Section 5 and essentially based on standard arguments from singular integrals. The construction of the solutions done in the subsection 6.3 in slightly intricate than the usual schemes used for transport equations. This is due to the fact that the establishment of the a priori estimates is not only purely energetic. First, at some levels we use some nonlinear rigidity of the equation like in Theorem 2.1-(3) where the factor $f'$ behind the operator should be the derivative of the function $f$ that appears inside the operator. Second, we use at some point the fact that the support is confined in time. Last we use at different steps the positivity of the solution. Hence it seems quite difficult to find a linear scheme taking into account of those constraints. The idea is to implement a nonlinear scheme with two regularizing parameters $\varepsilon$ and $n$. The first one is used to smooth out the singularity of the kernel and the second to smooth the solution through a nonlinear scheme. We first establish that one has uniform a priori estimates on $n$ but on some small interval depending on $\varepsilon$. We are also able to pass to the limit on $n$ and get a solution for a modified nonlinear problem. Second we check that the a priori estimates still be valid uniformly on $\varepsilon$. This ensures that the time existence can be in fact pushed up to the time given by the a priori estimates obtained for the initial equation (2.1). As a consequence we get a uniform time existence with respect to $\varepsilon$ and finally we establish the convergence towards a solution of the initial value problem using standard compactness arguments.

The global existence for small initial data requires much more careful analysis because there is no apparent dissipation or damping mechanisms in the equation. Moreover the estimates of the source term $G$ contains some linear parts as it is stated in Proposition 6.1. The basic ingredient to get rid of those linear parts is to use a hidden weak damping effect in $G$ that can just absorb the growth of the linear part. We do not know if the damping proved for lower regularity still happen in the resolution space. As to the nonlinear terms, they are always associated with some subcritical
norms and thus using an interpolation argument with the exponential decay of the $L^1$ norm we get a global in time control that leads to the global existence.

The second result that we shall discuss deals with the asymptotic behavior of the solutions to (1.2) and (2.1). We shall study the collapse of the support to a collection of disjoint segments located at the axis of symmetry. Another interesting issue that will covered by this discussion concerns the characterization of the limit behavior of the probability measure

\begin{equation}
(2.11)\quad dP_t \triangleq e^t \frac{1}{|D_0|} dA,
\end{equation}

with $dA$ being Lebesgue measure and $|D_0|$ denotes the Lebesgue measure of $D_0$. Our result reads as follows.

**Theorem 2.2.** Let $f_0$ be a positive compactly supported function such that $f_0' \in C^s(\mathbb{R})$, with $s \in (0,1)$. Assume that $\text{supp}f_0$ is the union of $n$-disjoint segments and satisfying the smallness condition (2.10). Then there exists a compact set $D_\infty \subset \mathbb{R}$ composed of exactly of $n$-disjoint segments and a constant $C > 0$ such that

$$
\forall t \geq 0, \quad d_H(D_t, D_\infty) \leq Ce^{-t}, \quad |D_\infty| \geq \frac{1}{2}|D_0|,
$$

with $d_H$ being the Hausdorff distance and $|D_\infty|$ is the one-dimensional Lebesgue measure of $D_\infty$. In addition, the probability measures $\{dP_t\}_{t \geq 0}$ defined in (2.11) converges weakly as $t$ goes to $+\infty$ to the probability measure

$$
dP_\infty := \Phi \delta_{D_\infty \otimes \{0\}},
$$

with $\Phi$ being a compactly supported function in $D_\infty$ belonging to $C^\alpha(\mathbb{R})$, for any $\alpha \in (0,1)$ and can be expressed in the form

\begin{equation}
(2.12)\quad \Phi(x) = \frac{f_0(\psi_\infty^{-1}(x))}{\|f_0\|_{L^1}} e^{g(x)},
\end{equation}

with $g$ a function that can be implicitly recovered from the full dynamics of solution $\{f_t, t \geq 0\}$ and

$$
\psi_\infty = \lim_{t \to +\infty} \psi(t).
$$

Note that $\psi(t)$ is the one-dimensional flow associated to $u_1$ defined in (6.26) and

$$
D_t = \left\{ (x,y), x \in \text{supp} f_t; -f_t(x) \leq y \leq f_t(x) \right\}.
$$

**Remark 2.3.** The regularity of the profile $\Phi$ might be improved and we expect that $\Phi$ keeps the same regularity as the graph.

The proof of the collapse of the support to a disjoint union of segments can be easily derived from the formula (2.12) which ensures that the support of the limit measure is exactly the image of the support of $f_0$ by the limit flow $\psi_\infty$ which is a homeomorphism of the real axis. To get the convergence with the Hausdorff distance we just use the exponential damping of the amplitude of the curve. As to the characterization of the limit measure it is based on the exponential decay decay of the amplitude of graph combined with the scattering as $t$ goes to infinity of the normalized solution $e^t f(t)$. In fact, we prove that the density is nothing but the formal quantity

$$
\Phi(x) = 2 \lim_{t \to +\infty} e^t f(t, x)
$$

whose existence is obtained using the transport structure of the equation through the characteristic method combined with the damping effects of the nonlinear source terms.
In this short section we shall discuss a simple result dealing with the role of symmetry in the structure of the limit shape $D_\infty$. Roughly speaking, we shall prove that thin initial domains along their axis of symmetry generate concentration to segments. Notice that

$$D_\infty \triangleq \left\{ \lim_{t \to +\infty} \psi(t, x), x \in D_0 \right\}$$

where $\psi$ is the flow associated to the velocity $v$ and defined through the ODE.

\begin{equation}
\begin{aligned}
&\partial_t \psi(t, x) = v(t, \psi(t, x)), t \geq 0, x \in \mathbb{R}^2, \\
&\psi(0, x) = x 
\end{aligned}
\end{equation}

The existence of the set $D_\infty$ will be proved below. We intend to prove the following.

**Proposition 3.1.** The following assertions hold.

1. If $D_0$ is a bounded domain of $\mathbb{R}^2$, then for any $x \in \mathbb{R}^2$ the quantity $\lim_{t \to +\infty} \psi(t, x)$ exists.
2. If $D_0$ is a simply connected bounded domain symmetric with respect to an axis $\Delta$. Denote by $d_0 = \text{Length}(D_0 \cap \Delta)$. There exists an absolute constant $C$ such that if $d_0 > C|D_0|^{\frac{1}{2}}$

then the shape $D_\infty$ contains an interval of the size $d_0 - C|D_0|^{\frac{1}{2}}$.

**Proof.** (1) Integrating in time the flow equation (3.1) yields

$$\psi(t, x) = x + \int_0^t v(\tau, \psi(\tau, x))d\tau.$$ 

Now observe that pointwisely

$$|v(t, x)| \leq \frac{1}{2\pi} \frac{1}{|\cdot|^{\frac{1}{2}}} |\rho(t)| \cdot |x|.$$ 

Thus interpolation inequalities combined with (1.3) lead to

\begin{equation}
\begin{aligned}
\|v(t)\|_{L^\infty} &\leq C\|\rho(t)\|_{L^1}^{\frac{1}{2}} \|\rho(t)\|_{L^\infty}^{\frac{1}{2}} \\
&\leq Ce^{-\frac{t}{2}}|D_0|^{\frac{1}{2}},
\end{aligned}
\end{equation}

with $C$ an absolute constant. This implies that the integral $\int_0^{+\infty} v(\tau, \psi(\tau, x))d\tau$ converges absolutely and therefore $\lim_{t \to +\infty} \psi(t, x)$ exists in $\mathbb{R}^2$. This allows to define the limit shape $D_\infty$ as follows:

$$D_\infty = \left\{ \lim_{t \to +\infty} \psi(t, x), \forall x \in D_0 \right\}.$$

(2) Without loss of generality we will suppose that the straight line $\Delta$ coincides with the real axis. Since $D$ is simply connected bounded domain, then there exist two different points $X_0^-, X_0^+ \in \mathbb{R}$ such that

$$\overline{D_0} \cap \Delta = [X_0^-, X_0^+]$$

Then it is clear that $\text{Length}(\overline{D_0} \cap \Delta) = X_0^+ - X_0^- := d_0$. By assumption $D_0$ is symmetric with respect to $\Delta$ then the domain $D_t$ remains also symmetric with respect to the same axis and the points $X_0^\pm$ move necessary along this axis. Denote by

$$X^\pm(t) = \psi(t, X_0^\pm)$$

then as the flow is an homeomorphism then

$$\overline{D_t} \cap \Delta = [X^-(t), X^+(t)].$$
Now we wish to follow the evolution of the distance \( d(t) := X^+(t) - X^-(t) \) and find a sufficient condition such that this distance remains away from zero up to infinity. Notice from the first point that \( \lim_{t \to +\infty} d(t) \) exists and equals to some positive number \( d_\infty \). From the triangular inequality, one easily gets that

\[
d(t) \geq d_0 - 2 \int_0^t \|v(\tau)\|_{L^\infty} d\tau.
\]

Inequality (3.2) ensures that

\[
d(t) \geq d_0 - C |D_0|^\frac{1}{2}
\]

and therefore \( d_\infty \geq d_0 - C |D_0|^\frac{1}{2} \). Consequently, if \( d_0 > C |D_0|^\frac{1}{2} \) then the points \( \{X^+(t)\} \) do not collide up to infinity and thus the set \( D_\infty \) contains a non trivial interval as claimed.

4. Basic properties of Dini and Hölder spaces

In this section we set up some function spaces that we shall use and review some of their important properties. Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous function, we define its modulus of continuity \( \omega_f : \mathbb{R}_+ \to \mathbb{R}_+ \) by

\[
\omega_f(r) = \sup_{|x-y| \leq r} |f(x) - f(y)|.
\]

This is a nondecreasing function satisfying \( \omega_f(0) = 0 \) and sub-additive, that is for \( r_1, r_2 \geq 0 \) we have

\[
\omega_f(r_1 + r_2) \leq \omega_f(r_1) + \omega_f(r_2).
\]

Now we intend to recall Dini and Hölder spaces. Dini space denoted by \( C^\star(\mathbb{R}) \) is the set of continuous bounded functions \( f \) such that

\[
\|f\|_{L^\infty} + \|f\|_D < \infty \quad \text{with} \quad \|f\|_D = \int_0^1 \frac{\omega_f(r)}{r} dr.
\]

Another space that we frequently use throughout this paper is Hölder space. Let \( s \in (0, 1) \) we denote by \( C^s(\mathbb{R}) \) the set of functions \( f : \mathbb{R} \to \mathbb{R} \) such that

\[
\|f\|_{L^\infty} + \|f\|_s < \infty \quad \text{with} \quad \|f\|_s = \sup_{0 < r < 1} \frac{\omega_f(r)}{r^s}.
\]

Let \( K \) be a compact set of \( \mathbb{R} \), we define \( C^\star_K \) as the subspace of \( C^\star(\mathbb{R}) \) whose elements are supported in \( K \). Note that \( C^\star_K \to L^\infty(\mathbb{R}) \) which means that a constant \( C \) depending only on the diameter of the compact \( K \) exists such that

\[
\forall f \in C^\star_K, \quad \|f\|_{L^\infty} \leq C \|f\|_D.
\]

This follows easily from the observation

\[
\forall r \in (0, 1/2], \quad \omega(r) \ln 2 \leq \|f\|_D.
\]

From (4.2) we deduce that for any \( A \geq 1 \)

\[
\int_0^A \frac{\omega_f(r)}{r} dr \leq \|f\|_D + 2 \|f\|_{L^\infty} \ln A
\]

\[
\leq C \|f\|_D (1 + \ln A).
\]

(4.3)

Coming back to the definition of Dini semi-norm one deduces the law products: for \( f, g \in C^\star_K \)

\[
\|fg\|_D \leq \|f\|_{L^\infty} \|g\|_D + \|g\|_{L^\infty} \|f\|_D
\]

and

\[
\|fg\|_D \leq C \|f\|_D \|g\|_D.
\]

(4.4)
Another useful space is \( C^s_K \) which is the subspace of \( C^s(\mathbb{R}) \) whose functions are supported on the compact \( K \). It is quite obvious that
\[
C^s_K \hookrightarrow C^{s+\varepsilon}_K \hookrightarrow L^\infty.
\]

We point out that all these spaces are complete. Another property which will be very useful is the following composition law. If \( f \in C^s(\mathbb{R}) \) with \( 0 < s < 1 \) and \( \psi : \mathbb{R} \rightarrow \mathbb{R} \) a Lipschitz function then \( f \circ \psi \in C^s(\mathbb{R}) \) and
\[
\| f \circ \psi \|_s \leq \| f \|_s + 2 \| f \|_{L^\infty} \| \nabla \psi \|_{L^\infty}.
\]

It is worth pointing out that in the case of Dini space \( C^*(\mathbb{R}) \) we get more precise estimate of logarithmic type,
\[
\| f \circ \psi \|_D \leq C(\| f \|_D + \| f \|_{L^\infty}) \left( 1 + \ln_+ (\| \nabla \psi \|_{L^\infty}) \right),
\]
with the notation
\[
\ln_+ x \triangleq \begin{cases} \ln x, & \text{if } x \geq 1 \\ 0, & \text{otherwise.} \end{cases}
\]

Another estimate of great interest is the following law product,
\[
\| fg \|_s \leq \| f \|_{L^\infty} \| g \|_s + \| g \|_{L^\infty} \| f \|_s.
\]

In the next task we will be concerned with a pointwise estimate connecting a positive smooth function to its derivative and explore how this property is affected by the regularity. This kind of property will be required in Section 5 in studying Cauchy operators with special forms.

**Lemma 4.1.** Let \( K \) be a compact set of \( \mathbb{R} \) and \( f' \in C^s(\mathbb{R}) \). Then we have,
\[
\forall x \in \mathbb{R}, \quad |f'(x)| \leq C \frac{\| f' \|_D + \| f' \|_{L^\infty}}{1 + \ln_+ \left( \frac{\| f' \|_{L^\infty}}{f(x)} \right)}.
\]

A weak version of this inequality is
\[
\forall x \in \mathbb{R}, \quad |f'(x)| \leq C \frac{\| f' \|_D + \| f' \|_{L^\infty}}{1 + \ln_+ \left( \frac{\| f' \|_{L^\infty}}{f(x)} \right)}
\]

with \( C \) an absolute constant. If in addition \( f' \in C^s(\mathbb{R}) \) with \( 0 < s < 1 \), then
\[
\forall x \in \mathbb{R}, \quad |f'(x)| \leq C \| f' \|_{L^\infty} \frac{1}{s} [f(x)]^{1+s}
\]

and the constant \( C \) depends only on \( s \).

**Proof.** Let \( x \) be a given point, without any loss of generality one can assume that \( f'(x) \geq 0 \). Now let \( h \in [0, 1] \) then using the mean value theorem, there exists \( c_h \in [x - h, x] \) such that
\[
f(x - h) = f(x) - h f'(c_h) \\
= f(x) - h f'(x) - h [f'(c_h) - f'(x)] \\
\leq f(x) - h f'(x) + h \omega_f(h).
\]

From the positivity of the function \( f \) we deduce that for any \( h \in [0, 1] \) one gets
\[
f(x) - h f'(x) + h \omega_f(h) \geq 0.
\]

Then dividing by \( h^2 \) and integrating in \( h \) between \( \varepsilon \) and \( 1 \), with \( \varepsilon \in (0, 1] \), we get
\[
f(x) \frac{1}{\varepsilon} + f'(x) \ln \varepsilon + \| f' \|_D \geq 0.
\]
Multiplying by \( \varepsilon \) we obtain
\[
(4.9) \quad \forall \varepsilon \in (0, 1), \quad f(x) + f'(x)\varepsilon \ln \varepsilon + \|f'||D\varepsilon \geq 0.
\]
By studying the variation with respect to \( \varepsilon \) we find that the suitable value of \( \varepsilon \) is given by
\[
\ln \varepsilon = -1 - \frac{\|f'||D}{f(x)}.
\]
Inserting this choice into (4.9) we find that
\[
\varepsilon f'(x) \leq f(x)
\]
that is
\[
e^{-1-\frac{\|f'||D}{f(x)}} f'(x) \leq f(x).
\]
From the inequality \( te^{-t} \leq e^{-1} \) we deduce that
\[
e^{-1} \geq \frac{\|f'||D}{f(x)} e^{-\frac{\|f'||D}{f(x)}}
\]
which implies in turn that
\[
\frac{1}{e^{-1}} \geq \frac{\|f'||D}{f(x)} \geq e^{-\frac{\|f'||D}{f(x)}} \|f'||D.
\]
Consequently we get
\[
e^{-2\frac{\|f'||D}{f(x)}} \|f'||D \leq f(x).
\]
Thus when \( \frac{f(x)}{\|f'||D} > 1 \) this estimate does not give any useful information and then we simply write
\[
f'(x) \leq \|f'||L^\infty.
\]
However for \( \frac{f(x)}{\|f'||D} < 1 \) we get
\[
f'(x) \leq C \frac{\|f'||D}{1 + \ln_+(\frac{\|f'||D}{f(x)})}.
\]
From which we deduce that
\[
f'(x) \leq C \frac{\|f'||D(1 + \ln_+(\frac{1}{f(x)})}{1 + \ln_+(\frac{1}{f(x)})}.
\]
Indeed, one may use the estimate
\[
\forall x > 0, \quad \frac{1 + \ln_+(1/x)}{1 + \ln_+(a/x)} \leq 1 + \ln_+(1/a),
\]
which can be checked easily by studying the variation of the fractional function.
Now let us move to the proof when \( f' \) is assumed to belong to Hölder space \( C^s \), with \( s \in (0, 1) \).
Following the same proof as before one deduces that under the assumption \( f'(x) \geq 0 \) one obtains for any \( h \in \mathbb{R}_+ \)
\[
f(x) - h f'(x) + h^{1+s} \|f'||s \geq 0.
\]
By studying the variation of this function with respect to \( h \) we find that the best choice of \( h \) is given by
\[
h^s = \frac{f'(x)}{(1 + s)\|f'||s},
\]
which implies the desired result, that is,
\[
f'(x) \leq C \|f'||_s^{\frac{1}{1+s}} [f(x)]^{\frac{1}{1+s}}.
\]
The proof is now achieved. \( \square \)
5. Modified curved Cauchy operators

This section is devoted to the study of some variants of Cauchy operators which are closely connected to the operators arising in (2.4) and (2.5). Let us first recall the classical Cauchy operator associated to the graph of a Lipschitz function $f : \mathbb{R} \to \mathbb{R}$,

$$C_f g(x) = \int_{\mathbb{R}} \frac{g(x + y) - g(x)}{y + i(f(x + y) - f(x))} \, dy.$$  \hspace{1cm} (5.1)

which is well-defined at least for smooth function $g$. According to a famous theorem of Coifman, McIntosh, and Meyer [17], this operator can be extended as a bounded operator from $L^p$ to $L^p$ for $1 < p < \infty$. By adapting the proof of the paper of Wittmann [34], this operator can also be extended continuously from $C^s_K$ to $C^s(\mathbb{R})$ for $0 < s < 1$, provided that $f$ belongs to $C^{1+s}(\mathbb{R})$. However this operator fails to be extended continuously from Dini space $C^*_K$ to itself as it can be checked from Hilbert transform. The structure of the operators that we have to deal with, as one may observe from the expression of $F$ following (2.6), is slightly different from the Cauchy operators. It can be associated to the truncated bilinear Cauchy operator defined as follows: for given $M > 0$, $\theta \in [0, 1]$,

$$C^\theta_f(g, h)(x) = \int_{-M}^{M} \frac{(g(x + \theta y) - g(x))(h(x + y) - h(x))}{y + i(f(x + y) - f(x))} \, dy.$$  \hspace{1cm} (5.2)

The real and imaginary parts of this operator are given respectively by

$$C^\theta_f^{\Re}(g, h)(x) = \int_{-M}^{M} \frac{y(g(x + \theta y) - g(x))(h(x + y) - h(x))}{y^2 + [f(x + y) - f(x)]^2} \, dy$$

and

$$C^\theta_f^{\Im}(g, h)(x) = -\int_{-M}^{M} \frac{(f(x + y) - f(x))(g(x + \theta y) - g(x))(h(x + y) - h(x))}{y^2 + [f(x + y) - f(x)]^2} \, dy.$$  

In what follows we denote by $X$ one of the spaces $C^s_K$, with $0 < s > 1$ or $C^*_K$. The result that we shall discuss deals with the continuity of the bilinear operator on the spaces $X$. This could have been discussed in the literature and as we need to control the continuity constant we shall give a detailed proof.

**Proposition 5.1.** Let $K$ be a compact set of $\mathbb{R}$ and $f$ be a compactly supported function such that $f' \in X$. Then the following assertions hold true.

1. The bilinear operator $C^\theta_f : X \times X \to X$ is well-defined and continuous. More precisely, there exits a constant $C$ independent of $\theta$ such that for any $g, h \in X$

$$\|C^\theta_f^{\Re}(g, h)\|_X \leq C \left(1 + \|f'\|_{L^\infty} \|f'\|_X\right) \left(\|g\|_D \|h\|_X + \|h\|_D \|g\|_X\right)$$

and

$$\|C^\theta_f^{\Im}(g, h)\|_X \leq C \|f'\|_X \left(1 + \|f'\|_{L^\infty}^2\right) \left(\|g\|_D \|h\|_X + \|g\|_X \|h\|_D\right).$$

**Proof.** We shall first establish the result for the real part operator given by (5.2). First we note that one may rewrite the expression using the notation (2.7) as follows

$$C^\theta_f^{\Re}(g, h)(x) = \int_{-M}^{M} \frac{y \Delta \theta y g(x) \Delta y h(x)}{y^2 + (\Delta y f(x))^2} \, dy$$
where we simply replace the notation $\Delta_y^-$ by $\Delta_y$. Using the law products (4.4) and (4.8) one obtains

$$
\|C^\theta,R_f(g, h)\|_{L^\infty} \leq \int_{-M}^{M} \left[ \|\Delta_{\theta y}g\|_{L^\infty}\|\Delta_yh\|_{L^\infty} + \|\Delta_{\theta y}g\|_{L^\infty}\|\Delta_yh\|_{L^\infty} \right] \frac{1}{y^2 + (\Delta_yf)^2} \, dy.
$$

Using once again those law products it comes

$$
\|\Delta_{\theta y}g\|_{L^\infty}\|\Delta_yh\|_{L^\infty} \leq \omega_y(|y|)\|\Delta_yh\|_{L^\infty} + 2\|g\|\|\Delta_yh\|_{L^\infty},
$$

where we have used that for $\theta \in [0, 1], y \in \mathbb{R}$

$$
(5.3) \quad \|\Delta_yh\|_{L^\infty} \leq 2\|h\|_{L^\infty}, \quad \|\Delta_{\theta y}g\|_{L^\infty} \leq \omega_y(|y|).
$$

Consequently

$$
(5.4) \quad \int_{-M}^{M} \|\Delta_{\theta y}g\|_{L^\infty}\|\Delta_yh\|_{L^\infty} \frac{dy}{|y|} \leq C(\|g\|_{L^\infty}\|h\|_{L^\infty} + \|h\|_{L^\infty}\|g\|_{L^\infty}).
$$

By the definition it is quite easy to check that for any function $\varphi \in X \cap L^\infty(\mathbb{R})$

$$
\left\| \frac{1}{y^2 + \varphi^2} \right\|_{L^\infty} \leq \frac{2\|\varphi\|_{L^\infty}}{y^4} \|\varphi\|_{L^\infty}.
$$

Hence we get

$$
(5.5) \quad \left\| \frac{1}{y^2 + (\Delta_yf)^2} \right\|_{L^\infty} \leq \frac{2\|\Delta_yf\|_{L^\infty}}{y^4} \|\Delta_yf\|_{L^\infty} \leq C y^{-2}\|f\|_{L^\infty}\|f\|_{L^\infty}.
$$

where we have used the inequalities

$$
\|\Delta_yf\|_{L^\infty} \leq |y|\|f\|_{L^\infty} \quad \text{and} \quad \omega_{\Delta_yf}(r) \leq |y|\omega_f(r).
$$

Therefore we get in view of (5.3),

$$
\int_{-M}^{M} \left[ \left\| \frac{1}{y^2 + (\Delta_yf)^2} \right\|_{L^\infty} \right] dy \leq C\|f\|_{L^\infty}\|f\|_{L^\infty} \int_{-M}^{M} \frac{\omega_y(|y|)}{|y|} \, dy \leq C\|f\|_{L^\infty}\|f\|_{L^\infty} \int_{-M}^{M} \frac{\omega_y(|y|)}{|y|} \, dy.
$$

Combining this last estimate with (5.4) we find that

$$
\|C^\theta,R_f(g, h)\|_{L^\infty} \leq C\|g\|_{L^\infty}\|h\|_{L^\infty} + \|h\|_{L^\infty}\|g\|_{L^\infty} + \|f\|_{L^\infty}\|f\|_{L^\infty}\|h\|_{L^\infty}\|g\|_{L^\infty}.
$$

To deduce the result it is enough to use (4.5).

We are left with the task of estimating the imaginary part which takes the form

$$
C^\theta,R_f(g, h)(x) = \int_{-M}^{M} \frac{\Delta_yf(x)\Delta_{\theta y}g(x)\Delta_yh(x)}{y^2 + (\Delta_yf(x))^2} \, dy.
$$

Note that we have dropped the sign minus before the integral which of course has no consequence on the computations. Using Taylor formula we get

$$
\Delta_yf(x) = y \int_{0}^{1} f'(x + \tau y) \, d\tau.
$$
and thus

\[ C_{f}^{\alpha,\beta}(g,h)(x) = \int_{-M}^{M} \int_{0}^{1} y f'(x + \tau y) \Delta_{\theta y} g \Delta_{\theta y} h(x) \frac{dyd\tau}{y^2 + (\Delta_y f(x))^2}. \]

It suffices to reproduce the preceding computations using in particular the estimates

\[ \| f' (\cdot + \tau y) \Delta_{\theta y} g \Delta_{\theta y} h \|_{L^\infty} \leq \| f' \|_{L^\infty} \| h \|_{L^\infty} \omega_{\theta}(|y|) \]

and

\[ \| f' (\cdot + \tau y) \Delta_{\theta y} g \Delta_{\theta y} h \|_X \leq \| f' \|_{L^\infty} \| \Delta_{\theta y} g \Delta_{\theta y} h \|_X + \| f' \|_X \| \Delta_{\theta y} g \Delta_{\theta y} h \|_{L^\infty} \]

\[ \leq \| f' \|_{L^\infty} \left( \omega_{\theta}(|y|) \| h \|_X + \| g \|_X \omega_{\theta}(|y|) \right) + 2 \| f' \|_X \| g \|_{L^\infty} \omega_{\theta h}(|y|) \]

This implies according to Sobolev embedding (4.5)

\[ \int_{-M}^{M} \int_{0}^{1} \| f' (\cdot + \tau y) \Delta_{\theta y} g \Delta_{\theta y} h \|_X \frac{dyd\tau}{|y|} \leq C \| f' \|_X \left( \| g \|_D \| h \|_X + \| g \|_X \| h \|_D \right). \]

Using (5.5) one may easily get

\[ \int_{-M}^{M} \| y \|_X \| f' (\cdot + \tau y) \Delta_{\theta y} g \Delta_{\theta y} h \|_{L^\infty} \frac{1}{y^2 + (\Delta_y f(x))^2} \|_X \frac{dy}{x} \leq \ C \| f' \|_{L^\infty}^2 \| f' \|_X \| h \|_{L^\infty} \| g \|_D \]

which gives the desired result using Sobolev embeddings (4.5). The proof of the proposition is now achieved.

The second kind of Cauchy integrals that we have to deal with and related to the integral terms in (2.4) and (2.5) is given by the following linear operators

\[ T_{f}^{\alpha,\beta} g(x) = \text{p.v.} \int_{R} \frac{y g(\alpha x + \beta y)}{y^2 + [f(x) + f(x + y)]^2} dy \]

with \( \alpha \) and \( \beta \) two parameters. The continuity of these operators in classical Banach spaces is not in general easy to establish and could fail for some special cases. We point out that it is not our purpose in this exposition to implement a complete study of those operators. A more complete theory may be achieved but this topics exceeds the scope of this paper and we shall restrict ourselves to some special configurations that fit with the application to the aggregation equation. Our result in this direction reads as follows.

**Theorem 5.2.** Let \( \alpha, \beta \in [0, 1] \), \( K \) be a compact set of \( R \) and \( f : R \to R^+ \) be a compactly supported continuous positive function such that \( f' \in C^*_K \). Then the following assertions hold true.

1. The operator \( T_{f}^{\alpha,\beta} : C^*_K \to L^\infty(R) \) is well-defined and continuous

\[ \| T_{f}^{\alpha,\beta} g \|_{L^\infty} \leq C \left( 1 + \| f' \|_{L^\infty}^2 + \| f' \|_{L^\infty} \| f' \|_D \right) \| g \|_D \]

with \( C \) a constant depending only on \( K \) and not on \( \alpha \) and \( \beta \).

2. The modified operator \( f'T_{f}^{\alpha,\beta} : C^*_K \to C^*_K \) is continuous. More precisely,

\[ \| f'T_{f}^{\alpha,\beta} g \|_D \leq C \| f' \|_D \left( C_{\beta} \ln_{+}(1/\| f' \|_D) + \| f' \|_D^4 \right) \| g \|_D \]

with \( C \) a constant depending only on \( K \) and

\[ C_{\beta} \triangleq \begin{cases} 1 - \ln(\beta), & \beta \in (0, 1] \\ 1, & \beta = 0 \end{cases} \]
(3) Let \( s \in (0, 1) \) and assume that \( f' \in C^s_K \), then \( f'^{\alpha, \beta}_f : C^s_K \rightarrow C^s_K(\mathbb{R}) \) is well-defined and continuous. More precisely, there exists a constant \( C \) depending only on the compact \( K \) and \( s \) such that

\[
\|f'^{\alpha, \beta}_f g\|_s \leq C \left( C_\beta \|f'\|_{L_\infty^{1/2}} + \|f'\|_{L_\infty}^{1/2} \right) \|g\|_s.
\]

In addition, one has the refined estimate

\[
\|f'^{\alpha, \beta}_f g\|_s \leq C \|f'\|_{L_\infty^{1/2}} [\|f'\|_{L_\infty}^{1/2} C_\beta + \|f'\|_{L_\infty}^{1/2}] \|g\|_s
\]

with

\[
C_\beta \triangleq \begin{cases} 
\beta^{-1/2}, \quad \beta \in (0, 1) \\
1, \quad \beta = 0.
\end{cases}
\]

Proof. To alleviate the notation we shall along this proof write \( Tfg \) instead of \( T_f^{\alpha, \beta} g \).

1) By symmetrizing we get

\[
Tfg(x) = \int_0^{+\infty} \frac{y [g(\alpha x + \beta y) - g(\alpha x - \beta y)]}{y^2 + \|f(x) + f(x + y)\|^2} dy + \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon}^{+\infty} \frac{y g(\alpha x - \beta y)[f(x-y)-f(x+y)][\Delta_+ f(x) + \Delta_- f(x)]}{\left(y^2 + [\Delta_+ f(x)]^2\right)} dy
\]

\[
\triangleq T_f^1 g(x) + T_f^2 g(x).
\]

Without loss of generality we can assume that \( K = [-1, 1] \) and \( \text{supp} g \subset [-1, 1] \) and deal only with \( x \geq 0 \). We shall distinguish two cases \( 0 \leq \alpha x \leq 2 \) and \( \alpha x \geq 2 \). In the first case reasoning on the support of \( g \) we simply get

\[
T_f^1 g(x) = \int_{\{0 \leq \beta y \leq 3\}} \frac{y [g(\alpha x + \beta y) - g(\alpha x - \beta y)]}{y^2 + \|f(x) + f(x + y)\|^2} dy.
\]

Hence we get by the definition of the modulus of continuity, a change of variables and (4.3)

\[
|T_f^1 g(x)| \leq \int_{\{0 \leq \beta y \leq 3\}} \frac{\omega_g(2\beta y)}{y} dy 
\]

\[
\leq C \|g\|_D.
\]

Coming back to the case \( \alpha x \geq 2 \) one may write

\[
|T_f^1 g(x)| \leq \int_{\{\alpha x - 1 \leq \beta y \leq 1+\alpha x\}} \frac{\omega_g(2\beta y)}{y} dy
\]

\[
\leq 2 \|g\|_{L_\infty} \int_{\alpha x - 1}^{1+\alpha x} \frac{1}{y} dy
\]

\[
\leq \|g\|_{L_\infty} \ln \left( \frac{1 + \gamma}{-1 + \gamma} \right), \quad \gamma = \alpha x \geq 2
\]

\[
\leq C \|g\|_{L_\infty}.
\]

Combining this last inequality with (5.9) we deduce that

\[
\|T_f^1 g\|_{L_\infty} \leq C \|g\|_D.
\]
For the second term $T_f^2 g$ we split it into two parts as follows

\[
T_f^2 g(x) = \lim_{\varepsilon \to 0} 4f(x) \int_{\varepsilon}^{+\infty} \frac{y g(\alpha x - \beta y) [f(x - y) - f(x + y)]}{\left( y^2 + |f(x) + f(x + y)|^2 \right)} dy + \int_0^{+\infty} \frac{y g(\alpha x - \beta y) [f(x - y) - f(x + y)] \psi(x, y)}{\left( y^2 + |f(x) + f(x + y)|^2 \right)} dy
\]

(5.11) \[ \triangleq T_f^{2,1} g(x) + T_f^{2,2} g(x) \]

with

\[
\psi(x, y) = f(x + y) + f(x - y) - 2f(x) = y \int_0^1 [f'(x + \theta y) - f'(x - \theta y)] d\theta.
\]

The first term $T_f^{2,1} g$ is easily estimated. Indeed, one can assume that $f(x) > 0$, otherwise the integral vanishes. Thus using the mean value theorem and a change of variables we obtain

\[
|T_f^{2,1} g(x)| \leq 8 \|g\|_{L^\infty} \|f'\|_{L^\infty} f(x) \int_0^{+\infty} \frac{y^2}{\left( y^2 + |f(x)|^2 \right)} dy \leq 8 \|g\|_{L^\infty} \|f'\|_{L^\infty} \int_0^{+\infty} \frac{y^2}{(y^2 + 1)^2} dy
\]

(5.12) \[ \leq C \|g\|_{L^\infty} \|f'\|_{L^\infty}. \]

As to the term $T_f^{2,2}$, straightforward arguments yield

\[
|T_f^{2,2} g(x)| \leq 8 \|g\|_{L^\infty} \|f\|_{L^\infty}^2 \int_{y \geq \frac{1}{2}} \frac{1}{y^3} dy + 2 \|g\|_{L^\infty} \|f'\|_{L^\infty} \int_0^{\frac{1}{2}} \frac{|\psi(x, y)|}{y^2} dy \leq C \|g\|_{L^\infty} \left( \|f\|_{L^\infty}^2 + C \|f'\|_{L^\infty} \int_0^{\frac{1}{2}} \frac{\omega_f(2y)}{y} dy \right) \leq C \|g\|_{L^\infty} \left( \|f'\|_{L^\infty}^2 + C \|f'\|_{L^\infty} \|f'\|_D \right)
\]

where we have used the fact

\[
|\psi(x, y)| \leq 2y \omega_f(2y).
\]

Consequently we obtain

(5.13) \[ \|T_f^2 g\|_{L^\infty} \leq C \|g\|_{L^\infty} \left( \|f'\|_{L^\infty}^2 + \|f'\|_{L^\infty} \|f'\|_D + \|f'\|_{L^\infty} \right). \]

Putting together this estimate with (5.12) and (4.2) we obtain the desired estimate.

2) First, recall from the point (1) of this proof the following decomposition

(5.14) \[ T_f g(x) = T_f^1 g(x) + T_f^{2,1} g(x) + T_f^{2,2} g(x). \]

The second term is more easier to deal with and one has

(5.15) \[ \|T_f^{2,1} g\|_D \leq C \|g\|_D \|f'\|_D (1 + \|f'\|_{L^\infty}^3). \]
This implies in view of the law products (4.4) and (5.12) that
\[
\|T_f^{2,1}g\|_D \leq C\|g\|_D \|f'\|_D (\|f'\|_{L^\infty} + \|f'\|_{L^\infty}^4).
\]
To establish (5.15) we first note that when \(f(x) = 0\) then \(T_f^{2,1}g(x) = 0\). However for \(f(x) > 0\), using the mean value theorem and a change of variables \(y \to f(x)y\) we get
\[
T_f^{2,1}g(x) = -4 \int_0^{+\infty} \frac{y^2 g(\alpha x - \beta f(x)y) \int_0^1 [f'(x + \theta f(x)y) + f'(x - \theta f(x)y)] d\theta}{\varphi(x,y) \varphi(x,-y)} dy
\]
with
\[
\varphi(x,y) = y^2 + \left[2 + y \int_0^1 f'(x + \theta f(x)y) d\theta\right]^2.
\]
Observe that the identity (5.17) is meaningful even for \(f(x) = 0\) and we can check easily that it vanishes. This follows from the fact that owing to the positivity of \(f\) when \(f(x) = 0\) then necessary \(f'(x) = 0\). To alleviate the expressions we introduce the functions
\[
N_1(x,y) = g(\alpha x - \beta f(x)y) \int_0^1 [f'(x + \theta f(x)y) + f'(x - \theta f(x)y)] d\theta
\]
and
\[
D_1(x,y) = \varphi(x,y) \varphi(x,-y).
\]
Then by (4.4) we obtain for fixed \(y\)
\[
\|N_1(\cdot,y)\|_D \leq 2\|g \circ (\alpha \text{Id} - \beta yf)\|_D \|f'\|_{L^\infty}
\]
\[
+ \|g\|_{L^\infty} \int_0^1 [\|f' \circ (\text{Id} + \theta yf)\|_D + \|f' \circ (\text{Id} - \theta yf)\|_D] d\theta.
\]
Using the composition law (4.7) we get successively
\[
\|g \circ (\alpha \text{Id} - \beta yf)\|_D \leq C\|g\|_D \left(1 + \ln(\alpha + \beta \|f'\|_{L^\infty} y)\right)
\]
and
\[
\|f' \circ (\text{Id} + \theta yf)\|_D \leq C\|f'\|_D \left(1 + \ln(1 + \theta \|f'\|_{L^\infty} y)\right).
\]
This implies that
\[
\|N_1(\cdot,y)\|_D \leq C\|g\|_D \left(1 + \ln + (\alpha + \beta \|f'\|_{L^\infty} y)\right) \|f'\|_{L^\infty}
\]
\[
+ C\|g\|_{L^\infty} \|f'\|_D \int_0^1 \left(1 + \ln(1 + \theta \|f'\|_{L^\infty} y)\right) d\theta.
\]
Since
\[
\ln(1 + \Pi_{i=1}^n x_i) \leq \sum_{i=1}^n \ln(1 + x_i), \forall x_i \geq 0
\]
then
\[
N_1(\cdot,y) \leq C\|g\|_D \|f'\|_D \left(1 + \ln + \|f'\|_{L^\infty} + \ln + y\right).
\]
On the other hand it is plain that
\[
\|N_1(\cdot,y)\|_{L^\infty} \leq C\|g\|_{L^\infty} \|f'\|_{L^\infty}.
\]
To estimate \(\frac{1}{D_1(\cdot,y)}\) in Dini space \(C_K^\infty\) we come back to the definition which implies that
\[
\|1/D_1(\cdot,y)\|_D \leq \|D_1(\cdot,y)\|_D \|1/D_1(\cdot,y)\|_{L^\infty}^2.
\]
Now using the law product (4.4) we deduce that
\[ \|D_1(\cdot, y)\|_D \leq \|\varphi(\cdot, y)\|_{L^\infty} \|\varphi(\cdot, -y)\|_D + \|\varphi(\cdot, y)\|_D \|\varphi(\cdot, -y)\|_{L^\infty}. \]

From simple calculations we get
\[ \|\varphi(\cdot, \pm y)\|_{L^\infty} \leq y^2 + (2 + y\|f^\prime\|_{L^\infty})^2 \leq C(1 + \|f^\prime\|_{L^\infty}^2)(1 + y^2). \]

Applying (4.4) and (4.7) to the expression of \( \varphi \) it is quite easy to check that
\[ \|\varphi(\cdot, \pm y)\|_D \leq C(y + y^2\|f^\prime\|_{L^\infty})\int_0^1 \|f^\prime \circ (\text{Id} \pm \theta yf)\|_D d\theta \leq C(y + y^2\|f^\prime\|_{L^\infty})\|f^\prime\|_D \left(1 + \ln_+ \|f^\prime\|_{L^\infty} + \ln_+ y\right). \]

Thus combining the preceding estimates we find
\[ \|D_1(\cdot, y)\|_D \leq C(y + y^2\|f^\prime\|_{L^\infty})\|f^\prime\|_D \left(1 + \ln_+ \|f^\prime\|_{L^\infty} + \ln_+ y\right)(1 + \|f^\prime\|_{L^\infty}^2)(1 + y^2) \]
\[ \leq C(1 + y^4 \ln_+ y)\|f^\prime\|_D \left(1 + \ln_+ \|f^\prime\|_{L^\infty}\right)(1 + \|f^\prime\|_{L^\infty}^3). \]

(5.21)

Now we shall use the following inequalities that can be proved in a straightforward way: for any \( y \in \mathbb{R}_+ \) and for any \( a, b \in \mathbb{R} \) with \( |a| \leq b \), one has
\[ y^2 + (2 + ya)^2 \geq y^2 + (2 - ya)^2 \geq \frac{1 + y^2}{1 + a^2} \geq \frac{1 + y^2}{1 + b^2}. \]

It follows that
\[ \left\| \frac{1}{\sqrt{\|f^\prime\|_{L^\infty}}} \right\|_{L^\infty} \leq \frac{1 + \|f^\prime\|_{L^\infty}^2}{1 + y^2}. \]

(5.22)

Putting this estimate together with (5.21) and (5.20) yields
\[ \|1/D_1(\cdot, y)\|_D \leq C \frac{1 + y^4 \ln_+ y}{1 + y^8} \|f^\prime\|_D \left(1 + \ln_+ \|f^\prime\|_{L^\infty}\right)(1 + \|f^\prime\|_{L^\infty}^1) \]
\[ \leq C \frac{1 + \ln_+ y}{1 + y^4} \|f^\prime\|_D (1 + \|f^\prime\|_{L^\infty}^2). \]

Therefore we obtain using (5.18), (5.19) and (5.22)
\[ \|(\mathcal{N}_1/D_1)(\cdot, y)\|_{L^\infty} \leq \|(\mathcal{N}_1(\cdot, y))_{L^\infty}\|_{L^\infty} \left\| \frac{1}{\sqrt{\|f^\prime\|_{L^\infty}}} \right\|_{L^\infty} \|\mathcal{N}_1(\cdot, y)\|_D \|1/D_1(\cdot, y)\|_D \]
\[ \leq C\|g\|_{L^\infty} \|f^\prime\|_{L^\infty} \frac{1 + \ln_+ y}{1 + y^4} \|f^\prime\|_D (1 + \|f^\prime\|_{L^\infty}^2) \]
\[ + C\|g\|_{L^\infty} \|f^\prime\|_D \frac{1 + \ln_+ y}{1 + y^4} \|f^\prime\|_{L^\infty} + \ln_+ y (1 + \|f^\prime\|_{L^\infty}^4) \]
\[ \leq C\|g\|_{L^\infty} \|f^\prime\|_D \frac{1 + \ln_+ y}{1 + y^4} (1 + \|f^\prime\|_{L^\infty}^3). \]
Plugging this estimate into (5.17) we find
\[ \|T^{2,1}_f g\|_D \leq 4 \int_0^{+\infty} y^2 \|(N_1/D_1)(\cdot,y)\|_D dy \]
(5.23)
\[ \leq C \|g\|_D \|f'\|_D (1 + \|f'\|_{L^\infty}^{13}). \]
This concludes the proof of (5.15).
Now we intend to estimate \( \|T^1_f g\|_D \) which is more tricky. Let \( r \in (0,1) \) and \( x_1, x_2 \in \mathbb{R} \) such that \( |x_1 - x_2| \leq r \). We shall decompose \( T^1_f g \) as follows
\[ T^1_f g = T^{r,1}_{f,int} g + T^{r,1}_{f,ext} g \]
(5.24)
with
\[ T^{r,1}_{f,int} g(x) = \int_0^r y \frac{[g(\alpha x + \beta y) - g(\alpha x - \beta y)]}{y^2 + [f(x) + f(x + y)]^2} dy \]
and
\[ T^{r,1}_{f,ext} g(x) = \int_r^{+\infty} y \frac{[g(\alpha x + \beta y) - g(\alpha x - \beta y)]}{y^2 + [f(x) + f(x + y)]^2} dy. \]
From the sub-additivity of the modulus of continuity we get
\[ |f'(x) T^{r,1}_{f,int} g(x)| \leq C |f'(x)| \int_0^r \frac{y \omega_g(y)}{y^2 + [f(x) + f(x + y)]^2} dy \]
\[ \leq C |f'(x)| \int_0^r \frac{\omega_g(y)}{y + f(x)} dy. \]
Using Lemma 4.1 we find
\[ |f'(x) T^{r,1}_{f,int} g(x)| \leq C \gamma(f) \frac{\omega_g(y)}{1 + \ln\left(\frac{1}{f'(x)}\right)} \int_0^r \frac{\omega_g(y)}{y + f(x)} dy \]
(5.25)
where
\[ \gamma(f) \triangleq \|f'||_D (1 + \|f'||_D). \]
(5.26)
Now we claim that: for \( y \in (0,1) \)
\[ \sup_{\varepsilon > 0} \frac{1}{1 + \ln(1/\varepsilon)} \frac{1}{y + \varepsilon} \leq C \frac{1}{y(1 + |\ln y|)} + \frac{1}{1 + y} \]
(5.27)
for some universal constant \( C > 0 \). To prove this result it is enough to get
\[ \sup_{\varepsilon \in (0,1)} \frac{1}{1 + \ln(1/\varepsilon)} \frac{1}{y + \varepsilon} \leq C \frac{1}{y(1 + |\ln y|)}. \]
Indeed, we shall consider the two cases \( \varepsilon \geq \sqrt{y} \) and \( \varepsilon \leq \sqrt{y} \). In the first case we observe
\[ \frac{1}{y + \varepsilon} \leq \frac{1}{\sqrt{y}} \quad \text{and} \quad \frac{1}{1 + \ln(1/\varepsilon)} \leq 1 \]
which implies that
\[ \frac{1}{1 + \ln(1/\varepsilon)} \frac{1}{y + \varepsilon} \leq \frac{1}{\sqrt{y}} \]
\[ \leq \frac{C}{y(1 + |\ln y|)}. \]
However in the second case $\varepsilon \leq \sqrt{y}$ we write simply that
\[
\frac{1}{y + \varepsilon} \leq \frac{1}{y} \quad \text{and} \quad \frac{1}{1 + \ln(1/\varepsilon)} \leq \frac{1}{1 + \frac{1}{2} \ln(1/y)}
\]
which gives the desired result. Coming back to (5.25) and using (5.27) we deduce that
\[
\sup_x |f'(x)T_{f,\text{int}}^r g(x)| \leq C\gamma(f) \int_0^r \sup_x \frac{\omega_g(y)}{(1 + \ln (f(x)))} (y + f(x)) \, dy
\]
\[
\leq C\gamma(f) \left( \int_0^r \frac{\omega_g(y)}{y(1 + |\ln y|)} \, dy + \int_0^r \frac{\omega_g(y)}{1 + y} \, dy \right).
\]
Consequently
\[
\sup_{|x_1 - x_2| \leq r} |f'(x_1)T_{f,\text{int}}^r g(x_1) - f'(x_2)T_{f,\text{int}}^r g(x_2)| \leq C\gamma(f) \left( \int_0^r \frac{\omega_g(y)}{y(1 + |\ln y|)} \, dy + \int_0^r \omega_g(y) \, dy \right).
\]
Therefore we get by using Fubini’s theorem
\[
\int_0^1 \sup_{|x_1 - x_2| \leq r} |f'(x_1)T_{f,\text{int}}^r g(x_1) - f'(x_2)T_{f,\text{int}}^r g(x_2)| \frac{dr}{r} \leq C\gamma(f) \int_0^1 \frac{\omega_g(y)}{y} \frac{|\ln y|}{(1 + |\ln y|)} \, dy
\]
\[
+ C\gamma(f) \int_0^1 |\ln y| \omega_g(y) \, dy
\]
\[
\leq C\gamma(f) \|g\|_D.
\]
As to $T_{f,\text{ext}}^r g$ we write
\[
f'(x_1)T_{f,\text{ext}}^r g(x_1) - f'(x_2)T_{f,\text{ext}}^r g(x_2) = (f'(x_1) - f'(x_2)) T_{f,\text{ext}}^r g(x_2)
\]
\[
+ f'(x_1) \left( T_{f,\text{ext}}^r g(x_1) - T_{f,\text{ext}}^r g(x_2) \right)
\]
\[
\triangleq \mu_1(x_1, x_2) + \mu_2(x_1, x_2).
\]
Our current goal is to prove that for $j \in \{1, 2\}$
\[
\int_0^1 \sup_{|x_1 - x_2| \leq r} \frac{\mu_j(x_1, x_2)}{r} \, dr
\]
is well-estimated. For the first term we use (5.10) leading to
\[
\int_0^1 \sup_{|x_1 - x_2| \leq r} \frac{\mu_1(x_1, x_2)}{r} \, dr \leq ||T_{f,\text{ext}}^r g||_{L^\infty} \int_0^1 \frac{\omega_f(r)}{r} \, dr
\]
\[
\leq C\|g\|_D \|f'\|_D.
\]
The second term is more subtle. First note that if $|x_1 - x_2| \leq 1$ then the quantity $f'(x_1)T_{f,\text{ext}}^r g(x_1) - f'(x_2)T_{f,\text{ext}}^r g(x_2)$ vanishes for $x_1, x_2$ outside a compact set related only to the support of $f$. Therefore the integrals defining $\mu_2(x_1, x_2)$ may be restricted to the set $\{3r \leq \beta y \leq B\}$ with $B$ being some constant related to the size of the supports of $f$ and $g$, and without loss of generality we can
take $B = 1$. It follows that

$$
\mu_2(x_1, x_2) = f'(x_1) \int_{\{\beta r \leq \beta y \leq 1\}} \frac{y [\tilde{g}(x_1, y) - \tilde{g}(x_2, y)]}{y^2 + [f(x_1) + f(x_1 + y)]^2} dy + f'(x_1) \int_{\{\beta r \leq \beta y \leq 1\}} \frac{y \tilde{g}(x_2, y) [\Delta_y^+ f(x_2) - \Delta_y^+ f(x_1)] [\Delta_y^+ f(x_2) + \Delta_y^+ f(x_1)]}{(y^2 + [\Delta_y^+ f(x_1)]^2)} dy
$$

(5.30)

with

$$
\tilde{g}(x, y) \triangleq g(\alpha x + \beta y) - g(\alpha x - \beta y) \quad \text{and} \quad \Delta_y^+ f(x) = f(x + y) + f(x).
$$

To estimate $\mu_{2,1}$ we shall use the next inequality which is a consequence of Lemma 4.1,

$$
\int_0^L \frac{|f'(x)|}{y + f(x)} dy = |f'(x)| \ln \left(1 + \frac{L}{f(x)}\right) \leq C \gamma(f)(1 + \ln \beta)
$$

(5.31)

with $C$ an absolute constant. This implies that

$$
\mu_{2,1}(x_1, x_2) \leq C \omega_g(\alpha|x_1 - x_2|)|f'(x_1)| \int_0^{1/\beta} \frac{1}{y + f(x_1)} dy \leq C \omega_g(|x_1 - x_2|) \gamma(f)(1 + |\ln \beta|).
$$

Consequently, we find that

$$
\sup_{|x_1 - x_2| \leq \beta} |\mu_{2,1}(x_1, x_2)| \leq C \omega_g(\beta) \gamma(f)(1 + |\ln \beta|)
$$

and therefore

$$
\int_0^1 \sup_{|x_1 - x_2| \leq \beta} |\mu_{2,1}(x_1, x_2)| \frac{dr}{r} \leq C \gamma(f)(1 + |\ln \beta|) \|g\|_D.
$$

We emphasize that for $\beta = 0$ one can still get an estimate since $\mu_{2,1}(x_1, x_2) = 0$ and therefore we get the desired estimate.

Now we shall move to the estimate of $\mu_{2,2}(x_1, x_2)$. We start with using the estimate

$$
\sup_{a > 0} \frac{a}{y^2 + a^2} \leq C
$$

which implies that

$$
\frac{y |\tilde{g}(x_2, y)| |\Delta_y^+ f(x_2) - \Delta_y^+ f(x_1)| |\Delta_y^+ f(x_2) + \Delta_y^+ f(x_1)|}{(y^2 + [\Delta_y^+ f(x_1)]^2)(y^2 + [\Delta_y^+ f(x_2)]^2)} \leq C |x_2 - x_1| \|f''\|_{L^\infty} \frac{\omega_g(2\beta y)}{y^2}.
$$

Thus

$$
\sup_{|x_1 - x_2| \leq \beta} \mu_{2,2}(x_1, x_2) \leq C r \|f''\|_{L^\infty} \int_r^\pi \frac{\omega_g(2\beta y)}{y^2}
$$
which yields in view of Fubini’s theorem

\[
\int_0^1 \sup_{|x_1-x_2| \leq r} \mu_{2,2}(x_1, x_2) \frac{dr}{r} \leq C\|f'\|_{L^\infty}^2 \int_0^1 \int_{\{\beta \leq \beta y \leq 1\}} \frac{\omega_g(2\beta y)}{y^2} dy dr
\]

\[
\leq C\|f'\|_{L^\infty}^2 \int_{\{0 \leq \beta y \leq 1\}} \frac{\omega_g(2\beta y)}{y} dy
\]

\[
\leq C\|f'\|_{L^\infty}^2 \int_0^2 \frac{\omega_g(y)}{y} dy
\]

\[
\leq C\|f'\|_{L^\infty}^2 \|g\|_D.
\]

Remark that the last constant does not depend on \(\beta\). Putting together the preceding estimates we find that

\[
\|f'T^1_f g\|_D \leq C\|g\|_D \left[ (1 + |\ln \beta|) \gamma(f) + \|f'\|_{L^\infty}^2 \right],
\]

where \(\gamma(f)\) has been defined in (5.26) As pointed before the case \(\beta = 0\) has a special structure and one gets

\[
\|f'T^1_f g\|_D \leq C\|g\|_D \left[ \gamma(f) + \|f'\|_{L^\infty}^2 \right].
\]

Now let us move to the estimate of \(f'(x)T^{2,2}_f g\) given by

\[
T^{2,2}_f g(x) = \int_0^{+\infty} \frac{y g(\alpha x - \beta y) [f(x - y) - f(x + y)] \psi(x, y)}{y^2 + [f(x) + f(x + y)]^2} dy
\]

\[
= T^{r,2,2}_{f,\text{int}} g(x) + T^{r,2,2}_{f,\text{ext}} g(x)
\]

(5.33)

where

\[
\psi(x, y) = y \int_0^1 [f'(x + \theta y) - f'(x - \theta y)] d\theta.
\]

and the cut-off operators are given by

\[
T^{r,2,2}_{f,\text{int}} g(x) \triangleq \int_0^r \frac{y g(\alpha x - \beta y) [f(x - y) - f(x + y)] \psi(x, y)}{y^2 + [\Delta^+_y f(x)]^2} dy
\]

and

\[
T^{r,2,2}_{f,\text{ext}} g(x) = \int_r^1 \frac{y g(\alpha x - \beta y) [f(x - y) - f(x + y)] \psi(x, y)}{y^2 + [\Delta^+_y f(x)]^2} dy
\]

(5.34)

\[
\triangleq \int_r^1 \frac{N(x, y)}{D(x, y)} dy.
\]

We shall proceed in a similar way to \(T^1_f g\). Let us start with \(f'(x)T^{r,2,2}_{f,\text{int}} g\). Since

\[
|\psi(x, y)| \leq 2y \omega_{f'}(y)
\]

(5.35)
then one has
\[ |f'(x)T_{f,\text{int}}^{r,2,2}g(x)| \leq C\|g\|_{L^\infty} \|f'\|_{L^\infty} |f'(x)| \int_0^r \frac{y^2 \omega_f(y)}{\left(y^2 + |f(x)|^2\right)^2} dy \]
\[ \leq C\|g\|_{L^\infty} \|f'\|_{L^\infty} |f'(x)| \int_0^r \frac{\omega_f(y)}{y} dy. \]

Thus following the same steps of (5.28) we obtain
\[ \sup_{|x_1 - x_2| \leq r} \left| f'(x_1)T_{f,\text{int}}^{r,2,2}g(x_1) - f'(x_2)T_{f,\text{int}}^{r,2,2}g(x_2) \right| \leq C\|g\|_{L^\infty} \|f'\|_{L^\infty} \gamma(f) \int_0^r \frac{\omega_f(y)}{y(1 + |\ln y|)} dy \]
\[ + C\|g\|_{L^\infty} \|f'\|_{L^\infty} \gamma(f) \int_0^r |\omega_f(y)| dy. \]

Thus by Fubini’s theorem and (4.2) we obtain
\[ \int_0^1 \sup_{|x_1 - x_2| \leq r} \left| f'(x_1)T_{f,\text{int}}^{r,2,2}g(x_1) - f'(x_2)T_{f,\text{int}}^{r,2,2}g(x_2) \right| \frac{dr}{r} \leq C\|g\|_{L^\infty} \|f'\|_{D}^2 \gamma(f). \]

What is left is to estimate the quantity \( f'(x)T_{f,\text{ext}}^{r,2,2}g \). First, it is obvious that
\[ f'(x_1)T_{f,\text{ext}}^{r,2,2}g(x_1) - f'(x_2)T_{f,\text{ext}}^{r,2,2}g(x_2) = (f'(x_1) - f'(x_2))T_{f,\text{ext}}^{r,2,2}(x_2) \]
\[ + f'(x_1) \left( T_{f,\text{ext}}^{r,2,2}(x_1) - T_{f,\text{ext}}^{r,2,2}(x_2) \right). \]

The first term of the right-hand side is easy to estimate. Indeed,
\[ |(f'(x_1) - f'(x_2))T_{f,\text{ext}}^{r,2,2}(x_2)| \leq \omega_f(|x_1 - x_2|) \|T_{f,\text{ext}}^{r,2,2}\|_{L^\infty}. \]

It is clear that
\[ |T_{f,\text{ext}}^{r,2,2}g(x)| \leq C\|g\|_{L^\infty} \|f'\|_{L^\infty} \int_0^1 \frac{\omega_f(y)}{y} dy \]
\[ \leq C\|g\|_{L^\infty} \|f'\|_{D}^2. \]

Hence
\[ \int_0^1 \sup_{|x_1 - x_2| \leq r} |(f'(x_1) - f'(x_2))T_{f,\text{ext}}^{r,2,2}(x_2)| \frac{dr}{r} \leq C\|g\|_{L^\infty} \|f'\|_{D}^3. \]

To deal with the second term we proceed as for the term \( \mu_2(x_1, x_2) \) in (5.30). From (5.34) one has
\[ f'(x_1) \left( T_{f,\text{ext}}^{r,2,2}(x_1) - T_{f,\text{ext}}^{r,2,2}(x_2) \right) = f'(x_1) \int_0^1 \frac{N(x_1, y) - N(x_2, y)}{D(x_1, y)} dy \]
\[ + f'(x_1) \int_0^1 \frac{N(x_2, y)(D(x_2, y) - D(x_1, y))}{D(x_1, y)D(x_2, y)} dy. \]

It is quite obvious from some straightforward computations using in particular (5.35) that for \(|x_1 - x_2| \leq r\)
\[ |N(x_1, y) - N(x_2, y)| \leq C\|f'\|_{L^\infty} y^2 \left[ \omega_2(\alpha r) \omega_f'(y) y + \|g\|_{L^\infty} \omega_f'(r) y + \|g\|_{L^\infty} r \omega_f(y) \right]. \]
Since
\[
\frac{1}{D(x,y)} \leq \frac{C}{|y + f(x)|} \leq \frac{C}{y^4}
\]
then we get
\[
\frac{N(x_1, y) - N(x_2, y)}{D(x_1, y)} \leq C \|f''\|_{L^\infty} \left[ \omega_g(\alpha r) \frac{\omega_f'(y)}{y} + \|g\|_{L^\infty} \frac{\omega_f'(r)}{y + f(x)} + \|g\|_{L^\infty} \frac{\omega_f'(y)}{y^2} \right].
\]

This leads in view of (4.2),
\[
|f'(x_1)| \int_r^1 \frac{|N(x_1, y) - N(x_2, y)|}{D(x_1, y)} dy \leq C \|f''\|_D \left[ \|f''\|_D \omega_g(\alpha r) + \|g\|_D \omega_f'(r) \int_0^1 \frac{|f'(x_1)|}{y + f(x_1)} dy \right]
\]
(5.38)
\[
+ \|g\|_D \|f''\|_D^2 \int_r^1 \frac{\omega_f'(y)}{y^2} dy,
\]
which implies according to (5.31)
\[
\int_0^1 \sup_{|x_1 - x_2| \leq r} |f'(x_1)| \int_r^1 \frac{|N(x_1, y) - N(x_2, y)|}{D(x_1, y)} dy dr \leq C \left( \|f''\|_D^2 + \|f''\|_D^2 \gamma(f) \right) \|g\|_D.
\]

Now straightforward computations show that
\[
\frac{|N(x_2, y)(D(x_2, y) - D(x_1, y))|}{D(x_1, y)D(x_2, y)} \leq C \|g\|_{L^\infty} \|f''\|_{L^\infty} |x_1 - x_2| \frac{\omega_f'(y)}{y^2}.
\]

Therefore using Fubini’s theorem we get
\[
\int_0^1 \sup_{|x_1 - x_2| \leq r} |f'(x_1)| \int_r^1 \frac{|N(x_2, y)(D(x_2, y) - D(x_1, y))|}{D(x_1, y)D(x_2, y)} dy dr \leq C \|f''\|_D \|g\|_D.
\]

Putting together the preceding estimates we find that
\[
\|f'T_f^{2,2}g\|_D \leq C \|g\|_D \left( \|f''\|_D^2 + \|f''\|_D^2 \gamma(f) + \|f''\|_D^2 \right)
\]
(5.40)
\[
\leq C \|g\|_D \left( \|f''\|_D^2 + \|f''\|_D^2 \right)
\]
with $C$ a constant depending only on the diameter of the compact $K$. To get the desired estimate it suffices to put together (5.16), (5.32) and (5.40).

3) We shall proceed as in the proof of the point 2) of the Theorem 5.2. We use exactly the same splitting with similar estimates and to avoid redundancy we shall only give the basic estimates with some details for the terms that require new treatment. We use the decomposition described in (5.14). To estimate $T_f^{2,1}g$ in $C^s$ we use the expression (5.17). Then following the same lines using in particular the law product (4.8) and the composition law (4.6), one has
\[
\|N_1(\cdot, y)\|_s \leq C \|g\|_s \left( \alpha^s + \beta^s \|f''\|_{L^\infty} y^s \right) \|f''\|_{L^\infty}
\]
\[
+ C \|g\|_{L^\infty} \|f''\|_s \int_0^1 \left( 1 + \theta^s \|f''\|_{L^\infty} y^s \right) d\theta.
\]

Since $\alpha, \beta \in [0, 1]$ we deduce
\[
\|N_1(\cdot, y)\|_s \leq C \left( \|g\|_s \|f''\|_{L^\infty} + \|g\|_{L^\infty} \|f''\|_s \right) \left( 1 + \|f''\|_{L^\infty} y^s \right).
\]
Similarly we get
\[ \|\varphi(\cdot, \pm y)\|_s \leq C(1 + y\|f'\|_{L^\infty}) y \int_0^1 \|f' \circ (\text{Id} \pm \theta y f)\|_s d\theta \leq C(y + y^2\|f'\|_{L^\infty})\|f'\|_s (1 + \|f'\|_{L^\infty}^s). \]
This implies
\[ \|\mathcal{D}_1(\cdot, y)\|_s \leq C(1 + y^{4+s})(1 + \|f'\|_{L^\infty}^{3+s})\|f'\|_s \]
and
\[ \|1/\mathcal{D}_1(\cdot, y)\|_s \leq \frac{C}{1 + y^{4-s}}(1 + \|f'\|_{L^\infty}^{1+s})\|f'\|_s. \]
Consequently for \( s \in (0, 1) \)
\[ \|(\mathcal{N}_1/\mathcal{D}_1)(\cdot, y)\|_s \leq \|(\mathcal{N}_1(\cdot, y)\|_{L^\infty}||1/\mathcal{D}_1(\cdot, y)\|_s + \|\mathcal{N}_1(\cdot, y)\|_s||1/\mathcal{D}_1(\cdot, y)\|_{L^\infty} \]
\[ \leq \frac{C}{1 + y^{4-s}}(1 + \|f'\|_{L^\infty}^{1+s})\|f'\|_s|g|_s. \]
Therefore we get similarly to (5.23)
\[ \|T_f^{2,1}g\|_s \leq C(1 + \|f'\|_{L^\infty}^{1+s})\|f'\|_s|g|_s \int_0^{+\infty} \frac{y^2}{1 + y^{4-s}} ds \leq C(1 + \|f'\|_{L^\infty}^{1+s})\|f'\|_s|g|_s. \]
Combining law products with Sobolev embeddings and (5.12) we get
\[ \|f'T_f^{2,1}g\|_s \leq \|f'\|_{L^\infty}||T_f^{2,1}g\|_s + \|f'\|_s||T_f^{2,1}g\|_{L^\infty} \]
\[ \leq C(1 + \|f'\|_{L^\infty}^{1+s})\|f'\|_s\|f'\|_{L^\infty}\|g\|_s + \|g\|_{L^\infty}\|f'\|_s\|f'\|_{D}\|g\|_s \]
\[ \leq C(1 + \|f'\|_{L^\infty}^{1+s})\|f'\|_s\|f'\|_{D}\|g\|_s. \]
Using once again Sobolev embeddings we get
\[ (5.41) \quad \|f'T_f^{2,1}g\|_s \leq C(\|f'\|_s + \|f'\|_{L^\infty}^{13})\|f'\|_D\|g\|_s. \]
Now to estimate \( T_f^2g \) we come back to the decomposition (5.24) and we easily get
\[ \|T_{f,\text{int}}^{r,1}g\|_{L^\infty} \leq C\|g\|_s \int_0^r y^{-1+s} dy \leq \|g\|_s r^s. \]
Hence we obtain, since \( r = |x_1 - x_2|, \)
\[ |T_{f,\text{int}}^{r,1}g(x_1) - T_{f,\text{int}}^{r,1}g(x_2)| \leq C\|g\|_s |x_1 - x_2|^s. \]
and we also get
\[ |f'(x_1)T_{f,\text{int}}^{r,1}g(x_1) - f'(x_2)T_{f,\text{int}}^{r,1}g(x_2)| \leq C\|f'\|_{L^\infty}\|g\|_s |x_1 - x_2|^s. \]
To estimate the term \( f'T_{f,\text{ext}}^{r,1}g \) we come back to (5.29) and (5.30) and following the same estimates one gets
\[ |\mu_1(x_1, x_2)| \leq |x_1 - x_2|^s\|f'\|_s\|T_{f,\text{ext}}^{r,1}g\|_{L^\infty} \]
\[ \leq C|x_1 - x_2|^s\|f'\|_s\|g\|_D. \]
Moreover
\[ |\mu_2(x_1, x_2)| \leq |\mu_{2,1}(x_1, x_2)| + |\mu_{2,2}(x_1, x_2)| \]
and

\[ |\mu_{2,2}(x_1, x_2)| \leq C|x_2 - x_1||f'||_{L_\infty}^2 \|g\|_s \int_{\{\beta r \leq \beta y \leq 1\}} (\beta y)^s y^{-2} \, dy \]
\[ \leq C\|f'||_{L_\infty}^2 \|g\|_s |x_1 - x_2|^s. \]

To deal with the term \( \mu_{2,1}(x_1, x_2) \) in (5.30) one obtains in view of (5.31)

\[ |\mu_{2,1}(x_1, x_2)| \leq |x_1 - x_2|^s \|g\|_s \|f'(x_1)| \int_{\{\beta r \leq \beta y \leq 1\}} \frac{y}{y^2 + f^2(x_1)} \, dy \]
\[ \leq |x_1 - x_2|^s \|g\|_s \|f'(x_1)| \int_0^{\frac{1}{\beta}} \frac{1}{y + f(x_1)} \, dy. \]

Using the second part of Lemma 4.1 one finds for \( s' \in (0, s) \)

\[ |f'(x_1)| \int_0^{\frac{1}{\beta}} \frac{1}{y + f(x_1)} \, dy \leq C\|f'||_{s'}^{\frac{1}{s'}} |f(x_1)|^{\frac{1}{s'}} \int_0^{\frac{1}{\beta}} \frac{1}{y + f(x_1)} \, dy. \]

Combining this inequality with

\[ \sup_{a > 0} \frac{a^{\frac{1}{s'}}}{y + a} \leq Cy^{-\frac{1}{s'}} \]

we get

\[ (5.42) \quad \sup_{x_1 \in \mathbb{R}} |f'(x_1)| \int_0^{\frac{1}{\beta}} \frac{1}{y + f(x_1)} \, dy \leq C\|f'||_{s'}^{\frac{1}{s'}} \beta^{-\frac{1}{s'}} \]

and therefore

\[ |\mu_{2,1}(x_1, x_2)| \leq |x_1 - x_2|^s \|g\|_s \|f'||_{s'}^{\frac{1}{s'}} \beta^{-\frac{1}{s'}}. \]

Hence

\[ |f'(x_1)T_{f, \text{ext}}^{r,1}g(x_1) - f'(x_2)T_{f, \text{ext}}^{r,1}g(x_2)| \leq C\|g\|_D \|f'||_{s'} |x_1 - x_2|^s + C\|f'||_{L_\infty}^2 \|g\|_s |x_1 - x_2|^s \]
\[ + C|x_1 - x_2|^s \|g\|_s \|f'||_{s'}^{\frac{1}{s'}} \beta^{-\frac{1}{s'}}. \]

It follows that

\[ (5.43) \quad \|f'T_f^g\|_s \leq C\|g\|_s \left[ \|f'||_{s'}^{\frac{1}{s'}} \beta^{-\frac{1}{s'}} + \|f'||_{L_\infty}^2 \right] + C\|g\|_D \|f'||_s. \]

It remains to estimate \( f'T_f^{2,2}g \) described in (5.33) and (5.34). First one may write

\[ |T_{f, \text{int}}^{r,2}g(x)| \leq C\|g\|_{L_\infty} \|f'||_{L_\infty} \|f'||_s \int_0^r y^{s-1} \, dy \]
\[ \leq C\|g\|_{L_\infty} \|f'||_{L_\infty} \|f'||_s |x_1 - x_2|^s. \]

Therefore

\[ |f'(x_1)T_{f, \text{int}}^{s,2,2}g(x_1) - f'(x_2)T_{f, \text{int}}^{s,2,2}g(x_2)| \leq C\|g\|_{L_\infty} \|f'||_{L_\infty}^2 \|f'||_s |x_1 - x_2|^s. \]

By Sobolev embeddings we get

\[ (5.44) \quad |f'(x_1)T_{f, \text{int}}^{r,2,2}g(x_1) - f'(x_2)T_{f, \text{int}}^{r,2,2}g(x_2)| \leq C\|g\|_s \|f'||_{L_\infty} \|f'||_s^2 |x_1 - x_2|^s. \]
From (5.36) and the analysis following this identity one has
\[ |(f'(x_1) - f'(x_2))T_{f,\text{ext}}^{r,2,2}(x_2)| \leq \|f'\|_s \|T_{f,\text{ext}}^{r,2,2}g\|_{L^\infty}|x_1 - x_2|^s \]
\[ \leq C\|g\|_{L^\infty}\|f'\|_s^2 \|f'\|_{L^\infty}|x_1 - x_2|^s \]
Using (5.37), (5.38) and (5.42) (with \(s' = s\)) combined with Sobolev embeddings one deduces
\[ |f'(x_1)| \int_0^1 \frac{|N(x_1, y) - N(x_2, y)|}{D(x_1, y)} dy \leq C\|f'\|_{L^\infty} \|g\|_s (\|f'|_s^2 + \|f'\|_s) \]
From (5.39) we get
\[ |N(x_2, y)(D(x_2, y) - D(x_1, y))| \leq C\|g\|_{L^\infty}\|f'\|_{L^\infty}\|f'\|_s|x_1 - x_2|^s \]
Therefore we get
\[ |f'(x_1)| \int_0^1 \frac{|N(x_2, y)(D(x_2, y) - D(x_1, y))|}{D(x_1, y)D(x_2, y)} dy \leq C\|g\|_{L^\infty}\|f'\|_{L^\infty}\|f'\|_s|x_1 - x_2|^s \]
Hence plugging the preceding estimates into (5.36) and (5.37), we find
\[ |f'(x_1)T_{f,\text{ext}}^{r,2,2}(x_1) - f'(x_2)T_{f,\text{ext}}^{r,2,2}(x_2)| \leq C\|g\|_s \|f'\|_{L^\infty}|x_1 - x_2|^s \]
\[ \quad + C\|f'\|_{L^\infty}\|g\|_s \left( \|f'|_s^2 + \|f'\|_s \right) |x_1 - x_2|^s \]
\[ \quad + C\|g\|_{L^\infty}\|f'\|_{L^\infty}\|f'\|_s|x_1 - x_2|^s \]
Using Standard embeddings we get
\[ |f'(x_1)T_{f,\text{ext}}^{r,2,2}(x_1) - f'(x_2)T_{f,\text{ext}}^{r,2,2}(x_2)| \leq C\|g\|_s \|f'\|_{L^\infty}|x_1 - x_2|^s \left( \|f'|_s + \|f'\|_s^2 \right) \]
Putting together (5.44), (5.45) and (5.33) we obtain
\[ \|f'T_f g\|_s \leq C\|g\|_s \|f'\|_{L^\infty} \left( \|f'|_s + \|f'\|_s^2 \right) \]
Combining (5.41), (5.43) and (5.46) we get for any \(s' \in (0, s]\)
\[ \|f'T_f g\|_s \leq C\|g\|_s \|f'\|_D \left( \|f'|_s + \|f'\|_s^{13} \right) \]
\[ \quad + C\|g\|_s \|f'\|_s \|f'|_s^{13} \beta^{-\frac{s'}{1+s'}} + C\|g\|_D \|f'\|_s \]
Now using the embedding \(C^s \hookrightarrow C^{s'} \hookrightarrow D\) we get
\[ \|f'T_f g\|_s \leq C\|g\|_s \left[ \beta^{-\frac{s'}{1+s'}} \|f'\|_s^{13} + \|f'\|_s^{14} \right] \]
\[ \leq C\|g\|_s \left[ \beta^{-\frac{s}{2}} \|f'\|_s^{13} + \|f'\|_s^{14} \right] \]
Another useful estimate that one can get from taking \(s' = s/2\) and using some interpolation inequalities
\[ \|f'\|_D \leq C\|f'|_s^{1+s} \leq \|f'\|_s^{\frac{1}{s+1-s}} \|f'\|_s^{1+s} \leq \|f'\|_s^{\frac{s}{2}} \leq C\|f'\|_{L^\infty} \|f'\|_s^{\frac{s}{2}} \]
\[ \beta^{-\frac{s}{1+s}} \leq \beta^{-\frac{s}{2}} \]
is the following
\[ \|f'T_f g\|_s \leq C\|g\|_s \|f'\|_{L^\infty} \left[ \|f'\|^{\frac{1}{s}} \beta^{-\frac{s}{2}} + \|f'\|_s^{14} \right] \]
\[ + C\|g\|_{L^\infty} \|g\|_s \|f'\|_s \]
This achieves the proof of Theorem 5.2.

6. Local well-posedness proof

The main objective of this section is to prove the local well-posedness result stated in the first part of Theorem 2.1. The approach that we shall follow is classical and will be done in several steps. We start with a priori estimates of smooth solutions in suitable Banach spaces and this will be the main concern of the Sections 6.1 and 6.2. The rigorous construction of classical solutions will be conducted in Section 6.3.

6.1. Estimates of the source terms. The main goal of this section is to establish the following a priori estimates for the source terms $F$ and $G$ described in (2.8) and (2.9).

Proposition 6.1. Let $K$ be a compact set of $\mathbb{R}$ and $s \in (0, 1)$. We denote by $X$ one of the space $C^s_K$ and $C^s_K$. There exists a constant $C > 0$ depending only on $K$ such that the following estimates hold true

1. For any $f \in X$ we have
   \[ \|F\|_{L^\infty} \leq C \|f'\|_{L^\infty} \|f'\|_D, \quad \|F\|_X \leq C \|f'\|_D (\|f'\|_X + \|f'\|_X^3). \]

2. For any $f \in X$ we have
   \[ \|G\|_{L^\infty} \leq C \|f'\|_{L^\infty} \left(1 + \|f'\|_D^3\right), \quad \|G\|_X \leq C \left(1 + \|f'\|_D^2\right) \left(\|f'\|_X + \|f'\|_X^6\right). \]

Proof. For simplicity we denote along this proof the operator $\Delta_y^{-1}$ by $\Delta_y$.

1. The estimate of $F$ in $L^\infty$ is quite easy. Indeed, it is obvious according to (4.3) that
   \[ \|F\|_{L^\infty} \leq C \int_{-M}^{M} \sup_{x \in \mathbb{R}} \frac{|f'(x + y) - f'(x)|}{|y|} dy \]
   \[ \leq C \|f'\|_{L^\infty} \int_{-M}^{M} \frac{\omega_f(|y|)}{|y|} dy \]
   \[ \leq C \|f'\|_{L^\infty} \|f'\|_D. \]

Now let us move to the estimate of $F$ in the function space $X$ which is Dini space $C^s_K$ or Hölder spaces $C^s_K$. For this purpose we shall transform slightly $F$ in order to apply Proposition 5.1. In fact from Taylor formula one can write

\[ F(x) = \int_{-M}^{M} \int_0^1 \frac{y \Delta_{xy} f'(x) \Delta_y f'(x)}{y^2 + (\Delta_y f(x))^2} dy d\theta. \]

From the notation (5.2) one has

\[ F(x) = \int_0^1 C_f^{\beta \gamma}(f', f')(x) d\theta. \]

At this stage it suffices to apply Proposition 5.1 which implies that

\[ \|F\|_X \leq C (\|f'\|_D \|f'\|_X + \|f'\|_{L^\infty} \|f'\|_D \|f'\|_X^3) \]

which in turn gives the desired result according to the embedding $X \hookrightarrow L^\infty$.

2. The expression of $G$ is given in (2.9) and for simplicity we shall assume along this part that $M = 1$. We shall first split $G$ as follows
\[ G(x) = \text{p.v.} \int_{-1}^{1} \frac{[2f(x) + \Delta_y f(x) + yf'(x)] (f'(x + y) + f'(x))}{y^2 + (f(x + y) + f(x))^2} dy \]

\[ = 2f(x) \text{p.v.} \int_{-1}^{1} \frac{f'(x + y) + f'(x)}{y^2 + (f(x + y) + f(x))^2} dy \]

\[ + \text{p.v.} \int_{-1}^{1} \frac{[\Delta_y f(x) + yf'(x)] (2f'(x) + \Delta_y f'(x))}{y^2 + (f(x + y) + f(x))^2} dy \]

\[ \triangleq G_1(x) + G_2(x). \] (6.1)

The estimate \( G_1 \) in \( L^\infty \) is quite easy. To see this we can first assume that \( f(x) > 0 \), otherwise the integral is vanishing. Thus by change of variables we get

\[ |G_1(x)| \leq 4\|f'\|_{L^\infty} \int_{-1}^{1} \frac{|f(x)|}{y^2 + f^2(x)} dy \]

\[ \leq C\|f'\|_{L^\infty}. \]

Note that for \( x \in \text{supp}\ f \) we have \( f(x + y) = 0, \forall y \notin [-1,1] \). Thus

\[ G_1(x) = 2f(x) \int_{\mathbb{R}} \frac{f'(x + y) + f'(x)}{y^2 + (f(x + y) + f(x))^2} dy - 4f(x)f'(x) \int_{1}^{+\infty} \frac{1}{y^2 + (f(x))^2} dy \]

\[ = 2f(x) \int_{\mathbb{R}} \frac{f'(x + y) + f'(x)}{y^2 + (f(x + y) + f(x))^2} dy - 4f'(x) \arctan(f(x)) \]

\[ \triangleq G_{11} + G_{12}. \] (6.2)

The estimate of \( G_{12} \) in \( L^\infty \) is elementray

\[ \|G_{12}\|_{L^\infty} \leq 4\|f'\|_{L^\infty} \|f\|_{L^\infty}. \] (6.3)

However, to estimate \( G_{12} \) in \( X \) we use the law product (4.3) leading to

\[ \|f' \arctan f\|_X \leq \|\arctan f\|_{L^\infty} \|f'\|_X + \|f'\|_{L^\infty} \|\arctan f\|_X. \]

It is easy to check from the mean value theorem that

\[ \|\arctan f\|_{L^\infty} \leq \|f\|_{L^\infty} \quad \text{and} \quad \omega_{\arctan f}(r) \leq \omega_f(r) \]

which implies in view of the embedding \( \text{Lip} \hookrightarrow X \) that

\[ \|\arctan f\|_X \leq \|f\|_X \leq C\|f'\|_{L^\infty}. \]

Therefore we obtain from the classical embeddings

\[ \|G_{12}\|_X \leq C(\|f\|_{L^\infty} \|f'\|_X + C\|f'\|_{L^\infty}^2) \]

\[ \leq C\|f'\|_{L^\infty} \|f'\|_X. \] (6.4)

We shall now estimate the term \( G_{11} \) in the spaces \( X \). First we use Taylor formula

\[ f(x + y) + f(x) = 2f(x) + y \int_{0}^{1} f'(x + \theta y) d\theta \]
which implies after a change of variables $y = f(x)z$ (assuming that $f(x) > 0$)

$$G_{11}(x) = 2f(x) \int_{\mathbb{R}} \frac{f'(x) + f'(x + y)}{y^2 + [2f(x) + yf_0' f'(x + \theta y)d\theta]_2} dy$$

(6.5)

$$= 2 \int_{\mathbb{R}} \frac{f'(x) + f'(x + f(x)z)}{\varphi(x, z)} dz$$

with

$$\varphi(x, z) = z^2 + \left(2 + z \int_0^1 f'(x + \theta f(x)z) d\theta \right)^2.$$  

Note that for $f(x) = 0$ we have from the definition $G_{11}(x) = 0$ which agrees with the expression (6.5) because $f'(x) = 0$. The estimate in $L^\infty$ is easy to get in view of (5.22)

$$\|G_{11}\|_{L^\infty} \leq 4\|f'\|_{L^\infty} \int_{\mathbb{R}} \|1/\varphi(\cdot, z)\|_{L^\infty} dz \leq C\left(\|f'\|_{L^\infty} + \|f''\|_{L^\infty}^3\right).$$

From the law products (4.4) and (4.8) we deduce that

$$\|G_{11}\|_X = 2 \int_{\mathbb{R}} \|f' + f' \circ (\text{Id} + zf)\|_X \|1/\varphi(\cdot, z)\|_{L^\infty} dz \leq 2 \int_{\mathbb{R}} \|f' + f' \circ (\text{Id} + zf)\|_{L^\infty} \|1/\varphi(\cdot, z)\|_X dz \leq \ell_1 + \ell_2.$$  

According to the law products (4.6) and (4.7) one may write

$$\|f' + f' \circ (\text{Id} + zf)\|_X \leq \|f'\|_X \left(1 + \mu(1 + |z|\|f'\|_{L^\infty})\right)$$

with

$$\mu(r) \triangleq \begin{cases} \ln r, & \text{if } X = C^*_K \\ r^\alpha, & \text{if } X = C^\alpha. \end{cases}$$

Observe that we can unify both cases through the estimate

$$\|f' + f' \circ (\text{Id} + zf)\|_X \leq C\|f'\|_X \left(1 + (1 + |z|\|f'\|_{L^\infty})^\alpha\right) \leq C\|f'\|_X \left(1 + |z|^\alpha\|f'\|_{L^\infty}^\alpha\right).$$

(6.6)

Putting together (6.6) and (5.22) we find for any $s \in (0, 1)$

$$\ell_1 \leq C\|f'\|_X \left(1 + \|f''\|_{L^\infty}^2\right) \int_{\mathbb{R}} \frac{1 + |z|^\alpha\|f'\|_{L^\infty}^\alpha}{1 + z^2} dz \leq C\|f'\|_X \left(1 + \|f''\|_{L^\infty}^3\right).$$

(6.7)

To estimate $\ell_2$ we use the elementary estimate

$$\|f' + f' \circ (\text{Id} + zf)\|_{L^\infty} \leq 2\|f'\|_{L^\infty}.$$  

Notice from the definition of the spaces $X$ and (5.22) that one can deduce

$$\|1/\varphi(\cdot, z)\|_X \leq \|1/\varphi(\cdot, z)\|_{L^\infty} \|\varphi(\cdot, z)\|_X$$

$$\leq C\frac{1 + \|f''\|_{L^\infty}^4}{1 + z^4} \|\varphi(\cdot, z)\|_X.$$  

(6.8)
Moreover by the law products
\[ \|\varphi(\cdot, z)\|_X \leq 2|z|\left(2 + |z|\|f'|_L^\infty\right) \int_0^1 \|f' \circ (\text{Id} + \theta z)\|_X d\theta, \]
which implies according to (6.6)
\[ \|\varphi(\cdot, z)\|_X \leq C|z|\left(2 + |z|\|f'|_L^\infty\right)\|f'|_X (1 + |z|^s\|f'|_L^\infty) \leq C(1 + |z|^{2+s})(1 + \|f'\|_{L^\infty}^{1+s})\|f'|_X. \]

Putting together this estimate with (6.8) we find
\[ (6.9) \quad \|1/\varphi(\cdot, z)\|_X \leq C\frac{(1 + \|f'\|_{L^\infty}^{5+s})\|f'|_X}{1 + |z|^{2-s}}. \]

Therefore we deduce that
\[ \ell_2 \leq C\|f'\|_{L^\infty} (1 + \|f'\|_{L^\infty}^{5+s})\|f'|_X \leq C(1 + \|f'\|_{L^\infty}^7)\|f'|_X. \]

Combining this estimate with (6.7) we obtain
\[ \|G_{11}\|_X \leq C(1 + \|f'\|_{L^\infty}^7)\|f'|_X. \]

It follows from this latter estimate, (6.4) and (6.2) that
\[ (6.10) \quad \|G_1\|_X \leq C(1 + \|f'\|_{L^\infty}^7)\|f'|_X. \]

What is left is to estimate \( G_2 \). For this purpose we write according to Taylor formula
\[ G_2(x) = \text{p.v.} \int_{\mathbb{R}} \frac{yf'(x)[f'(x)\chi(y) + 2\int_0^1 f'(x + \theta y)d\theta + f'(x + y)]}{y^2 + (f(x) + f(x + y))^2} dy \]
\[ + \text{p.v.} \int_{\mathbb{R}} \frac{\Delta_y f(x)\Delta_y f'(x)}{y^2 + (f(x) + f(x + y))^2^2} dy + 2f(x)f'(x)\int_1^{+\infty} \frac{dy}{y^2 + f'^2(x)} \]
\[ \Delta G_{2,1}(x) + G_{2,2}(x) + 2f'(x)\arctan(f(x)). \]

where \( \chi : \mathbb{R} \to \mathbb{R} \) is an even continuous compactly supported function belonging to \( X \) ad taking the value 1 on the neighborhood of \([-1, M1]\). Note that we have used in the first line the identity: for any \( x \in K \)
\[ \text{p.v.} \int_{-1}^{1} \frac{y}{y^2 + [f(x + y) + f(x)]^2} dy = \text{p.v.} \int_{\mathbb{R}} \frac{y\chi(y)}{y^2 + [f(x + y) + f(x)]^2} dy, \]
which follows from the fact that \( f(x + y) = 0, \forall y \notin [-1, 1] \). Therefore we may write
\[ G_{2,1}(x) = (f'(x))^2(T_{f}^{0,1}\chi)(x) + 2\int_0^1 f'(x)(T_{f}^{1,0}\chi)(x)d\theta + f'(x)(T_{f}^{1,1}\chi)(x) \]
where we use the notation \( T_{f}^{\alpha,\beta} \) from Theorem 5.2. The estimate of \( G_{2,1} \) in \( L^\infty \) is quite easy and follows from Theorem 5.2,
\[ \|G_{2,1}\|_{L^\infty} \leq C\|f'\|_{L^\infty}\|f'|_D(1 + \|f'\|_D^2). \]
However to estimate of $G_{2,2}$ in $L^\infty$ it is more convenient to write it in the form

$$G_{2,2}(x) = \text{p.v.} \int_{-1}^{1} \frac{\Delta_y f(x) \Delta_y f'(x)}{y^2 + (f(x) + f(x + y))^2} dy + 2f'(x) \arctan(f(x)).$$

Thus using the mean value theorem we find

$$\|G_{2,2}\|_{L^\infty} \leq C \|f'\|_{L^\infty} \|f'\|_D.$$

Combining these estimates with (4.2) we obtain

\begin{equation}
(6.11) \quad \|G_{2}\|_{L^\infty} \leq C \|f'\|_{L^\infty} \left(\|f'\|_D + \|f'\|_B^2\right).
\end{equation}

We shall now implement the estimates in $X$ and start with the term $G_{2,1}$. According to Theorem 5.2 we can unify the estimates in $C_K^*$ and $C^*$ and get the following weak estimate

\begin{equation}
(6.12) \quad \|f'T_{\alpha,\beta}^0 g\|_{X} \leq C \|g\|_{X} \left(\|f'\|_X^\frac{1}{2} \beta^{-\frac{1}{2}} + \|f'\|_X^{15}\right).
\end{equation}

From the law products (4.4) and (4.8) one has

$$\|(f')^2(T_{\alpha,\beta}^0 \chi)\|_{X} \leq \|f'\|_{L^\infty} \|f'T_{\alpha,\beta}^0 \chi\|_{X} + \|f'\|_{X} \|f'\|_{L^\infty} \|T_{\alpha,\beta}^0 \chi\|_{L^\infty}.$$ 

Hence we find

$$\|(f')^2(T_{\alpha,\beta}^0 \chi)\|_{X} \leq C \|f'\|_{L^\infty} \left(\|f'\|_X^\frac{1}{2} + \|f'\|_X^{15}\right) + \|f'\|_{X} \|f'\|_{L^\infty} \left(1 + \|f'\|_X^2\right)$$

\begin{equation}
\leq C \|f'\|_{L^\infty} \left(\|f'\|_X^\frac{1}{2} + \|f'\|_X^{15}\right).
\end{equation}

Using (6.12) we get successively

\begin{equation}
(6.13) \quad \|f'T_{\alpha,\beta}^{0,1} f'\|_{X} \leq C \|f'\|_{X} \left(\|f'\|_X^\frac{1}{2} \theta^{-\frac{1}{2}} + \|f'\|_X^{15}\right).
\end{equation}

and

$$\|f'T_{\alpha,\beta}^{1,1} f'\|_{X} \leq C \|f'\|_{X} \left(\|f'\|_X^\frac{1}{2} + \|f'\|_X^{15}\right).$$

Thus using the foregoing inequalities we deduce that

$$\|G_{2,1}\|_{X} \leq C \|f'\|_{L^\infty} \left(\|f'\|_X^\frac{1}{2} + \|f'\|_X^{15}\right) + C \|f'\|_{X} \left(\|f'\|_X^\frac{1}{2} + \|f'\|_X^{15}\right)$$

\begin{equation}
(6.14) \quad \leq C \left(\|f'\|_X^\frac{1}{2} + \|f'\|_X^{16}\right).
\end{equation}

When $X = C^s$ we can give a refined estimate for (6.13) using (5.7)

$$\|f'T_{\alpha,\beta}^{0,1} f'\|_{s} \leq C \|f'\|_X^\frac{1}{2} \left(\|f'\|_s^{\frac{3+2\alpha}{2+s}} \theta^{-\frac{1}{2}} + \|f'\|_s^{15}\right)$$

which implies that

$$\|G_{2,1}\|_{s} \leq C \|f'\|_{L^\infty} \left(\|f'\|_s^\frac{1}{2} + \|f'\|_s^{15}\right) + C \|f'\|_X^\frac{1}{2} \left(\|f'\|_s^{\frac{3+2\alpha}{2+s}} + \|f'\|_s^{15}\right)$$

\begin{equation}
(6.15) \quad \leq C \|f'\|_X^\frac{1}{2} \left(\|f'\|_s + \|f'\|_s^{16}\right).
\end{equation}

Hence one can unify (6.14) and (6.15)

\begin{equation}
(6.16) \quad \|G_{2,1}\|_{X} \leq C \|f'\|_X^\frac{1}{2} \left(\|f'\|_{X} + \|f'\|_X^{15}\right).
\end{equation}
As to the term \( G_{2,2} \) we may write
\[
G_{2,2}(x) = 2f'(x) \arctan(f(x)) + \int_{-M}^{M} \frac{[\Delta_y f(x) - yf'(x)] \Delta_y f'(x)}{y^2 + (f(x + y) + f(x))} dy
\]
\[+ \quad \text{p.v.} \int_{\mathbb{R}} \frac{yf'(x) \Delta_y f'(x)}{y^2 + (f(x + y) + f(x))} dy \]
\[\equiv 2f'(x) \arctan(f(x)) + G_{1,2}^1(x) + G_{2,2}^2(x).
\]
The last term was treated in the preceding estimates and we obtain as in (6.16)
\[
\|G_{2,2}^2\|_X \leq C\|f'\|_D^{16} \left( \|f'\|_X + \|f'\|_X^{16} \right).
\]
It remains to estimate \( G_{1,2}^1 \) which can be split into two terms
\[
G_{1,2}(x) = \hat{G}_{\text{int},r}(x) + \hat{G}_{\text{ext},r}(x)
\]
with
\[
\hat{G}_{\text{int},r}(x) = \int_{|y| \leq r} \frac{[\Delta_y f(x) - yf'(x)] \Delta_y f'(x)}{y^2 + (f(x + y) + f(x))} dy
\]
and
\[
\hat{G}_{\text{ext},r}(x) = \int_{M \geq |y| \geq r} \frac{[\Delta_y f(x) - yf'(x)] \Delta_y f'(x)}{y^2 + (f(x + y) + f(x))} dy.
\]
Now we shall proceed as in the proof of Theorem 5.2. Let \( r \in (0,1) \) and \( x_1, x_2 \in \mathbb{R} \) such that \( |x_1 - x_2| \leq r \). First it is clear that
\[
|\Delta_y f'(x)| \leq \omega_f(|y|).
\]
In addition, using Taylor formula we get
\[
|\Delta_y f(x) - yf'(x)| \leq |y| \omega_f(|y|).
\]
Therefore
\[
|\hat{G}_{\text{int},r}(x)| \leq \int_{|y| \leq r} \frac{[\omega_f(|y|)]^2}{|y|} dy.
\]
It follows that
\[
\sup_{|x_1 - x_2| \leq r} |\hat{G}_{\text{int},r}(x_2) - \hat{G}_{\text{int},r}(x_1)| \leq 4 \int_0^r \frac{[\omega_f(y)]^2}{y} dy.
\]
Hence by Fubini’s theorem
\[
\int_0^1 \sup_{|x_1 - x_2| \leq r} |\hat{G}_{\text{int},r}(x_1) - \hat{G}_{\text{int},r}(x_1)| \frac{dr}{r} \leq 4 \int_0^1 \frac{[\omega_f(y)]^2}{y} \ln y dy.
\]
From the definition and the monotonicity of the modulus of continuity one deduces that for any \( r \in (0,1) \)
\[
|\ln r \omega_f(r)| \leq \int_r^1 \frac{\omega_f(y)}{y} dy \leq \|f'\|_D
\]
which implies that

$$\int_0^1 \sup_{|x_1 - x_2| \leq r} |\hat{G}_{int,r}(x_1) - \hat{G}_{int,r}(x_1)| \frac{dr}{r} \leq 4\|f'\|_{D}^2.$$  

(6.21)

To get the suitable estimate in \(C^s\) we come back to (6.20) which gives

$$\sup_{|x_1 - x_2| \leq r} |\hat{G}_{int,r}(x_2) - \hat{G}_{int,r}(x_1)| \leq 4\|f'\|_{D}^2 \int_0^r y^{2s-1} dy \leq C\|f'\|_{D}^2 y^{2s}$$

and thus

$$\sup_{|x_1 - x_2| \leq 1} |\hat{G}_{int,r}(x_2) - \hat{G}_{int,r}(x_1)| \leq C\|f'\|_{D}^2.$$

(6.22)

As to \(\hat{G}_{ext,r}\) one writes

$$\hat{G}_{ext,r}(x_1) - \hat{G}_{ext,r}(x_2) = \int_{M \geq |y| \geq r} \frac{\mathcal{N}(x_1, y) - \mathcal{N}(x_2, y)}{\mathcal{K}(x_1)} dy + \int_{M \geq |y| \geq r} \frac{\mathcal{N}(x_2, y)\mathcal{K}(x_2, y) - \mathcal{K}(x_1, y)}{\mathcal{K}(x_1, y)\mathcal{K}(x_2, y)} dy$$

with

$$\mathcal{N}(x, y) = |\Delta_y f(x) - y f'(x)| \Delta_y f(x) \text{ and } \mathcal{K}(x, y) = y^2 + (f(x) + f(x + y))^2.$$ 

Notice that from (6.18) and (6.19) one gets

$$|\mathcal{N}(x, y) - \mathcal{N}(x_2, y)| \leq C|y| \omega_1(r) \omega_1(|y|) \text{ and } |\mathcal{N}(x, y)| \leq 2|y| \omega_1(|y|) \|f'\|_{L^\infty}.$$  

(6.23)

In addition using straightforward calculus we obtain

$$|\mathcal{K}(x_1, y) - \mathcal{K}(x_2, y)| \leq C r \|f'\|_{L^\infty} (\sqrt{|\mathcal{K}(x_1, y)|} + \sqrt{|\mathcal{K}(x_2, y)|}).$$

Thus

$$\sup_{|x_1 - x_2| \leq r} \frac{|\mathcal{N}(x_2, y)||\mathcal{K}(x_2, y) - \mathcal{K}(x_1, y)|}{\mathcal{K}(x_1, y)\mathcal{K}(x_2, y)} \leq C r \|f'\|_{L^\infty}^2 \frac{\omega_1(|y|)}{|y|^2}.$$ 

Hence we get by Fubini’s theorem and (4.3)

$$\int_0^1 \sup_{|x_1 - x_2| \leq r} \int_{\{M \geq |y| \geq r\}} \frac{|\mathcal{N}(x_1, y)|}{\mathcal{K}(x_1)} dy \frac{dr}{r} \leq \int_0^1 \int_{\{M \geq |y| \geq r\}} \frac{\omega_1(r)}{|y|^2} dy \frac{dr}{r} \leq C \|f'\|_{D}^2$$

and

$$\int_0^1 \sup_{|x_1 - x_2| \leq r} \int_{\{M \geq |y| \geq r\}} \frac{|\mathcal{N}(x_2, y)||\mathcal{K}(x_2, y) - \mathcal{K}(x_1, y)|}{\mathcal{K}(x_1, y)\mathcal{K}(x_2, y)} dy \frac{dr}{r} \leq C \|f'\|_{L^\infty}^2 \int_0^1 \int_{\{M \geq |y| \geq r\}} \frac{\omega_1(|y|)}{|y|^2} dy dr \leq C \|f'\|_{L^\infty}^2 \|f'\|_{D}.$$ 

Finally we obtain

$$\int_0^1 \sup_{|x_1 - x_2| \leq r} |\hat{G}_{ext,r}(x_1) - \hat{G}_{ext,r}(x_2)| \frac{dr}{r} \leq C \|f'\|_{D}^2 + C \|f'\|_{L^\infty}^2 \|f'\|_{D}.$$  

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As to the estimate in $C^s$ we use (6.23) which implies that
\[
\int_{\{r \leq |y| \leq M\}} \frac{|N(x_1, y) - N(x_2, y)|}{K(x_1)} \, dy \leq C\|f\|_{s} r^s \int_{\{r \leq |y| \leq M\}} \frac{\omega_f(|y|)}{|y|} \, dy 
\]
\[
\leq C\|f\|_{s} f' \|D r^s \|
\]
and
\[
\int_{\{M \geq |y| \geq r\}} \frac{|N(x_2, y)| |K(x_2, y) - K(x_1, y)|}{K(x_1, y)K(x_2, y)} \, dy \leq C\|f\|_{s} f' \|\hat{\omega}_f\|_{L^\infty} \int_{\{M \geq |y| \geq r\}} \frac{\hat{\omega}_f(|y|)}{|y|^{2-s}} \, dy 
\]
\[
\leq C\|f\|_{s} f' \|D r^s \|. 
\]
It follows from Sobolev embedding $C^s \hookrightarrow L^\infty$ that
\[
\frac{\sup_{|x_1 - x_2| \leq r} \left| \hat{G}_{ext,r}(x_1) - \hat{G}_{ext,r}(x_2) \right|}{|x_1 - x_2|^s} \leq C\|f\|_{D} \|f\|_{s} + C\|f\|_{D} \|f\|_{s}^2. 
\]
Combining the foregoing estimates with (6.21) and (6.22) we deduce
\[
\|G_{2,2}\|_{C^s} \leq C\|f\|_{D} \left(\|f\|_{C^s} + \|f\|_{C^s}^2\right). 
\]
Putting together this estimate with (6.16) and (6.17) we get
(6.24)
\[
\|G_{2}\|_{C^s} \leq C\|f\|_{D} \left(\|f\|_{C^s} + \|f\|_{C^s}^1 \|f\|_{C^s}^1\right). 
\]
Now using (6.10) and (6.24) we find
\[
\|G\|_{C^s} \leq C\|f\|_{D} \left(1 + \|f\|_{D}^2\right) + C\|f\|_{D} \left(\|f\|_{C^s} + \|f\|_{C^s}^1 \|f\|_{C^s}^1\right) 
\]
\[
\leq C\left(1 + \|f\|_{D}^2\right) \left(\|f\|_{C^s} + \|f\|_{C^s}^1 \|f\|_{C^s}^1\right) 
\]
which ends the proof of Proposition 6.1.

6.2. **A priori estimates.** The aim of this section is to establish weak and strong a priori estimates for solutions to the equation (2.1). This part is the cornerstone of the local well-posedness theory. The main result of this section reads as follows.

**Proposition 6.2.** Let $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ be a smooth solution for the graph equation (2.1). Assume that the initial data is positive and with compact support $K_0$. Then the following assertions hold true.

1. For any $t \in [0, T]$, the function $f_t$ is positive and
   \[
   \forall t \in [0, T], \quad \|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty}.
   \]
2. For any $t \in [0, T]$, we have
   \[
   \|f(t)\|_{L^1} = \|f_0\|_{L^1} e^{-t}.
   \]
3. The support supp $f_t$ is contained in the convex hull of $K_0$, that is
   \[
   \forall t \in [0, T], \quad \text{supp } f(t) \subset \text{Conv } K_0.
   \]
4. Set $X = C^s_{K}$ or $X = C^s_{K}$, with $s \in (0, 1)$. If $f_0 \in X$ then there exists $T$ depending only on $\|f_0\|_{X}$ such that $f' \in L^\infty([0, T]; X)$. 37
Proof. (1) To get the first part about the persistence of the positivity of we shall prove that

\[ u_2(t, x) = f(t, x)U(t, x) \quad \forall x \in \mathbb{R}, \]

with

\[ \|U(t)\|_{L^\infty} \leq C(1 + \|f'(t)\|_D^6) \]

and \( C \) being a constant depending only on the size of the support of \( f_t \). Note from the point (2) of the current proposition that the support of \( f_t \) is contained in a fixed compact and therefore the constant \( C \) can be taken independent of the time variable. Assume for a while (6.25) and let us see how to propagate the positivity. Denote by \( \psi \) the flow associated to the velocity \( u_1 \), that is, the solution of the ODE

\[ \partial_t \psi(t, x) = u_1(t, \psi(t, x)), \quad \psi(0, x) = x. \]

Recall that

\[ u_1(t, x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ \arctan \left( \frac{f(t, x + y) - f(t, x)}{y} \right) - \arctan \left( \frac{f(t, x + y) + f(t, x)}{y} \right) \right\} dy. \]

Set

\[ \eta(t, x) = f(t, \psi(t, x)) \]

then

\[ \partial_t \eta(t, x) = u_2(t, \psi(t, x)) = \eta(t, x)U(t, \psi(t, x)). \]

Consequently

\[ \eta(t, x) = f_0(x)e^{\int_0^t U(\tau, \psi(\tau, x))d\tau}. \]

Since the flow \( \psi(t) : \mathbb{R} \to \mathbb{R} \) is a diffeomorphism we get the representation

\[ f(t, x) = f_0(\psi^{-1}(t, x))e^{\int_0^t U(\tau, \psi(\tau, \psi^{-1}(t, x)))d\tau}. \]

As an immediate consequence we get the persistence through the time of the positivity of the solution. Let us now come back to the proof of the identity (6.25). To alleviate the notation we remove the variable \( t \) from the functions. Applying Taylor formula to the function

\[ \tau \in [0, f(x)] \mapsto g(\tau) \triangleq \log \frac{y^2 + (\tau - f(x + y))^2}{y^2 + (\tau + f(x + y))^2} \]

yields to

\[-2\pi u_2(x) = f(x) \int_0^1 \int_{-M}^M \frac{f(x + y) - \tau f(x)}{y^2 + [f(x + y) - \tau f(x)]^2} d\tau dy \]

\[ + f(x) \int_0^1 \int_{-M}^M \frac{f(x + y) + \tau f(x)}{y^2 + [f(x + y) + \tau f(x)]^2} d\tau dy \]

\[ \triangleq f(x)V_1(x) + f(x)V_2(x). \]
Using once again Taylor formula we get the following expressions

\[
V_1(x) = \int_0^1 \int_{-M}^M \frac{(1-\tau)f(x)}{y^2 + [(1-\tau)f(x) + y \int_0^1 f'(x + \theta y)d\theta]^2} \, d\tau dy + \text{p.v.} \int_0^1 \int_{-M}^M \frac{y \int_0^1 f'(x + \theta y)d\theta}{y^2 + [f(x + y) - \tau f(x)]^2} \, d\tau dy \\
\triangleq V_{1,1}(x) + V_{1,2}(x)
\]

and

\[
V_2(x) = \int_0^1 \int_{-M}^M \frac{(1+\tau)f(x)}{y^2 + [(1+\tau)f(x) + y \int_0^1 f'(x + \theta y)d\theta]^2} \, d\tau dy + \text{p.v.} \int_0^1 \int_{-M}^M \frac{y \int_0^1 f'(x + \theta y)d\theta}{y^2 + [f(x + y) + \tau f(x)]^2} \, d\tau dy \\
\triangleq V_{2,1}(x) + V_{2,2}(x).
\]

To estimate \(V_{1,1}\) and \(V_{2,1}\) we can assume that \(f(x) > 0\). Then making the change of variables \(z \mapsto y = (1-\tau)f(x)z\) leads to

\[
(6.29) \quad V_{1,1}(x) = \int_0^1 \int_{-M}^M \frac{d\tau dz}{z^2 + [1 + z \int_0^1 f'(x + \theta(1-\tau)f(x)z)d\theta]^2}.
\]

Using (5.22) we deduce that

\[
(6.30) \quad \|V_{1,1}\|_{L^\infty} \leq C(1 + \|f'\|^2_{L^\infty}).
\]

Similarly we get

\[
(6.31) \quad \|V_{2,1}\|_{L^\infty} \leq C(1 + \|f'\|^2_{L^\infty}).
\]

Let us now bound \(V_{j,2}, j = 1, 2\) First by symmetry we write

\[
V_{1,2}(x) = \int_0^1 \int_{-M}^M \frac{y \int_0^1 f'(x + \theta y)d\theta [f(x - y) - f(x + y)] \psi_\tau(x, y)}{y^2 + [f(x + y) - \tau f(x)]^2} \, dyd\tau \\
+ \int_0^1 \int_{-M}^M \frac{y \int_0^1 [f'(x + \theta y) - f'(x - \theta y)]d\theta}{y^2 + [f(x - y) - \tau f(x)]^2} \, dyd\tau
\]

where

\[
\psi_\tau(x, y) = f(x + y) + f(x - y) - 2\tau f(x) \\
= 2(1 - \tau)f(x) + y \int_0^1 [f'(x + \theta y) - f'(x - \theta y)]d\theta.
\]

Thus

\[
\|V_{1,2}\|_{L^\infty} \leq C \int_0^1 \int_{-M}^M \frac{\|f'\|_{L^\infty}^2 y^2 [(1 - \tau)f(x) + y \omega f'(y)]}{y^2 + [f(x + y) - \tau f(x)]^2} \, dyd\tau \\
+ C \int_0^1 \int_{-M}^M \frac{\omega f'(y)}{y} \, dyd\tau.
\]
Similarly to $V_{1,1}$ one gets
\[
\int_0^1 \int_0^M \frac{y^2(1 - \tau)f(x) \, dy \, d\tau}{\left(y^2 + [f(x + y) - \tau f(x)]^2\right)\left(y^2 + [f(x - y) - \tau f(x)]^2\right)} \leq C\left(1 + \|f'\|^4_L\right).
\]
It follows that
\[
\|V_{1,2}\|_{L^\infty} \leq C\|f'\|^2_L\left(1 + \|f'\|^4_L + \int_0^M \frac{\omega f(y) \, dy}{y}\right) + C\|f'\|_D.
\]
(6.32)
The estimate of $V_{2,2}$ can be done in a similar way and one obtains
\[
\|V_{2,2}\|_{L^\infty} = C\|f'\|^2_L\left(1 + \|f'\|^4_L + \|f'\|_D\right) + C\|f'\|_D.
\]
(6.33) Combining both last estimates with (6.30) and (6.31) we finally get according to the embedding (4.2)
\[
\|U\|_{L^\infty} \leq C\left(1 + \|f'\|^6_D\right)
\]
where the constant $C$ depends only on the size of the support of $f$.

Now let us establish the maximum principle. From (2.2) combined with the positivity of $f_t$ one gets
\[
\forall t \in [0, T], \forall x \in \mathbb{R} \quad u_2(t, x) \leq 0.
\]
Coming back to (6.27) we deduce that
\[
\partial_t \eta(t, x) \leq 0
\]
which implies in turn that
\[
\forall t \in [0, T], \forall x \in \mathbb{R} \quad f(t, x) \leq f_0(\psi^{-1}(t, x)).
\]
Combined with the positivity of $f(t)$ we deduce immediately the maximum principle
\[
\forall t \in [0, T], \quad \|f(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty}.
\]
Now we intend to provide more refined identity that we shall use later in studying the asymptotic behavior of the solution. Actually we have
\[
u_2(t, x) = -f(t, x)(1 + R(t, x)),
\]
with
\[
\|R(t)\|_{L^\infty} \leq C\|f'(t)\|_{D}\left(1 + \|f'(t)\|^5_{L^\infty}\right).
\]
First note that $R = \sum_{i,j=1}^2 V_{i,j}$. The estimates of $V_{1,2}$ and $V_{2,2}$ are done in (6.32) and (6.33). However to deal with $V_{1,1}$ and similarly $V_{2,1}$ we return to the expression (6.29). Set
\[
\tau \mapsto K(\tau) = \frac{1}{z^2 + [1 + z\tau]^2}.
\]
Easy computations using (5.22) show the existence of a positive constant $C$ such that
\[
\forall \tau, z \in \mathbb{R}, \quad |K'(\tau)| = \frac{2|z||1 + z\tau|}{(z^2 + [1 + z\tau]^2)^2} \leq \frac{1}{z^2 + [1 + z\tau]^2} \leq C\frac{1 + \tau^2}{1 + z^2}.
\]
Applying the mean value theorem yields
\[ |K(\tau) - \frac{1}{1 + z^2}| \leq C|\tau|\frac{1 + \tau^2}{1 + z^2}. \]
Therefore we get
\[ \left| V_{1,1}(x) - \int_0^1 \int M(1-\tau)f(x) \frac{dzd\tau}{1 + z^2} \right| \leq C\|f'\|_{L\infty} \left(1 + \|f'\|_{L\infty}^2\right), \]
which implies that
\[ |V_{1,1}(x) - \pi| \leq C\|f'\|_{L\infty} \left(1 + \|f'\|_{L\infty}^2\right) + C\|f\|_{L\infty}. \]
Similarly we obtain
\[ |V_{2,1}(x) - \pi| \leq C\|f'\|_{L\infty} \left(1 + \|f'\|_{L\infty}^2\right) + C\|f\|_{L\infty}. \]
Putting together (6.32), (6.33), (6.35), (6.36) we get (6.34).

(2) Integrating the equation (1.2) in the space variable we get after integration by parts
\[ \frac{d}{dt} \int_{\mathbb{R}} \rho(t, x) dx = \int_{\mathbb{R}} \text{div} v(t, x) \rho(t, x) dx \]
\[ = - \int_{\mathbb{R}} \rho^2(t, x) dx \]
\[ = - \int_{\mathbb{R}} \rho(t, x) dx \]
where in the last line we have used that for the characteristic function one has \( \rho^2 = \rho. \) The time decay follows then easily.

(3) According to the representation of the solution given by (6.28) we have easily that the support of \( f(t) \) is the image by the flow \( \psi(t) \) of the initial support, that is,
\[ K_t = \psi(t, K_0). \]
We have to check that if \( K_0 \subset [a, b] \), with \( a < b \), then \( K_t \subset [a, b] \). To do so it is enough to prove that
\[ \psi(t, [a, b]) \subset [a, b]. \]
This means somehow that the flow is contractive. As \( \psi(t) \) is an homeomorphism then necessary \( \psi(t, [a, b]) = [\psi(t, a), \psi(t, b)]. \) Hence to get the desired inclusion it suffices to establish that
\[ a_t \triangleq \psi(t, a) \geq a \quad \text{and} \quad b_t \triangleq \psi(t, b) \leq b. \]
This reduces to study the derivative in time of \( a_t \) and \( b_t \). First one has
\[ a_t = u_1(t, a_t) \quad \text{and} \quad b_t = u_1(t, b_t). \]
Since \( f(t, y) = 0, \forall y \notin (a_t, b_t) \) and \( f_t \) is positive everywhere then
\[ u_1(t, a_t) = \frac{1}{\pi} \int_0^{b_t - a_t} \arctan \left( \frac{f_t(a_t + y)}{y} \right) dy \geq 0. \]
Hence \( a_t \geq 0 \) and therefore \( a_t \geq a \), for any \( t \in [0; T] \).
Similarly we get
\[ u_1(t, b_t) = -\frac{1}{\pi} \int_0^{b_t - a_t} \arctan \left( \frac{f_t(b_t - y)}{y} \right) dy \leq 0. \]
which implies that $b_i \leq b$, for any $t \in [0; T]$. This ends the proof of the point (2).

(4) Recall from (2.3) and (2.6) that $g \triangleq f'$ satisfies the equation
\[ \partial_t g + u_1 \partial_t g = \frac{1}{2\pi} (F - G). \]
Set $h(t, x) = g(t, \psi(t, x))$, where $\psi$ is the flow defined in (6.26). Then
\[ \partial_t h(t, x) = \frac{1}{2\pi} \left( F(t, \psi(t, x)) - G(t, \psi(t, x)) \right). \]
Thus
\[ g(t, x) = g_0(\psi^{-1}(t, x)) + \frac{1}{2\pi} \int_0^t (F - G)(\tau, \psi(\tau, \psi^{-1}(t, x))) d\tau. \]
Recall the classical estimate
\[ \left\| \partial_x \left[ \psi(\tau, \psi^{-1}(t, \cdot)) \right] \right\|_{L^\infty} \leq e^{\int_0^\tau \| \partial_x u_1(t', \cdot) \|_{L^\infty} d\tau} \]
that we may combine with the composition laws (4.6) and (4.7) to get
\[ \| g(t) \|_X \leq C e^{V(t) \left[ \| g_0 \|_X + \int_0^t \| (F - G)(\tau) \|_X d\tau \right]}, \quad V(t) \triangleq \int_0^t \| \partial_x u_1(\tau) \|_{L^\infty} d\tau. \]
To estimate $\| \partial_x u_1(t) \|_{L^\infty}$ we come back to (2.4). The first integral term can be restricted to a compact set $[-M, M]$ and thus
\[ \left| \text{p.v.} \int_{-M}^M \frac{f'(x + y) - f'(x)}{y^2 + (f(x + y) + f(x))^2} dy \right| \leq 2 \int_0^M \frac{\omega f'(y)}{y} dy \leq C \| f' \|_D. \]
As to the second term, the integral can be restricted to $[-M, M]$ and we simply write
\[ \text{p.v.} \int_{\mathbb{R}} \frac{f'(x + y) + f'(x)}{y^2 + (f(x + y) + f(x))^2} dy = \text{p.v.} \int_{-M}^M \frac{f'(x + y) - f'(x)}{y^2 + (f(x + y) + f(x))^2} dy \]
\[ + \quad \text{p.v.} \int_{\mathbb{R}} \frac{2f'(x)}{y^2 + (f(x + y) + f(x))^2} dy. \]
The first term of the right-hand side is controlled as before
\[ \left| \text{p.v.} \int_{-M}^M \frac{f'(x + y) - f'(x)}{y^2 + (f(x + y) + f(x))^2} dy \right| \leq C \| f' \|_D. \]
However for the last term it can be estimated as in the proof of Theorem 5.2-(1). One gets in view of (5.8), (5.11) and (5.12)
\[ \left| \text{p.v.} \int_{\mathbb{R}} \frac{y}{y^2 + (f(x + y) + f(x))^2} dy \right| \leq C \left( \| f' \|_{L^\infty}^2 + \| f' \|_{L^\infty} \| f' \|_D + \| f' \|_{L^\infty} \right). \]
Hence using the embedding $X \hookrightarrow C^*_K \hookrightarrow L^\infty$ we find
\[ \| \partial_x u_1(t) \|_{L^\infty} \leq C \left( \| f' \|_D + \| f' \|_{L^\infty} \| f' \|_D \right) \]
(6.40)
\[ \leq C \left( \| f'(t) \|_X + \| f'(t) \|_{L^\infty}^2 \right), \]
which implies that
\[ V(t) \leq Ct \left( \| f' \|_{L^\infty}^2 X + \| f' \|_{L^\infty}^2 X \right). \]
Using Proposition 6.1 we obtain
\begin{equation}
\| (F - G)(t) \|_X \leq C \left( \| f'(t) \|_X + \| f'(t) \|_{L^\infty} \right).
\end{equation}
Plugging (6.41) and (6.42) into (6.39) we obtain
\[ \| f' \|_{L^\infty} \leq e^{CT} \left( \| f' \|_{L^\infty} + \| f' \|_{L^\infty} \right) \left[ \| f_0' \|_X + T \left( \| f' \|_{L^\infty} + \| f' \|_{L^\infty} \right) \right]. \]
This shows the existence of small $T$ depending only on $\| f_0' \|_X$ and such that
\[ \| f' \|_{L^\infty} \leq 2 \| f_0' \|_X, \]
which ends the proof of the proposition. □

6.3. **Scheme construction of the solutions.** This section is devoted to the construction of the solutions to (2.3) in short time. Before giving a precise description about the method used here and based on a double regularization, let us explain the big lines of the strategy. The a priori estimates developed in the previous sections require some rigid properties like the confinement of the support, the positivity of the solution and some nonlinear effects in order to control some singular terms as it was mentioned in Theorem 5.2. So it appears so hard to find a linear scheme that respects all of those constraints. The idea is to proceed with a nonlinear double regularization scheme. First, we fix a small parameter $\varepsilon > 0$ used to regularize the singularity of the kernels around the origin, and second we elaborate an iterative nonlinear scheme giving rise to a family of solutions $(f_\varepsilon^n)_n$ that may violate some of the mentioned constraints. With this scheme we are able to derive a priori estimates uniformly with respect to $n$ during a short time $T_\varepsilon > 0$, but this time may shrink to zero as $\varepsilon$ goes to zero. By compactness arguments we prove that this approximate solutions $(f_\varepsilon^n)_n$ converges as $n$ goes to infinity to a solution $f^\varepsilon$ living in our function space during the time interval $[0, T_\varepsilon]$. Now the function $f^\varepsilon$ satisfies a modified nonlinear problem but the important fact is that all the a priori estimates developed in the preceding sections hold uniformly on $\varepsilon$. This allows by a classical procedure to implement the bootstrap argument and prove that the family $(f^\varepsilon)_\varepsilon$ is actually defined on some time interval $[0, T]$ independently on $\varepsilon$. To conclude it remains to pass to the limit when $\varepsilon$ goes to zero and this allows to construct a solution for our initial problem.

Let us now give more details about this double scheme regularization. Consider the iterative scheme
\begin{equation}
\begin{cases}
\partial_t f_\varepsilon^{n+1} + u_1^\varepsilon(f_\varepsilon^n) \partial_x f_\varepsilon^{n+1} = u_2^\varepsilon(f_\varepsilon^{n+1}), & n \in \mathbb{N}, \\
f_\varepsilon^n(t, x) = f_0(x) \\
\quad f_\varepsilon^{n+1}(0, x) = f_0(x)
\end{cases}
\end{equation}
with
\begin{equation}
\begin{align*}
u_1^\varepsilon(g)(t, x) & \triangleq \frac{1}{2\pi} \chi(x) \int_{|y| \geq \varepsilon} \chi(y) \left\{ \arctan \left( \frac{g(t, x + y) - g(t, x)}{y} \right) + \arctan \left( \frac{g(t, x + y) + g(t, x)}{y} \right) \right\} dy \\
u_2^\varepsilon(g)(t, x) & \triangleq \frac{1}{4\pi} \int_{|y| \geq \varepsilon} \chi(y) \log \left( \frac{y^2 + (g(t, x + y) - g(t, x))^2}{y^2 + (g(t, x + y) + g(t, x))^2} \right) dy.
\end{align*}
\end{equation}
The function $\chi$ is a positive smooth cut-off function taking the value 1 on some interval $[-M, M]$ such that
\[ K_0, K_0 - K_0 \subset [-M, M] \]
with $K_0$ being the convex hull of supp $f_0$. The function $\chi$ is introduced in order to guarantee the convergence of the integrals. We shall see later by using the support structure of the solutions that one can in fact remove this cut-off function. Denote by
\[ \mathcal{E}_T = \left\{ f; f \in L^\infty([0, T] \times \mathbb{R}), f' \in L^\infty([0, T], X) \right\} \]
equipped with the norm
\[ \|f\|_{\mathcal{E}_T} = \|f\|_{L^\infty([0,T] \times \mathbb{R})} + \|\partial_x f\|_{L^\infty([0,T],X)} \]
where \(X\) denotes Dini space \(C^s\) or Hölder spaces \(C^s(\mathbb{R}), 0 < s < 1\) and for the simplicity we shall during this part work only with Hölder space. We intend to explain the approach without giving all the details, because some of them are classical. Using the characteristics method, one can transform the equation (6.43) into a fixed pint problem
\[ f_{n+1} = N^\varepsilon_n(f_{n+1}) \quad \text{with} \quad N(f)(t, x) = f_0(\psi_{n,\varepsilon}^{-1}(t, x)) + \int_0^t u_2^\varepsilon(f)\left(\tau, \psi_{n,\varepsilon}(\tau, \psi_{n,\varepsilon}^{-1}(t, x))\right) d\tau \]
with \(\psi_{n,\varepsilon}\) being the one-dimensional flow associated to \(u_1^\varepsilon(f_n^\varepsilon)\), that is, the solution of the ODE
(6.45)
\[ \psi_{n,\varepsilon}(t, x) = x + \int_0^t u_1^\varepsilon(f_n^\varepsilon)(\tau, \psi_{n,\varepsilon}(\tau, x)) d\tau. \]
It is plain that
\[ \|N(f)(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty} + \int_0^t \|u_2^\varepsilon(f)(\tau)\|_{L^\infty} d\tau. \]
Applying the elementary inequality: for \(a > 0, b, c \in \mathbb{R}_+\)
\[ \left| \log \left( \frac{a + b}{a + c} \right) \right| \leq \frac{b + c}{a} \]
we get from (6.44) that
\[ |u_2^\varepsilon(f)(t, x)| \leq \frac{1}{4\pi} \int_{|y| \geq \varepsilon} \chi(y) \frac{f^2(t, x + y) + f^2(t, x)}{y^2} dy \leq C\varepsilon^{-2} \|f(t)\|^2_{L^\infty}. \]
It follows that
(6.46)
\[ \|N(f)\|_{L^\infty} \leq \|f_0\|_{L^\infty} + C\varepsilon^{-2} T \|f\|^2_{L^\infty}. \]
We shall move to the estimate of \(\|\partial_x N(f)\|_{L^\infty T X}\). Let us first start with the estimate of \(\|\partial_x f_0(\psi_{n,\varepsilon}^{-1})\|_{L^\infty T X}\). By straightforward computations using law products (4.8), composition laws (4.6) in Hölder spaces and the following classical estimates on the flow,
\[ \|\partial_x \psi_{n,\varepsilon}^{\pm 1}\|_{L^\infty T X} \leq C e^{C\|\partial_x (u_1^\varepsilon(f_n^\varepsilon))\|_{L^1 T X} \left( 1 + \|\partial_x (u_1^\varepsilon(f_n^\varepsilon))\|_{L^1 T X} \right)} \]
once we get
\[ \|\partial_x f_0(\psi_{n,\varepsilon}^{-1})\|_{L^\infty T X} \leq \|\partial_x f_0\|_{L^\infty T X} \|\partial_x (\psi_{n,\varepsilon}^{-1})\|_{L^\infty T X} \leq C \|\partial_x f_0\|_{L^\infty T X} \left( 1 + \|\partial_x (u_1^\varepsilon(f_n^\varepsilon))\|_{L^1 T X} \right). \]
Differentiating the expression of \(u_1^\varepsilon(f_n^\varepsilon)\) in (6.44) and making standard estimates we get easily
\[ \|\partial_x \{u_1^\varepsilon(f_n^\varepsilon)(t)\}\|_{X} \leq C + C\varepsilon^{-1} \|\partial_x f_n^\varepsilon(t)\|_{X} + C\varepsilon^{-3} \|\partial_x f_n^\varepsilon(t)\|_{L^\infty} \|f_n^\varepsilon(t)\|^2_{X} \]
\[ \leq C + C\varepsilon^{-1} \|f_n^\varepsilon\|_{X} + C\varepsilon^{-3} \|f_n^\varepsilon\|_{X}^3, \]
where we have used
\[ \left\| \frac{1}{y^2 + f^2} \right\|_{X} \leq C \|f\|^2_{X} y^{-4}. \]
Therefore
(6.47)
\[ \|\partial_x f_0(\psi_{n,\varepsilon}^{-1})\|_{L^\infty T X} \leq C \|\partial_x f_0\|_{X} e^{CT + C\varepsilon^{-1} T \|f_n^\varepsilon\|_{X} + C\varepsilon^{-3} T \|f_n^\varepsilon\|_{X}^3}. \]
In order to satisfy mutually the conditions (6.49) and (6.50) it suffices to take
then we get the uniform estimates
\[ \forall (6.53) \]
Combining this estimate with law products and (6.48) we deduce that
\[ \|N\|_{x} \leq C \left( \|f_{0}\|_{L^\infty} + \|\partial_{x}f_{0}\|_{x} + T \varepsilon^{-2} \|f\|_{\xi_{T}}^{2} + T \varepsilon^{-4} \|f\|_{\xi_{T}}^{4} \right) e^{CT\varepsilon^{-3} \|f_{n}\|_{\xi_{T}}^{3}}. \]

Putting together this estimate with (6.47) we find that
\[ \|N(f)\|_{x} \leq C \left( \|f_{0}\|_{L^\infty} + \|\partial_{x}f_{0}\|_{x} + T \varepsilon^{-2} \|f\|_{\xi_{T}}^{2} + T \varepsilon^{-4} \|f\|_{\xi_{T}}^{4} \right) e^{CT\varepsilon^{-3} \|f_{n}\|_{\xi_{T}}^{3}}. \]

We can assume that \( 0 < T \leq 1 \) and then
\[ \|N(f)\|_{x} \leq C \left( \|f_{0}\|_{L^\infty} + \|\partial_{x}f_{0}\|_{x} + T \varepsilon^{-2} \|f\|_{\xi_{T}}^{2} + T \varepsilon^{-4} \|f\|_{\xi_{T}}^{4} \right) e^{CT\varepsilon^{-3} \|f_{n}\|_{\xi_{T}}^{3}}. \]

Consider now the closed ball
\[ B = \left\{ f \in E_{T}, \|f\|_{x} \leq 2C \left( \|f_{0}\|_{L^\infty} + \|\partial_{x}f_{0}\|_{x} \right) e^{CT\varepsilon^{-3} \|f_{n}\|_{\xi_{T}}} \right\}, \]
then if we choose \( T \) such that
\[ (6.49) \quad 16C^{3}\varepsilon^{-4}T \left( \|f_{0}\|_{L^\infty} + \|\partial_{x}f_{0}\|_{x} \right)^{3} e^{5CT\varepsilon^{-3} \|f_{n}\|_{\xi_{T}}^{3}} \leq 1 \]
then \( N : B \to B \) is well-defined and proceeding as before we can show under this condition that it is also a contraction. This implies the existence in this ball of a unique solution to the fixed point problem and so one can construct a solution \( f_{n+1}^{\varepsilon} \in E_{T} \) to (6.43) and we have the estimates
\[ \forall n \in \mathbb{N}, \|f_{n+1}^{\varepsilon}\|_{x} \leq 2C \left( \|f_{0}\|_{L^\infty} + \|\partial_{x}f_{0}\|_{x} \right) e^{CT\varepsilon^{-3} \|f_{n}\|_{\xi_{T}}^{3}}. \]

Now we select \( T \) such that it satisfies also
\[ (6.50) \quad 64C^{4} \left( \|f_{0}\|_{L^\infty} + \|\partial_{x}f_{0}\|_{x} \right)^{3} T \varepsilon^{-3} \leq \ln 2 \]
then we get the uniform estimates
\[ \forall n \in \mathbb{N}, \|f_{n}\|_{x} \leq 4C \left( \|f_{0}\|_{L^\infty} + \|\partial_{x}f_{0}\|_{x} \right). \]

In order to satisfy mutually the conditions (6.49) and (6.50) it suffices to take
\[ (6.51) \quad T_{\varepsilon} := C_{0}\varepsilon^{2} \]
with \( C_{0} \) depending only on \( \|f_{0}\|_{L^\infty} + \|\partial_{x}f_{0}\|_{x} \) such that
\[ (6.52) \quad \forall n \in \mathbb{N}, \|f_{n}\|_{x} \leq 4C \left( \|f_{0}\|_{L^\infty} + \|\partial_{x}f_{0}\|_{x} \right). \]

Now we shall check that we can remove the localization in space through the cut-off function \( \chi \). To do so, it suffices to get suitable information on the support of \( (f_{n}^{\varepsilon}) \). We shall prove that
\[ (6.53) \quad \forall n \in \mathbb{N}, \text{ supp } f_{n}^{\varepsilon}(t) \subset K_{0} \]
where $K_0$ is the convex hull of the support of $f_0$. Before giving the proof let us assume for a while this property and see how to get rid of the localizations in the velocity fields. From the expression of $u^\varepsilon_n(t)$ one has

$$ u^\varepsilon_n(t) = \frac{1}{4\pi} \int_{|y| \geq \varepsilon} \log \left( \frac{y^2 + (f_{n+1}^\varepsilon(t,x+y) - f_{n+1}^\varepsilon(t,x))^2}{y^2 + (f_{n+1}^\varepsilon(t,x+y) + f_{n+1}^\varepsilon(t,x))^2} \right) dy $$

$$ - \frac{1}{4\pi} \int_{|y| \geq \varepsilon} [1 - \chi(y)] \log \left( \frac{y^2 + (f_{n+1}^\varepsilon(t,x+y) - f_{n+1}^\varepsilon(t,x))^2}{y^2 + (f_{n+1}^\varepsilon(t,x+y) + f_{n+1}^\varepsilon(t,x))^2} \right) dy. $$

Since $\forall x \notin K_0$ we have $f_{n+1}(t,x) = 0$ and hence $u^\varepsilon_n(t) = 0$. Thus $\forall x \in K_0$ one has

$$ \int_{|y| \geq \varepsilon} [1 - \chi(y)] \log \left( \frac{y^2 + (f_{n+1}^\varepsilon(t,x+y) - f_{n+1}^\varepsilon(t,x))^2}{y^2 + (f_{n+1}^\varepsilon(t,x+y) + f_{n+1}^\varepsilon(t,x))^2} \right) dy = 0 $$

because $\chi = 1$ on $K_0 - K_0$. Now we claim that in the advection term $u^\varepsilon_n(t) \partial_x f_{n+1}^\varepsilon(t)$ of the equation (6.43) one can remove the cut-off function. Since $\partial_x f_{n+1}^\varepsilon(t) = 0$ outside $K_0$ then one gets immediately $\chi(x) \partial_x f_{n+1}^\varepsilon(t) = \partial_x f_{n+1}^\varepsilon(t)$. Similarly one has

$$ u^\varepsilon_n(t,x) = \frac{1}{2\pi} \int_{|y| \geq \varepsilon} \left\{ \arctan \left( \frac{g(t,x+y) - g(t,x)}{y} \right) + \arctan \left( \frac{g(t,x+y) + g(t,x)}{y} \right) \right\} dy $$

$$ - \frac{1}{2\pi} \int_{|y| \geq \varepsilon} (1 - \chi(y)) \left\{ \arctan \left( \frac{g(t,x+y) - g(t,x)}{y} \right) + \arctan \left( \frac{g(t,x+y) + g(t,x)}{y} \right) \right\} dy $$

and for $x \in K_0$ it is clear that

$$ \int_{|y| \geq \varepsilon} (1 - \chi(y)) \left\{ \arctan \left( \frac{g(t,x+y) - g(t,x)}{y} \right) + \arctan \left( \frac{g(t,x+y) + g(t,x)}{y} \right) \right\} dy = 0 $$

Now let us come back to the proof of (6.53) and provide further qualitative properties. Now similarly to the identity (6.25) one obtains

$$ u^\varepsilon_n(t,x) = f_{n+1}^\varepsilon(t,x)(1 + U_{n+1}(t,x)), \quad \|U_{n+1}(t)\|_{L^\infty} \leq C(1 + \|f_{n+1}(t)\|^6_D). $$

So following the same line of the proof of Proposition 6.2 we get a similar formula to (6.28) which implies that the positivity result,

$$ f_{n+1}(t,x) \geq 0 $$

where we have used in particular that the initial data $f_{n+1}^\varepsilon(0,x) = f_0(x) \geq 0$. Thus we obtain

$$ \forall n \in \mathbb{N}, \quad f_n(t,x) \geq 0. $$

As $u^\varepsilon_n(t,x) \leq 0$ then following the same proof of Proposition 6.2 we get the maximum principle

$$ \forall n \in \mathbb{N}, \quad \|f_n(t)\|_{L^\infty} \leq \|f_0\|_{L^\infty}. $$
The proof of the confinement of the support (6.53) follows exactly the same lines of the proof of Proposition 6.2-(3). Now we shall study the strong convergence of the sequence \((f_n^\epsilon)\). Set 
\[ \theta_n^\epsilon(t, x) := f_{n+1}^\epsilon(t, x) - f_n^\epsilon(t, x). \]

Then
\[ \partial_t \theta_n^\epsilon + u_1^\epsilon(f_{n+1}^\epsilon) \partial_x \theta_n^\epsilon = - \left[ u_1^\epsilon(f_{n+1}^\epsilon) - u_1^\epsilon(f_n^\epsilon) \right] \partial_x f_{n+1}^\epsilon + u_2^\epsilon(f_{n+2}^\epsilon) - u_2^\epsilon(f_{n+1}^\epsilon). \]

According to the mean value theorem one has for \(a > 0, x, y \in \mathbb{R}\)
\[ |\arctan(x) - \arctan(y)| \leq |x - y| \quad \text{and} \quad |\log(a + |x|) - \log(a + |y|)| \leq |x - y|a^{-1}, \]

which imply that
\[ \|u_1^\epsilon(f_{n+1}^\epsilon)(t) - u_1^\epsilon(f_n^\epsilon)(t)\|_{L^\infty} \leq C\|f_{n+1}^\epsilon(t) - f_n^\epsilon(t)\|_{L^\infty} \int_{|y| \geq \epsilon} \frac{\chi(y)}{|y|} dy. \]

Similarly, we obtain
\[ \|u_2^\epsilon(f_{n+1}^\epsilon)(t) - u_2^\epsilon(f_n^\epsilon)(t)\|_{L^\infty} \leq C\epsilon^{-2}(\|f_{n+1}^\epsilon(t)\|_{L^\infty} + \|f_n^\epsilon(t)\|_{L^\infty})\|f_{n+1}^\epsilon(t) - f_n^\epsilon(t)\|_{L^\infty}. \]

Using the uniform estimates (6.52) we get for any \(t \in [0, T^\epsilon]\)
\[ \|u_2^\epsilon(f_{n+1}^\epsilon)(t) - u_2^\epsilon(f_n^\epsilon)(t)\|_{L^\infty} \leq C\|f_0^\epsilon\|_{X} \epsilon^{-2}\|f_{n+1}^\epsilon(t) - f_n^\epsilon(t)\|_{L^\infty}, \quad \|\partial_x f_{n+1}^\epsilon(t)\|_{L^\infty} \leq C\|f_0^\epsilon\|_{X}. \]

Using the maximum principle for transport equation allows to get for any \(t \in [0, T^\epsilon]\)
\[ \|\theta_{n+1}(t)\|_{L^\infty} \leq C\epsilon^{-2}\|f_0^\epsilon\|_{X} \int_0^t \left[ \|\theta_{n+1}(\tau)\|_{L^\infty} + \|\theta_n(\tau)\|_{L^\infty} \right] d\tau. \]

By virtue of Gronwall lemma one finds that for any \(t \in [0, T^\epsilon]\)
\[ \|\theta_{n+1}(t)\|_{L^\infty} \leq e^{C\epsilon^{-2}\|f_0^\epsilon\|_{X} T^\epsilon} \int_0^t \|\theta_n(\tau)\|_{L^\infty} d\tau. \]

Hence we obtain in view of (6.51)
\[ \|\theta_{n+1}(t)\|_{L^\infty} \leq C_0 \int_0^t \|\theta_n(\tau)\|_{L^\infty} d\tau. \]

By induction we find
\[ \forall n \in \mathbb{N}, \forall t \in [0, T^\epsilon], \quad \|\theta_n\|_{L^\infty L^\infty} \leq C_0 \frac{t n}{m} \|\theta_0\|_{L^\infty L^\infty}. \]

This implies the convergence of the series
\[ \sum_{n \in \mathbb{N}} \|\theta_{n+1}\|_{L^\infty L^\infty} < \infty. \]

Therefore \((f_n^\epsilon)\) converges strongly in \(L^\infty [0, T^\epsilon]\) to an element \(f^\epsilon \in L^\infty [0, T^\epsilon]\). From the uniform estimates (6.52) we deduce that \(f^\epsilon \in \mathcal{E}_{T^\epsilon}\). This allows to pass to the limit in the equation (6.43) and obtain that \(f^\epsilon\) is solution to
\[
(6.54) \quad \begin{cases} 
\partial_t f^\epsilon + u_1^\epsilon(f^\epsilon) \partial_x f^\epsilon = u_2^\epsilon(f^\epsilon), \\
f_0^\epsilon(t, x) = f_0(x) 
\end{cases}
\]
with
\[ u_1^\varepsilon(f^\varepsilon)(t, x) \triangleq \frac{1}{2\pi} \int_{|y| \geq \varepsilon} \left\{ \arctan \left( \frac{f^\varepsilon(t, x + y) - f^\varepsilon(t, x)}{y} \right) + \arctan \left( \frac{f^\varepsilon(t, x + y) + f^\varepsilon(t, x)}{y} \right) \right\} dy \]
\[ u_2^\varepsilon(f^\varepsilon)(t, x) \triangleq \frac{1}{4\pi} \int_{|y| \geq \varepsilon} \log \left( \frac{y^2 + (f^\varepsilon(t, x + y) - f^\varepsilon(t, x))^2}{y^2 + (f^\varepsilon(t, x + y) + f^\varepsilon(t, x))^2} \right) dy. \]

(6.55)

Now, looking to the proofs used to get the a priori estimates, they can be adapted to the equation (6.54) supplemented with (6.55). For instance the a priori estimates obtained in Proposition 6.2 hold for the modified equation (6.54) independently on vanishing \( \varepsilon \). In particular one can bound uniformly in \( \varepsilon \) the solution \( f^\varepsilon \) in the space \( X_T^s \) and therefore \( T_\varepsilon \) is not maximal and by a standard bootstrap argument we can continue the solution up to the local time \( T \) constructed in Proposition 6.2. It follows that \( f^\varepsilon \) belongs to \( \mathcal{E}_T \) uniformly with respect to small \( \varepsilon \). This yields according once again to the Proposition 6.2 and the inequalities (6.25) and (6.40)
\[ \sup_{\varepsilon \in [0,1]} \| \partial_t f^\varepsilon \|_{L^\infty_T L^\infty} \leq \| u_1^\varepsilon(f^\varepsilon) \|_{L^\infty_T L^\infty} \| \partial_x f^\varepsilon \|_{L^\infty_T L^\infty} + \| u_2^\varepsilon(f^\varepsilon) \|_{L^\infty_T L^\infty} \leq C_0, \]
and \( C_0 \) is a constant depending on the size of the initial data. Now from the compact embedding \( C_K^s \rightarrow C_b \) and Ascoli lemma we deduce that up to a sequence \( (f^\varepsilon) \) converges strongly in \( L^\infty_T L^\infty \) to some element \( f \) which belongs in turn to \( \mathcal{E}_T \). This allows to pass to the limit in (6.54) and (6.55) and find a solution to the initial value problem (6.43). We point out that by working more one may obtain the strong convergence of the full sequence \( (f^\varepsilon) \) to \( f \). Note finally that the uniqueness follows easily from the arguments used to prove that \( (\theta_n) \) is a Cauchy sequence.

7. Global well-posedness

We are concerned here with the global existence of strong solutions already constructed in Theorem 2.1. This will be established under a smallness condition on the initial data and it is probable that for arbitrary large initial data the graph structure might be destroyed in finite time. The basic ingredient which allows to balance the energy amplification during the time evolution is a damping effect generated by the source terms. Note that this damping effect is plausible from the graph equation (2.1) according to the identity (6.34). However, as we shall see in the next section, it is quite complicate to extend this behavior for higher regularity at the level of the resolution space due to the existence of linear part in the source term governing the motion of the slope (2.3). This part could in general bring an amplification in time of the energy. To circumvent this difficulty we establish a weakly dissipative property of the linearized operator associated to the source term that we combine with the time decay of the solution for weak regularity using an interpolation argument.

7.1. Weak and strong damping behavior of the source term. Note from Proposition 6.1 that \( F \) does not contribute at the linear level which is not the case of the functional \( G \). We shall prove that actually there is no linear contribution for \( G \). This will be done by establishing a dissipative property that occurs at least at the linear level. This is described by the following proposition.

Proposition 7.1. Let \( K \) be a compact set of \( \mathbb{R} \) and \( s \in (0,1) \). Then for any \( f \in C_K^s \) we have the decomposition
\[ G(x) = 2\pi f'(x) + L(x) + N(x) \]
Performing the composition law (4.6) we deduce that
\[ L \leq 2\pi (\|f\|_s + 2\|f\|_{L_\infty}) + C\|f\|_s^{\frac{1}{2}} \|f\|_{s^6} \quad \text{and} \quad N \leq C\|f\|_s^{\frac{1}{2}} (\|f\|_s + \|f\|_{s^6}^{10}) , \]
where \( C > 0 \) is a constant depending only on \( K \). Moreover,
\[ \|L\|_{L_\infty} \leq C \min (\|f\|_s^{\frac{1}{4}} \|f\|_s^{\frac{1}{2}} \|f\|_{L_\infty}) \quad \text{and} \quad \|N\|_{L_\infty} \leq C\|f\|_s^{\frac{1}{2}} (\|f\|_{D} + \|f\|_{D}^{10}) . \]

**Proof.** In view of (6.1), (6.2), (6.4), (6.16), (6.17) and (6.24) one gets
\[ G(x) = G_{11}(x) + H(x), \quad H = G_{12} + G_2 \]
with
\[ ||H||_s \leq C\|f\|_s^{\frac{1}{2}} (\|f\|_s + \|f\|_{s^6}^{10}) . \]
Note also that from (6.3) and (6.11) we get
\[ \|H\|_{L_\infty} \leq C\|f\|_s^{\frac{1}{2}} (\|f\|_s + \|f\|_{s^6}^{10}) . \]
Now from (6.5) we get
\[ G_{11}(x) = 2 \int_{\mathbb{R}} \frac{f(x) + f(x + f(x)z)}{\varphi(x, z)} dz \]
with
\[ \varphi(x, z) = z^2 + \left(2 + z \int_{0}^{1} f(x + \theta f(x)z) d\theta \right)^2 . \]
We shall split again \( G_{11} \) as follows
\[ G_{11}(x) = 2 \int_{\mathbb{R}} \frac{f(x) + f(x + f(x)z)}{z^2 + 4} dz - 2 \int_{\mathbb{R}} \frac{[f(x) + f(x + f(x)z)] \psi(x, z)}{\varphi(x, z)(z^2 + 4)} dz \]
\[ \triangleq \mathcal{L}(x) + \mathcal{N}(x) , \]
with
\[ \psi(x, z) \triangleq 4z \int_{0}^{1} f(x + f(x)z) d\theta + z^2 \left( \int_{0}^{1} f(x + f(x)z) d\theta \right)^2 . \]
From (5.22) one gets
\[ ||\mathcal{N}||_{L_\infty} \leq C\|f\|_s^{\frac{1}{2}} (1 + \|f\|_{L_\infty}^{10}) . \]
Using the law product (4.8) we get
\[ \left\| \frac{[f' + f' \circ (\text{Id} + zf)] \psi(\cdot, z)}{\varphi(\cdot, z)} \right\|_s \leq 2\|f\|_s^{\frac{1}{2}} \|\psi(\cdot, z)\|_{L_\infty} \|1/\varphi(\cdot, z)\|_s + \frac{1}{2}\|f\|_s^{\frac{1}{2}} \|\psi(\cdot, z)\|_s \|1/\varphi(\cdot, z)\|_{L_\infty} + \left[ \|f\|_s + \|f' \circ (\text{Id} + zf)\|_s \right] \|\psi(\cdot, z)\|_{L_\infty} \|1/\varphi(\cdot, z)\|_{L_\infty} . \]
In addition, it is clear that
\[ \|\psi(\cdot, z)\|_{L_\infty} \leq 4|z|\|f\|_{L_\infty} + |z|^2\|f'\|_{L_\infty} . \]
Performing the composition law (4.6) we deduce that
\[ \|\psi(\cdot, z)\|_s \leq C|z|\|f\|_s [1 + |z|^2\|f'\|_{L_\infty}] + C|z|^2\|f'\|_{L_\infty}\|f'\|_s [1 + |z|^2\|f'\|_{L_\infty}] . \]
Combining this latter estimate with (6.9) and (5.22) yields
\[
\left\| \frac{\left[ f' + f' \circ \left( \text{Id} + zf \right) \right] \psi(\cdot, z)}{\varphi(\cdot, z)} \right\|_s \leq C \| f' \|_{L^\infty} \| f' \|_s [1 + \| f' \|_{L^\infty}^3] (1 + |z|^s).
\]

Hence we get according to the embedding \( C^s_K \hookrightarrow L^\infty \)
\[
\| \mathcal{N} \|_s \leq C \| f' \|_{L^\infty} \| f' \|_s [1 + \| f' \|_{L^\infty}^3] \\
\leq C \| f' \|_{L^\infty}^\frac{1}{3} \left[ \| f' \|_s^\frac{3}{5} + \| f' \|_{L^\infty}^{\frac{26}{5} + 8} \right] \\
\leq C \| f' \|_{L^\infty}^\frac{1}{3} \left[ \| f' \|_s + \| f' \|_{L^\infty}^{10} \right].
\]

Setting \( N = \mathcal{N} + H \) and combining the latter estimate with (7.1) we find the desired estimate for \( N \) stated in the proposition. Putting together (7.2) and (7.3) combined with Sobolev embedding we find
\[
\| N \|_{L^\infty} \leq C \| f' \|_{L^\infty} \left( \| f' \|_s + \| f' \|_{L^\infty}^4 \right).
\]

Coming back to \( \mathcal{L} \) one may write
\[
L(x) = 4 f'(x) \int_{\mathbb{R}} \frac{dz}{z^2 + 4} + 2 \int_{\mathbb{R}} \frac{f'(x + f(x)z) - f'(x)}{z^2 + 4} dz
\]
\[
(7.4) \quad \triangleq \quad 2 \pi f'(x) + L(x).
\]

To estimate \( L \) in \( C^s \) we simply write
\[
\| L \|_s \leq 2 \int_{\mathbb{R}} \frac{\| f' \circ \left( \text{Id} + zf \right) \|_s + \| f' \|_s dz}{z^2 + 4}.
\]

Combined with (4.6) we find
\[
\| f' \circ \left( \text{Id} + zf \right) \|_s \leq \left( \| f' \|_s + 2 \| f' \|_{L^\infty} \right) (1 + |z| \| f' \|_{L^\infty})^s \\
\leq \left( \| f' \|_s + 2 \| f' \|_{L^\infty} \right) (1 + |z|^s \| f' \|_{L^\infty}^s),
\]

where in the last line we have use the inequality: \( \forall s \in (0, 1), \forall x, y \geq 0 \) one has
\[
(x + y)^s \leq x^s + y^s.
\]

Using (4.2), it follows that
\[
\| L \|_s \leq 2 \pi \left( \| f' \|_s + 2 \| f' \|_{L^\infty} \right) + C \| f' \|_s \| f' \|_{L^\infty}^s.
\]

The estimate of \( L \) in \( L^\infty \) is easier and one gets according to (7.4),
\[
|L(x)| \leq 2 |f(x)|^s \| f' \|_s \int_{\mathbb{R}} \frac{|z|^s}{z^2 + 4} dz \\
\leq C |f(x)|^s \| f' \|_s.
\]

Therefore we obtain
\[
\| L \|_{L^\infty} \leq C \| f \|_{L^\infty}^s \| f' \|_s.
\]

We point out that we have obviously
\[
\| L \|_{L^\infty} \leq 2 \pi \| f' \|_{L^\infty}.
\]

Therefore we find
\[
(7.5) \quad \| L \|_{L^\infty} \leq C \min \left( \| f \|_{L^\infty}^s \| f' \|_s, \| f' \|_{L^\infty} \right).
\]

This achieved the proof of Proposition 7.1. □
7.2. **Global a priori estimates.** The main goal of this section is to show how we may use the weakly damping effect of the source terms stated in Proposition 7.1 in order to get global a priori estimates when the initial data is small enough. The basic result reads as follows.

**Proposition 7.2.** Let $K$ be a compact set of $\mathbb{R}$ and $s \in (0,1)$. There exists a constant $\varepsilon > 0$ such that if $\|f_0\|_s \leq \varepsilon$ then the equation (2.1) admits a unique global solution

$$f' \in L^\infty(\mathbb{R}_+; C^*_K).$$

Moreover, there exists a constant $C_0$ depending on the initial data such that

$$\forall t \geq 0, \quad \|f'(t)\|_{L^\infty} \leq C_0 e^{-t}.$$  

**Proof.** According to the decomposition of Proposition 7.1 combined with the equation (2.3) and (2.6) we get that $g = \partial_x f$ satisfies the equation

$$(7.6) \quad \partial_t g(t,x) + u_1(t,x) \partial_1 g(t,x) + g(t,x) = \mathcal{R}(t,x), \quad \mathcal{R} \triangleq \frac{1}{2\pi} (F - L - N).$$

Using Proposition 6.1 and Proposition 7.1 combined with the (4.2) we find

$$\|\mathcal{R}\|_s \leq \|f\|_s + 2\|f\|_{L^\infty} + C\|f\|_D \left(\|f\|_s + \|f\|_s^3\right)$$

$$+ \ C\|f\|_s^8 \|f\|_s + C\|f\|_s^4 \left(\|f\|_s + \|f\|_s^6\right).$$

The embedding $C^5_K \subset C^*_K$ combined with interpolation inequalities in Hölder spaces yield

$$(7.7) \quad \|f\|_D \leq C\|f\|_s^{2/3} \|f\|_s^{1/3}.$$  

Set $s_0 = \min(s, \frac{1}{3})$ then it is easy to get

$$(7.8) \quad \|\mathcal{R}\|_s \leq \|f\|_s + 2\|f\|_{L^\infty} + C\|f\|^8_{L^\infty} \left(\|f\|_s + \|f\|_s^6\right).$$

Let $h(t,x) \triangleq g(t,\psi(t,x))$, where $\psi$ is the flow introduced in (6.26). Then it is obvious that

$$\partial_1 h(t,x) + h(t,x) = \mathcal{R}(t,\psi(t,x)).$$

This allows to deduce the following Duhamel integral representation

$$e^t g(t,x) = g_0(\psi^{-1}(t,x)) + \int_0^t e^{\tau} \mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t,x))) d\tau.$$  

Thus

$$e^t \|g(t)\|_s \leq \|g_0(\psi^{-1}(t))\|_s + \int_0^t e^\tau \|\mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t)))\|_s d\tau.$$  

According to (6.38) and (4.6) we obtain

$$\|g_0(\psi^{-1}(t))\|_s \leq C\|g_0\|_s e^{V(t)}, V(t) = \int_0^t \|\partial_x u_1(\tau)\|_{L^\infty} d\tau$$  

and

$$\|\mathcal{R}(\tau,\psi(\tau,\psi^{-1}(t)))\|_s \leq \left[\|\mathcal{R}(\tau)\|_s + 2\|\mathcal{R}(\tau)\|_{L^\infty}\right] e^{V(t)-V(\tau)}.$$  

Note that the estimate of $\mathcal{R}$ in $C^n$ has been already stated in (7.8). However to get a suitable estimate in $L^\infty$ we use Proposition 6.1 and Proposition 7.1 combined with Sobolev embedding,

$$\|\mathcal{R}(t)\|_{L^\infty} \leq C\|f'(t)\|_{L^\infty} \left(\|f'(t)\|_D + \|f'(t)\|_s^3\right) + C \min \left(\|f(t)\|_{L^\infty}, \|f'(t)\|_s, \|f'(t)\|_{L^\infty}\right)$$

$$\leq C\|f'(t)\|^8_{L^\infty} \left(\|f'(t)\|_s + \|f'(t)\|_s^3\right)$$

$$\leq C\|f'(t)\|^8_{L^\infty} \left(\|f'(t)\|_s + \|f'(t)\|_s^3\right).$$
It follows that
\[
\|R(\tau, \psi(\tau, \psi^{-1}(t)))\|_s \leq (\|f'\|_s + 2\|f'\|_{L^\infty})e^{V(t)-V(\tau)} + C\|f'\|_s^{s_0}(\|f'\|_s + \|f'\|_{s_0}^{16})e^{V(t)-V(\tau)}.
\]

Set \(K(t) = e^{-V(t)}e^t\|f'(t)\|_s\) and
\[
S(t) = Ce^t e^{-V(t)}\left(\|f'(t)\|_{L^\infty} + \|f'(t)\|_{L^\infty}^{s_0}(\|f'\|_s + \|f'(t)\|_{s_0}^{16})\right)
\]
then
\[
K(t) \leq CK(0) + \int_0^t K(\tau)d\tau + \int_0^t S(\tau)d\tau.
\]
By virtue of Gronwall lemma we deduce that
\[
K(t) \leq Ce^t K(0) + \int_0^t e^{t-\tau}S(\tau)d\tau.
\]
This implies that
\[
\|f'(t)\|_s \leq Ce^{V(t)}\|f_0\|_s + Ce^{V(t)}\int_0^t \|f'(\tau)\|_{L^\infty}d\tau + e^{V(t)}\int_0^t \|f'(\tau)\|_{L^\infty}^{s_0}(\|f'\|_s + \|f'(t)\|_{s_0}^{16})d\tau.
\]
(7.10)
Combining the following interpolation inequality
\[
\|f_0'\|_{L^\infty} \leq C\|f_0\|_{L^1}^{\frac{s}{2+s}}\|f_0'\|_{L^s}^{\frac{2}{2+s}},
\]
with Proposition 6.2-(2) we obtain
\[
\|f'(t)\|_{L^\infty} \leq Ce^{-\frac{s}{2+s}t}\|f_0\|_{L^1}^{\frac{s}{2+s}}\|f'(t)\|_{L^s}^{\frac{2}{2+s}}.
\]
(7.11)
Plugging this estimate into (6.40) we find
\[
\|\partial_x u_1(t)\|_{L^\infty} \leq Ce^{-\frac{s}{2+s}t}\|f_0\|_{L^1}^{\frac{s}{2+s}}\left(\|f'(t)\|_{L^s}^{\frac{2}{2+s}} + \|f'(t)\|_{L^s}^{\frac{2+s}{2+s}}\right).
\]
(7.12)
It is quite obvious from (4.2) and the compactness of the support that
\[
\|f_0\|_{L^1} \leq C\|f_0'\|_s
\]
with \(C\) a constant depending on the size of the support of \(f_0\). Set
\[
\rho(T) = \sup_{t\in[0,T]} \|f'(t)\|_s
\]
then combining (7.10) with (7.11) and (7.12) yields to
\[
\rho(T) \leq Ce^C\|f_0'\|_{L^s}^{\frac{2+s}{2+s}}\left(\rho(T)\|f_0'\|_{L^s}^{\frac{2}{2+s}} + \rho(T)\|f_0'\|_{L^s}^{\frac{2+s}{2+s}}\right)\mu(T)
\]
with
\[
\mu(T) = \|f_0'\|_s + \|f_0'\|_{L^s}^{\frac{2}{2+s}}\rho(T)\|f_0'\|_{L^s}^{\frac{s}{2+s}} + \|f_0'\|_{L^s}^{\frac{2+s}{2+s}}\rho(T)\|f_0'\|_{L^s}^{\frac{2+s}{2+s}}\rho(T) + \rho(T)^{16}
\]
This implies the existence of small number \(\epsilon > 0\) depending only on \(C\) and therefore on the size of the support of \(f_0\) such that if
\[
\|f_0'\|_s \leq \epsilon \implies \forall T > 0, \quad \rho(T) \leq \delta(\|f_0'\|_s)
\]
with \(\lim_{x\to0} \delta(x) = 0\). This gives the global a priori estimates.
What is left is to establish the precise time decay of \( \|f'(t)\|_{L^\infty} \) stated in Proposition 7.2. From the equation (7.6) it is easy to establish the following estimate using the characteristic method,

\[
(7.14) \quad \|g(t)\|_{L^\infty} \leq e^{-t}\|g_0\|_{L^\infty} + \int_0^t e^{-(t-\tau)}\|R(\tau)\|_{L^\infty}d\tau.
\]

According to (7.9) we obtain

\[
e^t\|f'(t)\|_{L^\infty} \leq \|f'_0\|_{L^\infty} + C\int_0^t e^\tau\|f' (\tau)\|_{L^\infty} (\|f'(\tau)\|_D + \|f'\|_D^2) d\tau.
\]

Using Gronwall lemma we obtain

\[
e^t\|f'(t)\|_{L^\infty} \leq \|f'_0\|_{L^\infty} e^{W(t)}, \quad W(t) = C\int_0^t (\|f' (\tau)\|_D + \|f'\|_D^2) d\tau.
\]

Putting together (7.7) with (7.11) we obtain

\[
\|f'(t)\|_D \leq Ce^{-\frac{2}{1+2s}t}\|f'_0\|_{L^1} \|f'(t)\|_D^{\frac{4+2s}{3+2s}}.
\]

Hence we deduce from (7.13) that

\[
\forall t \geq 0, W(t) \leq C_0
\]

and therefore

\[
(7.15) \quad \forall t \geq 0, \quad \|f'(t)\|_{L^\infty} \leq C_0 e^{-t}, \quad \|f'(t)\|_D \leq C_0 e^{-\frac{2}{1+2s}t},
\]

for a suitable constant \( C_0 \) depending on the initial data. Inserting these estimates into (7.9) we obtain

\[
(7.16) \quad \forall t \geq 0, \quad \|R(t)\|_{L^\infty} \leq C_0 e^{-t}.
\]

Since \( f_t \) is compactly supported in a fixed compact then

\[
(7.17) \quad \forall t \geq 0, \quad \|f(t)\|_{L^\infty} \leq C_1 e^{-t}.
\]

Finally, we point out that all the constants involved in the preceding estimates and related to the support of \( f_t \) are actually independent of the time due to the fact that the support of \( f_t \) is confined in the convex hull of the support of the initial data, as it has been stated in Proposition 6.2-(3).

\[\Box\]

8. SCATTERING AND COLLAPSE TO SINGULAR MEASURE

The aim of the last section is to analyze and identify the longtime behavior of the global solutions stated in Theorem 2.2. It attempts to investigate the time evolution of the following probability measure,

\[
dP_t(x) \triangleq \frac{\rho(t, x)}{\|\rho_t\|_{L^1}} dA(x) = e^t1_{\mathcal{D}_t}(x)dA(x)
\]

where \( dA \) denotes the usual Lebesgue measure. Note that without loss of generality we have assumed in the last line that \( \|\rho_0\|_{L^1} = 1 \). As we shall see this measure converges weakly as \( t \) goes to infinity to a probability measure concentrated on the real line and absolutely continuous with respect to Lebesgue measure on the real line. The description of the density and the support of this limiting measure will be the subject of the next two sections.
8.1. Structure of the singular measure. In this section we shall prove the part of Theorem 2.2 dealing with the weak convergence of the measure $dP_t$ when $t$ goes to $+\infty$. First, it is obvious that the probability measure $dP_t$ is absolutely continuous with respect to the Lebesgue measure. The convergence of the family $\{dP_t, t \geq 0\}$ will be done in a weak sense as follows. Let $\varphi \in D(\mathbb{R}^2)$ be a test function, one can write using Fubini’s theorem

$$I_t \triangleq \int_{\mathbb{R}^2} \varphi(x,y) dP_t = e^t \int_{\mathbb{R}} \int_{-f_t(x)}^{f_t(x)} \varphi(x,y) dy.$$

According to Taylor expansion in the second variable one gets

$$\forall (x, y) \in \mathbb{R}^2, \quad \varphi(x,y) = \varphi(x,0) + y\psi(x,y) \quad \text{and} \quad \|\psi\|_{L^\infty} \leq C.$$

This implies that

$$I_t = 2e^t \int_{\mathbb{R}} f_t(x) \varphi(x,0) dx + I^1_t, \quad I^1_t \triangleq e^t \int_{\mathbb{R}} \int_{-f_t(x)}^{f_t(x)} y\psi(x,y) dy.$$

We shall check that the term $I^1_t$ does not contribute in the limiting behavior. Actually it vanishes for $t$ going to infinity. Indeed,

$$|I^1_t| \leq e^t \|\psi\|_{L^\infty} \int_{\mathbb{R}} [f_t(x)]^2 dx.$$

Using (7.17) and the localization of the support of $f_t$ in the convex hull of the initial support, we deduce that

$$|I^1_t| \leq Ce^{-t}$$

and thus

$$\lim_{t \to +\infty} I^1_t = 0.$$

Combining (2.1), (6.34), (7.15), (7.17) and (7.13) we deduce that

$$\partial_t f(t,x) + u_1 \partial_x f(t,x) + f(t,x) = -f(t,x) R(t,x)$$

with

$$\|R(t)\|_{L^\infty} \leq \|f'(t)\|_D (1 + \|f'(t)\|_\infty^5) \leq Ce^{-\frac{t}{1+2s}}.$$

From the characteristic method developed in studying (7.6) we get the representation

$$e^t f(t, \psi(t,x)) = f_0(x)e^{\int_0^t R(\tau, \psi(\tau,x)) d\tau}.$$

From the integrability property (8.3) we deduce that $\{e^t f(t, \psi(t))\}$ converges uniformly as $t$ goes to $+\infty$ to the positive function

$$x \mapsto f_0(x)e^{\int_0^{+\infty} R(\tau, \psi(\tau,x)) d\tau} \triangleq R_2(x).$$

More precisely, using straightforward computations we easily get

$$\|e^t f_t \circ \psi(t) - R_2\|_{L^\infty} \leq \|R_2\|_{L^\infty} \int_t^{+\infty} \|R(\tau)\|_{L^\infty} d\tau \leq Ce^{-\frac{t}{1+2s}}.$$
The next goal is to prove that the flow \( \psi(t) \) converges uniformly as \( t \) goes to infinity to some homeomorphism \( \psi_\infty : \mathbb{R} \to \mathbb{R} \) which belongs to the bi-Lipschitz class. First, recall from the definition (6.26) that

\[
\psi(t, x) = x + \int_0^t u_1(\tau, \psi(\tau, x)) \, d\tau.
\]

Recall from Section 2 that \( u_1(x) = v_1(x, f(x)) \) and the velocity is computed from the density \( \rho \) according to the second equation of (1.2). Hence we get

\[
\|u_1(t)\|_{L^\infty} \leq \|\Delta^{-1} \nabla \rho\|_{L^\infty}.
\]

Now using the classical interpolation inequality

\[
\|\Delta^{-1} \nabla \rho\|_{L^\infty} \leq C \|\rho\|_{L^1}^{\frac{1}{2}} \|\rho\|_{L^\infty}^{\frac{1}{2}}
\]

combined with the decay rate stated in Proposition 6.2-(2) we deduce that

\[
\|u_1(t)\|_{L^\infty} \leq Ce^{-t/2}
\]

Consequently, it follows that \( \psi(t) \) converges uniformly to the function

\[
\psi_\infty(x) \triangleq x + \int_0^{+\infty} u_1(\tau, \psi(\tau, x)) \, d\tau.
\]

More precisely, we have

\[
\|\psi(t) - \psi_\infty\|_{L^\infty} \leq \int_t^{+\infty} \|u_1(\tau)\|_{L^\infty} \, d\tau \leq Ce^{-t/2}.
\]

It remains to check that \( \psi_\infty \) is bi-Lipschitz. First we know that

\[
\|\partial_\tau \psi(t)\|_{L^\infty} \leq e^{V(t)}, \quad V(t) = \int_0^t \|\partial_\tau u_1(\tau)\|_{L^\infty} \, d\tau.
\]

Using (7.12) and (7.13) we deduce that

\[
\forall t \geq 0, \quad \|\partial_\tau \psi(t)\|_{L^\infty} \leq C, \quad \|\partial_\tau u_1(t)\|_{L^\infty} \leq C\varepsilon^{\frac{1}{4+t^2}} e^{-\frac{1}{4+t^2}t}.
\]

Differentiating \( \psi_\infty \) and using the triangle inequality we get

\[
1 - \int_0^{+\infty} \|\partial_\tau u_1(\tau)\|_{L^\infty} \|\partial_\tau \psi(\tau)\|_{L^\infty} \, d\tau \leq \psi'_\infty(x) \leq 1 + \int_0^{+\infty} \|\partial_\tau u_1(\tau)\|_{L^\infty} \|\partial_\tau \psi(\tau)\|_{L^\infty} \, d\tau.
\]

Therefore we obtain

\[
\forall x \in \mathbb{R}, \quad 1 - C\varepsilon^{\frac{1}{4+t^2}} \leq \psi'_\infty(x) \leq 1 + C\varepsilon^{\frac{1}{4+t^2}}.
\]

Taking \( \varepsilon \) small enough, meaning that the initial data is very small, we get

\[
\forall x \in \mathbb{R}, \quad \frac{1}{2} \leq \psi'_\infty(x) \leq \frac{3}{2}.
\]

This shows that \( \psi_\infty \) is a bi-Lipschitz function from \( \mathbb{R} \) to \( \mathbb{R} \). Further, it is obvious that

\[
\psi_\infty(x) = \psi(t, x) + \int_t^{+\infty} u_1(\tau, \psi(\tau, x)) \, d\tau
\]

and hence

\[
\psi_\infty(\psi^{-1}(t, x)) = x + \int_t^{+\infty} u_1(\tau, \psi(\tau, \psi^{-1}(t, x))) \, d\tau.
\]
Combined this identity with \( \psi_\infty \circ \psi^{-1}_\infty = \text{Id} \) and (8.7) yields
\[
|\psi_\infty((\psi^{-1}(t, x)) - \psi_\infty(\psi^{-1}_\infty x)| \leq \int_t^{t+\infty} \|u_1(\tau)\|_{L^\infty} d\tau \\
\leq Ce^{-t/2}.
\]
Applying (8.10) we deduce that
\[
\|\psi^{-1}(t) - \psi^{-1}_\infty\|_{L^\infty} \leq Ce^{-t/2}.
\]
This shows that \( \psi^{-1}(t) \) converges uniformly to \( \psi^{-1}_\infty \) with an exponential rate. Set
\[
\Phi = R_2 \circ \psi^{-1}_\infty
\]
and assume for a while that \( R_2 \) belongs to \( C^\alpha \) for any \( \alpha \in (0, 1) \), then we deduce from the preceding estimates, especially (8.6) and (8.4), that
\[
\|e^{t}f(t) - \Phi\|_{L^\infty} \leq \|e^{t}f(t) - R_2 \circ \psi^{-1}(t)\|_{L^\infty} + \|R_2 \circ \psi^{-1}(t) - R_2 \circ \psi^{-1}_\infty\|_{L^\infty}
\leq Ce^{-\frac{t}{4+2s}} + \|R_2\|_\alpha \|\psi^{-1}(t) - \psi^{-1}_\infty\|_{L^\infty}^\alpha
\leq Ce^{-\frac{t}{4+2s}} + Ce^{-\alpha t/2}.
\]
Taking \( \alpha = \frac{2s}{4+2s} \) we get
\[
\|e^{t}f(t) - \Phi\|_{L^\infty} \leq Ce^{-\frac{t}{4+2s}}.
\]
Let us now check that \( R_2 \) belongs to \( C^\alpha \) for any \( \alpha \in (0, 1) \). For this goal we shall express differently the function \( R_2 \). Set \( R_1(t, x) = -f(t, x)R(t, x) \) then from the characteristic method the solution to (8.2) may be recovered as follows
\[
e^{t}f(t, \psi(t, x)) = f_0(x) + \int_0^t e^\tau R_1(\tau, \psi(\tau, x)) d\tau.
\]
Putting together (8.3) and (7.17) we deduce that
\[
\|R_1(\tau, \psi(\tau))\|_{L^\infty} \leq Ce^{-\frac{t+4s}{4+2s}}.
\]
Therefore we find the identity
\[
R_2(x) = f_0(x) + \int_0^{t+\infty} e^\tau R_1(\tau, \psi(\tau, x)) d\tau.
\]
We shall now study the regularity of \( R_2 \) through the use of this representation. Differentiating in \( x \) the equation (8.2) and comparing it to the equation (7.6) we get the identity
\[
\partial_x R_1(t, x) = R(t, x) + \partial_x u_1(t, x) \partial_x f(t, x).
\]
Using (7.15), (7.16) and (8.9) we find
\[
\forall t \geq 0, \quad \|\partial_x R_1(t)\|_{L^\infty} \leq Ce^{-t}.
\]
Combining this latter estimate with Leibniz formula and (8.9) implies
\[
\forall t \geq 0, \quad \|\partial_x (R_1(t, \psi(t, \cdot)))\|_{L^\infty} \leq Ce^{-t}.
\]
It suffices now to apply the following classical interpolation inequality: for any \( \alpha \in (0, 1) \) there exists \( C > 0 \) such that
\[
\|h\|_\alpha \leq C\|h\|_{L^\infty}^{\frac{1-\alpha}{\alpha}}\|h'\|_{L^\infty}^\alpha
\]
which implies that according to (8.13) and (8.15)
\[
\forall t \geq 0, \quad \|R_1(t, \psi(t, \cdot))\|_\alpha \leq Ce^{-t}e^{-\frac{t+4s}{4+2s}}.
\]
Returning to the identity (8.14), one obtains in view of (8.16)
\[ \|R_2\|_\alpha \leq \|f_0\|_\alpha + \int_0^{+\infty} e^\tau \|R_1(\tau, \psi(\tau, \cdot))\|_\alpha d\tau \]
\[ \leq C, \]
for any \( \alpha \in (0, 1) \). As an immediate consequence of (8.11), (8.10) and (4.6) we find that \( \Phi \) belongs to \( C^\alpha \) for any \( \alpha \in (0, 1) \).

We guess the profile \( \Phi \) to keep the same regularity as \( f_0 \), that is, in \( C^{1+s} \) but this could require much more refined analysis.

Now coming back to (8.1) we find in view of (8.12) and Lebesgue theorem
\[ \lim_{t \to +\infty} I(t) = 2 \int_\mathbb{R} \Phi(x) \varphi(x, 0) dx. \]
This is equivalent to write in the weak sense
\[ \lim_{t \to +\infty} dP_t = 2\Phi \delta_\mathbb{R} \otimes \{0\} \]
\[ \cong dP_\infty. \]

Now we shall discuss some properties of \( \Phi \). From (8.5) and (8.11) we have
\[ \text{supp } \Phi = \psi_\infty(K_0), \quad K_0 = \text{supp } f_0. \]
According to (8.10), the measure of \( \text{supp } \Phi \) is strictly positive with
\[ |\text{supp } \Phi| \geq \frac{1}{2} |K_0|. \]
It remains to check that \( dP_\infty \) is a probability measure on the real axis, which reduces to verify that
\[ 2 \int_\mathbb{R} \Phi(x) dx = 1. \]
First note that using Proposition 6.2-(2) one obtains for any \( t \geq 0, \)
\[ 1 = 2 \int_\mathbb{R} e^t f(t, x) dx. \]
To exchange limit and integral it suffices to apply Lebesgue theorem thanks to the conditions (8.12) and the fact that \( \text{supp } f_t \in \text{Conv } K_0 \), recall that for simplicity we have assumed that \( \|\rho_0\|_{L^1} = 1, \)
\[ \lim_{t \to +\infty} \int_\mathbb{R} e^t f(t, x) dx = \int_\mathbb{R} \Phi(x) dx, \]
which provides the desired result. We point out that with the normalization \( \|\rho_0\|_{L^1} = 1 \) one gets instead of (8.17)
\[ dP_\infty = \frac{\Phi}{\|f_0\|_{L^1}} \delta_\mathbb{R} \otimes \{0\}, \]
which gives the structure of the limiting measure stated in Theorem 2.2 thanks to (8.5) and (8.11).

8.2. **Concentration of the support.** In this section we shall complete the study of the limiting measure \( dP_\infty \) and identify its support denoted by \( K_\infty \). What is left to conclude the proof of Theorem 2.2 is just to check that the support \( D_t \) of the solution \( \rho_t \) converges in the Hausdorff sense to \( K_\infty \). Recall that \( K_0 \) is the support of \( f_0 \) and is assumed to be is a finite collection of increasing segments \( [a_i; b_i], i = 1, \ldots, n \), such that \( a_i < b_i < a_{i+1} \). According to (8.18) one has
\[ \text{supp } \Phi = \psi_\infty(K_0) \cong K_\infty. \]
Since \( \Psi_\infty \) is strictly increasing due to (8.10) one deduces easily that
\[ \text{supp } \Phi = \bigcup_{i=1}^n [a_i^\infty, b_i^\infty], \quad a_i^\infty \cong \psi_\infty(a_i), \quad b_i^\infty \cong \psi_\infty(b_i). \]
Using more again (8.10) one may easily obtain that
\[ \forall \, i, \quad |a_i^\infty - b_i^\infty| \geq \frac{1}{2}|a_i - b_i|. \]
Now to establish the convergence in the Hausdorff sense of \( D_t \) towards \( K_\infty \) it suffices to prove the result for each connected component, that is,
\[ \forall i = 1, \ldots, n, \quad d_H(\Gamma^i_t, [a_t^i, b_t^i]) \leq Ce^{-t} \]
with
\[ \Gamma^i_t \triangleq \{(x, f_t(x)), x \in [a^i_t, b^i_t]\}. \]

By straightforward analysis using (7.17) one obtains
\[ d_H(\Gamma^i_t, [a_t^i, b_t^i]) \leq Ce^{-t} + C \max(|a_t^i - a_t^\infty|, |b_t^i - b_t^\infty|). \]
From (8.8) one gets
\[ \max(|a_t^i - a_t^\infty|, |b_t^i - b_t^\infty|) \leq Ce^{-t} \]
and therefore
\[ \forall t \geq 0, \quad d_H(D_t, K_\infty) \leq Ce^{-t}. \]
The proof of Theorem 2.1 is now achieved.

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