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► **To cite this version:**

El Houssaine Quenjel, Mazen Saad, Mustapha Ghilani, Marianne Bessemoulin-Chatard. Convergence of a monotone nonlinear DDFV scheme for degenerate parabolic equations. 2018. hal-01757025v1

HAL Id: hal-01757025

<https://hal.science/hal-01757025v1>

Preprint submitted on 3 Apr 2018 (v1), last revised 25 May 2020 (v3)

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Convergence of a monotone nonlinear DDFV scheme for degenerate parabolic equations

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April 3, 2018

Abstract

In this work, we carry out the convergence analysis of an efficient DDFV method for approximating solutions of degenerate parabolic equations. The basic idea rests upon different approximations of the fluxes on the same interface of the control volume. Precisely, the approximate flux is split into two terms corresponding to the primal normal and the dual normal components. Then the first term is discretized using a centered scheme whereas the second one is approximated in a non evident way with the help of an upwind scheme. Therefore the discrete maximum principle is respected and a uniform bound on the discrete velocity is established. As a consequence, the scheme has an approximate solution. Thanks to some compactness arguments, this discrete solution converges to that of the continuous problem provided the size of the mesh tends to zero. Some numerical tests are then presented and they show that the scheme in question turns out to be robust and efficient with an accuracy of second order.

Keywords. Finite volume scheme, monotone, degenerate parabolic equations, convergence

1 Problem statement and goal of the paper

Nonlinear degenerate parabolic equations are the main core to study some complex problems arising from petroleum engineering, hydrology and biology. Hence, seeking for analytical or approximate solutions of these equations is of immense advantage. Throughout this work, we will be interested in approximating the most commune nonlinear unsteady diffusion equation, which reads

$$\begin{cases} \partial_t u - \nabla \cdot (f(u)\Lambda \nabla u) = 0 & \text{in } \Omega \times (0, T) \\ u = 0 & \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) = u^0 & \text{in } \Omega \end{cases} \quad (1.1)$$

where Ω is a bounded polygonal open of \mathbb{R}^2 and T a fixed positive integer, $\partial\Omega$ is the boundary of Ω , and f is a given nonnegative function. In the context of porous media flows, the function f is usually called the mobility. More precisely, the problem (1.1) describes the infiltration of a single fluid through a porous medium with no gravity effects [18]. It is derived from the Darcy law together with the mass conservation equation. On the other hand, this problem is known under the name of the porous medium equation [43]

This work is supported by: Ministère de l'Enseignement Supérieur, de la Recherche Scientifique et de la Formation des Cadres du Maroc, CNRST and l'Institut Français au Maroc, and Project MoHyCon – ANR-17-CE40-0027-01

whenever $f(u) = u^m$, for some nonnegative real number m . In view of theoretical study, the elliptic term of (1.1) can be formulated otherwise by introducing the so-called Kirchhoff transformation F . With some general assumptions on F , this formulation is sometimes said to be the simplified Stefan problem [26], which is used to model free boundary value problems. Even if this function seems to have no physical interpretation, it will play a remarkable role to carry out the analysis of the scheme we consider here. It is then defined by

$$F(u) = \int_0^u f(s) \, ds, \quad \forall u \in \mathbb{R}. \quad (1.2)$$

In this discretization, we will also introduce the feature of the "squared-Kirchoff" transform, denoting ξ , which is given by a nondecreasing Lipschitz continuous function from \mathbb{R} to \mathbb{R} and it reads

$$\xi(u) = \int_0^u \sqrt{f(s)} \, ds. \quad (1.3)$$

Different approximations, with various assumptions on the data, have been conducted to discretize problems involving nonlinear diffusion equations of type (1.1). For upstream finite difference method, we cite the work [36]. Concerning finite volume schemes, we refer to this battery of contributions [2, 5, 6, 7, 9, 22, 29, 28, 31, 38]. Plenty of these discretizations stipulate an orthogonality condition on the mesh in the sense of Eymard *et al.* [27], which excludes a large variety of interesting meshes. For example, in Hydrology, most of geological layers are quite deformed thus the meshes used to discretize the field are somehow distorted. In this case, the orthogonality condition is impossible to be satisfied for most of the edges. In addition, in the presence of anisotropic media, we may encounter the same difficulties. Yet, some works have combined between finite volume and finite element methods [1, 6, 24, 30, 35, 40]. Carrying out the analysis of this schemes, the authors required a positivity assumption on the stiffness coefficients that does not hold for any sort of meshes. To overcome this issue, positive schemes with their convergence studies have been proposed in [14, 16]. More generally, a gradient scheme [23, 26] has been suggested to discretize the Stefan problem, which is an equivalent formulation of (1.1) using the Kirchhoff transform. Nevertheless, no proof on the physical bound of the approximate solution is provided.

In this paper, we are concerned in the Discrete Duality Finite Volume (DDFV) method for the discretization of the problem (1.1). This is viewed as a class of the finite volume methods but discretized on two different kind of meshes. It has been firstly introduced in [19, 33, 34] and the method is proved to be quite equivalent to that of finite element. The convergence analysis of the DDFV scheme is carried out later for many types of partial differential equations of second order in several works [4, 13, 17, 20, 21, 37]. The strength of this discretization consists in producing consistent discrete whole gradient without any major restriction on the mesh, meaning that distorted meshes are included with more flexible cells. This reconstruction of the gradient verifies the discrete Stokes formula, which is a powerful tool to make the analysis of a such scheme. Moreover, we point out that the DDFV method is unconditionally coercive, which ensures the stability of the scheme.

Theoretically and up to our knowledge the DDFV schemes fail to satisfy an explicit discrete maximum principle. This property is crucial whenever we deal with positive physical quantities like saturation and concentration. As to be more precise, let us consider the DDFV discretization of the linear diffusion equation $-\Delta u = f$ with Dirichlet boundary conditions. It yields a stiffness matrix which is not monotone in the case of non admissible meshes in the sense of Eymard *et al.* [27]. By choosing an appropriate positive source term, we acquire a solution with some negative values. As a result, the question of monotonicity of the DDFV scheme has been a drawback of the method since it has been appeared. However, in the work of [11] the authors were able to design a monotone nonlinear DDFV scheme for the diffusion equation. It basically rests upon the DDFV idea together with nonlinear monotone two-point finite volumes methods [32, 39, 45]. So far and unfortunately, there is no convergence proof of the numerical schemes proposed in [11, 32, 39, 45] since they suffer from the lack of the coercivity as pointed out in [12, 22]. Recently, in [13] the authors have employed a nonlinear technique to establish the nonnegativity of the approximate solution in the case of a linear drift equation enclosed by Neumann boundary conditions. So, the contribution of this paper is to propose a new monotone scheme that obviously fulfills the physical range of the discrete

solution even on almost general meshes and even if the considered medium could be anisotropic. The key point of this approximation consists in making use of the degeneracy of the mobility function to remove the terms which generate undershoots or overshoots.

The remainder of this paper is structured as follows. In Section 2, we give the DDFV discretization, some related notations and the definition of discrete operators. In Section 3, we sketch out how to derive the proposed DDFV scheme. In Section 4, we prove that this scheme preserves the physical range of the approximate solution and we derive some energy estimate on the discrete gradient of the squared-Kirchoff function. In Section 5, we establish that the nonlinear algebraic system has a solution using a monotony criterion. In Section 6, we show some compactness properties. In Section 7, we apply the Kolmogorov's theorem to ensure the existence of a convergent subsequence of the discrete solution. In Section 8, we demonstrate that this subsequence tends to a weak solution of the continuous problem. In Section 9, we exhibit some numerical results to show the efficiency and the stability of the scheme.

Let us now formulate the main assumptions on the data. Let us also denote $Q_T = \Omega \times (0, T)$.

(A₁) The initial condition u^0 is assumed to be in $L^\infty(\Omega)$.

(A₂) The function f belongs to $C^1([0, 1], \mathbb{R})$ with

$$\begin{cases} f(u) > 0, & \text{for all } u \in (0, 1), \\ f(u) = 0, & \text{for all } u \in \mathbb{R} \setminus (0, 1). \end{cases} \quad (1.4)$$

(A₄) The tensor Λ is defined as follows

$$\begin{aligned} \Lambda: L^\infty(\Omega) &\rightarrow \mathcal{S}_2(\mathbb{R}) \cap [L^\infty(\Omega)]^{2 \times 2} \\ x &\mapsto \Lambda(x), \end{aligned}$$

where $\mathcal{S}_2(\mathbb{R})$ is the 2-square symmetric matrices space. Furthermore, Λ verifies the ellipticity condition

$$\underline{\Lambda} |\zeta|^2 \leq \Lambda(x) \zeta \cdot \zeta \leq \bar{\Lambda} |\zeta|^2, \text{ for all } \zeta \in \mathbb{R}^2 \text{ and a.e. } x \in \Omega. \quad (1.5)$$

for some positive constants $\underline{\Lambda}$ and $\bar{\Lambda}$.

(A₅) ξ is a $C^0(\mathbb{R}, \mathbb{R})$ nondecreasing function. It is a bijection and its inverse is assumed to be θ -Hölder function with $\theta \in (0, 1)$. This means that for all $a, b \in \mathbb{R}$, $|a - b| \leq C |\xi(a) - \xi(b)|^\theta$.

We next define the natural space $L^2(0, T; H_0^1(\Omega))$ where the solution of the problem (1.1) will be sought

$$H_0^1(\Omega) = \{v \in H^1(\Omega) / v = 0 \text{ on } \partial\Omega\}.$$

Moreover, $H_0^1(\Omega)$ is a Hilbert space endowing with the norm

$$\|v\|_{H_0^1(\Omega)} = \|\nabla v\|_{(L^2(\Omega))^2}.$$

This leads us to the definition of the weak solution

Definition 1.1. (*Weak solution*) A measurable function $u : \Omega \times (0, T) \rightarrow [0, 1]$ is called a weak solution of the problem (1.1) provided

$$\xi(u) \in L^2(0, T; H_0^1(\Omega)) \quad (1.6)$$

$$\begin{aligned} & - \int_{Q_T} u \partial_t \varphi \, dx \, dt + \int_{Q_T} \nabla F(u) \cdot \nabla \varphi \, dx \, dt \\ & - \int_{\Omega} u^0 \varphi(\cdot, 0) \, dx = 0, \quad \forall \varphi \in C_c^\infty(\bar{\Omega} \times [0, T]). \end{aligned} \quad (1.7)$$

The existence of a weak solution of the problem (1.1) has been investigated in [44] while the proof of uniqueness is addressed in [42].

2 DDFV discretization

For the simplicity of the exposition, we follow most of the notations given in the works [3, 17].

2.1 Meshes and notations

A DDFV discretization requires three kinds of meshes, the primal mesh, the dual mesh and the diamond mesh. The primal mesh is denoted by $\overline{\mathfrak{M}} = \mathfrak{M} \cup \partial\mathfrak{M}$, where \mathfrak{M} is a partition of Ω with control volumes and $\partial\mathfrak{M}$ is the set of boundary edges, which are viewed as degenerate control volumes. For every $K \in \overline{\mathfrak{M}}$, the center of K is denoted by x_K . We define \mathcal{V} as the family of these centers.

We designate by \mathcal{V}^* the set of all the vertices of the mesh \mathfrak{M} . It is composed of inner vertices \mathcal{V}_{int}^* and boundary ones \mathcal{V}_{ext}^* . For each $x_{K^*} \in \mathcal{V}_{int}^*$ (resp. $x_{K^*} \in \mathcal{V}_{ext}^*$), a dual control volume K^* is a polygon whose vertices are given by the set $\{x_K \in \mathcal{V} / x_{K^*} \in \overline{K}, K \in \mathfrak{M}\}$ (resp. $\{x_{K^*}\} \cup \{x_K \in \mathcal{V} / x_{K^*} \in \overline{K}, K \in \partial\mathfrak{M}\}$). With these dual sub-domains, we define the dual mesh that is denoted by $\overline{\mathfrak{M}^*} = \mathfrak{M}^* \cup \partial\mathfrak{M}^*$.

For every couple of neighboring primal (resp. dual) control volumes K and L (resp. K^* and L^*), we assume that $\partial K \cap \partial L$ (resp. $\partial K^* \cap \partial L^*$) is an edge of the mesh \mathfrak{M} (resp. $\mathfrak{M}^* \cup \partial\mathfrak{M}^*$) that is denoted by $\sigma = K|L$ (resp. $\sigma^* = K^*|L^*$). We mean by \mathcal{E} (resp. \mathcal{E}^*) the set of all these edges.

We define the diamond mesh $\mathfrak{D} = (\mathcal{D}_{\sigma,\sigma^*})_{(\sigma,\sigma^*) \in \mathcal{E} \times \mathcal{E}^*}$ such that $\mathcal{D}_{\sigma,\sigma^*}$ is a quadrilateral obtained by connecting the end points of σ and σ^* . In the case where $\sigma \in \mathcal{E} \cap \partial\overline{\Omega}$, the quadrilateral $\mathcal{D}_{\sigma,\sigma^*}$ reduces to a triangle as depicted in Figure 1.

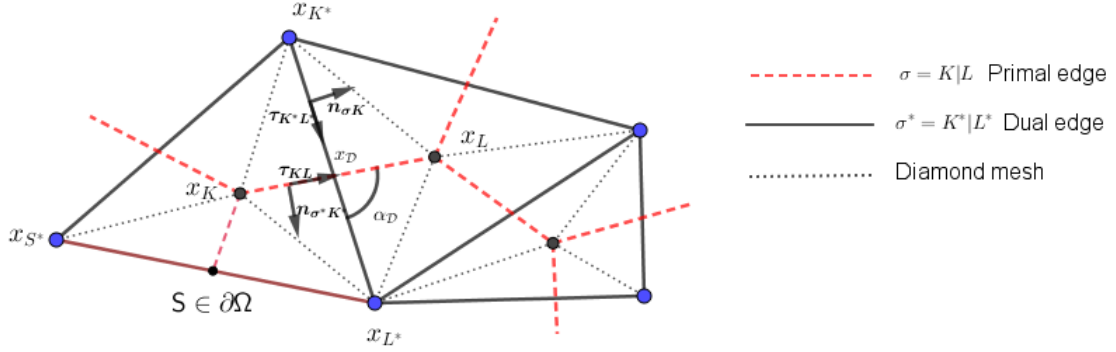


Figure 1: Illustration of the DDFV meshes

The DDFV mesh is defined then as the union of $\mathcal{T} = (\overline{\mathfrak{M}}, \overline{\mathfrak{M}^*})$ and \mathfrak{D} . For every $M \in \mathcal{T}$, the notation m_M represents the measure of M , \mathcal{E}_M contains the edges of M (if $M \in \partial\mathfrak{M}$ we have $\sigma = M$), \mathcal{D}_M is made of all the diamonds $\mathcal{D}_{\sigma,\sigma^*}$ such that $m(\mathcal{D}_{\sigma,\sigma^*} \cap M) > 0$, and d_M refers to the diameter of M .

For each $\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}$, the vertices of $\mathcal{D}_{\sigma,\sigma^*}$ are $(x_K, x_{K^*}, x_L, x_{L^*})$. The center x_D of $\mathcal{D}_{\sigma,\sigma^*} =: \mathcal{D}$ is defined as the intersection of its main diagonals. $m_{\mathcal{D}}$ stands for the measure of \mathcal{D} , $d_{\mathcal{D}}$ its diameter, and $\alpha_{\mathcal{D}}$ is the angle between (x_K, x_L) and (x_{K^*}, x_{L^*}) . For every $e \in \mathcal{E} \cup \mathcal{E}^*$, we define m_e the measure of e . By $\mathbf{n}_{\sigma K}$ (resp. $\mathbf{n}_{\sigma^* K^*}$) we mean the unit normal to σ (resp. σ^*) outward K (resp. K^*), $\boldsymbol{\tau}_{K,L}$ (resp. $\boldsymbol{\tau}_{K^*,L^*}$) the unit tangent vector to σ (resp. σ^*), which is oriented from K (resp. K^*) to L (resp. L^*).

Now, we define the regularity of the mesh that determines the flattening of the diamonds cells and the difference between the size of a primal (resp. dual) control volume and the size of a diamond cell whenever they have a nonempty intersection. This regularity must be controlled in order to perform the convergence analysis of the scheme. So, let $h_{\mathfrak{D}}$ be the largest of the diameters of the diamond cells, $\alpha_{\mathcal{T}}$ the unique real number in $]0, \frac{\pi}{2}]$ such that

$$\sin(\alpha_{\mathcal{T}}) := \min_{\mathcal{D} \in \mathfrak{D}} |\sin(\alpha_{\mathcal{D}})|$$

and ρ_K (resp. ρ_{K^*}) the radius of the biggest inscribed ball in K (resp. K^*) whose center is x_K (resp. x_{K^*}). Thus, the regularity of the mesh is defined by

$$\text{reg}(\mathcal{T}) = \max \left(\frac{1}{\sin(\alpha_{\mathcal{T}})}, \max_{\mathcal{D} \in \mathfrak{D}} \frac{h_{\mathcal{D}}}{\sqrt{m_{\mathcal{D}}}}, \max_{K \in \mathfrak{M}} \frac{d_K}{\sqrt{m_K}}, \max_{K^* \in \mathfrak{M}^*} \frac{d_{K^*}}{\sqrt{m_{K^*}}}, \right. \\ \left. \max_{K \in \mathfrak{M}} \left(\frac{d_K}{\rho_K} + \frac{\rho_K}{d_K} \right), \max_{K^* \in \mathfrak{M}^*, x_{K^*} \neq \text{corner}} \left(\frac{d_{K^*}}{\rho_{K^*}} + \frac{\rho_{K^*}}{d_{K^*}} \right) \right) \quad (2.1)$$

It follows from this definition that there exists a positive constant C depending only on $\text{reg}(\mathcal{T})$ such that

$$m_{\sigma} m_{\sigma^*} \leq C m_K, \quad m_{\sigma^*}^2 \leq C m_{\mathcal{D}}, \quad m_{\sigma}^2 \leq C m_{\mathcal{D}}, \quad m_{\sigma} m_{\sigma^*} \leq C m_{\mathcal{D}}.$$

The time discretization is an increasing sequence of real numbers $(t^n)_{n=0, \dots, N}$ such that

$$t^0 = 0 < t^1 < \dots < t^N = T.$$

For every $n \in \{0, \dots, N-1\}$, we denote $\delta t^n = t^{n+1} - t^n$ and we define $\delta t = \max_{0 \leq n \leq N-1} \delta t^n$. Without loss of generality, we can assume that the step δt^n is uniform. Then $\delta t = \delta t^n$.

2.2 Discrete operators

We now survey the discrete version of the unknowns and operators that will allow us to define the DDFV method to discretize the problem (1.1). To begin with, let us set for every $u_{\mathcal{T}} \in \mathbb{R}^{\#\tau}$

$$u_{\mathcal{T}} = \left((u_K)_{K \in \mathfrak{M}}, (u_{K^*})_{K^* \in \mathfrak{M}^*} \right).$$

We next define the following scalar product on $\mathbb{R}^{\#\tau}$

$$\llbracket u_{\mathcal{T}}, v_{\mathcal{T}} \rrbracket_{\tau} = \frac{1}{2} \left(\sum_{K \in \mathfrak{M}} m_K u_K v_K + \sum_{K^* \in \mathfrak{M}^*} m_{K^*} u_{K^*} v_{K^*} \right), \quad \forall u_{\mathcal{T}}, v_{\mathcal{T}} \in \mathbb{R}^{\#\tau}.$$

The set $(\mathbb{R}^2)^{\#\mathfrak{D}}$ represents the space of vector fields that are constant on the diamond cells of the form $\zeta_{\mathcal{D}} = (\zeta_{\mathcal{D}})_{\mathcal{D} \in \mathfrak{D}}$. This space is endowed by the inner product $(\zeta_{\mathcal{D}}, \varphi_{\mathcal{D}})_{\mathfrak{D}}$, which is defined as

$$(\zeta_{\mathcal{D}}, \varphi_{\mathcal{D}})_{\mathfrak{D}} = \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} \zeta_{\mathcal{D}} \cdot \varphi_{\mathcal{D}}.$$

2.2.1 Discrete gradient

In the framework of the DDFV method, the discrete gradient is a linear mapping from $\mathbb{R}^{\#\tau}$ to $(\mathbb{R}^2)^{\#\mathfrak{D}}$. It is defined for every $u_{\mathcal{T}} \in \mathbb{R}^{\#\tau}$ by

$$\nabla^{\mathfrak{D}} u_{\mathcal{T}} = \sum_{\mathcal{D} \in \mathfrak{D}} \nabla^{\mathcal{D}} u_{\mathcal{T}} \mathbf{1}_{\mathcal{D}},$$

where $\mathbf{1}_{\mathcal{D}}$ is the characteristic function. This definition has been invented in such a way that

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \boldsymbol{\tau}_{K,L} = \frac{u_L - u_K}{m_{\sigma^*}}, \quad \nabla^{\mathcal{D}} u_{\mathcal{T}} \cdot \boldsymbol{\tau}_{K^*,L^*} = \frac{u_{L^*} - u_{K^*}}{m_{\sigma}}.$$

or equivalently,

$$\nabla^{\mathcal{D}} u_{\mathcal{T}} = \frac{1}{\sin(\alpha_{\mathcal{D}})} \left(\frac{u_L - u_K}{m_{\sigma^*}} \mathbf{n}_{\sigma_K} + \frac{u_{L^*} - u_{K^*}}{m_{\sigma}} \mathbf{n}_{\sigma^* K^*} \right), \quad \forall \mathcal{D} \in \mathfrak{D}.$$

Notice that the two components of the discrete gradients are reproduced so that we can have a consistent approximation of the continuous gradient. This has required supplementary unknowns that we introduced

in the dual mesh.

Consider $u_{\mathcal{T}}, v_{\mathcal{T}} \in \mathbb{R}^{\#\mathcal{T}}$ and $\mathcal{D} \in \mathcal{D}$, we define $\delta^{\mathcal{D}}u_{\mathcal{T}} = \begin{bmatrix} u_K - u_L \\ u_{K^*} - u_{L^*} \end{bmatrix}$. Thus, one puts

$$(\nabla^{\mathcal{D}}u_{\mathcal{T}}, \nabla^{\mathcal{D}}v_{\mathcal{T}})_{\mathcal{D}, \Lambda} = \sum_{\mathcal{D} \in \mathcal{D}} \delta^{\mathcal{D}}u_{\mathcal{T}} \cdot \mathbb{A}^{\mathcal{D}, \Lambda} \delta^{\mathcal{D}}v_{\mathcal{T}}, \quad (2.2)$$

where the local matrix $\mathbb{A}^{\mathcal{D}, \Lambda}$ reads

$$\mathbb{A}^{\mathcal{D}, \Lambda} = \frac{1}{4m_{\mathcal{D}}} \begin{bmatrix} m_{\sigma}^2 \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma K} & m_{\sigma} m_{\sigma^*} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} \\ m_{\sigma} m_{\sigma^*} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} & m_{\sigma^*}^2 \Lambda \mathbf{n}_{\sigma^* K^*} \cdot \mathbf{n}_{\sigma^* K^*} \end{bmatrix}, \quad \forall \mathcal{D} \in \mathcal{D}. \quad (2.3)$$

One also defines

$$\mathbb{A}^{\mathcal{D}} = \frac{1}{4m_{\mathcal{D}}} \begin{bmatrix} m_{\sigma}^2 & m_{\sigma} m_{\sigma^*} \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} \\ m_{\sigma} m_{\sigma^*} \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} & m_{\sigma^*}^2 \end{bmatrix}, \quad \forall \mathcal{D} \in \mathcal{D}. \quad (2.4)$$

These matrices verify prominent inequalities, which are stated and proved in Lemma A.2.

2.2.2 Discrete divergence

The discrete divergence has been introduced in order to reproduce a discrete counterpart of Green's formula. It is defined by a mapping from $(\mathbb{R}^2)^{\#\mathcal{D}}$ to $\mathbb{R}^{\#\mathcal{T}}$ such that: for all $\Psi_{\mathcal{D}} \in (\mathbb{R}^2)^{\#\mathcal{D}}$

$$\operatorname{div}^{\mathcal{T}} \Psi_{\mathcal{D}} = \left(\operatorname{div}^{\mathfrak{M}} \Psi_{\mathcal{D}}, \operatorname{div}^{\mathfrak{M}^*} \Psi_{\mathcal{D}}, \operatorname{div}^{\partial \mathfrak{M}^*} \Psi_{\mathcal{D}} \right)$$

with $\operatorname{div}^{\mathfrak{M}} \Psi_{\mathcal{D}} = (\operatorname{div}_K \Psi_{\mathcal{D}})_{K \in \mathfrak{M}}$, $\operatorname{div}^{\mathfrak{M}^*} \Psi_{\mathcal{D}} = (\operatorname{div}_{K^*} \Psi_{\mathcal{D}})_{K^* \in \mathfrak{M}^*}$ and $\operatorname{div}^{\partial \mathfrak{M}^*} \Psi_{\mathcal{D}} = (\operatorname{div}_{K^*} \Psi_{\mathcal{D}})_{K^* \in \partial \mathfrak{M}^*}$. Each component is explicitly given by

$$\begin{aligned} \operatorname{div}_K \Psi_{\mathcal{D}} &= \frac{1}{m_K} \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} m_{\sigma} \Psi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma K}, & \forall K \in \mathfrak{M}, \\ \operatorname{div}_{K^*} \Psi_{\mathcal{D}} &= \frac{1}{m_{K^*}} \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{K^*}} m_{\sigma^*} \Psi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^* K^*}, & \forall K^* \in \mathfrak{M}^*, \\ \operatorname{div}_{K^*} \Psi_{\mathcal{D}} &= \frac{1}{m_{K^*}} \left(\sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{K^*}} m_{\sigma^*} \Psi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^* K^*} + \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{K^*} \cap \partial \Omega} \frac{m_{\sigma^*}}{2} \Psi_{\mathcal{D}} \cdot \mathbf{n}_{\sigma^* K^*} \right), & \forall K^* \in \partial \mathfrak{M}^*. \end{aligned}$$

2.3 Approximation spaces

This subsection is devoted to describe the discrete spaces together with some related notations. Firstly we have seen that a DDFV mesh is composed of three different partitions. We therefore require three piecewise reconstructions for any vector $u_{\mathcal{T}} \in \mathbb{R}^{\#\mathcal{T}}$. More Precisely

- (i) We will denote $u_{\mathfrak{M}}$ (resp. $u_{\mathfrak{M}^*}$) the first (resp. second) reconstruction on the primal (dual) mesh, which is a piecewise constant function with

$$u_{\mathfrak{M}} = \sum_{K \in \mathfrak{M}} u_K \mathbf{1}_K, \quad u_{\mathfrak{M}^*} = \sum_{K^* \in \mathfrak{M}^*} u_{K^*} \mathbf{1}_{K^*}$$

In addition, it is worthy to point out that $u_{\mathcal{T}}$ can be viewed as a function of $L^1(\Omega)$ and it can be rewritten as the mean value of $u_{\mathfrak{M}}$ and $u_{\mathfrak{M}^*}$, we then set

$$u_{\mathcal{T}} = \frac{1}{2} \left(\sum_{K \in \mathfrak{M}} u_K \mathbf{1}_K + \sum_{K^* \in \mathfrak{M}^*} u_{K^*} \mathbf{1}_{K^*} \right).$$

We henceforth denote $X_{\mathcal{T}}$ the set of all these functions $u_{\mathcal{T}}$. In fact $X_{\mathcal{T}}$ and $\mathbb{R}^{\#\mathcal{T}}$ can be identified. Thus $u_{\mathcal{T}} \in \mathbb{R}^{\#\mathcal{T}}$ if and only if $u_{\mathcal{T}} \in X_{\mathcal{T}}$.

- (ii) The third reconstruction concerns the diamond mesh. It is also a piecewise constant function on the diamond cells and defined by $u_{\mathcal{D}} =: \sum_{\mathcal{D} \in \mathcal{D}} u_{\mathcal{D}} \mathbf{1}_{\mathcal{D}}$. The set of all these functions will be denoted with $X_{\mathcal{D}}$.

As a consequence, we define the following approximation spaces

$$\begin{aligned} X_{\mathcal{T}, \delta t} &= \left\{ u_{\mathcal{T}, \delta t} \in L^1(Q_T) : u_{\mathcal{T}, \delta t}(x, t) = u_{\mathcal{T}}^{n+1}(x) / u_{\mathcal{T}} \in X_{\mathcal{T}}, \forall t \in (t^n, t^{n+1}], \forall n = 0, \dots, N-1 \right\} \\ X_{\mathcal{D}, \delta t} &= \left\{ u_{\mathcal{D}, \delta t} \in L^1(Q_T) : u_{\mathcal{D}, \delta t}(x, t) = u_{\mathcal{D}}^{n+1}(x) / u_{\mathcal{D}} \in X_{\mathcal{D}}, \forall t \in (t^n, t^{n+1}], \forall n = 0, \dots, N-1 \right\}. \end{aligned}$$

For each function $u_{\mathcal{D}, \delta t} \in X_{\mathcal{T}, \delta t}$, its gradient $\nabla^{\mathcal{D}} u_{\mathcal{T}, \delta t} \in X_{\mathcal{D}, \delta t} \times X_{\mathcal{D}, \delta t}$ is written by

$$\nabla^{\mathcal{D}} u_{\mathcal{T}, \delta t}(x, t) = \nabla^{\mathcal{D}} u_{\mathcal{T}}^{n+1}(x), \quad \forall t \in (t^n, t^{n+1}], \forall n = 0, \dots, N-1.$$

As for $u_{\mathcal{T}, \delta t} \in X_{\mathcal{T}, \delta t}$, we define

$$u_{\mathfrak{M}, \delta t}(x, t) = u_{\mathfrak{M}}^{n+1}(x), \quad u_{\overline{\mathfrak{M}^*}, \delta t}(x, t) = u_{\overline{\mathfrak{M}^*}}^{n+1}(x) \quad \forall t \in (t^n, t^{n+1}], \forall n = 0, \dots, N-1.$$

Let us now consider a nonlinear function $F : \mathbb{R} \rightarrow \mathbb{R}$. We will denote $F_{\mathcal{T}, \delta t}$ as the mean value of $F(u_{\mathfrak{M}, \delta t})$ and $F(u_{\overline{\mathfrak{M}^*}, \delta t})$. In other words

$$F_{\mathcal{T}, \delta t} = \frac{1}{2} (F(u_{\mathfrak{M}, \delta t}) + F(u_{\overline{\mathfrak{M}^*}, \delta t})).$$

We next equip the finite dimensional space $X_{\mathcal{T}}$ with the norm $|\cdot|_{p, \mathcal{T}}$. For every $u_{\mathcal{T}} \in \mathbb{R}^{\#\mathcal{T}}$, we define

$$|u_{\mathcal{T}}|_{p, \mathcal{T}} = \begin{cases} \left(\frac{1}{2} \sum_{K \in \mathfrak{M}} m_K |u_K|^p + \frac{1}{2} \sum_{K^* \in \overline{\mathfrak{M}^*}} m_{K^*} |u_{K^*}|^p \right)^{1/p} & \text{if } 1 \leq p < +\infty \\ \max \left(\max_{K \in \mathfrak{M}} |u_K|, \max_{K^* \in \overline{\mathfrak{M}^*}} |u_{K^*}| \right) & \text{if } p = +\infty \end{cases}.$$

This leads us to define the following discrete Sobolev norm

$$\|u_{\mathcal{T}}\|_{1, p, \mathcal{T}} = \begin{cases} \left(|u_{\mathcal{T}}|_{p, \mathcal{T}}^p + \|\nabla^{\mathcal{D}} u_{\mathcal{T}}\|_p^p \right)^{1/p} & \text{if } 1 \leq p < +\infty \\ |u_{\mathcal{T}}|_{\infty, \mathcal{T}} + \|\nabla^{\mathcal{D}} u_{\mathcal{T}}\|_{\infty} & \text{if } p = +\infty \end{cases},$$

where we recall that the norm of the discrete gradient is

$$\|\nabla^{\mathcal{D}} u_{\mathcal{T}}\|_p^p = \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} |\nabla^{\mathcal{D}} u_{\mathcal{T}}|^p, \quad \forall 1 \leq p < +\infty, \quad \text{and} \quad \|\nabla^{\mathcal{D}} u_{\mathcal{T}}\|_{\infty} = \max_{\mathcal{D} \in \mathcal{D}} |\nabla^{\mathcal{D}} u_{\mathcal{T}}|.$$

Observe that

$$\|\nabla^{\mathcal{D}} u_{\mathcal{T}}\|_2^2 = \sum_{\mathcal{D} \in \mathcal{D}} \delta^{\mathcal{D}} u_{\mathcal{T}} \cdot \mathbb{A}^{\mathcal{D}} \delta^{\mathcal{D}} u_{\mathcal{T}}.$$

Finally we can define the discrete counterpart of the $L^q(0, T; W^{1, p}(\Omega))$ -norm

$$\|u_{\mathcal{T}, \delta t}\|_{q; 1, p, \mathcal{T}} = \begin{cases} \left(\sum_{n=1}^N \delta t \|u_{\mathcal{T}}^n\|_{1, p, \mathcal{T}}^q \right)^{1/q} & \text{if } 1 \leq p, q < +\infty \\ \max_{n=1, \dots, N} \|u_{\mathcal{T}}^n\|_{1, \infty, \mathcal{T}} & \text{if } p = q = +\infty \end{cases}.$$

3 Numerical scheme

Belonging to the family of finite volume methods, the DDFV scheme is basically obtained by integrating the first equation of (1.1) over $M \times]t^n, t^{n+1}]$, where M is a primal or dual cell. Performing the Green's formula, yields the balance equation. Then the resulting fluxes are approximated by introducing the definition of the discrete gradient and by that of the numerical flux function.

For the convenience of the reader, we briefly look at the discretization of (1.1) on the primal mesh and it is deduced similarly in the case of the dual mesh. So, let $n \in \{0, \dots, N-1\}$ and K be a primal control volume, then

$$\int_K \left(u(x, t^{n+1}) - u(x, t^n) \right) dx - \sum_{\sigma \in \mathcal{E}_K} \int_{t^n}^{t^{n+1}} \int_{\sigma} f(u) \Lambda \nabla u \cdot \mathbf{n}_{\sigma K} d\sigma dt = 0. \quad (3.1)$$

The evolution term is approximated with the aid of Euler scheme

$$\int_K \left(u(x, t^{n+1}) - u(x, t^n) \right) dx \approx m_K \left(u_K^{n+1} - u_K^n \right), \quad (3.2)$$

where u_K^m is the mean value of $u(\cdot, t^m)$ over K for $m = n, n+1$. Concerning the diffusion part, it is discretized as follows

$$\begin{aligned} - \int_{t^n}^{t^{n+1}} \int_{\sigma} f(u) \Lambda \nabla u \cdot \mathbf{n}_{\sigma K} d\sigma dt \approx & \frac{\delta t}{\sin(\alpha_D)} \left(\frac{m_{\sigma}}{m_{\sigma^*}} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma K} (F(u_K^{n+1}) - F(u_L^{n+1})) \right. \\ & \left. + v_{KL}^{n+1} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} (\xi(u_{K^*}^{n+1}) - \xi(u_{L^*}^{n+1})) \right) \end{aligned}$$

where F (resp. ξ) is the Kirchoff (resp. squared-Kirchoff) function and v_{KL}^{n+1} is an approximation of $\sqrt{f(u)} =: v(u)$ on the primal edge σ so that the term $v_{KL}^{n+1} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} (\xi(u_{K^*}^{n+1}) - \xi(u_{L^*}^{n+1}))$ forms a numerical flux function, which is computed at $(u_K^{n+1}, u_L^{n+1}, \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} (\xi(u_{K^*}^{n+1}) - \xi(u_{L^*}^{n+1})))$. More generally, a function G of arguments $(a, b, c) \in \mathbb{R}^3$ is said to be numerical flux if the following assertions are satisfied:

$$\begin{cases} (A_1) & G(\cdot, b, c) \text{ is nondecreasing for all } b, c \in \mathbb{R}, \\ & \text{and } G(a, \cdot, c) \text{ is nonincreasing for all } a, c \in \mathbb{R}; \\ (A_2) & G(a, b, c) = -G(a, b, -c) \text{ for all } a, b, c \in \mathbb{R}; \\ (A_3) & G(a, a, c) = (\sqrt{f(a)})c = v(a)c \text{ for all } a, c \in \mathbb{R}. \end{cases} \quad (3.3)$$

In order to make a conspicuous scheme, we will denote

$$a_{KL} := \frac{1}{\sin(\alpha_D)} \frac{m_{\sigma}}{m_{\sigma^*}} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma K} > 0, \quad \eta_{\sigma\sigma^*}^D := \frac{1}{\sin(\alpha_D)} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} \in \mathbb{R} \quad (3.4)$$

$$g_M^{n+1} := g(u_M^{n+1}), \quad \forall M \in \{K, L, K^*, L^*\} \text{ and } g \in \{F, \xi\} \quad (3.5)$$

$$\delta_{LK}^{n+1} u := u_L^{n+1} - u_K^{n+1}, \quad \delta_{L^*K^*}^{n+1} u := u_{L^*}^{n+1} - u_{K^*}^{n+1}. \quad (3.6)$$

We next provide a central option concerning the constructions of v_{KL}^{n+1} in a such way that the numerical flux function obeys the properties (3.3) and will be the cornerstone for proving the energy estimate result. This choice consists in considering the Engquist-Osher scheme [41], which reads

$$v_{KL}^{n+1} = \begin{cases} v_{\downarrow}(u_L^{n+1}) + v_{\uparrow}(u_K^{n+1}) & \text{if } \eta_{\sigma\sigma^*}^D (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}) \geq 0 \\ v_{\downarrow}(u_K^{n+1}) + v_{\uparrow}(u_L^{n+1}) & \text{else,} \end{cases} \quad (3.7)$$

where the functions $v_{\downarrow}, v_{\uparrow}$ are given by

$$v_{\uparrow}(u) := \int_0^u \left(v'(s) \right)^+ ds, \quad v_{\downarrow}(u) := - \int_0^u \left(v'(s) \right)^- ds$$

with the following convention $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$ that will be adopted hereafter. We wish to emphasize that one can write $G(u_K^{n+1}, u_{L^*}^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u))$ as follows

$$G(u_K^{n+1}, u_{L^*}^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u)) = v_{KL}^{n+1} \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}).$$

As stressed in [3, 17], we require a penalization operator, which is crucial when passing to limit in the convergence. It permits to check that the approximate solution on the primal mesh and the dual mesh tend to the same limit. This will be also a key point in our study of convergence of the diffusive term. For this purpose, let $\varepsilon \in]0, 2[$ and $u_{\mathcal{T}} \in \mathbb{R}^{\mathcal{T}}$. The penalization $\mathcal{P}^{\mathcal{T}}$ is a map from $\mathbb{R}^{\#\mathcal{T}}$ to $\mathbb{R}^{\#\mathcal{T}}$ which is defined, for all $u_{\mathcal{T}}$, by

$$\mathcal{P}^{\mathcal{T}} u_{\mathcal{T}} = \left(\mathcal{P}^{\mathfrak{M}} u_{\mathcal{T}}, \mathcal{P}^{\mathfrak{M}^*} u_{\mathcal{T}}, \mathcal{P}^{\partial\mathfrak{M}^*} u_{\mathcal{T}} \right),$$

where $\mathcal{P}^{\mathfrak{M}} u_{\mathcal{T}} = (\mathcal{P}_K u_{\mathcal{T}})_{K \in \mathfrak{M}}$, $\mathcal{P}^{\mathfrak{M}^*} u_{\mathcal{T}} = (\mathcal{P}_{K^*} u_{\mathcal{T}})_{K^* \in \mathfrak{M}^*}$, $\mathcal{P}^{\partial\mathfrak{M}^*} u_{\mathcal{T}} = (\mathcal{P}_{K^*} u_{\mathcal{T}})_{K^* \in \partial\mathfrak{M}^*}$ such that

$$\mathcal{P}_K u_{\mathcal{T}} = \frac{1}{m_K} \frac{1}{h_{\mathfrak{D}}^{\varepsilon}} \sum_{K^* \in \overline{\mathfrak{M}^*}} m_{K \cap K^*} \left(F(u_K) - F(u_{K^*}) \right), \quad \forall K \in \mathfrak{M}, \quad (3.8)$$

$$\mathcal{P}_{K^*} u_{\mathcal{T}} = \frac{1}{m_{K^*}} \frac{1}{h_{\mathfrak{D}}^{\varepsilon}} \sum_{K \in \mathfrak{M}} m_{K \cap K^*} \left(F(u_{K^*}) - F(u_K) \right), \quad \forall K^* \in \overline{\mathfrak{M}^*}. \quad (3.9)$$

Based on the following inequality

$$(F(a) - F(b))(a - b) \geq (\xi(a) - \xi(b))^2, \quad \forall a, b \in \mathbb{R}, \quad (3.10)$$

one can check that

$$\begin{aligned} \llbracket \mathcal{P} u_{\mathcal{T}}, u_{\mathcal{T}} \rrbracket_{\mathcal{T}} &= \frac{1}{2} \frac{1}{h_{\mathfrak{D}}^{\varepsilon}} \sum_{K^* \in \overline{\mathfrak{M}^*}} \sum_{K \in \mathfrak{M}} m_{K \cap K^*} (F(u_K) - F(u_{K^*})) (u_K - u_{K^*}) \\ &\geq \frac{1}{2} \frac{1}{h_{\mathfrak{D}}^{\varepsilon}} \left\| \xi(u_{\mathfrak{M}}) - \xi(u_{\overline{\mathfrak{M}^*}}) \right\|_{L^2(\Omega)}^2. \end{aligned} \quad (3.11)$$

Thanks to the DDFV discretization, an approximate solution of problem (1.1) is defined as a function $u_{\mathcal{T}, \delta t} \in X_{\mathcal{T}, \delta t}$ and satisfies the set of equations

$$u_M^0 = \frac{1}{m_M} \int_M u^0(x) dx, \quad \forall M \in \mathcal{T}, \quad (3.12)$$

$$\begin{aligned} &\frac{m_K}{\delta t} \left(u_K^{n+1} - u_K^n \right) \\ &+ \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} \left(a_{KL} (F_K^{n+1} - F_L^{n+1}) + G(u_K^{n+1}, u_L^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u)) \right) \\ &+ \gamma \mathcal{P}_K u_{\mathcal{T}}^{n+1} = 0, \quad \forall K \in \mathfrak{M}, \quad n \geq 0, \end{aligned} \quad (3.13)$$

$$\begin{aligned} &\frac{m_{K^*}}{\delta t} \left(u_{K^*}^{n+1} - u_{K^*}^n \right) \\ &+ \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{K^*}} \left(a_{K^*L^*} (F_{K^*}^{n+1} - F_{L^*}^{n+1}) + G^*(u_{K^*}^{n+1}, u_{L^*}^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{LK}^{n+1} \xi(u)) \right) \\ &+ \gamma \mathcal{P}_{K^*} u_{\mathcal{T}}^{n+1} = 0, \quad \forall K \in \overline{\mathfrak{M}^*}, \quad n \geq 0. \end{aligned} \quad (3.14)$$

By G, G^* we mean numerical flux functions and γ a positive parameter. Let us next check that G and G^* are well defined. This is the object of the following result.

Lemma 3.1. *The numerical flux functions G, G^* are well defined, meaning that assertions $(A_1), (A_2)$ and (A_3) of (3.3) are fulfilled.*

Proof. Let us focus on G and it is similar in the case of G^* . Observe that items $(A_1), (A_3)$ of Definition 3.3 are direct consequences of the expression of v_{KL}^{n+1} given in (3.7). It remains to check that the assertion (A_2) holds. We firstly point out that the discrete gradient on a fixed diamond, which we recall below, is uniquely defined

$$\nabla^{\mathcal{D}} u_{\tau} = \frac{1}{\sin(\alpha_{\mathcal{D}})} \left(\frac{u_L - u_K}{m_{\sigma^*}} \mathbf{n}_{\sigma K} + \frac{u_{L^*} - u_{K^*}}{m_{\sigma}} \mathbf{n}_{\sigma^* K^*} \right).$$

In other word, we associate to the primal interface $\sigma = K|L$ a unique dual interface $\sigma = K^*|L^*$. Now if permute K, L then K^*, L^* are automatically permuted, but the coefficient $\eta_{\sigma\sigma^*}^{\mathcal{D}}$ keeps the same sign. In particular, this asserts that $\eta_{\sigma\sigma^*}^{\mathcal{D}} = \eta_{\sigma^*\sigma}^{\mathcal{D}}$. Consequently

$$\eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}) = - \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{L^*}^{n+1} - \xi_{K^*}^{n+1}).$$

According to this identity and the definition of v_{KL}^{n+1} introduced in (3.7), one finds

$$v_{KL}^{n+1} = v_{LK}^{n+1}.$$

Hence

$$G(u_K^{n+1}, u_L^{n+1}, \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1})) = - G(u_L^{n+1}, u_K^{n+1}, \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{L^*}^{n+1} - \xi_{K^*}^{n+1})).$$

□

Remark 3.1. In case of $\Lambda = Id$, the coefficient $\eta_{\sigma\sigma^*}^{\mathcal{D}}$ measures the flattening of the diamond cells with eventual anisotropy. In case of $\eta_{\sigma\sigma^*}^{\mathcal{D}} \equiv 0$ for all \mathcal{D} , which means that the mesh is orthogonal [27], the above discretization reduces to the VF4 scheme of the problem (1.1) on the primal mesh and on the dual mesh separately. Its convergence analysis can be found in [29].

4 Maximum principle and a priori estimates

In this section, we show that any solution of the proposed scheme verifies a L^∞ bound. In addition, we derive some a priori estimates on the discrete gradient of the squared-Kirchoff function. These materials are of an immense importance when we prove the convergence.

4.1 Boundedness of discrete solution

Lemma 4.1. (*Maximum Principle*)

For each fixed integer $0 \leq n \leq N - 1$, let $(u_{\mathcal{T}}^n)$ be a sequence such that the DDFV scheme (3.13)-(3.14) holds. If $u_{\mathfrak{M}}^0, u_{\mathfrak{M}^*}^0$ belongs to $[0, 1]$ then $u_{\mathfrak{M}}^{n+1}, u_{\mathfrak{M}^*}^{n+1}$ remain also in $[0, 1]$.

Proof. The proof is carried out by induction on n . Fix $n \in \{0, \dots, N - 1\}$. Let us assume that the claim is true for $u_{\mathfrak{M}}^n, u_{\mathfrak{M}^*}^n$ and check that it is so for $u_{\mathfrak{M}}^{n+1}, u_{\mathfrak{M}^*}^{n+1}$. For this purpose, we perform the proof in two steps.

Step 1 : We consider $u_K^{n+1} = \min_{L \in \mathfrak{M}} (u_L^{n+1})$. Multiplying (3.13) by $-(u_K^{n+1})^-$ yields

$$\begin{aligned} - \frac{m_K}{\delta t} (u_K^{n+1} - u_K^n) (u_K^{n+1})^- - \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} \left(a_{KL} (F_K^{n+1} - F_L^{n+1}) + G(u_K^{n+1}, u_L^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u)) \right) (u_K^{n+1})^- \\ - \gamma \mathcal{P}_K u_{\tau}^{n+1} (u_K^{n+1})^- = 0. \end{aligned}$$

Since F is a nondecreasing function, we obtain $a_{KL} (F_K^{n+1} - F_L^{n+1}) \leq 0$. Furthermore

$$G(u_K^{n+1}, u_L^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u)) (u_K^{n+1})^- \leq 0.$$

Indeed, if $0 \leq u_K^{n+1}$ then $(u_K^{n+1})^- = 0$. Otherwise, we introduce the fact that the numerical flux function is nonincreasing with respect to the second argument and that is consistent

$$\begin{aligned} G(u_K^{n+1}, u_L^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u))(u_K^{n+1})^- &\leq G(u_K^{n+1}, u_K^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{K^*L^*}^{n+1} \xi(u))(u_K^{n+1})^- \\ &= v(u_K^{n+1}) \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u)(u_K^{n+1})^- = 0. \end{aligned}$$

The previous equality holds thanks to the degeneracy of the function v for every $u \leq 0$. Let us next demonstrate that

$$-\mathcal{P}_K u_\tau^{n+1} (u_K^{n+1})^- \geq 0. \quad (4.1)$$

It follows from the definition of the penalization term highlighted in (3.8) that

$$\begin{aligned} -\mathcal{P}_K u_\tau^{n+1} (u_K^{n+1})^- &= \frac{1}{m_K} \frac{1}{h_{\mathfrak{D}}^\varepsilon} \sum_{K^* \in \mathfrak{M}^*} m_{K \cap K^*} \left(-F(u_K^{n+1})(u_K^{n+1})^- + F(u_{K^*}^{n+1})(u_K^{n+1})^- \right) \\ &= \frac{1}{m_K} \frac{1}{h_{\mathfrak{D}}^\varepsilon} \sum_{K^* \in \mathfrak{M}^*} m_{K \cap K^*} \left((F(0) - F(u_K^{n+1}))(u_K^{n+1})^- + (F(u_{K^*}^{n+1}) - F(0))(u_K^{n+1})^- \right). \end{aligned}$$

Thanks to the uniform continuity of the function F , one finds $z_0 \in]u_K^{n+1}, 0[$ and $z_0^* \in] - (u_{K^*}^{n+1})^-, (u_{K^*}^{n+1})^+[$ such that

$$-\mathcal{P}_K u_\tau^{n+1} (u_K^{n+1})^- = \frac{1}{m_K} \frac{1}{h_{\mathfrak{D}}^\varepsilon} \sum_{K^* \in \mathfrak{M}^*} m_{K \cap K^*} \left(f(z_0)(u_K^{n+1})^{-,2} + f(z_0^*) u_{K^*}^{n+1} (u_K^{n+1})^- \right).$$

Observe that $f(z_0^*) \geq 0$ if $u_{K^*}^{n+1} \geq 0$ and $f(z_0^*) = 0$ otherwise. Hence, inequality (4.1) holds. As a consequence

$$- \left(u_K^{n+1} - u_K^n \right) (u_K^{n+1})^- = (u_K^{n+1})^{-,2} + (u_K^{n+1})^- u_K^n \leq 0,$$

which implies, using the induction assumption, that $(u_K^{n+1})^- = 0$. Hence, $u_K^{n+1} \geq 0$.

Step 2 : We here switch the role of the control volume K and take now $u_K^{n+1} = \max_{L \in \mathfrak{M}} (u_L^{n+1})$. Multiplying (3.13) by $(u_K^{n+1} - 1)^+$ gives

$$\begin{aligned} \frac{m_K}{\delta t} \left(u_K^{n+1} - u_K^n \right) (u_K^{n+1} - 1)^+ &+ \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} \left(a_{KL} (F_K^{n+1} - F_L^{n+1}) + G(u_K^{n+1}, u_L^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u)) \right) (u_K^{n+1} - 1)^+ \\ &+ \gamma \mathcal{P}_K u_\tau^{n+1} (u_K^{n+1} - 1)^+ = 0. \end{aligned}$$

It is now evident that $a_{KL} (F_K^{n+1} - F_L^{n+1})(u_K^{n+1} - 1)^+ \geq 0$. Next, let us establish that the following inequality holds

$$G(u_K^{n+1}, u_L^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u))(u_K^{n+1} - 1)^+ \leq 0.$$

So, if $u_K^{n+1} \leq 1$ then $(u_K^{n+1} - 1)^+ = 0$. Otherwise, $u_K^{n+1} \geq 1$, we utilize once again the consistency and the decrease of the function G with respect to the second variable. Therefore

$$\begin{aligned} G(u_K^{n+1}, u_L^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u))(u_K^{n+1} - 1)^+ &\geq G(u_K^{n+1}, u_K^{n+1}; \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{K^*L^*}^{n+1} \xi(u))(u_K^{n+1} - 1)^+ \\ &= v(u_K^{n+1}) \eta_{\sigma\sigma^*}^{\mathcal{D}} \delta_{L^*K^*}^{n+1} \xi(u)(u_K^{n+1} - 1)^+ = 0. \end{aligned}$$

One can see in a straightforward way that $\mathcal{P}_K u_\tau (u_K^{n+1} - 1)^+ \geq 0$ as we detailed in *Step 1*. Utilizing the identity

$$\left(u_K^{n+1} - u_K^n \right) (u_K^{n+1} - 1)^+ = (u_K^{n+1} - 1)^{+,2} + (u_K^{n+1} - 1)^+ (1 - u_K^n),$$

we deduce that $(u_K^{n+1} - 1)^+ = 0$, which yields $u_K^{n+1} \leq 1$.

Similarly, we mimic the same steps so that we prove the property in the case of the dual mesh. Hence, the proof of the Lemma is concluded. \square

Remark 4.1. *It follows from the above proof of the discrete maximum principle that the degeneracy of the function v absorbs the terms, which may engender any undershoots or overshoots. This function is of great advantage to derive some energy estimate as we will see next.*

In the sequel, we will denote by C different constants in various occurrences, which depend only on the physical data and are independent of the discretization parameters δt , $h_{\mathfrak{D}}$.

4.2 Estimates on the discrete gradients

We need first to introduce some remarkable lemmas in the framework of DDFV methods.

Lemma 4.2. *(Discrete integration by parts) Let \mathcal{M} be a primal or a dual mesh of the domain Ω . For every $K \in \mathcal{M}$, we denote by $N(K)$ the set of neighbors of K . Let A_{KL} , $K \in \mathcal{M}$ and $L \in N(K)$ be a real value such that $A_{KL} = -A_{LK}$, and let φ be a piecewise constant function on the cells of \mathcal{M} . Then*

$$\sum_{K \in \mathcal{M}} \sum_{L \in N(K)} A_{KL} \varphi_K = -\frac{1}{2} \sum_{K \in \mathcal{M}} \sum_{L \in N(K)} A_{KL} (\varphi_L - \varphi_K). \quad (4.2)$$

Particularly, if $A_{KL} = T_{KL}(c_L - c_K)$, with $T_{KL} = T_{LK}$, one infers

$$\sum_{K \in \mathcal{M}} \sum_{L \in N(K)} T_{KL}(c_L - c_K) \varphi_K = -\frac{1}{2} \sum_{K \in \mathcal{M}} \sum_{L \in N(K)} T_{KL}(c_L - c_K) (\varphi_L - \varphi_K). \quad (4.3)$$

Proof. *The proof of this lemma is omitted since it is similar to that one given in [7].*

The convergence analysis of the DDFV schemes has been based on the following discrete duality formula called sometimes discrete Stokes formula. One can look for its proof for instance in [21].

Lemma 4.3. *(Discrete Stokes formula) For all $(\zeta_{\mathfrak{D}}, w_{\mathcal{T}}) \in (\mathbb{R}^2)^{\#\mathfrak{D}} \times \mathbb{R}^{\#\mathcal{T}}$, the following relationships holds*

$$\llbracket \operatorname{div}^{\mathcal{T}} \zeta_{\mathfrak{D}}, w_{\mathcal{T}} \rrbracket_{\mathcal{T}} = -\left(\zeta_{\mathfrak{D}}, \nabla^{\mathfrak{D}} w_{\mathcal{T}} \right)_{\mathfrak{D}}. \quad (4.4)$$

We refer to [8, 27] for the proof of the following fundamental inequality.

Lemma 4.4. *(Discrete Poincaré inequality) Consider \mathcal{T} a mesh of Ω . Then there exists a constant C , only depending on the diameter of Ω , such that for every $w_{\mathcal{T}} \in \mathbb{R}^{\#\mathcal{T}}$, one has*

$$|w_{\mathcal{T}}|_{2, \mathcal{T}} \leq \|w_{\mathfrak{M}}\|_{L^2(\Omega)} + \|w_{\mathfrak{M}^*}\|_{L^2(\Omega)} \leq \|\nabla^{\mathfrak{D}} w_{\mathcal{T}}\|_2$$

Proposition 4.1. *(Discrete gradient estimate) Let $(u_{\mathcal{T}}^n)$, for $n = 0, \dots, N$, such that the DDFV scheme (3.12)-(3.14) holds. Then*

$$\sum_{n=0}^{N-1} \delta t \|\nabla^{\mathfrak{D}} \xi_{\mathcal{T}}^{n+1}\|_2^2 + \frac{1}{2} \frac{1}{h_{\mathfrak{D}}^\varepsilon} \sum_{n=0}^{N-1} \delta t \left\| \xi(u_{\mathfrak{M}}^{n+1}) - \xi(u_{\mathfrak{M}^*}^{n+1}) \right\|_{L^2(\Omega)}^2 \leq C, \quad (4.5)$$

for some appropriate positive constant C .

Proof. We multiply the first (resp. second) equation of the DDFV scheme (3.13)-(3.14) by u_K^{n+1} (resp. $u_{K^*}^{n+1}$) and sum up over all the primal (resp. dual) cells and the integers n . Adding together the obtained equations leads to

$$T_1 + T_2 + T_3 = 0,$$

where we have put

$$\begin{aligned}
T_1 &= \sum_{n=0}^{N-1} \sum_{K \in \mathfrak{M}} m_K (u_K^{n+1} - u_K^n) u_K^{n+1} + \sum_{n=0}^{N-1} \sum_{K^* \in \overline{\mathfrak{M}^*}} m_{K^*} (u_{K^*}^{n+1} - u_{K^*}^n) u_{K^*}^{n+1}, \\
T_2 &= \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathfrak{M}} \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} \left(a_{KL} (F_K^{n+1} - F_L^{n+1}) + v_{KL}^{n+1} \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{K^*}^{n+1} - \xi_L^{n+1}) \right) u_K^{n+1} \\
&\quad + \sum_{n=0}^{N-1} \delta t \sum_{K^* \in \overline{\mathfrak{M}^*}} \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{K^*}} \left(a_{K^*L^*} (F_{K^*}^{n+1} - F_{L^*}^{n+1}) + v_{K^*L^*}^{n+1} \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}) \right) u_{K^*}^{n+1} \\
T_3 &= \sum_{n=0}^{N-1} \delta t \gamma \llbracket \mathcal{P} u_{\mathcal{T}}^{n+1}, u_{\mathcal{T}}^{n+1} \rrbracket_{\mathcal{T}}.
\end{aligned}$$

First of all, observe that

$$x(x - y) \geq \frac{1}{2}(x^2 - y^2), \quad \forall x, y \in \mathbb{R}.$$

As a consequence of the above inequality, one can underestimate T_1

$$\frac{1}{2} \sum_{K \in \mathfrak{M}} m_K \left((u_K^N)^2 - (u_K^0)^2 \right) + \frac{1}{2} \sum_{K^* \in \overline{\mathfrak{M}^*}} m_{K^*} \left((u_{K^*}^N)^2 - (u_{K^*}^0)^2 \right) \leq T_1. \quad (4.6)$$

Let us now turn our attention to the term T_2 . For this goal, we perform a discrete integration by parts as given in Lemma 4.2, hence we obtain

$$T_2 = T_{21} + T_{22},$$

with

$$\begin{aligned}
T_{21} &= \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}} \left(a_{KL} (F_K^{n+1} - F_L^{n+1}) (u_K^{n+1} - u_L^{n+1}) + a_{K^*L^*} (F_{K^*}^{n+1} - F_{L^*}^{n+1}) (u_{K^*}^{n+1} - u_{L^*}^{n+1}) \right), \\
T_{22} &= \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}} \left(v_{KL}^{n+1} \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{K^*}^{n+1} - \xi_L^{n+1}) (u_K^{n+1} - u_L^{n+1}) + v_{K^*L^*}^{n+1} \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}) (u_{K^*}^{n+1} - u_{L^*}^{n+1}) \right).
\end{aligned}$$

The practical inequality (3.10) implies that

$$T_{21} \geq \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}} a_{KL} (\xi_K^{n+1} - \xi_L^{n+1})^2 + \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}} a_{K^*L^*} (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1})^2.$$

Now, the monotonicity of the functions $v_{\uparrow}, v_{\downarrow}$ gives

$$\left(v_{\uparrow}(u_K^{n+1}) + v_{\downarrow}(u_L^{n+1}) \right) \left(u_K^{n+1} - u_L^{n+1} \right) \geq \left(\xi_K^{n+1} - \xi_L^{n+1} \right).$$

Thanks to the above inequality and the definition of v_{KL}^{n+1} , we find

$$v_{KL}^{n+1} \left(u_K^{n+1} - u_L^{n+1} \right) \eta_{\sigma\sigma^*}^{\mathcal{D}} \left(\xi_{K^*}^{n+1} - \xi_L^{n+1} \right) \geq \eta_{\sigma\sigma^*}^{\mathcal{D}} \left(\xi_K^{n+1} - \xi_L^{n+1} \right) \left(\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1} \right).$$

In the same manners, one may write

$$v_{K^*L^*}^{n+1} \left(u_{K^*}^{n+1} - u_{L^*}^{n+1} \right) \eta_{\sigma\sigma^*}^{\mathcal{D}} \left(\xi_K^{n+1} - \xi_L^{n+1} \right) \geq \eta_{\sigma\sigma^*}^{\mathcal{D}} \left(\xi_K^{n+1} - \xi_L^{n+1} \right) \left(\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1} \right).$$

As a result we get

$$T_{22} \geq \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}} 2 \times \eta_{\sigma\sigma^*}^{\mathcal{D}} \left(\xi_K^{n+1} - \xi_L^{n+1} \right) \left(\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1} \right).$$

We deduce that

$$T_2 \geq \sum_{n=0}^{N-1} \delta t \left(\nabla^{\mathfrak{D}} \xi_{\mathcal{T}}^{n+1}, \nabla^{\mathfrak{D}} \xi_{\mathcal{T}}^{n+1} \right)_{\mathfrak{D}, \Lambda}. \quad (4.7)$$

In view of the relationship (2.2) and Lemma A.1 we assert

$$T_2 \geq C \sum_{n=0}^{N-1} \delta t \left\| \nabla^{\mathfrak{D}} \xi_{\mathcal{T}}^{n+1} \right\|_2^2,$$

for some constant $C > 0$. Next, owing to (3.11), we write

$$T_3 \geq \frac{1}{2} \frac{1}{h_{\mathfrak{D}}^{\varepsilon}} \sum_{n=0}^{N-1} \delta t \left\| \xi(u_{\mathfrak{M}}^{n+1}) - \xi(u_{\mathfrak{M}^*}^{n+1}) \right\|_{L^2(\Omega)}^2. \quad (4.8)$$

Combining between (4.6)-(4.8), one proves the energy estimate (4.5) follows as required. \square

Corollary 4.1. *From the previous Proposition, one gets*

$$\sum_{n=0}^{N-1} \delta t \left\| \nabla^{\mathfrak{D}} F_{\mathcal{T}}^{n+1} \right\|_2^2 \leq C, \quad (4.9)$$

Proof. This result is a direct consequence of Lemma A.1 together with inequality (4.5). It is sufficient to observe that

$$F(a) - F(b) = v(x_0) \left(\xi(a) - \xi(b) \right),$$

for some $x_0 \in]\min(a, b), \max(a, b)[$ and notice that the function v is bounded. \square

5 Existence of discrete solutions

In this section, we prove that the nonlinear algebraic system, which comes from the DDFV scheme, admits a solution. To this end, we will need the following fundamental lemma, that can be found in [25]. This result ensures the existence of at least one zero of some specific vector fields.

Lemma 5.1. *Let \mathcal{A} be a finite dimensional Hilbert space with inner product (\cdot, \cdot) and norm $\|\cdot\|$, and let \mathcal{L} be a continuous mapping from \mathcal{A} into itself such that*

$$(\mathcal{L}(x), x) > 0 \quad \text{for } \|x\| = r > 0.$$

Then there exists $x^ \in \mathcal{A}$ with $\|x^*\| < r$ such that*

$$\mathcal{L}(x^*) = 0.$$

We now state the existence result in the proposition below.

Proposition 5.1. *The DDFV scheme (3.12)-(3.14) has at least one solution $(u_{\mathcal{T}}^{n+1})_{n=0, \dots, N}$.*

Proof. For sake of shortness we drop the superscript n from time to time. We proceed by induction on n . We then assume that $u_{\mathcal{T}}^n$ is given and prove the existence of $u_{\mathcal{T}}^{n+1}$ satisfying the numerical scheme (3.13)-(3.14). We define the mapping $\mathcal{L} : \mathbb{R}^{\#\mathcal{T}} \rightarrow \mathbb{R}^{\#\mathcal{T}}$ such that

$$\mathcal{L}(u_{\mathcal{T}}) = \left(\{\mathcal{L}_M\}_{M \in \mathcal{T}} \right),$$

where

$$\begin{aligned}
\mathcal{L}_K &= \frac{m_K}{\delta t} (u_K^{n+1} - u_K^n) \\
&\quad + \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_K} \left(a_{KL} (F_K^{n+1} - F_L^{n+1}) + v_{KL}^{n+1} \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}) \right) + \gamma \mathcal{P}_K u_\tau^{n+1}, \quad \text{if } M = K, \\
\mathcal{L}_{K^*} &= \frac{m_{K^*}}{\delta t} (u_{K^*}^{n+1} - u_{K^*}^n) \\
&\quad + \sum_{\mathcal{D}_{\sigma, \sigma^*} \in \mathcal{D}_{K^*}} \left(a_{K^*L^*} (F_{K^*}^{n+1} - F_{L^*}^{n+1}) + v_{KL}^{n+1} \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_K^{n+1} - \xi_L^{n+1}) \right) + \gamma \mathcal{P}_{K^*} u_\tau^{n+1}, \quad \text{if } M = K^*.
\end{aligned}$$

\mathcal{L} is well-defined and continuous. It remains to demonstrate that

$$\left(\mathcal{L}(u_\tau^{n+1}), u_\tau^{n+1} \right) > 0, \quad \text{for } \|u_\tau^{n+1}\|_{\mathbb{R}^{\#\tau}} = r, \quad (5.1)$$

for some sufficiently large r . It follows from the calculation of the previous section that

$$\begin{aligned}
\left(\mathcal{L}(u_\tau^{n+1}), u_\tau^{n+1} \right) &\geq \frac{1}{\delta t} \sum_{K \in \mathfrak{M}} m_K \left((u_K^{n+1})^2 - (u_K^n)^2 \right) + \frac{1}{\delta t} \sum_{K^* \in \overline{\mathfrak{M}^*}} m_{K^*} \left((u_{K^*}^{n+1})^2 - (u_{K^*}^n)^2 \right) \\
&\quad + \|\nabla^{\mathcal{D}} \xi_\tau^{n+1}\|_2^2 - C \\
&\geq \frac{1}{\delta t} |u_\tau^{n+1}|_{2, \tau}^2 - C_n,
\end{aligned}$$

for some constants $C, C_n > 0$. Thanks to the equivalence of the usual norms $\|\cdot\|_{\mathbb{R}^{\#\tau}}, |\cdot|_{2, \tau}$ on the finite dimensional space $\mathbb{R}^{\#\tau}$, inequality (5.1) is fulfilled provided a large r . We therefore obtain the existence of at least one solution of the DDFV scheme (3.13)-(3.14). Hence, the proof is ended. \square

6 Compactness results

We here highlight some compactness properties. Let us begin with space translates.

Lemma 6.1. (*Space Translates*)

Let $u_{\tau, \delta t}$ be a discrete solution of the DDFV scheme (3.13)-(3.14). Then

$$\int_0^T \int_{\Omega'} |\xi_{\tau, \delta t}(x+y, t) - \xi_{\tau, \delta t}(x, t)| dx dt \leq \omega(|y|), \quad \text{for every } y \in \mathbb{R}^2, \quad (6.1)$$

where $\Omega' = \{x \in \Omega / x+y \in \Omega\}$ and ω is a modulus of continuity independent of $\delta t, h_{\mathfrak{D}}$, verifying $\omega(|y|) \rightarrow 0$ as $|y| \rightarrow 0$.

Proof. The proof of this claim is made in case of the primal mesh and it is similar for the dual mesh, i.e. it is sufficient to prove that

$$\int_0^T \int_{\Omega'} |\xi(u_{\mathfrak{M}, \delta t}(x+y, t)) - \xi(u_{\mathfrak{M}, \delta t}(x, t))| dx dt \leq \omega(|y|), \quad \text{for every } y \in \mathbb{R}^2. \quad (6.2)$$

Now, for every $x, y \in \mathbb{R}^2$ and $\sigma = K|L$, we define the characteristic function χ_σ as

$$\chi_\sigma(x, y) = \begin{cases} 1, & \text{if } [x, x+y] \cap \sigma \neq \emptyset, \\ 0, & \text{else.} \end{cases}$$

We know that $\int_{\Omega'} \chi_\sigma(x, y) dx \leq m_\sigma |y|$ (see [27] for more details). As a consequence, since we have

$$\left| \xi(u_{\mathfrak{M}, \delta t}(x+y, t)) - \xi(u_{\mathfrak{M}, \delta t}(x, t)) \right| \leq \sum_{\sigma=K|L} \chi_\sigma(x, y) |\xi_L^{n+1} - \xi_K^{n+1}|,$$

this gives

$$\begin{aligned} \int_{\Omega'} |\xi(u_{\mathfrak{M},\delta t}(x+y,t)) - \xi(u_{\mathfrak{M},\delta t}(x,t))| dx &\leq |y| \sum_{\sigma=K|L} m_\sigma |\xi_L^{n+1} - \xi_K^{n+1}| \\ &\leq C |y| \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}} m_{\mathcal{D}} \left| \frac{\xi_L^{n+1} - \xi_K^{n+1}}{m_{\sigma^*}} \right| \end{aligned}$$

for some appropriate C depending on the regularity of the mesh. On the other hand, we have

$$\left| \frac{\xi_L^{n+1} - \xi_K^{n+1}}{m_{\sigma^*}} \right| \leq |\nabla^{\mathcal{D}} \xi_\tau^{n+1}|.$$

As a result of the Cauchy-Schwarz inequality and the energy estimate (4.5), one infers

$$\int_0^T \int_{\Omega'} |\xi(u_{\mathfrak{M},\delta t}(x+y,t)) - \xi(u_{\mathfrak{M},\delta t}(x,t))| dx dt \leq C |y|.$$

This completes the proof. \square

Lemma 6.2. (*Time translates*)

Let $u_{\tau,\delta t}$ be a solution of the DDFV scheme (3.12)-(3.14). Then there exists a constant C that does not depend neither on $h_{\mathfrak{D}}$ nor on δt such that

$$\begin{aligned} &\int_0^{T-\tau} \int_{\Omega} \left| \xi(u_{\mathfrak{M},\delta t}(x,t+\tau)) - \xi(u_{\mathfrak{M},\delta t}(x,t)) \right|^2 dx dt \\ &+ \int_0^{T-\tau} \int_{\Omega} \left| \xi(u_{\mathfrak{M}^*,\delta t}(x,t+\tau)) - \xi(u_{\mathfrak{M}^*,\delta t}(x,t)) \right|^2 dx dt \leq C (\tau + \delta t), \end{aligned} \quad (6.3)$$

for all $\tau \in (0, T)$.

Proof. The proof follows the main steps of [27, Lemma 4.6]. We will provide the proof of the first integral in (6.3) and that of the second one is proved in similar way. To begin with, let $\tau \in (0, T)$ and $t \in (0, T - \tau)$. We set

$$A = \int_0^{T-\tau} \int_{\Omega} \left| \xi(u_{\mathfrak{M},\delta t}(x,t+\tau)) - \xi(u_{\mathfrak{M},\delta t}(x,t)) \right|^2 dx.$$

We next define $n_0(t) \in \{0, \dots, N-1\}$ such that $t^{n_0(t)} < t \leq t^{n_0(t)} + 1$ and $n_1(t) \in \{0, \dots, N-1\}$ such that $t^{n_1(t)} < t + \tau \leq t^{n_1(t)} + 1$. One then can rewrite A as follows

$$\begin{aligned} A &= \int_0^{T-\tau} \sum_{K \in \mathfrak{M}} m_K \left| \xi(u_K^{n_1(t)}) - \xi(u_K^{n_0(t)}) \right|^2 \\ &\leq C \int_0^{T-\tau} \sum_{K \in \mathfrak{M}} \left(\left(\xi(u_K^{n_1(t)}) - \xi(u_K^{n_0(t)}) \right) \times \sum_{t \leq n\delta t < t+\tau} m_K (u_K^{n+1} - u_K^n) \right) dt, \end{aligned}$$

for some constant $C > 0$ depending only on ξ . In light of the definition of the DDFV scheme, one gets

$$\begin{aligned} A &\leq L \int_0^{T-\tau} \sum_{K \in \mathfrak{M}} \left(\xi(u_K^{n_1(t)}) - \xi(u_K^{n_0(t)}) \right) \\ &\times \sum_{t \leq n\delta t < t+\tau} \delta t \left(- \sum_{\mathcal{D}_{\sigma,\sigma^*} \in \mathfrak{D}_K} \left(a_{KL} (F_K^{n+1} - F_L^{n+1}) + v_{KL}^{n+1} \eta_{\sigma\sigma^*}^{\mathcal{D}} (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}) \right) + \gamma \mathcal{P}_K u_\tau^{n+1} \right) dt. \end{aligned}$$

Applying the integration by parts and the first mean value theorem ensures the existence of a positive constant C that depends only on the regularity of the mesh, ξ , $\|\xi'\|_\infty$ and on $\bar{\Lambda}$ with

$$\begin{aligned} A \leq & C \int_0^{T-\tau} \sum_{t \leq n\delta t < t+\tau} \delta t \sum_{\mathcal{D} \in \mathfrak{D}} \left(|\xi_K^{n+1} - \xi_L^{n+1}| \left| \xi(u_K^{n_1(t)}) - \xi(u_L^{n_1(t)}) \right| \right. \\ & + |\xi_K^{n+1} - \xi_L^{n+1}| \left| \xi(u_L^{n_0(t)}) - \xi(u_K^{n_0(t)}) \right| \\ & + |\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}| \left| \xi(u_K^{n_1(t)}) - \xi(u_L^{n_1(t)}) \right| \\ & \left. + |\xi_{L^*}^{n+1} - \xi_{K^*}^{n+1}| \left| \xi(u_L^{n_0(t)}) - \xi(u_K^{n_0(t)}) \right| \right) dt \\ & - \gamma \int_0^{T-\tau} \sum_{t \leq n\delta t < t+\tau} \delta t \sum_{K \in \mathfrak{M}} \left(\xi(u_K^{n_1(t)}) - \xi(u_K^{n_0(t)}) \right) \mathcal{P}_K u_\tau^{n+1} dt. \end{aligned}$$

Let us now introduce the characteristic function β which is defined by (see [27])

$$\beta(n, t) = \begin{cases} 1 & \text{if } t \leq (n+1)\delta t < t + \tau, \\ 0 & \text{otherwise} \end{cases},$$

Using the elementary inequality $ab \leq \frac{1}{2}a^2 + \frac{1}{2}b^2$ in the previous estimate leads to

$$A \leq \frac{C}{2} (E_1 + E_2 + E_3 + E_4) + E_5$$

where we have obtained

$$\begin{aligned} E_1 &= \sum_{n=0}^{N-1} \delta t \int_0^{T-\tau} \beta(n, t) \sum_{\mathcal{D} \in \mathfrak{D}} \left(\xi_K^{n+1} - \xi_L^{n+1} \right)^2 dt, \\ E_2 &= \sum_{n=0}^{N-1} \delta t \int_0^{T-\tau} \beta(n, t) \sum_{\mathcal{D} \in \mathfrak{D}} \left(\xi(u_K^{n_1(t)}) - \xi(u_L^{n_1(t)}) \right)^2 dt, \\ E_3 &= \sum_{n=0}^{N-1} \delta t \int_0^{T-\tau} \beta(n, t) \sum_{\mathcal{D} \in \mathfrak{D}} \left(\xi(u_K^{n_0(t)}) - \xi(u_L^{n_0(t)}) \right)^2 dt, \\ E_4 &= \sum_{n=0}^{N-1} \delta t \int_0^{T-\tau} \beta(n, t) \sum_{\mathcal{D} \in \mathfrak{D}} \left(\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1} \right)^2 dt, \\ E_5 &= -\gamma \sum_{n=0}^{N-1} \delta t \int_0^{T-\tau} \beta(n, t) \sum_{K \in \mathfrak{M}} \left(\xi(u_K^{n_1(t)}) - \xi(u_K^{n_0(t)}) \right) \mathcal{P}_K u_\tau^{n+1} dt. \end{aligned}$$

On the other hand, observe that

$$\sum_{\mathcal{D} \in \mathfrak{D}} \left(\xi_K^{n+1} - \xi_L^{n+1} \right)^2 \leq C_1 \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} \left| \frac{\xi_K^{n+1} - \xi_L^{n+1}}{m_{\sigma^*}} \right|^2 \leq C_1 \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} |\nabla^{\mathcal{D}} \xi_\tau^{n+1}|^2,$$

where $C_1 > 0$ depends only on the regularity of the mesh. By virtue of the energy estimate (4.5) and since $\int_0^{T-\tau} \beta(n, t) \leq \tau$, there exists an appropriate constant C such that $E_1 \leq C\tau$ and $E_4 \leq C\tau$. Next, following [27, Lemma 4.6], one claims that

$$E_2 \leq C_1 \sum_{m=0}^{N-1} \int_{t^m}^{t^{m+1}} \delta t \sum_{n=0}^{N-1} \beta(n, t) \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} |\nabla^{\mathcal{D}} \xi_\tau^m|^2 dt \leq C\tau.$$

In an analogous way, one has $E_3 \leq C\tau$. Finally, to treat E_5 , we use the maximum principle and the estimate (4.5) to get that $E_5 \leq C\tau$. Hence, the proof of the lemma is concluded. \square

7 Convergence of the approximate solution

We now claim a weak convergence of the discrete gradient and a strong convergence of $u_{\mathcal{T}_h, \delta t}$ in the following Proposition.

Proposition 7.1. *Let \mathcal{T}_h be a sequence of DDFV meshes such that $h_{\mathcal{D}}, \delta t$ tend to zero and $\text{reg}(\mathcal{T}_h)$ is bounded. Then*

$$u_{\mathcal{T}_h, \delta t}, u_{\mathfrak{M}_h, \delta t}, u_{\overline{\mathfrak{M}}^*_h, \delta t} \longrightarrow u \quad \text{a.e. in } Q_T, \quad (7.1)$$

and up to a subsequence, we have

$$\nabla^{\mathcal{D}} F_{\mathcal{T}_h, \delta t} \longrightarrow \nabla F(u) \quad \text{weakly in } L^2(Q_T)^2. \quad (7.2)$$

Moreover

$$0 \leq u \leq 1 \quad \text{a.e. in } Q_T. \quad (7.3)$$

Proof. Thanks to the Kolmogorov's compactness theorem [10], the sequences $\xi(u_{\mathfrak{M}_h, \delta t}), \xi(u_{\overline{\mathfrak{M}}^*_h, \delta t})$ are relatively compact in $L^1(Q_T)$. This ensures the existence of unlabeled subsequences of $\xi(u_{\mathfrak{M}_h, \delta t}), \xi(u_{\overline{\mathfrak{M}}^*_h, \delta t})$ such that

$$\xi(u_{\mathfrak{M}_h, \delta t}) \longrightarrow \xi_1 \quad \text{a.e. in } Q_T, \quad \text{and} \quad \xi(u_{\overline{\mathfrak{M}}^*_h, \delta t}) \longrightarrow \xi_2 \quad \text{a.e. in } Q_T.$$

Since ξ^{-1} is continuous, we deduce that

$$u_{\mathfrak{M}_h, \delta t} \longrightarrow u_1 := \xi^{-1}(\xi_1) \quad \text{a.e. in } Q_T, \quad \text{and} \quad u_{\overline{\mathfrak{M}}^*_h, \delta t} \longrightarrow u_2 := \xi^{-1}(\xi_2) \quad \text{a.e. in } Q_T.$$

In light of Proposition 4.1, we assert

$$\left\| \xi(u_{\mathfrak{M}_h, \delta t}) - \xi(u_{\overline{\mathfrak{M}}^*_h, \delta t}) \right\|_{L^2(Q_T)}^2 \leq 2C h_{\mathcal{D}}^\varepsilon. \quad (7.4)$$

Thus, up to unlabeled subsequence, we get

$$\xi(u_{\mathfrak{M}_h, \delta t}) - \xi(u_{\overline{\mathfrak{M}}^*_h, \delta t}) \longrightarrow 0, \quad \text{a.e. in } Q_T.$$

Therefore

$$u_{\mathfrak{M}_h, \delta t} - u_{\overline{\mathfrak{M}}^*_h, \delta t} \longrightarrow 0, \quad \text{a.e. in } Q_T.$$

We then verify that $u_1 = u_2 := u$. Consequently

$$u_{\mathcal{T}_h, \delta t} \longrightarrow u \quad \text{a.e. in } Q_T, \quad \text{and} \quad \xi_{\mathcal{T}_h, \delta t} \longrightarrow \xi(u) \quad \text{a.e. in } Q_T.$$

Thanks to the maximum principle given in Lemma 4.1, we deduce from Lebesgue's dominated convergence theorem that

$$\lim_{h_{\mathcal{D}}, \delta t \rightarrow 0} \left\| u_{\mathcal{T}_h, \delta t} - u \right\|_{L^2(Q_T)} = 0.$$

As a consequence

$$\lim_{h_{\mathcal{D}}, \delta t \rightarrow 0} \left\| F_{\mathcal{T}_h, \delta t} - F(u) \right\|_{L^2(Q_T)} = 0.$$

Since the problem (1.1) possesses a unique solution, a classical argument confirms that the whole sequence converges to u . Next, thanks to Corollary 4.1, the sequence $(\nabla^{\mathcal{D}} F_{\mathcal{T}_h, \delta t})$ is bounded in $(L^2(Q_T))^2$. As a result, there exists a function $G \in (L^2(Q_T))^2$ such that

$$\nabla^{\mathcal{D}} F_{\mathcal{T}_h, \delta t} \longrightarrow G \quad \text{weakly in } (L^2(Q_T))^2. \quad (7.5)$$

Let us now show that $\nabla F(u) = G$ is the sense of distribution. To do this, let $\varphi \in (\mathcal{C}^\infty(\bar{\Omega} \times [0, T]))^2$. Due to the weak convergence of $(\nabla^{\mathcal{D}} F_{\mathcal{T}_h, \delta t})$ and the strong one of $(F_{\mathcal{T}_h, \delta t})$, one can pass to limit in

$$\begin{aligned} I_{\mathcal{T}_h, \delta t} &:= \int_{Q_T} \nabla^{\mathcal{D}} F_{\mathcal{T}_h, \delta t} \cdot \varphi \, dx \, dt + \int_{Q_T} F_{\mathcal{T}_h, \delta t} \operatorname{div} \varphi \, dx \, dt \\ &\longrightarrow \int_{Q_T} G \cdot \varphi \, dx \, dt + \int_{Q_T} F(u) \operatorname{div} \varphi \, dx \, dt. \end{aligned}$$

The definition of the discrete gradient allows us to write

$$\int_{Q_T} \nabla^{\mathcal{D}} F_{\mathcal{T}_h, \delta t} \cdot \varphi \, dx \, dt = \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} \nabla^{\mathcal{D}} F_{\mathcal{T}_h, \delta t} \cdot \varphi_{\mathcal{D}}^{n+1}, \quad (7.6)$$

where $\varphi_{\mathcal{D}}^{n+1} = \frac{1}{\delta t m_{\mathcal{D}}} \int_{t^n}^{t^{n+1}} \int_{\mathcal{D}} \varphi \, dx \, dt$. For every diamond $\mathcal{D} = \mathcal{D}_{\sigma, \sigma^*}$, we introduce $\varphi_{\sigma}^{n+1}, \varphi_{\sigma^*}^{n+1}$ and $\tilde{\varphi}_{\mathcal{D}}^{n+1}$ as follows

$$\begin{aligned} \varphi_{\sigma}^{n+1} &= \frac{1}{\delta t m_{\sigma}} \int_{t^n}^{t^{n+1}} \int_{\sigma} \varphi(s, t) \, ds \, dt, \quad \varphi_{\sigma^*}^{n+1} = \frac{1}{\delta t m_{\sigma^*}} \int_{t^n}^{t^{n+1}} \int_{\sigma^*} \varphi(s, t) \, ds \, dt, \\ \tilde{\varphi}_{\mathcal{D}}^{n+1} \cdot \mathbf{n}_{\sigma_K} &= \varphi_{\sigma}^{n+1} \cdot \mathbf{n}_{\sigma_K}, \quad \tilde{\varphi}_{\mathcal{D}}^{n+1} \cdot \mathbf{n}_{\sigma^* K^*} = \varphi_{\sigma^*}^{n+1} \cdot \mathbf{n}_{\sigma^* K^*}. \end{aligned}$$

Note that $\tilde{\varphi}_{\mathcal{D}}^{n+1}$ is uniquely defined. Thanks to the smoothness of φ , we derive the estimate

$$\begin{aligned} |\varphi_{\mathcal{D}}^{n+1} - \tilde{\varphi}_{\mathcal{D}}^{n+1}| &\leq \frac{1}{\sin(\alpha_{\mathcal{T}_h})} \left(|\varphi_{\mathcal{D}}^{n+1} - \varphi_{\sigma}^{n+1}| + |\varphi_{\mathcal{D}}^{n+1} - \varphi_{\sigma^*}^{n+1}| \right) \\ &\leq 2 \operatorname{reg}(\mathcal{T}_h) h_{\mathcal{D}} \|\nabla \varphi\|_{L^\infty}. \end{aligned} \quad (7.7)$$

Now, the expression (7.6) becomes

$$\begin{aligned} \int_{Q_T} \nabla^{\mathcal{D}} F_{\mathcal{T}_h, \delta t} \cdot \varphi \, dx \, dt &= \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} \nabla^{\mathcal{D}} F_{\mathcal{T}_h}^{n+1} \cdot \tilde{\varphi}_{\mathcal{D}}^{n+1} \\ &\quad + \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}} m_{\mathcal{D}} \nabla^{\mathcal{D}} F_{\mathcal{T}_h}^{n+1} \cdot (\varphi_{\mathcal{D}}^{n+1} - \tilde{\varphi}_{\mathcal{D}}^{n+1}) \\ &=: A_{\mathcal{T}_h, \delta t} + B_{\mathcal{T}_h, \delta t}. \end{aligned}$$

In addition, inequality (7.7) and the energy estimate (4.5) lead to

$$\lim_{h_{\mathfrak{D}}, \delta t \rightarrow 0} B_{\mathcal{T}_h, \delta t} = 0.$$

We next return to the definition of the discrete gradient. It implies

$$\begin{aligned} A_{\mathcal{T}_h, \delta t} &= \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}} m_{\sigma} m_{\sigma^*} \left(\frac{F_L^{n+1} - F_K^{n+1}}{m_{\sigma^*}} \mathbf{n}_{\sigma_K} + \frac{F_{L^*}^{n+1} - F_{K^*}^{n+1}}{m_{\sigma}} \mathbf{n}_{\sigma^* K^*}, \tilde{\varphi}_{\mathcal{D}}^{n+1} \right) \\ &= -\frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K \in \mathfrak{M}} F_K^{n+1} \sum_{\sigma \in \mathcal{E}_K} m_{\sigma} \left(\tilde{\varphi}_{\mathcal{D}}^{n+1}, \mathbf{n}_{\sigma_K} \right) \\ &\quad - \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{K^* \in \overline{\mathfrak{M}^*}} F_{K^*}^{n+1} \sum_{\sigma^* \in \mathcal{E}_{K^*}} m_{\sigma^*} \left(\tilde{\varphi}_{\mathcal{D}}^{n+1}, \mathbf{n}_{\sigma^* K^*} \right), \end{aligned}$$

where we used discrete integration by parts (4.2). By virtue of the expression of $\tilde{\varphi}_D^{n+1}$, one gets

$$\begin{aligned} A_{\mathcal{T}_h, \delta t} &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathfrak{M}} F_K^{n+1} \sum_{\sigma \in \mathcal{E}_K} \int_{t^n}^{t^{n+1}} \int_{\sigma} \varphi(s) \cdot \mathbf{n}_{\sigma_K} \, ds \, dt \\ &\quad - \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K^* \in \overline{\mathfrak{M}^*}} F_{K^*}^{n+1} \sum_{\sigma^* \in \mathcal{E}_{K^*}} \int_{t^n}^{t^{n+1}} \int_{\sigma^*} \varphi(s) \cdot \mathbf{n}_{\sigma^*_{K^*}} \, ds \, dt. \end{aligned}$$

Stokes formula entails

$$\begin{aligned} A_{\mathcal{T}_h, \delta t} &= -\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathfrak{M}} F_K^{n+1} \int_{t^n}^{t^{n+1}} \int_K \operatorname{div} \varphi \, dx \, dt \\ &\quad - \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K^* \in \overline{\mathfrak{M}^*}} F_{K^*}^{n+1} \int_{t^n}^{t^{n+1}} \int_{K^*} \operatorname{div} \varphi \, dx \, dt \\ &= - \int_{Q_T} F_{\mathcal{T}_h, \delta t} \operatorname{div} \varphi \, dx \, dt. \end{aligned}$$

As a consequence

$$\lim_{h_{\mathfrak{D}}, \delta t \rightarrow 0} I_{\mathcal{T}_h, \delta t} = 0.$$

Thereby we proved that $F(u) \in L^2(0, T; H_0^1(\Omega))$ and $\nabla F(u) = G$. This finishes up the proof. \square

8 Passage to limit

This section is dedicated to prove that any limit of the approximate solution converges toward the weak solution of the main problem.

Theorem 8.1. *Assuming a uniform boundedness of the mesh regularity, the limit function u of Proposition 7.1 is the weak solution of the problem (1.1) in the sense of Definition 1.1.*

Proof. Let $\psi \in C_c^\infty(\overline{\Omega} \times [0, T])$, we denote by $\psi_K^{n+1} = \psi(x_K, t^{n+1})$ and $\psi_{K^*}^{n+1} = \psi(x_{K^*}, t^{n+1})$. Multiply the equations (3.13), (3.14) by $\frac{1}{2} \delta t \psi_K^{n+1}$, $\frac{1}{2} \delta t \psi_{K^*}^{n+1}$ respectively, summing over K , K^* and n . Next, one performs an integration by parts, adds and subtracts $\sum_{n=0}^{N-1} \delta t \left(\nabla^{\mathfrak{D}} F_{\mathcal{T}_h}^{n+1}, \nabla^{\mathfrak{D}} \psi_{\mathcal{T}_h}^{n+1} \right)_{\mathfrak{D}, \Lambda}$ to get

$$\mathcal{S}_{\mathcal{T}_h, \delta t}^1 + \mathcal{S}_{\mathcal{T}_h, \delta t}^2 + \mathcal{S}_{\mathcal{T}_h, \delta t}^3 + \mathcal{S}_{\mathcal{T}_h, \delta t}^4 = 0, \quad (8.1)$$

where

$$\begin{aligned} \mathcal{S}_{\mathcal{T}_h, \delta t}^1 &= \sum_{n=0}^{N-1} \llbracket u_{\mathcal{T}_h}^{n+1} - u_{\mathcal{T}_h}^n, \psi_{\mathcal{T}_h}^{n+1} \rrbracket_{\mathcal{T}_h}, \\ \mathcal{S}_{\mathcal{T}_h, \delta t}^2 &= \sum_{n=0}^{N-1} \delta t \left(\nabla^{\mathfrak{D}} F_{\mathcal{T}_h}^{n+1}, \nabla^{\mathfrak{D}} \psi_{\mathcal{T}_h}^{n+1} \right)_{\mathfrak{D}, \Lambda} = \int_{Q_T} \Lambda \nabla^{\mathfrak{D}} F_{\mathcal{T}_h}^{n+1} \cdot \nabla^{\mathfrak{D}} \psi_{\mathcal{T}_h}^{n+1} \, dx \, dt, \\ \mathcal{S}_{\mathcal{T}_h, \delta t}^3 &= \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}_h} \eta_{\sigma\sigma^*}^{\mathcal{D}} \left[v_{KL}^{n+1} \left(\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1} \right) - \left(F_{K^*}^{n+1} - F_{L^*}^{n+1} \right) \right] \left(\psi_K^{n+1} - \psi_{L^*}^{n+1} \right) \\ &\quad + \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}_h} \eta_{\sigma\sigma^*}^{\mathcal{D}} \left[v_{K^*L^*}^{n+1} \left(\xi_K^{n+1} - \xi_L^{n+1} \right) - \left(F_K^{n+1} - F_L^{n+1} \right) \right] \left(\psi_{K^*}^{n+1} - \psi_{L^*}^{n+1} \right), \\ \mathcal{S}_{\mathcal{T}_h, \delta t}^4 &= \gamma \sum_{n=0}^{N-1} \delta t \llbracket \mathcal{P} F_{\mathcal{T}_h}^{n+1}, \psi_{\mathcal{T}_h}^{n+1} \rrbracket_{\mathcal{T}_h}. \end{aligned}$$

Let us start off by establishing

$$\lim_{h_{\mathfrak{D}}, \delta t \rightarrow 0} \mathcal{S}_{\tau_h, \delta t}^1 = - \int_{\Omega} u^0 \psi(\cdot, 0) dx - \int_{Q_T} u \partial_t \psi dx dt.$$

Using a summation by parts and the fact that $\psi_K^N = \psi_{K^*}^N = 0$, yields

$$\begin{aligned} \mathcal{S}_{\tau_h, \delta t}^1 &= - \llbracket u_{\tau_h}^0, \psi_{\tau_h}(\cdot, 0) \rrbracket_{\tau_h} - \sum_{n=0}^{N-1} \llbracket u_{\tau_h}^{n+1}, \psi_{\tau_h}^{n+1} - \psi_{\tau_h}^n \rrbracket_{\tau_h} \\ &=: \mathcal{S}_{\tau_h, \delta t}^{1,1} + \mathcal{S}_{\tau_h, \delta t}^{1,2}. \end{aligned}$$

Thanks to the strong convergence of $(\psi_{\tau_h}(\cdot, 0))$, one obtains Consequently

$$\lim_{h_{\mathfrak{D}}, \delta t \rightarrow 0} \mathcal{S}_{\tau_h, \delta t}^{1,1} = - \int_{\Omega} u^0 \psi(\cdot, 0) dx.$$

Expanding the term $\mathcal{S}_{\tau_h, \delta t}^{1,2}$ entails

$$\begin{aligned} \mathcal{S}_{\tau_h, \delta t}^{1,2} &= - \sum_{n=0}^{N-1} \llbracket u_{\tau_h}^{n+1}, \psi_{\tau_h}^{n+1} - \psi_{\tau_h}^n \rrbracket_{\tau_h} \\ &= - \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathfrak{M}} m_K \int_{t^n}^{t^{n+1}} u_K^{n+1} \partial_t \psi(x_K, t) dx dt - \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K^* \in \overline{\mathfrak{M}^*}} m_{K^*} \int_{t^n}^{t^{n+1}} u_{K^*}^{n+1} \partial_t \psi(x_{K^*}, t) dx dt. \end{aligned}$$

Bearing in mind that $(\partial_t \psi(x_K, \cdot))_{K \in \mathfrak{M}}$ and $(\partial_t \psi(x_{K^*}, \cdot))_{K^* \in \overline{\mathfrak{M}^*}}$ converge uniformly toward $\partial_t \psi$, we apply the Lebesgue dominated convergence theorem to find

$$\lim_{h_{\mathfrak{D}}, \delta t \rightarrow 0} \mathcal{S}_{\tau_h, \delta t}^{1,2} = \lim_{h_{\mathfrak{D}}, \delta t \rightarrow 0} \mathcal{S}_{\tau_h, \delta t}^{1,2} = - \int_{Q_T} u \partial_t \psi dx dt.$$

Let us next prove the convergence of the diffusion part. To do so, we recall that the sequence $(\nabla^{\mathfrak{D}} F_{\tau_h}^{n+1})$ converges weakly toward $\nabla F(u)$ whereas $(\Lambda \nabla^{\mathfrak{D}} \psi_{\tau_h}^{n+1})$ converges uniformly toward $\Lambda \nabla \psi$. Thereby

$$\lim_{h_{\mathfrak{D}}, \delta t \rightarrow 0} \mathcal{S}_{\tau_h, \delta t}^2 = \int_{Q_T} \Lambda \nabla F(u) \cdot \nabla \psi dx dt.$$

Let us turn our attention to the convergence of $\mathcal{S}_{\tau_h, \delta t}^3$. This term can be split up into two parts as follows

$$\mathcal{S}_{\tau_h, \delta t}^3 = \mathcal{S}_{\tau_h, \delta t}^{3,1} + \mathcal{S}_{\tau_h, \delta t}^{3,2},$$

where we have set

$$\begin{aligned} \mathcal{S}_{\tau_h, \delta t}^{3,1} &= \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}_h} \eta_{\sigma\sigma^*}^{\mathcal{D}} \left[v_{KL}^{n+1} (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}) - (F_{K^*}^{n+1} - F_{L^*}^{n+1}) \right] (\psi_K^{n+1} - \psi_L^{n+1}), \\ \mathcal{S}_{\tau_h, \delta t}^{3,2} &= \frac{1}{2} \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}_h} \eta_{\sigma\sigma^*}^{\mathcal{D}} \left[v_{K^*L^*}^{n+1} (\xi_K^{n+1} - \xi_L^{n+1}) - (F_K^{n+1} - F_L^{n+1}) \right] (\psi_{K^*}^{n+1} - \psi_{L^*}^{n+1}). \end{aligned}$$

Next, the first mean value theorem guarantees the existence of a constant $u_{K^*L^*} \in I_{K^*L^*}^{n+1}$ such that

$$F_{K^*}^{n+1} - F_{L^*}^{n+1} = v(u_{K^*L^*}) (\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}).$$

Thus, using assumption (A_4) on the tensor Λ and the regularity of the mesh we get

$$\left| \mathcal{S}_{\tau_h, \delta t}^{3,1} \right| \leq C \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathfrak{D}_h} m_{\mathcal{D}} |v_{KL}^{n+1} - v(u_{K^*L^*})| |\nabla^{\mathcal{D}} \xi_{\tau, \delta t}| \left| \nabla^{\mathcal{D}} \psi_{\tau_h, \delta t} \right|.$$

for some constant $C > 0$. We set

$$\begin{aligned}\bar{\xi}_{\mathcal{D}}^{n+1} &:= \max_{M \in \mathcal{V}_{\mathcal{D}}} \{\xi(u_M^{n+1})\}, & \underline{\xi}_{\mathcal{D}}^{n+1} &:= \min_{M \in \mathcal{V}_{\mathcal{D}}} \{\xi(u_M^{n+1})\} \\ \bar{\xi}_{\mathcal{T}, \delta t |_{\mathcal{D} \times (t^n, t^{n+1})}} &:= \bar{\xi}_{\mathcal{D}}^{n+1}, & \underline{\xi}_{\mathcal{T}, \delta t |_{\mathcal{D} \times (t^n, t^{n+1})}} &:= \underline{\xi}_{\mathcal{D}}^{n+1},\end{aligned}$$

where $\mathcal{V}_{\mathcal{D}}$ stands for the set of vertices of the diamond \mathcal{D} . Applying once more the fact that ξ^{-1} is a θ -Hölder function, the Cauchy-Schwarz inequality and Hölder inequality since $\theta \in (0, 1]$, we discover

$$\begin{aligned}\left| \mathcal{S}_{\mathcal{T}_h, \delta t}^{3,1} \right| &\leq C \|v\|_{\infty} \|\nabla \psi\|_{\infty} \sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathcal{D}_h} m_{\mathcal{D}} \left| \bar{\xi}_{\mathcal{D}}^{n+1} - \underline{\xi}_{\mathcal{D}}^{n+1} \right|^{\theta} |\nabla^{\mathcal{D}} \xi_{\mathcal{T}, \delta t}| \\ &\leq C \left(\sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathcal{D}_h} m_{\mathcal{D}} \left| \bar{\xi}_{\mathcal{D}}^{n+1} - \underline{\xi}_{\mathcal{D}}^{n+1} \right|^{2\theta} \right)^{1/2} \times \left(\sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathcal{D}_h} m_{\mathcal{D}} |\nabla^{\mathcal{D}} \xi_{\mathcal{T}, \delta t}|^2 \right)^{1/2} \\ &\leq C \left(\sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} \left| \bar{\xi}_{\mathcal{D}}^{n+1} - \underline{\xi}_{\mathcal{D}}^{n+1} \right|^2 \right)^{\theta/2} \times \left(\sum_{n=0}^{N-1} \delta t \sum_{\mathcal{D} \in \mathcal{D}_h} m_{\mathcal{D}} |\nabla^{\mathcal{D}} \xi_{\mathcal{T}, \delta t}|^2 \right)^{1/2},\end{aligned}$$

for some positive constant C . In view of Lemma A.2 together with (4.5), we deduce that

$$\lim_{h_{\mathcal{D}}, \delta t \rightarrow 0} \mathcal{S}_{\mathcal{T}_h, \delta t}^{3,1} = 0.$$

Similarly, we establish that

$$\lim_{h_{\mathcal{D}}, \delta t \rightarrow 0} \mathcal{S}_{\mathcal{T}_h, \delta t}^{3,2} = 0.$$

Finally, let us demonstrate that

$$\lim_{h_{\mathcal{D}}, \delta t \rightarrow 0} \mathcal{S}_{\mathcal{T}_h, \delta t}^4 = 0.$$

Owing to the definition of the penalization term we explore

$$\begin{aligned}\left| \llbracket \mathcal{P} F_{\mathcal{T}_h}^{n+1}, \psi_{\mathcal{T}_h}^{n+1} \rrbracket_{\mathcal{T}_h} \right| &= \left| \frac{1}{2} \frac{1}{h_{\mathcal{D}}^{\varepsilon}} \sum_{K^* \in \overline{\mathfrak{M}^*}} \sum_{K \in \mathfrak{M}} m_{K \cap K^*} \left(F(u_K^{n+1}) - F(u_{K^*}^{n+1}) \right) \left(\psi_K^{n+1} - \psi_{K^*}^{n+1} \right) \right| \\ &\leq \frac{1}{2} \frac{\|v\|_{\infty}}{h_{\mathcal{D}}^{\varepsilon}} \sum_{K^* \in \overline{\mathfrak{M}^*}} \sum_{K \in \mathfrak{M}} m_{K \cap K^*} \left| \xi_K^{n+1} - \xi_{K^*}^{n+1} \right| \left| \psi_K^{n+1} - \psi_{K^*}^{n+1} \right| \\ &\leq \frac{1}{2} \frac{\|v\|_{\infty}}{h_{\mathcal{D}}^{\varepsilon}} \left\| \xi_{\mathfrak{M}_h, \delta t} - \xi_{\overline{\mathfrak{M}^*}_h, \delta t} \right\|_{L^2(Q_T)} \left\| \psi_{\mathfrak{M}_h, \delta t} - \psi_{\overline{\mathfrak{M}^*}_h, \delta t} \right\|_{L^2(Q_T)}.\end{aligned}$$

On the other hand, the regularity of the function ψ ensures the existence of a constant C depending on the regularity of the mesh such that (see [17] for deep details)

$$\left\| \psi_{\mathfrak{M}_h, \delta t} - \psi_{\overline{\mathfrak{M}^*}_h, \delta t} \right\|_{L^2(Q_T)} \leq Ch_{\mathcal{D}} \|\psi\|_{W^{1, \infty}(\Omega)}.$$

Utilizing the energy estimate (4.5) and the fact that $\varepsilon < 2$ to obtain

$$\left| \mathcal{S}_{\mathcal{T}_h, \delta t}^4 \right| \leq Ch_{\mathcal{D}}^{1-\varepsilon/2} \longrightarrow 0, \quad h_{\mathcal{D}}, \delta t \longrightarrow 0.$$

This ends the proof of the theorem. \square

9 Numerical results

In this section, we present some numerical tests so that we can show the efficiency and the stability of the DDFV scheme that we have proposed. As highlighted in the introduction of this work, this method

will allow us to take into account almost general meshes and any tensor. However, we restrict our study to diagonal tensors with weak anisotropy. We also stress that boundary conditions of Dirichlet type are prescribed. It is sufficient to take the trace of a given exact solution on the boundary. This particularity provides analytical solutions of the continuous problem and enable us to compare them with the discrete ones.

To begin with, let us consider the unit square $\Omega = [0, 1]^2$ as the porous medium. Next, the primal meshes are given by a sequence of a perturbed quadrangulation of the domain Ω , known as mesh_quad.i of the FVCA8 benchmark on incompressible flows, while the dual mesh is constructed as described in Section 2.

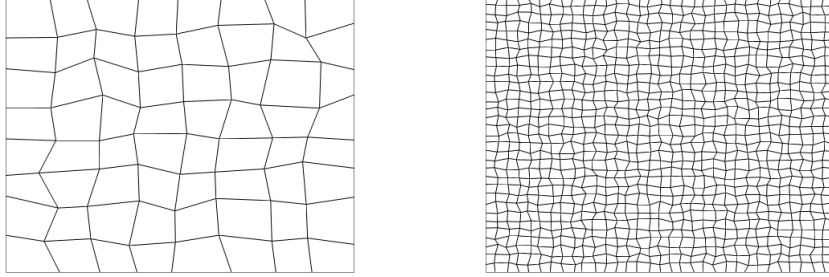


Figure 2: Quadrangle meshes

Furthermore the mobility function is chosen as follows

$$f(u) = u^m(1-u)^m, \quad \forall u \in [0, 1] \quad \text{and} \quad m \in \{1, 2\}.$$

Notice that this function presents some degeneracy in $u = 0$ and in $u = 1$. Additionally, in order to calculate the numerical flux, we require the computation of the functions $v_{\uparrow}(u)$ and $v_{\downarrow}(u)$. In our study, the function v admits a unique global maximum $\bar{u} = 1/2$. Hence, one gets in a straightforward way that

$$v_{\uparrow}(u) = v\left(\min\left\{u, \frac{1}{2}\right\}\right), \quad \text{and} \quad v_{\downarrow}(u) = v\left(\max\left\{u, \frac{1}{2}\right\}\right) - v\left(\frac{1}{2}\right), \quad \text{for all } u \in (0, 1)^2.$$

We are also interested in case of anisotropic media to verify the validity of our discretization. So we select a diagonal tensor Λ such that

$$\Lambda = \begin{pmatrix} \Lambda_{xx} & 0 \\ 0 & \Lambda_{yy} \end{pmatrix}.$$

So, the DDFV scheme is formulated in a nonlinear algebraic system, which is solved thanks to the Newton's method with a given tolerance $\varepsilon = 1.e^{-10}$. We underline that the numerical scheme (3.13)-(3.14) is fully implicit in time and then it is unconditionally stable and convergent. Yet, we require the time step to be proportional to the square of the mesh size as mentioned in [13].

As we are focused in the accuracy of scheme as well, we need to evaluate the error of the proposed discretization. In all the tests, we denote by $ErrLI2u$ the difference between the analytical solution and the numerical one in $L^\infty(0, T; L^2(\Omega))$. Moreover we study the error between the corresponding gradients in $L^2(\Omega \times (0, T))^2$, which is denoted by $ErrL2gu$. The corresponding convergence rate will be designated by $RLI2u$ and $RL2gu$ respectively. More precisely

$$\begin{aligned} ErrLI2u &= \|u_{\text{ex}} - u_{\mathcal{T}, \delta t}\|_{L^\infty(0, T; L^2(\Omega))}, & ErrL2u &= \|\nabla u_{\text{ex}} - \nabla u_{\mathcal{T}, \delta t}\|_{L^2(\Omega \times (0, T))^2}, \\ RLI2u &= \frac{\log\left(ErrLI2u^{i+1}/ErrLI2u^i\right)}{\log\left(h_{\mathcal{D}}^{i+1}/h_{\mathcal{D}}^i\right)}, & RL2u &= \frac{\log\left(ErrL2gu^{i+1}/ErrL2gu^i\right)}{\log\left(h_{\mathcal{D}}^{i+1}/h_{\mathcal{D}}^i\right)}, \end{aligned}$$

where the index i is the space discretization \mathcal{T}_i for $i = 1, \dots, 5$. In all the tables below u_{\min} (resp. u_{\max}) stands for the minimum (resp. maximum) of the computed solution.

Test 1

In this test, we investigate the numerical convergence of the DDFV scheme (3.12)-(3.14) using the following exact solution

$$u_{\text{ex}}(x, t) = 80x_1^2(1 - x_1)^2 \times t, \quad \forall x = (x_1, x_2) \in \Omega, \quad t \in (0, T).$$

Substituting this expression in the main problem (1.1) yields a source term, which is nonnegative. One notices that this solution degenerates at the line $\{x_1 = 0\}$ and at $\{x_1 = 1\}$. The mobility function $f(u) = u^2(1 - u)^2$ is considered. Here the final time is fixed to $T = 0.15$.

h	# Unknowns	$\gamma = 0$		$\gamma = 0.5$	
		$\ u_{\mathfrak{M}, \delta t} - u_{\overline{\mathfrak{M}}^*, \delta t}\ $	Rate	$\ u_{\mathfrak{M}, \delta t} - u_{\overline{\mathfrak{M}}^*, \delta t}\ $	Rate
0.3420	41	0.111 E-01	-	0.110 E-01	-
0.1740	145	0.575 E-02	0.974	0.574 E-02	0.973
0.0920	545	0.296 E-02	1.034	0.295 E-02	1.034
0.0470	2113	0.146 E-02	1.059	0.146 E-02	1.059
0.0195	8321	0.705 E-03	0.823	0.705 E-03	0.823

Table 1: The norm $\|u_{\mathfrak{M}, \delta t} - u_{\overline{\mathfrak{M}}^*, \delta t}\|_{L^2(Q_T)}$ with and without penalization term and $n = m = 2$.

We have seen that the penalization term has played a crucial role to establish that the two reconstructions of the solution on the primal and dual meshes converge to the same limit. We now study this numerically. More precisely, we compute the difference $\|u_{\mathfrak{M}, \delta t} - u_{\overline{\mathfrak{M}}^*, \delta t}\|_{L^2(\Omega \times (0, T))}$ in $L^2(\Omega \times (0, T))$ norm between the approximate solution on the primal mesh and that on the dual mesh. For this, we consider two values of the stabilization parameter $\gamma = 0$ and $\gamma = 0.5$ with a fixed $\varepsilon = 1$. As shown in Table 1, the presence or the absence of the penalization term does not influence the convergence of the sequence $\|u_{\mathfrak{M}, \delta t} - u_{\overline{\mathfrak{M}}^*, \delta t}\|_{L^2(\Omega \times (0, T))}$. From this table, one can as well check the rate convergence is almost one.

In the sequel, we set the parameter γ to zero. In Table 2 we list the obtained results with an isotropic tensor $\Lambda_{xx} = \Lambda_{yy} = 1$. We can observe that the convergence rate of the solution is almost of second order. We thus reach the well known order of DDFV schemes for linear problems [13, 19, 20, 34]. Despite of being of order between 1 and 1.5 for linear problem, the convergence rate of the discrete gradient may be deteriorated with respect to the nonlinearity, the anisotropy and/or the discretization. For instance we refer to [3] where the authors have found an accuracy of order 0.4 for anisotropic Laplace equation. Here for our nonlinear problem, we observe that the convergence rate of the gradient is between 0.6 and 1.2. We also verify that the computed solution preserves the maximum principle property. Table 3 gives the errors in the anisotropic case. Thus, the tensor entries are $\Lambda_{xx} = 1$ and $\Lambda_{yy} = 0.01$. It demonstrates that the numerical solution is always nonnegative with convergence rates which are slightly similar with respect to the isotropic case.

h	ERL2	Rate	ERGL2	Rate	u _{min}	u _{max}
0.3420	0.127 E-01	-	0.161 E-01	-	0	0.703
0.1740	0.629 E-02	1.048	0.130 E-01	0.324	0	0.747
0.0920	0.216 E-02	1.669	0.937 E-02	0.509	0	0.748
0.0470	0.665 E-03	1.766	0.653 E-02	0.543	0	0.750
0.0195	0.126 E-03	1.880	0.210 E-02	1.280	0	0.750

Table 2: Numerical convergence with isotropic tensor and $n = m = 2$.

h	ERL2	Rate	ERGL2	Rate	u_{\min}	u_{\max}
0.342	0.130 E-01	-	0.163 E-01	-	0	0.705
0.174	0.649 E-02	1.030	0.135 E-01	0.291	0	0.748
0.092	0.244 E-02	1.529	0.111 E-01	0.307	0	0.749
0.047	0.898 E-03	1.499	0.924 E-02	0.270	0	0.751
0.0195	0.180 E-03	1.812	0.353 E-02	1.085	0	0.750

Table 3: Numerical convergence with anisotropic tensor and $n = m = 2$.

Test 2

We now test the accuracy and the stability of our scheme thanks to the analytical solution

$$u_{\text{ex}}(x, t) = 6x_1^2 \times t, \quad \forall x = (x_1, x_2) \in \Omega, \quad t \in (0, T).$$

Here the mobility function is chosen to be $f(u) = u(1 - u)$. Note that this function is not a perfect square with $f(0) = f(1) = 0$. This solution fulfills the continuous problem (1.1) with a corresponding source term, which is also nonnegative. It vanishes at the line $\{x_1 = 0\}$. The final time is taken as $T = 0.15$. Tables 4 and 5 present the convergence of the scheme including isotropic tensor, meaning that $\Lambda_{xx} = \Lambda_{yy} = 1$, and anisotropic one, $\Lambda_{xx} = 1$ and $\Lambda_{yy} = 0.001$, respectively. The first table shows that the numerical scheme is accurate of almost second order whereas the second one exhibits an accuracy of order 1.5 which might be explained by the impact of anisotropy. In both cases we have not recorded any undershoots nor overshoots.

h	ERL2	Rate	ERGL2	Rate	u_{\min}	u_{\max}
0.342	0.104 E-01	-	0.220 E-01	-	0	0.840
0.174	0.425 E-02	1.335	0.162 E-01	0.460	0	0.895
0.092	0.132 E-02	1.821	0.102 E-01	0.730	0	0.897
0.047	0.365 E-03	1.933	0.445 E-02	1.236	0	0.900
0.0195	0.114 E-03	1.312	0.292 E-02	0.475	0	0.900

Table 4: Numerical convergence with isotropic tensor and $n = m = 1$.

h	ERL2	Rate	ERGL2	Rate	u_{\min}	u_{\max}
0.342	0.116 E-01	-	0.224 E-01	-	0	0.840
0.174	0.506 E-02	1.245	0.189 E-01	0.258	0	0.895
0.092	0.199 E-02	1.459	0.162 E-01	0.235	0	0.897
0.047	0.754 E-03	1.453	0.106 E-01	0.637	0	0.900
0.0195	0.207 E-03	1.459	0.604 E-02	0.638	0	0.900

Table 5: Numerical convergence with anisotropic tensor and $n = m = 1$.

Test 3

This test concerns the porous medium equation. We test our scheme with the following exact solution [15] of the main problem (1.1)

$$u_{\text{ex}}(x, t) = \frac{\lambda_1(x_1 - 0.5)^2 + \lambda_2(x_2 - 0.5)^2}{1 - t}, \quad \forall x = (x_1, x_2) \in \Omega, t \in (0, T),$$

with $\lambda_1 = \frac{1}{16\Lambda_{xx}}$ and $\lambda_2 = \frac{1}{16\Lambda_{yy}}$. The mobility function now is $f(u) = 2u$. Note that this choice does not match with the assumption (A_2) . We then record the numerical convergence results in Table 6 and Table 7 with a final time setting to $T = 0.2$. From Table 6, one can check that the method is accurate of second order and it produces no undershoots. Moreover, in the presence of anisotropy with $\Lambda_{xx} = 10$ and $\Lambda_{yy} = 0.1$, one can observe, as shown in Table 7, similar remarks with a relative difference with respect to the last mesh.

h	ERL2	Rate	ERGL2	Rate	u_{\min}
0.342	0.426 E-03	-	0.974 E-03	-	0.206 E-03
0.174	0.260 E-03	0.733	0.111 E-02	- 0.196	0.243 E-04
0.092	0.789 E-04	1.860	0.684 E-03	0.759	0.304 E-05
0.047	0.213 E-04	1.965	0.354 E-03	0.990	0.612 E-06
0.0195	0.450 E-05	1.755	0.104 E-03	1.371	0.234 E-06

Table 6: Numerical convergence of the scheme with $\Lambda_{xx} = \Lambda_{yy} = 1$.

h	ERL2	Rate	ERGL2	Rate	u_{\min}
0.342	0.340 E-01	-	0.991 E-01	-	0.734 E-03
0.174	0.123 E-01	1.516	0.611 E-01	0.719	0.131 E-03
0.092	0.336 E-02	2.022	0.375 E-01	0.764	0.178 E-04
0.047	0.847 E-03	2.068	0.176 E-01	1.136	0.449 E-05
0.0195	0.222 E-03	1.509	0.991 E-02	0.646	0.234 E-06

Table 7: Numerical convergence of the scheme with $\Lambda_{xx} = 0.1$ and $\Lambda_{yy} = 10$.

Conclusion

We have constructed and investigated a DDFV scheme preserving the maximum principle for nonlinear parabolic equations. The basic idea of this scheme consists in considering the elliptic equation as a hyperbolic one. Thanks to nonstandard approximations of the fluxes, we have derived some energy estimates. The existence of the solution is proved by an inheriting result of the Brouwer's fixed theorem. With the help of some compactness properties, we have established the convergence of a sequence of approximate solutions toward the weak one of the continuous problem. Numerical tests showed that our method is almost of second order accurate and that the computed solutions are always nonnegative. Consequently, our theoretical predictions are confirmed as required. As far as we know this work is the first one to prove the discrete maximum principle together with the convergence of the numerical scheme.

A Appendix: Technical lemmas

Let \mathcal{D} be a fixed diamond cell. We define the following 2×2 matrices

$$\mathbb{A}^{\mathcal{D}} = \frac{1}{4m_{\mathcal{D}}} \begin{bmatrix} m_{\sigma}^2 & m_{\sigma}m_{\sigma^*} \\ m_{\sigma}m_{\sigma^*} & m_{\sigma^*}^2 \end{bmatrix} =: \begin{bmatrix} \mathbb{A}_{\sigma}^{\mathcal{D}} & \mathbb{A}_{\sigma,\sigma^*}^{\mathcal{D}} \\ \mathbb{A}_{\sigma,\sigma^*}^{\mathcal{D}} & \mathbb{A}_{\sigma^*}^{\mathcal{D}} \end{bmatrix} \quad (\text{A.1})$$

$$\begin{aligned} \mathbb{A}^{\mathcal{D},\Lambda} &= \frac{1}{4m_{\mathcal{D}}} \begin{bmatrix} m_{\sigma}^2 \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma K} & m_{\sigma}m_{\sigma^*} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} \\ m_{\sigma}m_{\sigma^*} \Lambda \mathbf{n}_{\sigma K} \cdot \mathbf{n}_{\sigma^* K^*} & m_{\sigma^*}^2 \Lambda \mathbf{n}_{\sigma^* K^*} \cdot \mathbf{n}_{\sigma^* K^*} \end{bmatrix} \\ &=: \begin{bmatrix} \mathbb{A}_{\sigma}^{\mathcal{D},\Lambda} & \mathbb{A}_{\sigma,\sigma^*}^{\mathcal{D},\Lambda} \\ \mathbb{A}_{\sigma,\sigma^*}^{\mathcal{D},\Lambda} & \mathbb{A}_{\sigma^*}^{\mathcal{D},\Lambda} \end{bmatrix} \end{aligned} \quad (\text{A.2})$$

and

$$\mathbb{B}^{\mathcal{D},\Lambda} = \begin{bmatrix} |\mathbb{A}_{\sigma}^{\mathcal{D},\Lambda}| + |\mathbb{A}_{\sigma,\sigma^*}^{\mathcal{D},\Lambda}| & 0 \\ 0 & |\mathbb{A}_{\sigma,\sigma^*}^{\mathcal{D},\Lambda}| + |\mathbb{A}_{\sigma^*}^{\mathcal{D},\Lambda}| \end{bmatrix}, \quad \forall \mathcal{D} \in \mathfrak{D}. \quad (\text{A.3})$$

The following lemma claims a crucial property of the matrix $\mathbb{A}^{\mathcal{D},\Lambda}$. In particular, it states that $\mathbb{A}^{\mathcal{D},\Lambda}$ is positive definite.

Lemma A.1. *There exist some positive constants λ_0 and λ_1 depending only on the mesh regularity and on $\underline{\Lambda}$, $\bar{\Lambda}$ so that the following inequalities hold*

$$\mathbb{A}^{\mathcal{D},\Lambda} x \cdot x \leq \mathbb{B}^{\mathcal{D},\Lambda} x \cdot x \leq \lambda_1 \mathbb{A}^{\mathcal{D},\Lambda} x \cdot x, \quad \forall x \in \mathbb{R}^2, \quad (\text{A.4})$$

$$\lambda_0 \mathbb{A}^{\mathcal{D}} x \cdot x \leq \mathbb{A}^{\mathcal{D},\Lambda} x \cdot x, \quad \forall x \in \mathbb{R}^2. \quad (\text{A.5})$$

Proof. For the sake of completeness we reproduce the same proof as given in [13]. Let $x = (x_1, x_2)$ be a fixed vector of \mathbb{R}^2 . Thus, for every $\mathcal{D} \in \mathfrak{D}$, we have

$$\mathbb{A}^{\mathcal{D},\Lambda} x \cdot x \leq \mathbb{B}^{\mathcal{D},\Lambda} x \cdot x \leq \|\mathbb{A}^{\mathcal{D},\Lambda}\|_1 |x|^2.$$

where $\|\cdot\|_p$ is the usual p -norm matrix, $p = 1, 2$. Thanks to the equivalence of norms property on $\mathbb{R}^2 \times \mathbb{R}^2$, ensures the existence of a coefficient $L \geq 1$ such that

$$\|\mathbb{A}^{\mathcal{D},\Lambda}\|_1 |x|^2 \leq L \|\mathbb{A}^{\mathcal{D},\Lambda}\|_2 |x|^2 \leq L \text{Cond}_2(\mathbb{A}^{\mathcal{D},\Lambda}) \mathbb{A}^{\mathcal{D},\Lambda} x \cdot x, \quad (\text{A.6})$$

where Cond_2 stands for the condition number with respect to the 2-norm. In addition, this number can be overestimated as follows

$$\text{Cond}_2(\mathbb{A}^{\mathcal{D},\Lambda}) \leq \text{Cond}_2(\Lambda^{\mathcal{D}}) \left(\mathcal{Q}_{\mathcal{D}}^2 + \sqrt{\mathcal{Q}_{\mathcal{D}}^2 - \frac{1}{\text{Cond}_2(\Lambda^{\mathcal{D}})}} \right)^2 \leq 4 \text{reg}(\mathcal{T})^2 \times \frac{\bar{\Lambda}}{\underline{\Lambda}}.$$

where we have set

$$\mathcal{Q}_{\mathcal{D}} = \frac{1}{2 \sin(\alpha_{\mathcal{D}})} \left(\frac{m_{\sigma}}{m_{\sigma^*}} + \frac{m_{\sigma^*}}{m_{\sigma}} \right) \geq 1.$$

This proves the inequality

$$\mathbb{A}^{\mathcal{D},\Lambda} x \cdot x \leq \mathbb{B}^{\mathcal{D},\Lambda} x \cdot x \leq \lambda_1 \mathbb{A}^{\mathcal{D},\Lambda} x \cdot x,$$

with $\lambda_1 = 4L \text{reg}(\mathcal{T})^2 \times \left(\frac{\bar{\Lambda}}{\underline{\Lambda}} \right)$. Using the elementary inequality

$$|\mathbb{A}_{\sigma,\sigma^*}^{\mathcal{D}}| x_1 x_2 \leq \frac{1}{2} \left(\mathbb{A}_{\sigma}^{\mathcal{D}} x_1^2 + \mathbb{A}_{\sigma^*}^{\mathcal{D}} x_2^2 \right),$$

one deduces that

$$\mathbb{A}^{\mathcal{D}} x \cdot x \leq 2 \left(\mathbb{A}_{\sigma}^{\mathcal{D}} x_1^2 + \mathbb{A}_{\sigma^*}^{\mathcal{D}} x_2^2 \right).$$

Now the ellipticity of the tensor Λ implies

$$\mathbb{A}^{\mathcal{D}} x \cdot x \leq \frac{2}{\underline{\Lambda}} \left(\mathbb{A}_{\sigma}^{\mathcal{D},\Lambda} x_1^2 + \mathbb{A}_{\sigma^*}^{\mathcal{D},\Lambda} x_2^2 \right) \leq \frac{2}{\underline{\Lambda}} \mathbb{B}^{\mathcal{D},\Lambda} x \cdot x.$$

Thanks to inequality (A.4), one gets

$$\mathbb{A}^{\mathcal{D}} x \cdot x \leq \frac{2\lambda_1}{\underline{\Lambda}} \mathbb{A}^{\mathcal{D},\Lambda} x \cdot x.$$

Hence, the second inequality follows by setting $\lambda_0 = \frac{\underline{\Lambda}}{2\lambda_1}$. \square

Lemma A.2. Consider the following piecewise constant functions

$$\begin{aligned}\bar{\xi}_{\mathcal{D}}^{n+1} &:= \max_{M \in \mathcal{V}_{\mathcal{D}}} \{\xi(u_M^{n+1})\}, & \underline{\xi}_{\mathcal{D}}^{n+1} &:= \min_{M \in \mathcal{V}_{\mathcal{D}}} \{\xi(u_M^{n+1})\}, \\ \bar{\xi}_{\mathcal{T}, \delta t |_{\mathcal{D} \times (t^n, t^{n+1}]}} &:= \bar{\xi}_{\mathcal{D}}^{n+1}, & \underline{\xi}_{\mathcal{T}, \delta t |_{\mathcal{D} \times (t^n, t^{n+1}]}} &:= \underline{\xi}_{\mathcal{D}}^{n+1},\end{aligned}$$

where we denote $\mathcal{V}_{\mathcal{D}} = \{K, L, K^*, L^*\}$. Then

$$\lim_{h_{\mathcal{D}}, \delta t \rightarrow 0} \left\| \bar{\xi}_{\mathcal{T}, \delta t} - \underline{\xi}_{\mathcal{T}, \delta t} \right\|_{L^2(Q_T)} = 0. \quad (\text{A.7})$$

Proof. We firstly observe that

$$\left| \bar{\xi}_{\mathcal{D}}^{n+1} - \underline{\xi}_{\mathcal{D}}^{n+1} \right|^2 \leq |\xi(u_K) - \xi(u_L)|^2 + |\xi(u_{K^*}) - \xi(u_{L^*})|^2 + |\xi(u_K) - \xi(u_{K^*})|^2 + |\xi(u_L) - \xi(u_{L^*})|^2.$$

This implies

$$\begin{aligned}& \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} \left| \bar{\xi}_{\mathcal{D}}^{n+1} - \underline{\xi}_{\mathcal{D}}^{n+1} \right|^2 \\ & \leq \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} \left(|\xi_K^{n+1} - \xi_L^{n+1}|^2 + |\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}|^2 + |\xi_K^{n+1} - \xi_{K^*}^{n+1}|^2 + |\xi_L^{n+1} - \xi_{L^*}^{n+1}|^2 \right) \\ & \leq h_{\mathcal{D}}^2 \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} \left(\left| \frac{\xi_K^{n+1} - \xi_L^{n+1}}{m_{\sigma^*}} \right|^2 + \left| \frac{\xi_{K^*}^{n+1} - \xi_{L^*}^{n+1}}{m_{\sigma}} \right|^2 \right) \\ & \quad + \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} |\xi_K^{n+1} - \xi_{K^*}^{n+1}|^2 + \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} |\xi_L^{n+1} - \xi_{L^*}^{n+1}|^2 \\ & \leq h_{\mathcal{D}}^2 \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} \left(|\nabla^{\mathcal{D}} \xi_{\mathcal{T}}^{n+1} \cdot \tau_{K,L}|^2 + |\nabla^{\mathcal{D}} \xi_{\mathcal{T}}^{n+1} \cdot \tau_{K^*,L^*}|^2 \right) \\ & \quad + \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} |\xi_K^{n+1} - \xi_{K^*}^{n+1}|^2 + \sum_{\mathcal{D} \in \mathcal{D}} m_{\mathcal{D}} |\xi_L^{n+1} - \xi_{L^*}^{n+1}|^2.\end{aligned}$$

Due to the estimates (4.5) and (7.4) one concludes that

$$\lim_{h_{\mathcal{D}}, \delta t \rightarrow 0} \left\| \bar{\xi}_{\mathcal{T}, \delta t} - \underline{\xi}_{\mathcal{T}, \delta t} \right\|_{L^2(Q_T)} = 0.$$

□

References

- [1] Afif, M. and Amaziane, B. (2002). Convergence of finite volume schemes for a degenerate convection–diffusion equation arising in flow in porous media. *Computer methods in applied mechanics and engineering*, 191(46):5265–5286.
- [2] Andreianov, B., Bendahmane, M., and Saad, M. (2011). Finite volume methods for degenerate chemotaxis model. *Journal of computational and applied mathematics*, 235(14):4015–4031.
- [3] Andreianov, B., Boyer, F., and Hubert, F. (2007). Discrete duality finite volume schemes for Leray–Lions–type elliptic problems on general 2D meshes. *Numerical Methods for Partial Differential Equations*, 23(1):145–195.
- [4] Andreianov, B., Cancès, C., and Moussa, A. (2017). A nonlinear time compactness result and applications to discretization of degenerate parabolic–elliptic PDEs. *Journal of Functional Analysis*, 273(12):3633–3670.

- [5] Angelini, O., Brenner, K., and Hilhorst, D. (2013). A finite volume method on general meshes for a degenerate parabolic convection–reaction–diffusion equation. *Numerische Mathematik*, 123(2):219–257.
- [6] Barrett, J. W. and Knabner, P. (1997). Finite element approximation of the transport of reactive solutes in porous media. Part II: Error estimates for equilibrium adsorption processes. *SIAM journal on numerical analysis*, 34(2):455–479.
- [7] Bendahmane, M., Khalil, Z., and Saad, M. (2014). Convergence of a finite volume scheme for gas–water flow in a multi-dimensional porous medium. *Mathematical Models and Methods in Applied Sciences*, 24(01):145–185.
- [8] Bessemoulin-Chatard, M., Chainais-Hillairet, C., and Filbet, F. (2014). On discrete functional inequalities for some finite volume schemes. *IMA Journal of Numerical Analysis*, 35(3):1125–1149.
- [9] Bessemoulin-Chatard, M. and Filbet, F. (2012). A finite volume scheme for nonlinear degenerate parabolic equations. *SIAM Journal on Scientific Computing*, 34(5):B559–B583.
- [10] Brezis, H. (2010). *Functional analysis, Sobolev spaces and partial differential equations*. Springer Science & Business Media.
- [11] Camier, J.-S. and Hermeline, F. (2016). A monotone nonlinear finite volume method for approximating diffusion operators on general meshes. *International Journal for Numerical Methods in Engineering*, 107(6):496–519.
- [12] Cancès, C., Cathala, M., and Le Potier, C. (2013). Monotone corrections for generic cell-centered finite volume approximations of anisotropic diffusion equations. *Numerische Mathematik*, 125(3):387–417.
- [13] Cancès, C., Chainais-Hillairet, C., and Krell, S. (2017a). Numerical analysis of a nonlinear free-energy diminishing Discrete Duality Finite Volume scheme for convection diffusion equations. *arXiv preprint arXiv:1705.10558*.
- [14] Cancès, C. and Guichard, C. (2016). Convergence of a nonlinear entropy diminishing control volume finite element scheme for solving anisotropic degenerate parabolic equations. *Mathematics of Computation*, 85(298):549–580.
- [15] Cancès, C. and Guichard, C. (2017). Numerical analysis of a robust free energy diminishing finite volume scheme for parabolic equations with gradient structure. *Foundations of Computational Mathematics*, 17(6):1525–1584.
- [16] Cancès, C., Ibrahim, M., and Saad, M. (2017b). Positive nonlinear CVFE scheme for degenerate anisotropic Keller-Segel system. *SMAI Journal of Computational Mathematics*, 3:1–28.
- [17] Chainais-Hillairet, C., Krell, S., and Mouton, A. (2015). Convergence analysis of a DDFV scheme for a system describing miscible fluid flows in porous media. *Numerical Methods for Partial Differential Equations*, 31(3):723–760.
- [18] Chavent, G. and Jaffré, J. (1986). *Mathematical models and finite elements for reservoir simulation: single phase, multiphase and multicomponent flows through porous media*. Elsevier.
- [19] Coudière, Y., Vila, J.-P., and Villedieu, P. (1999). Convergence rate of a finite volume scheme for a two dimensional convection-diffusion problem. *ESAIM: Mathematical Modelling and Numerical Analysis*, 33(3):493–516.
- [20] Delcourte, S., Domelevo, K., and Omnès, P. (2005). Discrete duality finite volume method for second order elliptic problems. *Hermes Science publishing*, pages 447–458.
- [21] Domelevo, K. and Omnès, P. (2005). A finite volume method for the Laplace equation on almost arbitrary two-dimensional grids. *ESAIM: Mathematical Modelling and Numerical Analysis*, 39(6):1203–1249.

- [22] Droniou, J. (2014). Finite volume schemes for diffusion equations: introduction to and review of modern methods. *Mathematical Models and Methods in Applied Sciences*, 24(08):1575–1619.
- [23] Droniou, J., Eymard, R., Gallouët, T., Guichard, C., and Herbin, R. (2016). The gradient discretisation method. *ESAIM: Mathematical Modelling and Numerical Analysis*, 50(3):749–781.
- [24] Ebmeyer, C. (1998). Error estimates for a class of degenerate parabolic equations. *SIAM journal on numerical analysis*, 35(3):1095–1112.
- [25] Evans, L. C. (2010). Partial differential equations. *2nd edn. Amer. Math. Soc.*
- [26] Eymard, R., Féron, P., Gallouët, T., Herbin, R., and Guichard, C. (2013). Gradient schemes for the Stefan problem. *International Journal On Finite Volumes*, 10.
- [27] Eymard, R., Gallouët, T., and Herbin, R. (2000). Finite volume methods. *Handbook of numerical analysis*, 7:713–1018.
- [28] Eymard, R., Gallouët, T., Herbin, R., and Michel, A. (2002). Convergence of a finite volume scheme for nonlinear degenerate parabolic equations. *Numerische Mathematik*, 92(1):41–82.
- [29] Eymard, R., Gallouët, T., Hilhorst, D., and Slimane, Y. N. (1998). Finite volumes and nonlinear diffusion equations. *ESAIM: Mathematical Modelling and Numerical Analysis*, 32(6):747–761.
- [30] Eymard, R., Hilhorst, D., and Vohralík, M. (2006). A combined finite volume–nonconforming/mixed-hybrid finite element scheme for degenerate parabolic problems. *Numerische Mathematik*, 105(1):73–131.
- [31] Filbet, F. (2006). A finite volume scheme for the Patlak–Keller–Segel chemotaxis model. *Numerische Mathematik*, 104(4):457–488.
- [32] Gao, Z. and Wu, J. (2015). A second-order positivity-preserving finite volume scheme for diffusion equations on general meshes. *SIAM Journal on Scientific Computing*, 37(1):A420–A438.
- [33] Hermeline, F. (1998). Une méthode de volumes finis pour les équations elliptiques du second ordre. *Comptes Rendus de l’Académie des Sciences-Series I-Mathematics*, 326(12):1433–1436.
- [34] Hermeline, F. (2000). A finite volume method for the approximation of diffusion operators on distorted meshes. *Journal of computational Physics*, 160(2):481–499.
- [35] Jäger, W. and Kačur, J. (1991). Solution of porous medium type systems by linear approximation schemes. *Numerische Mathematik*, 60(1):407–427.
- [36] Karlsen, K., Risebro, N., and Towers, J. (2002). Upwind difference approximations for degenerate parabolic convection–diffusion equations with a discontinuous coefficient. *IMA journal of numerical analysis*, 22(4):623–664.
- [37] Krell, S. and Manzini, G. (2012). The Discrete Duality Finite Volume method for the Stokes equations on 3-D polyhedral meshes. *SIAM Journal on Numerical Analysis*, 50(2):808–837.
- [38] Lazarov, R., Mishev, I. D., and Vassilevski, P. S. (1996). Finite volume methods for convection-diffusion problems. *SIAM Journal on Numerical Analysis*, 33(1):31–55.
- [39] Lipnikov, K., Shashkov, M., Svyatskiy, D., and Vassilevski, Y. (2007). Monotone finite volume schemes for diffusion equations on unstructured triangular and shape-regular polygonal meshes. *Journal of Computational Physics*, 227(1):492–512.
- [40] Nochetto, R., Schmidt, A., and Verdi, C. (2000). A posteriori error estimation and adaptivity for degenerate parabolic problems. *Mathematics of Computation of the American Mathematical Society*, 69(229):1–24.

- [41] Osher, S. and Solomon, F. (1982). Upwind difference schemes for hyperbolic systems of conservation laws. *Mathematics of computation*, 38(158):339–374.
- [42] Otto, F. (1996). L^1 -contraction and uniqueness for quasilinear elliptic–parabolic equations. *Journal of differential equations*, 131(1):20–38.
- [43] Vázquez, J. L. (2007). *The porous medium equation: mathematical theory*. Oxford University Press.
- [44] Wilhelm Alt, H. and Luckhaus, S. (1983). Quasilinear elliptic-parabolic differential equations. *Mathematische Zeitschrift*, 183(3):311–341.
- [45] Yuan, G. and Sheng, Z. (2008). Monotone finite volume schemes for diffusion equations on polygonal meshes. *Journal of computational physics*, 227(12):6288–6312.