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Analysis and control of multi-levelled opinions spreading in social networks

Vineeth S. Varma*, Irinel Constantin Morarescu* and Yezekael Hayel‡

Abstract—This paper proposes and analyzes a stochastic multi-agent opinion dynamics model. We are interested in a multi-levelled opinion of each agent which is randomly influenced by the binary actions of its neighbors. It is shown that, as far as the number of agents in the network is finite, the model asymptotically produces consensus. The consensus value corresponds to one of the absorbing states of the associated Markov system. However, when the number of agents is large, we emphasize that partial agreements are reached and these transient states are metastable, i.e., the expected persistence duration is arbitrarily large. These states are characterized using an N-intertwined mean field approximation (NIMFA) for the Markov system. Moreover we analyze a simple and easily implementable way of controlling the opinion in the network. Numerical simulations validate the proposed analysis.

Index Terms—Opinion dynamics, Social computing and networks, Markov chains, agent based models

I. INTRODUCTION

The analysis and control of complex sociological phenomena as consensus, clustering and propagation are challenging scientific problem. In the past decades much progress has been made both on the development and the analysis of new models that capture more features characterizing the social network behavior. A possible classification of the existing models can be done by looking at the evolution space of the opinions. Precisely we find models in which the opinions evolve in a discrete set of values, they come from statistical physics, and the most employed are the Ising [1], voter [2] and Sznaid [3] models. A second class is given by the models that consider a continuous set of opinion's values [4], [5], [6]. While in some models the interaction network is fixed in some others it is state-dependent. Although some studies propose repulsive interactions [7] the predominant tendency of empirical studies emphasize the attractive action of the neighbors opinions. We can also emphasize that many studies in the literature focus on the emergence of consensus in social networks [8], [9], [10] while some others point out local agreements leading to clustering [5], [6].

Most of the existing models including the aforementioned ones share the idea that an individual's opinion is influenced by the opinions of his neighbors. Nevertheless, it is very hard to estimate these opinions and often one may access only a quantized version of them. Following this idea a mix of continuous opinion with discrete actions (CODA) was proposed in [11]. This model reflects the fact that even if we often face binary choices or actions which are visible

by our neighbors, the opinions evolve in a continuous space of values which are not explicitly visible to the neighbors. A multi-agent system with a CODA model was proposed and analyzed in [12]. It was shown that this deterministic model leads to a variety of asymptotic behaviors including consensus and clustering.

In [13] the model in [12] was reformulated as a discrete interactive Markov chain. One advantage of this approach is that, it also allows analysis of the behavior of infinite populations partitioned into a certain number of opinion classes. Due to the complexity of the opinion dynamics, we believe that stochastic models are more suitable than deterministic ones. Indeed, we can propose a realistic deterministic update rule but many random events will still influence the interaction network and consequently the opinion dynamics. Following the development in [13] we propose here a continuous-time interactive Markov chain modeling that approximates the model in [12]. Although the asymptotic behavior of the model can be given by characterizing the absorbing states of the Markov chain, the convergence time can be arbitrarily large and transient but persistent local agreements, called metastable equilibria, can appear. These equilibria are very interesting because they describe the finite-time behavior of the network. Consequently, we consider in this paper an N-intertwined mean field approximation (NIMFA) based approach in order to characterize the metastable equilibria of the Markov system. It is noteworthy that NIMFA was successfully used to analyze and validate some epidemiological models [14], [15].

In this work, we model the social network as a multi-agent system in which each agent represents an individual whose state is his opinion. This opinion can be understood as the preference of the agent towards performing a binary action, i.e. action can be 0 or 1. These agents are interconnected through an interaction directed graph, whose edge weights represent the trust given by an agent to his neighbor. We propose continuous time opinion dynamics in which the opinions are discrete and belong to a given set that is fixed a priori. Each agent is influenced randomly by the *actions* of his neighboring agents and consequently influences its neighbors. Therefore the opinions of agents are an intrinsic variable that is hidden from the other agents, the only visible variable is the action. As an example, consider the opinion of users regarding two products red and blue cars. A user may prefer red cars strongly, while some other users might be more indifferent. However, what the other users see (and is therefore influenced by) is only what the user buys, which is the *action* taken.

The contributions of this paper can be summarized as

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follows. Firstly, we formulate and analyze a stochastic version of the CODA model proposed in [12]. Secondly, we characterize the local agreements which are persistent for a long duration by using the NIMFA for the original Markov system. Thirdly, we provide conditions for the preservation of the main action inside one cluster as well as for the propagation of actions. Finally, we study how an external entity can control the opinion dynamics by manipulating the network edge weights. In particular, we study how such a control can be applied to a cluster for preservation or propagation of its opinion.

The rest of the paper is organized as follows. Section II introduces the main notation and concepts and provides a description of the model used throughout the paper. The analysis of the asymptotic behavior of opinions described by this stochastic model is provided in Section III. The presented results are valid for any connected networks with a finite number of agents. Moreover Section III contains the description of the NIMFA model and an algorithm to compute its equilibria. In Section IV we emphasize conditions for the preservation of the main action (corresponding to a metastable state) in some clusters as well as conditions for the action propagation, both without any control and in the presence of some control. The results of our work are numerically illustrated in Section V. The paper ends with some concluding remarks and perspectives for further developments.

Preliminaries: We use \mathbb{E} for the expectation of a random variable, $\mathbf{1}_A(x)$ the indicator function which takes the value 1 when $x \in A$ and 0 otherwise, \mathbb{R}_+ the set of non-negative reals and $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers..

II. MODEL

Throughout the paper we consider $N \in \mathbb{N}$ an *even* number of possible opinion levels, and the set of agents $\mathcal{K} = \{1, 2, \dots, K\}$ with $K \in \mathbb{N}$. Each agent i is characterized at time $t \in \mathbb{R}_+$ by its opinion represented as a scalar $X_i(t) \in \Theta$ where $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$ is the discrete set of possible opinions, such that $\theta_n \in (0, 1) \setminus \{0.5\}$ and $\theta_n < \theta_{n+1}$ for all $n \in \{1, 2, \dots, N\}$. Moreover Θ is constructed such that $\theta_{N/2} < 0.5$ and $\theta_{N/2+1} > 0.5$. In the following let us introduce some graph notions allowing us to define the interaction structure in the social network under consideration.

Definition 1 (Directed graph): A weighted directed graph G is a couple (\mathcal{K}, A) with \mathcal{K} being a finite set denoting the vertices, and A being a $K \times K$ matrix, with elements a_{ij} denoting the trust given by i on j . We say that agent j is a neighbor of agent i if $a_{ij} > 0$.

We denote by τ_i the total trust in the network for agent i as $\tau_i = \sum_{j=1}^K a_{ij}$. Agent i is said to be *connected* with agent j if G contains a directed path from i to j , i.e. if there exists at least one sequence $(i = i_1, i_2, \dots, i_{p+1} = j)$ such that $a_{i_k, i_{k+1}} > 0, \forall k \in \{1, 2, \dots, p\}$.

Definition 2 (Strongly connected): The graph G is strongly connected if any two distinct agents $i, j \in \mathcal{K}$ are connected.

In the sequel we suppose the following holds true.

Assumption 1: The graph $(\mathcal{K}, \mathcal{E})$ modeling the interaction in the network is strongly connected.

The action $Q_i(t)$ taken by agent i at time t is defined by the opinion $X_i(t)$ through the following relation

$$Q_i(t) = \lfloor X_i(t) \rfloor,$$

where $\lfloor \cdot \rfloor$ is the nearest integer function. This means that if an agent has an opinion more than 0.5, it will take the action 1 and 0 otherwise. This kind of opinion quantization is suitable for many practical applications. For example, an agent may support the left or right political party, with various opinion levels (opinions close to 0 or 1 represents a stronger preference), however, in the election, the agent's action is to vote with exactly two choices (left or right). Similarly, an agent might have to choose between two cars or other types of merchandise like cola as mentioned in the introduction. Although its preference for one product is not of the type 0 or 1, its action will be, since it cannot buy fractions of cars, but one of them.

A. Opinion dynamics

In this work, we look at the evolution of opinions of the agents based on their mutual influence. We also account for the *inertia* of opinion, i.e., when the opinion of the agent is closer to 0.5, he is more likely to shift as he is less decisive, whereas someone with a strong opinion (close to 1 or 0) is less likely to shift his opinion as he is more convinced by his opinion. The opinion of agent j may shift towards the actions of its neighbors with a rate β_n while $X_j(t) = \theta_n$. If no action is naturally preferred by the opinion dynamics, then we construct $\theta_n = 1 - \theta_{N+1-n}$ and assume that $\beta_n = \beta_{N+1-n}$ for all $n \in \{1, 2, \dots, N\}$. At each time $t \in \mathbb{R}_+$ we denote the vector collecting all the opinions in the network by $X(t) = (X_1(t), \dots, X_K(t))$. Notice that the evolution of $X(t)$ is described by a continuous time Markov process with N^K states and its analysis is complicated even for small number opinion levels and relatively small number of agents. The stochastic transition rate of agent i shifting its opinion to the right, i.e. to have opinion θ_{n+1} when at opinion θ_n , with $n \in \{1, 2, \dots, N-1\}$, is given by

$$\beta_n \sum_{j=1}^N a_{ij} \mathbf{1}_{(0.5, 1]}(X_j(t)) = \beta_n \sum_{j=1}^K a_{ij} Q_j(t) = \beta_n R_i(t).$$

Similarly, the transition rate to the left, i.e. to shift from θ_n to θ_{n-1} is given by

$$\beta_n \sum_{j=1}^N a_{ij} \mathbf{1}_{[0, 0.5)}(X_j(t)) = \beta_n \sum_{j=1}^N a_{ij} (1 - Q_j(t)) = \beta_n L_i(t).$$

for $n \in \{2, \dots, N\}$. Therefore, we can write the infinitesimal generator $M_{i,t}$ (a tri-diagonal matrix of size $N \times N$) for an agent i as:

$$M_{i,t} = \begin{pmatrix} -\beta_1 R_i(t) & \beta_1 R_i(t) & 0 & \dots \\ \beta_2 L_i(t) & -\beta_2 \tau_i & \beta_2 R_i(t) & \dots \\ \vdots & & & \end{pmatrix} \quad (1)$$

with elements corresponding the n -th row and m -th column given by $M_{i,t}(m,n)$ and

$$\begin{aligned} \forall n \in \{1, \dots, N-1\}, \quad M_{i,t}(n, n+1) &= \beta_n R_i(t), \\ \forall n \in \{2, \dots, N\}, \quad M_{i,t}(n, n-1) &= \beta_n L_i(t), \\ \forall |m-n| > 1, \quad M_{i,t}(m, n) &= 0 \end{aligned}$$

and

$$M_{i,t}(n, n) = \begin{cases} -\beta_1 R_i(t) & \text{for } n = 1, \\ -\beta_n \tau_i & \text{for } n \in \{2, \dots, N-1\}, \\ -\beta_N L_i(t) & \text{for } n = N. \end{cases}$$

Let $v_{i,n}(t) := \mathbb{E}[\mathbf{1}_{\{\theta_n\}}(X_i(t))] = \Pr(X_i(t) = \theta_n)$ be the probability for an opinion level θ_n for user i at time t . Then, in order to propose an analysis of the stochastic process introduced above, we may consider the mean-field approximation by replacing the transitions by their expectations. Then, the expected transition rate from state n to state $n+1$ for $K \rightarrow \infty$, is given by:

$$\beta_n \sum_{j=1}^K a_{ij} \mathbb{E}[\mathbf{1}_{(0.5,1)}(X_j(t))] = \beta_n \sum_{j=1}^K a_{ij} \sum_{n=N/2+1}^N v_{j,n}(t).$$

We have similar expression for transition between state n and $n-1$.

III. STEADY STATE ANALYSIS

Define by $\bar{\theta}_- = (\theta_1, \dots, \theta_1)$ and $\bar{\theta}_+ = (\theta_N, \dots, \theta_N)$ the states where all the agents in the network have an identical opinion, which correspond to the two extreme opinions.

Proposition 1: Under Assumption 1, the continuous time Markov process $X(t)$, with (1) as the infinitesimal generators corresponding to each agent, has exactly two absorbing states $X(t) = \bar{\theta}_+$ and $X(t) = \bar{\theta}_-$.

Proof: Due to space limitation the proof is omitted. ■

Considering the NIMFA approximation, we get that the dynamics of the opinion for an agent i are given by:

$$\begin{aligned} \dot{v}_{i,1} &= -\beta_1 r_i v_{i,1} + \beta_2 l_i v_{i,2} \\ \dot{v}_{i,n} &= -\beta_n r_i v_{i,n} + \beta_{n+1} l_i v_{i,n+1} \\ &\quad -\beta_n l_i v_{i,n} + \beta_{n-1} r_i v_{i,n-1} \\ \dot{v}_{i,N} &= -\beta_N l_i v_{i,N} + \beta_{N-1} r_i v_{i,N-1} \end{aligned} \quad (2)$$

for all $i \in \mathcal{K}$ and $1 < n < N$ where

$$\begin{aligned} l_i &= \sum_{j \in \mathcal{K}} a_{ij} E[1 - Q_j] \\ &= \sum_{j \in \mathcal{K}} \sum_{n=1}^{N/2} a_{ij} v_{j,n}, \\ r_i &= \sum_{j \in \mathcal{K}} a_{ij} E[Q_j] \\ &= \sum_{j \in \mathcal{K}} \sum_{n=N/2+1}^N a_{ij} v_{j,n}. \end{aligned} \quad (3)$$

and $\sum_n v_{i,n} = 1$.

We can easily verify that $X_i = \theta_1$, i.e. $v_{i,1} = 1$ for all i is an equilibrium for the above set of equations. When $v_{i,1} = 1$ for all i , $v_{i,n} = 0$ for all $n \geq 2$ and as a result, $l_i = \tau_i$ and $r_i = 0$ for all i which gives $\dot{v}_{i,n} = 0$ for all i, n . Excluding the extreme solutions $\bar{\theta}_+$ and $\bar{\theta}_-$, the non-linearity of system (2) could give rise to the existence of interior rest points which are locally stable. Such rest points are referred to as metastable states in Physics. Metastability of Markov

processes is precisely defined in [16], where the exit times from these metastable states are shown to approach infinity when the network size is arbitrarily large.

A. Rest points of the dynamics

For a given $r_i = E[R_i(t)]$, the equilibrium state $v_{i,n}^*$ must satisfy the following conditions

$$\begin{aligned} 0 &= -\beta_1 \frac{r_i}{\tau_i} v_{i,1}^* + \beta_2 \left(\frac{\tau_i - r_i}{\tau_i} \right) v_{i,2}^* \\ 0 &= -\beta_n v_{i,n}^* + \beta_{n+1} \left(\frac{\tau_i - r_i}{\tau_i} \right) v_{i,n+1}^* \\ &\quad + \beta_{n-1} \frac{r_i}{\tau_i} v_{i,n-1}^* \\ 0 &= -\beta_N \left(\frac{\tau_i - r_i}{\tau_i} \right) v_{i,N}^* + \beta_{N-1} \frac{r_i}{\tau_i} v_{i,N-1}^* \end{aligned} \quad (4)$$

We can write any $v_{i,n}^*$ based on $v_{i,1}^*$, by simplification as

$$v_{i,n}^* = \frac{\beta_1}{\beta_n} \left(\frac{r_i}{\tau_i - r_i} \right)^{n-1} v_{i,1}^*. \quad (5)$$

As the sum of $v_{i,n}$ over n must be 1, we can solve for $v_{i,1}^*$ as

$$v_{i,1}^* = \frac{1}{\sum_{n=1}^N \frac{\beta_1}{\beta_n} \left(\frac{r_i}{\tau_i - r_i} \right)^{n-1}}. \quad (6)$$

We then can use this relationship to construct a fixed-point algorithm that computes a rest-point of the global opinion dynamics for all users.

Data: Number of agents K , the edge weights $a_{i,j}$ for all $i, j \in \mathcal{K}$, initial values $v_{i,n}(0)$, convergence factor $\epsilon \ll 1$, opinion levels N and the jump rates β_n for all $n \in \{1, \dots, N\}$.

Result: $v(m)$ at the end of the loop is close to a fixed point of the opinion dynamics

do

$m \leftarrow m + 1$;

Set

$$r_i(m) = \sum_k \sum_{n=N/2+1}^N a_{i,k} v_{k,n}(m) \quad (7)$$

for all $i \in \mathcal{K}$;

Set

$$v_{i,n}(m) = \frac{\frac{\beta_1}{\beta_n} \left(\frac{r_i(m)}{\tau_i - r_i(m)} \right)^{n-1}}{\sum_{l=1}^N \frac{\beta_1}{\beta_l} \left(\frac{r_i(m)}{\tau_i - r_i(m)} \right)^{l-1}} \quad (8)$$

for all $n \in \{1, \dots, N\}, i \in \mathcal{K}$;

while $\|v^m - v^{m-1}\| \geq \epsilon$;

Algorithm 1: Algorithm to find a fixed point of the NIMFA.

Additionally, we can obtain some nice properties on the relation between r_i and $v_{i,n}$ by studying the following function.

Lemma 1: Consider the function $f : [0, 1] \rightarrow [0, 1]$ defined as

$$f(x) := \frac{\sum_{n=N/2+1}^N \frac{\beta_1}{\beta_n} \left(\frac{x}{1-x}\right)^{n-1}}{\sum_{n=1}^N \frac{\beta_1}{\beta_n} \left(\frac{x}{1-x}\right)^{n-1}} \quad (9)$$

for all $x \in [0, 1)$ and with $f(1) = 1$. We have that $f(x)$ is a monotonically increasing continuous function and takes the values $f(0) = 0$, $f(0.5) = 0.5$ and $\lim_{x \rightarrow 1} f(x) = 1$.

Proof: Due to space limitation the proof is omitted. ■

We can use $f(\frac{\tau_i}{\tau_i - r_i})$ to calculate the probability that an agent i will take the action 1, i.e., $\sum_{n=N/2+1}^N v_{i,n}^* = f(\frac{\tau_i}{\tau_i - r_i})$ from (6) and (5).

IV. OPINION SPREADING

A way to model generic interaction networks is to consider that they are the union of a number of clusters (see for instance [6] for a cluster detection algorithm). Basically a cluster C is a group of agents in which the opinion of any agent in C is influenced more by the other agents in C , than agents outside C . When the interactions between cluster are deterministic and very weak, we can use a two time scale-modeling as in [17] to analyze the overall behavior of the network. In the stochastic framework and knowing only a quantized version of the opinions we propose here a development aiming at characterizing the majority actions in clusters. The notion of cluster can be mathematically formalized as follows.

Definition 3 (Cluster): A subset of agents $C \subset \mathcal{K}$ defines a cluster when, for all $i, j \in C$ and some $\lambda > 0.5$ the following inequality holds

$$a_{ij} \geq \lambda \frac{\tau_i}{|C|}. \quad (10)$$

The maximum λ which satisfies this inequality for all $i, j \in C$ is called the cluster coefficient.

For any given set of agents $C \subset \mathcal{K}$, let us denote that

$$\nu_-^C = \frac{\sum_{j \in C} \sum_{n=1}^{N/2} v_{j,n}}{|C|},$$

and

$$\nu_+^C = \frac{\sum_{j \in C} \sum_{n=N/2+1}^N v_{j,n}}{|C|}.$$

Those values represent the expected fraction of agents within a set C with action 0 and 1, respectively. We also denote by ν_n^C , the average probability of agents in a cluster to have opinion θ_n , i.e., $\nu_n^C = \frac{\sum_{i \in C} v_{i,n}}{|C|}$. Now we can use the definition of a cluster given in (10) to obtain the following proposition.

Proposition 2: The dynamics of the average opinion probabilities in a cluster $C \subset \mathcal{K}$ can be written as:

$$\begin{aligned} \frac{\dot{\nu}_1^C}{\kappa} &= -\beta_1 \nu_1^C (\lambda \nu_+^C + (1-\lambda)\delta) \\ &\quad + \beta_2 \nu_2^C (\lambda \nu_-^C + (1-\lambda)(1-\delta)) \\ \frac{\dot{\nu}_n^C}{\kappa} &= -\beta_n \nu_n^C + \beta_{n+1} \nu_{n+1}^C (\lambda \nu_-^C + (1-\lambda)(1-\delta)) \\ &\quad + \beta_{n-1} \nu_{n-1}^C (\lambda \nu_+^C + (1-\lambda)\delta) \\ \frac{\dot{\nu}_N^C}{\kappa} &= -\beta_N \nu_N^C (\lambda \nu_-^C + (1-\lambda)(1-\delta)) \\ &\quad + \beta_{N-1} \nu_{N-1}^C (\lambda \nu_+^C + (1-\lambda)\delta) \end{aligned} \quad (11)$$

where $\kappa = \frac{\sum_{i \in C} \tau_i}{|C|}$ and $\delta \in [0, 1]$.

Proof: Due to space limitation the proof is omitted. ■

The result above shows that instead of looking at individual opinions of agents inside a cluster, we can provide equation (11) for the dynamics of the expected fraction of agents in a cluster with certain opinions.

A. Action preservation

One question that can be asked in this context is what are the sufficient conditions for the preservation of actions in a cluster, i.e. regardless of external opinions, agents preserve the majority action inside the cluster C for long time. At the limit, all the agents will have identical opinions corresponding to an absorbing state of the network but, clusters with large enough λ may preserve their action in metastable states (long time).

Proposition 3: If $\exists x \in (0.5, 1)$ such that $x = f(\lambda x)$ than cluster C with coefficient λ preserves its action in a metastable state. If no such x exists, then the only equilibrium when the perturbation term $\delta = 1$ from (11) is at $\nu_+^C = 1$, and when $\delta = 0$, the equilibrium is at $\nu_+^C = 0$.

Proof: Due to space limitation the proof is omitted. ■

B. Propagation of actions

In the previous subsection, we have seen that a cluster C can preserve its majority action regardless of external opinion if it has a sufficiently large λ . If there are agents outside C with some connections to agents in C , then this action can be propagated. Let $\tau_i^C = \sum_{j \in C} a_{ij}$ denote the total trust of agent i in the cluster C . Let the cluster C be such that it has λ large enough so that $\bar{\nu}_+^C > 0.5$ exists where $\bar{\nu}_+^C = f(\lambda \bar{\nu}_+^C)$.

Proposition 4: If the cluster C is preserving an action 1 with at least a fraction $\bar{\nu}_+^C$ of the population in C having action 1, then the probability of any agent $i \in \mathcal{K} \setminus C$ to chose action 1 at equilibrium is bounded as follows.

$$f\left(\bar{\nu}_+^C \frac{\tau_i^C}{\tau_i}\right) \leq \Pr(Q_i^* = 1) \leq 1 - f\left(\bar{\nu}_-^C \frac{\tau_i^C}{\tau_i}\right). \quad (12)$$

where Q_i^* is the action taken by i when the system is in a non-absorbing NIMFA equilibrium state.

Proof: Due to space limitation the proof is omitted. ■

C. Control of opinion spreading

Consider that an external entity, a firm for example, wants to control the actions of the agents in the network. In particular, the external agency wants to ensure that a cluster C preserves its initial action. Practically, a set of consumers in C might prefer the product of the firm, and this firm wants to ensure that other competing firms do not sway their opinions and convert their opinions.

For this purpose, the firm tries to reinforce the opinion spread within C and from C to its followers, by strengthening the influence of agents in C . As an example, the party might pay the social network, such as Facebook or Twitter, to make messages, in the form of wall posts or tweets, to be more visible when the source of these messages are within C . This implies that a_{ij} for any $i \in \mathcal{K}$ and $j \in C$, is now modified to $a'_{ij} = a_{ij}(1 + u)$, where u denotes the control parameter. Here, $u = 0$ represents the normal state of the network and $u > 0$ implies that agents in the network will view messages from agents in C more frequently or more easily (with a higher priority).

Thus, the cluster coefficient of C is no longer the initial cluster coefficient λ , but the new coefficient is increased depending on u .

Proposition 5: Let λ be the cluster coefficient of C . Then a control which strengthens the connections from C from a_{ij} to $a_{ij}(1 + u)$ for all $j \in C$ and $i \in \mathcal{K}$ results in a new cluster coefficient $\lambda' \geq \lambda$ given by

$$\lambda' \geq \lambda \frac{1 + u}{1 + \lambda u} \quad (13)$$

This results in a threshold on the control required for preservation as the smallest $u^* \geq 0$ satisfying

$$x = f\left(\lambda \frac{1 + u^*}{1 + \lambda u^*} x\right) \quad (14)$$

for some $x \in (0.5, 1]$. Additionally, any follower of C has its probability of action 1 modified to

$$f\left(\bar{v}_+^C \frac{\tau_i^C(1+u)}{\tau_i + \tau_i^C u}\right) \leq \Pr(Q_i^* = 1) \leq 1 - f\left(\bar{v}_-^C \frac{\tau_i^C(1+u)}{\tau_i + \tau_i^C u}\right) \quad (15)$$

Proof: Due to space limitation the proof is omitted. ■

As $\lambda \leq 1$, $\lambda' \geq \lambda$ always, and we also have $\lim_{u \rightarrow \infty} \lambda' = 1$. This implies that as long as we apply a sufficiently large control u , any cluster C will be able to preserve its action (as a result of Corollary 1). Additionally, this also means that agents who are followers of C will now be influenced to a greater degree by C .

V. NUMERICAL RESULTS

For all simulations studied, unless otherwise mentioned, we take $\Theta = \{0.2, 0.4, 0.6, 0.8\}$, $\beta_1 = \beta_4 = 0.01$ per unit of time and $\beta_2 = \beta_3 = 0.02$ per unit of time.

For the first set of simulations, we take another graph structure, but with the same \mathcal{K} as indicated in Figure 1. We first randomly make links between any $i, j \in \mathcal{K}$ with a probability of 0.05. When such a link exists, $a_{ij} = 1$

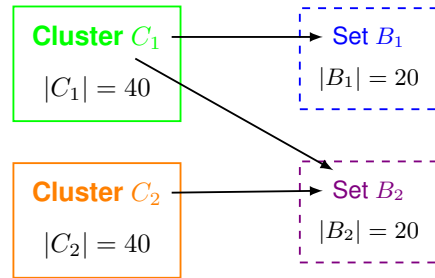


Fig. 1: Structure of the graph. Any two agents in \mathcal{K} may be connected with a 0.05 probability. All agents within a cluster are connected, and the arrows indicate directed connections.

and $a_{ij} = 0$ otherwise. Then we construct a cluster C_1 , with the agents $i = 1, 2, \dots, 40$, and C_2 with agents $i = 81, 82, \dots, 120$. We also label agents $40 < i \leq 60$ as set B_1 and $60 < j \leq 80$ as set B_2 . To provide the relevant cluster structure, we made the edge weights $a_{ij} = 1$ for all

- $i, j \in C_1$ or $i, j \in C_2$, making C_1 and C_2 clusters with coefficients $\lambda_1 = 0.833$ and $\lambda_2 = 0.816$ (for the particular random graph generated for this simulation).
- $40 < i \leq 60$ and $1 \leq j \leq 20$, making agents in B_1 trust C_1 with $\frac{\tau_i^{C_1}}{\tau_i} \geq 0.714$ for all $40 < i \leq 60$.
- $60 < i \leq 80$ and $1 \leq j \leq 20$ or $80 < j \leq 120$, making agents in B_2 trust both C_1 and C_2 , with $\frac{\tau_i^{C_1}}{\tau_i}, \frac{\tau_i^{C_2}}{\tau_i} \geq 0.444$ for all $60 < i \leq 80$.

A. Propagation of actions

We find that the largest x satisfying $x = f(\lambda_1 x)$ is 0.95 and that satisfying $x = f(\lambda_2 x)$ is 0.94. Therefore, if all agents in C_1 start with opinion 0.2 and all agents in cluster 2 start with opinion 0.8, we predict from proposition 3 that $\nu_-^{C_1} \geq 0.95$ and $\nu_+^{C_2} \geq 0.94$ in the metastable state. Additionally, applying proposition 4 yields $\nu_-^{B_1} \geq f(0.95 \times 0.714) = 0.85$, $\nu_-^{B_2} \geq f(0.95 \times 0.444) = 0.324$ and $\nu_+^{B_2} \geq f(0.94 \times 0.444) = 0.315$.

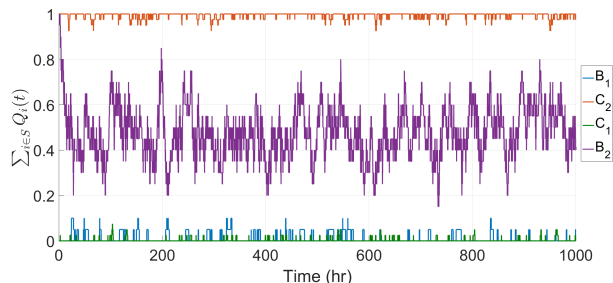
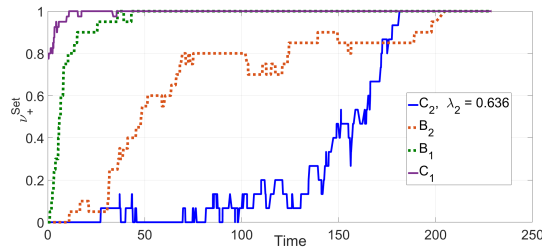


Fig. 2: Simulation of $\sum_{i \in S} Q_i(t)$ for $S = C_1, C_2, B_1, B_2$. We see that C_1 and C_2 preserve their initial actions as given by proposition 3. We also see that as B_1 follows only C_1 , it's action is close to C_1 . As B_2 follows both C_1 and C_2 who have contradicting actions, it has a very mixed opinion which keeps changing randomly in time.

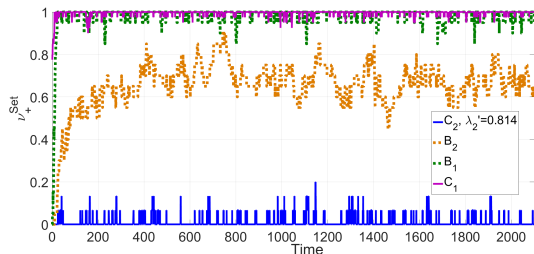
Simulations of the continuous time Markov chain show that our theoretical results are valid even when the cluster size is 40. Figure 2 plots the population fraction of agents with action 1 within a certain set for one simulation. We look for this value in the clusters C_1 and C_2 as well as the sets B_1 and B_2 . C_1 and C_2 are seen to preserve their actions which are opposite to each other. Since B_1 has a significant trust in C_1 alone, the opinion of C_1 is propagated to B_1 . However, as B_2 trusts both C_1 and C_2 , its opinion is influenced by the two contradicting actions resulting in having some agents with action 1 and the rest with action 0.

B. Control of opinion spreading

We consider the same graph structure used in Fig. 2 illustrated in Fig. 1, but with Cluster C_2 having its first 25 agents removed. As a result, the cluster coefficient of this cluster becomes $\lambda_2 = 0.636$, which no longer allows for preservation of its action as seen in Figure 3a. From proposition 3, we find that a $\lambda > 0.8$ ensures preservation. Therefore, we introduce a control $u = 1.5$ which enhances the visibility of actions spread by agents in C_2 by a factor of 2.5, resulting in $a'_{ij} = 2.5a_{ij}$ for all $j \in C, i \in \mathcal{K}$. This results in a new $\lambda'_2 \geq 0.8$ according to proposition 5 and verified by numerical calculation on the graph to be $\lambda'_2 = 0.814$. As a result, we observe from Figure 3b that the cluster 2 action is preserved for a long duration.



(a) Simulation with no control implemented.



(b) Simulation with $u = 1$ on cluster 2.

Fig. 3: Average population with action 1 plotted vs time for each set.

VI. CONCLUSION

In this paper, we have proposed a stochastic multi-agent opinion dynamics model with binary actions. Agents interact through a network and individual opinions are influenced by neighbors action. Our analysis based on a Markov model of opinion dynamics show a consensus like limiting behavior

for a finite number of agents. Whereas, when this number becomes large enough, the stochastic system can enter into a quasi-stationary regime in which partial agreements are reached. This type of phenomenon has been observed in all-to-all and cluster type topologies. Additionally, we have also studied the impact of an external entity which tries to control the actions of the users in a social network by manipulating the network connections. This can be interpreted as a company paying a social platform to make content from certain groups of agents more visible or frequent. We have shown how such a control can enable a community or cluster of agents to preserve or propagate its opinion better.

REFERENCES

- [1] E. Ising, "Contribution to the theory of ferromagnetism," Ph.D. dissertation, University of Hamburg, 1924.
- [2] P. Clifford and A. Sudbury, "A model for spatial conflict," *Biometrika*, vol. 60, no. 3, pp. 581–588, 1973.
- [3] K. Sznajd-Weron and J. Sznajd, "Opinion evolution in closed community," *International Journal of Modern Physics C*, vol. 11, no. 6, pp. 1157–1165, 2000.
- [4] M. H. DeGroot, "Reaching a consensus," *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 118–121, 1974.
- [5] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis, and simulation," *Journal of Artificial Societies and Social Simulation*, vol. 5, no. 3, 2002.
- [6] I.-C. Morărescu and A. Girard, "Opinion dynamics with decaying confidence: Application to community detection in graphs," *IEEE Trans. on Automatic Control*, vol. 56, no. 8, pp. 1862 – 1873, 2011.
- [7] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 935–946, 2013.
- [8] S. Galam and S. Moscovici, "Towards a theory of collective phenomena: Consensus and attitude changes in groups," *European Journal of Social Psychology*, vol. 21, no. 1, pp. 49–74, 1991.
- [9] R. Axelrod, "The dissemination of culture: A model with local convergence and global polarization," *The Journal of Conflict Resolution*, vol. 41, no. 2, pp. 203–226, 1997.
- [10] S. Fortunato, V. Latora, A. Pluchino, and A. Rapisarda, "Vector opinion dynamics in a bounded confidence consensus model," *International Journal of Modern Physics C*, vol. 16, no. 10, pp. 1535–1551, 2005.
- [11] A. Martins, "Continuous opinions and discrete actions in opinion dynamics problems," *International Journal of Modern Physics C*, vol. 19, no. 4, pp. 617–625, 2008.
- [12] N. R. Chowdhury, I.-C. Morărescu, S. Martin, and S. Srikant, "Continuous opinions and discrete actions in social networks: a multi-agent system approach," in *Proceedings 55th IEEE Conference on Decision and Control*, 2016.
- [13] V. Varma and I.-C. Morărescu, "Modeling stochastic dynamics of agents with multi-leveled opinions and binary actions," in *Proceedings of the 55th IEEE Conference on Decision and Control*, 2017.
- [14] P. Van Mieghem, J. Omic, and R. Kooij, "Virus spread in networks," *IEEE/ACM Transactions on Networking*, vol. 17, no. 1, pp. 1–14, 2009.
- [15] S. Trajanovski, Y. Hayel, E. Altman, H. Wang, and P. Van Mieghem, "Decentralized protection strategies against sis epidemics in networks," *IEEE Transactions on Control of Network Systems*, vol. 2, no. 4, pp. 406–419, 2015.
- [16] W. Huisinga, S. Meyn, C. Schutt, H. Wang, and P. Van Mieghem, "Phase transitions and metastability in markovian and molecular systems," *The Annals of Applied Probability*, vol. 14, no. 1, pp. 419–458, 2004.
- [17] S. Martin, I. C. Morărescu, and D. Nešić, "Time scale modeling for consensus in sparse directed networks with time-varying topologies," in *Proceedings 55th IEEE Conference on Decision and Control*, 2016, pp. 7–12.