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# On the model completion of scaled lattices and the co-Heyting algebras of closed $p$ -adic semi-algebraic sets\*

Luck Darnière

September 11, 2018

## Abstract

Let  $p$  be prime number,  $K$  be a  $p$ -adically closed field,  $X \subseteq K^m$  a semi-algebraic set defined over  $K$  and  $L(X)$  the lattice of semi-algebraic subsets of  $X$  which are closed in  $X$ . We prove that the complete theory of  $L(X)$  eliminates the quantifiers in a certain language  $\mathcal{L}_{\text{ASC}}$ , the  $\mathcal{L}_{\text{ASC}}$ -structure on  $L(X)$  being an extension by definition of the lattice structure. Moreover it is decidable, contrary to what happens over a real closed field. We classify these  $\mathcal{L}_{\text{ASC}}$ -structures up to elementary equivalence, and get in particular that the complete theory of  $L(K^m)$  only depends on  $m$ , not on  $K$  nor even on  $p$ . As an application we obtain a classification of semi-algebraic sets over countable  $p$ -adically closed fields up to so-called “pre-algebraic” homeomorphisms.

## 1 Introduction

This paper explores the model-theory of various classes of lattices coming from algebraic geometry, real geometry or  $p$ -adic geometry, with special emphasis on the  $p$ -adic case. We obtain model-completion and decidability results for some of them. Before entering in technical details let us present the main motivations for this, coming from geometry and model-theory of course, but also from proof theory and non-classical logics.

Given an expansion of a topological field  $K$  and a definable<sup>1</sup> set  $X \subseteq K^n$ , we consider the lattice  $L(X)$  of all definable subsets of  $X$  which are closed in  $X$ , and the ring  $\mathcal{C}(X)$  of all continuous definable functions from  $X$  to  $K$ . These rings are central objects nowadays in functional analysis, topology and geometry. To name an example, they are rings of sections for the sheaf of continuous (say, real valued) functions on a topological space and as such play the algebraic part in the study of topological (Hausdorff) spaces. In most cases  $L(X)$  is interpretable in  $\mathcal{C}(X)$ , and the prime filter spectrum of  $L(X)$  is homeomorphic to the prime ideal spectrum of  $\mathcal{C}(X)$ . Thus  $L(X)$  is a first-order structure interpretable in  $\mathcal{C}(X)$ , which captures all the topological (hence second-order) information on the spectrum of the ring  $\mathcal{C}(X)$ . For the real field  $\mathbf{R}$  for example, it is known since [Grz51] that  $L(\mathbf{R}^n)$  is undecidable for every  $n \geq 2$ , hence can be held liable for the undecidability of  $\mathcal{C}(\mathbf{R}^n)$ . Recently this has been strengthened and widely generalised in [Tre17]. However these undecidability results for  $L(X)$  strongly depend on the

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<sup>1</sup>We assume the reader to be familiar with basic notions from model theory, in particular definable sets and functions. In simplest cases “definable” boils down to “semi-algebraic” over the field  $\mathbf{R}$  of real numbers, or the field  $\mathbf{Q}_p$  of  $p$ -adic numbers.

existence of irreducible or connected components, hence do not apply to the  $p$ -adic case. But even in that case it is proved in [DT18] that  $\mathcal{C}(\mathbf{Q}^n)$  is undecidable, this time for every  $n \geq 1$ . At the contrary, our main result implies that for every  $n \geq 1$ :

- $L(\mathbf{Q}_p^n)$  is decidable, and;
- The theory of  $L(\mathbf{Q}_p^n)$  eliminates the quantifier in a natural expansion by definition of the lattice language.

In another direction, the model-theory of these geometric lattices  $L(X)$  is tightly connected to the existence of uniform interpolants for propositional calculus in certain intermediate<sup>2</sup> and modal logics. Indeed, thanks to the one-to-one correspondence between intermediate logics and varieties of Heyting algebras, the existence of uniform interpolants for a logic  $\mathcal{L}$  can be rephrased, *mutatis mutandis*, as the existence of a model-completion for the theory of the corresponding variety  $\mathcal{V}(\mathcal{L})$  (see [GZ97]). As lattices of closed sets, all our lattices  $L(X)$  are co-Heyting algebras, that is Heyting algebras with the order reversed. Moreover, their structure is mostly determined by the *geometry* of  $X$ . This geometric intuition coming from  $X$  was essential to our model-completion results, with natural axiomatisations, for certain theories of (expansions of) co-Heyting algebras (theorems B and C below). See also [CG17], [DJ18] for related results.

Now we are going to present our results in more detail. They are based on a fine-grain study of certain expansions of lattices, all inspired by the following geometric example.

**Example 1.1** Let  $\mathcal{K}$  be an  $o$ -minimal,  $P$ -minimal or  $C$ -minimal expansion<sup>3</sup> of a field  $K$ . For every definable sets  $A, B \subseteq X \subseteq K^m$  let  $A - B = \overline{A \setminus B} \cap X$  where the overline stands for the topological closure. For every  $a \in A$ , the **local dimension** of  $A$  at  $a$  is the maximum of the dimensions of the definable neighborhoods of  $a$  in  $A$ , and  $A$  is called **pure dimensional** if it has the same local dimension at every point. For every non-negative integer  $i$  let

$$C^i(A) = \overline{\{a \in A \mid \dim(A, a) = i\}} \cap A.$$

This is a definable subset of  $A$ , closed in  $A$ , called the  **$i$ -pure component** of  $A$ . We call  $L_{\text{def}}(X)$  the lattice of all the definable subsets of  $X$  which are closed in  $X$ , enriched with the above functions “ $-$ ” and  $C^i$  for every  $i$ .

Let  $\mathcal{L}_{\text{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$  be the language of lattices, and  $\mathcal{L}_{\text{SC}} = \mathcal{L}_{\text{lat}} \cup \{-, (C^i)_{i \geq 0}\}$  be its expansion by the above function symbols. Finally let  $\text{SC}_{\text{def}}(\mathcal{K}, d)$  be the class of the  $\mathcal{L}_{\text{SC}}$ -structures  $L_{\text{def}}(X)$  of Example 1.1, for all the sets  $X$  of dimension at most  $d$  definable over  $\mathcal{K}$ . A similar construction can be done over a pure field  $K$ , with the Zariski topology on  $K^m$ . We let  $\text{SC}_{\text{Zar}}(K, d)$  denote the corresponding class of  $\mathcal{L}_{\text{SC}}$ -structures. Surprisingly enough, we prove that in most cases the universal theory of  $\text{SC}_{\text{def}}(\mathcal{K}, d)$  (resp.  $\text{SC}_{\text{Zar}}(K, d)$ ) does not depend on  $\mathcal{K}$  (resp.  $K$ )!

**Theorem A** *Given any non-negative integer  $d$ , the universal theories of  $\text{SC}_{\text{def}}(\mathcal{K}, d)$  (resp.  $\text{SC}_{\text{Zar}}(K, d)$ ) in the language  $\mathcal{L}_{\text{SC}}$  are the same for every  $o$ -minimal,  $C$ -minimal or  $P$ -minimal expansion  $\mathcal{K}$  of a field  $K$  (resp. for every infinite field  $K$ ).*

<sup>2</sup>An intermediate logic is logic which stands between classical and intuitionist logic.

<sup>3</sup>More generally we may consider a dp-minimal expansion of a field with a definable uniform structure (dp-minimality is a stability-theoretic notion which contains the above mentioned minimality notions) provided that: (i)  $K$  has no isolated point, and; (ii) every infinite definable subset of  $K$  has non-empty interior. It then follows from [SW18] that every definable set over  $\mathcal{K}$  decomposes in pure-dimensional components, and this decomposition has very nice properties. This generalises known results for  $o$ -minimal fields [vdD98] and  $P$ -minimal fields [CKDL17].

In order to prove this we give in Section 2 an explicit list of universal axioms for a theory  $T_d$  in  $\mathcal{L}_{\text{SC}}$ , the models of which we call  $d$ -subscaled lattices. All the examples given above are  $d$ -scaled lattices, a natural subclass of  $d$ -subscaled lattices (the class of  $d$ -scaled lattices is elementary but not universal). After some technical preliminaries in Section 3 we prove in Section 4 that every finitely generated  $d$ -subscaled lattice is finite. Combining this with a linear representation for finite  $d$ -subscaled lattices (Proposition 5.3) and with the model-theoretic compactness theorem, we then prove in Section 5 that, whatever is  $K$  or  $\mathcal{K}$  in Example 1.1, the theory of  $d$ -subscaled lattices is exactly the universal theory of  $\text{SC}_{\text{def}}(\mathcal{K}, d)$  and of  $\text{SC}_{\text{Zar}}(K, d)$  (Theorem 5.3).

A detailed study of the minimal finitely generated extensions of finite  $d$ -subscaled lattices, achieved in Section 6, leads us in Section 7 to the next result (Theorem 7.3 and Corollary 7.5).

**Theorem B** *For every non-negative integer  $d$ , the theory of  $d$ -subscaled lattices admits a model-completion  $\bar{T}_d$  which is finitely axiomatizable and  $\aleph_0$ -categorical. Moreover  $\bar{T}_d$  has finitely many prime models, hence it is decidable as well as all its completions.*

The axiomatization of  $\bar{T}_d$  given in Section 7 consists in a pair of axioms expressing a ‘‘Catenarity’’ and a ‘‘Splitting’’ property which both have a natural topological and geometric meaning. In particular the Splitting Property expresses a very strong form of disconnectedness, which implies that the models of  $\bar{T}_d$  are atomless.

**Remark 1.2** Since 0-subscaled lattices are exactly non-trivial boolean algebras the above model-completion result for subscaled lattices is a generalisation to arbitrary finite dimension  $d$  of the classical theorem on the model-completion of boolean algebras.

We develop in Sections 8 and 9 a variant of this quantifier elimination result in a language  $\mathcal{L}_{\text{ASC}} = \mathcal{L}_{\text{SC}} \cup \{\text{At}_k\}_{k \geq 1}$ , where each  $\text{At}_k$  is a unary predicate symbol, to be interpreted as the set of elements which are the join of exactly  $k$  atoms. The model-completion  $\bar{T}_d^{\text{At}}$  that we obtain is axiomatized by the Catenarity property and a little restriction of the Splitting property which preserves the atoms. This theory  $\bar{T}_d^{\text{At}}$  has  $\aleph_0$  prime models which can easily be classified in terms of the prime models of  $\bar{T}_d$ , from which it follows that it is decidable as well as all its completions (Theorem 9.4).

In the initial version of this paper [Dar06] we conjectured that  $L_{\text{def}}(\mathbf{Q}_p^d)$  might be a natural model of  $\bar{T}_d^{\text{At}}$ . This intuition proved to be crucial in the proof of the triangulation of semi-algebraic sets over a  $p$ -adically closed field [Dar17]. Conversely it follows from this triangulation that  $L_{\text{def}}(X)$  is indeed a model of  $\bar{T}_d^{\text{At}}$ , for every semi-algebraic<sup>4</sup> set  $X \subseteq K^m$  of dimension  $\leq d$ , from which the next result follows in the last section (Theorem 10.2).

**Theorem C** *Let  $K$  be a  $p$ -adically closed field,  $X \subseteq K^m$  a semi-algebraic set. Then the complete theory of  $L(X)$  is decidable, and eliminates the quantifier in  $\mathcal{L}_{\text{ASC}}$ .*

The prime  $\mathcal{L}_{\text{ASC}}$ -substructure of  $L_{\text{def}}(X)$  (which is generated by the empty set) is finite. By Theorem C it determines the complete theory of  $L_{\text{def}}(X)$ . We expect this invariant to play also as decisive role in the classification of semi-algebraic sets over  $p$ -adically closed fields up to semi-algebraic homeomorphisms. Such a classification is far from being achieved, but a weaker classification, up to ‘‘pre-algebraic’’ homeomorphisms over countable  $p$ -adically closed fields, is done here by means of this invariant (Theorem 10.5).

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<sup>4</sup>A generalization to definable sets over more general  $P$ -minimal fields, if possible, has still to be done.

## 2 Notation and definitions

$\mathbf{N}$  denotes the set of non-negative integers, and  $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$ . If  $\mathcal{N}$  is an unbounded non-empty subset of  $\mathbf{N}$  (resp. the empty subset) we set  $\max \mathcal{N} = +\infty$  (resp.  $\max \mathcal{N} = -\infty$ ). The symbols  $\subseteq$  and  $\subset$  denote respectively the inclusion and the strict inclusion. The logical connectives ‘or’, ‘and’ and their iterated forms will be denoted by  $\vee$ ,  $\wedge$ ,  $\bigvee$  and  $\bigwedge$  respectively.

### 2.1 Lattices and dimension

In this paper a **lattice** is a partially ordered set in which every finite subset has a greatest lower element and a least greater element. This applies in particular to the empty subset, hence our lattices must have a least and a greatest element.  $\mathcal{L}_{\text{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$  is the language of lattices, each symbol having its obvious meaning. As usually  $b \leq a$  is an abbreviation for  $a \vee b = a$  and similarly for  $b < a$ ,  $b \geq a$  and  $b > a$ . Iterated  $\vee$  and  $\wedge$  operations are denoted by  $\bigvee_{i \in I} a_i$  and  $\bigwedge_{i \in I} a_i$  respectively. If the index set  $I$  is empty then  $\bigvee_{i \in I} a_i = \mathbf{0}$  and  $\bigwedge_{i \in I} a_i = \mathbf{1}$ . Given a subset  $S$  of a lattice  $L$ , the **upper semi-lattice** generated in  $L$  by  $S$  is the set of finite joins of elements of  $L$ . We will make intensive use of the following definable relation:

$$b \ll a \iff \forall c (c < a \Rightarrow b \vee c < a)$$

This is a strict order on  $L \setminus \{\mathbf{0}\}$  (but not on  $L$  because  $\mathbf{0} \ll a$  for every  $a$  including  $a = \mathbf{0}$ ).

The **spectrum** of a lattice  $L$  is the set  $\text{Spec}(L)$  of all prime filters of  $L$ , endowed with the so-called Zariski topology, defined by taking as a basis of closed sets all the sets

$$P(a) = \{\mathfrak{p} \in \text{Spec}(L) \mid a \in \mathfrak{p}\}$$

for  $a$  ranging over  $L$ . Stone’s duality asserts that if  $L$  is distributive (which is always the case in this paper) the map  $a \mapsto P(a)$  is an isomorphism between  $L$  and the lattice of closed subsets  $S$  of  $\text{Spec}(L)$  such that the complement of  $S$  in  $\text{Spec}(L)$  is compact.

We call a lattice **noetherian** if it is isomorphic to the lattice of closed sets of a noetherian topological space. By Stone’s duality a lattice  $L$  is noetherian if and only if its spectrum is a noetherian topological space. In such a lattice every filter is principal and every element  $a$  writes uniquely as the join of its (finitely many)  **$\vee$ -irreducible components**, which are the maximal elements in the set of  $\vee$ -irreducible<sup>5</sup> elements of  $L$  smaller than  $a$ . We denote by  $\mathcal{I}(L)$  the set of all  $\vee$ -irreducible elements of  $L$ .

We define the **dimension of an element  $a$  in a lattice  $L$** , denoted  $\dim_L a$ , as the least upper bound (in  $\mathbf{N} \cup \{-\infty, +\infty\}$ ) of the set of non-negative integers  $n$  such that

$$\exists \mathfrak{p}_0 \subset \cdots \subset \mathfrak{p}_n \in P(a).$$

This is nothing but the ordinary topological or Krull dimension (defined by chains of irreducible closed subsets) of the spectral space  $P(a)$ . By construction  $\dim_L \mathbf{0} = -\infty$  and  $\dim_L a \vee b = \max(\dim_L a, \dim_L b)$ . The index  $L$  is necessary since  $\dim_L a$  is not preserved by  $\mathcal{L}_{\text{lat}}$ -embeddings, but we will omit it whenever the ambient lattice is clear from the context. We let the **dimension of  $L$**  be the dimension of  $\mathbf{1}_L$  in  $L$ .

In all the geometric examples given in the introduction, a set  $A$  is pure dimensional if and only if  $\dim U = \dim A$  for every non-empty definable subset  $U$  of  $A$  which is open in  $A$ . This

<sup>5</sup>An element  $c$  of a lattice  $L$  is  $\vee$ -irreducible if it is *non-zero* and if  $a \vee b = c$  implies  $a = c$  or  $b = c$ .

motivates the next definition: given an integer  $k$  we say that an element  $a$  of a distributive lattice  $L$  is  **$k$ -pure in  $L$**  if and only if

$$\forall b \in L (a - b \neq \mathbf{0} \Rightarrow \dim_L a - b = k).$$

Then either  $a = \mathbf{0}$  or  $\dim_L a = k$ . In the latter case we say that  $a$  **has pure dimension  $k$  in  $L$** .

**Fact 2.2** *If  $L = L(X)$  is any of the lattices of the introduction, then for any  $A \in L(X)$ ,  $\dim_{L(X)} A$  is exactly the usual dimension of  $A$  as a semi-algebraic set over  $K$ , and  $A$  is pure dimensional in  $L(X)$  if and only if it so in the geometric sense.*

The second assertion in Fact 2.2 follows from the first, which itself follows from Fact 2.4 below.

### 2.3 Co-Heyting algebras

We let  $\mathcal{L}_{\text{TC}} = \mathcal{L}_{\text{lat}} \cup \{-\}$  with ‘ $-$ ’ a binary function symbol. An  $\mathcal{L}_{\text{TC}}$ -structure  $L$  is a **co-Heyting algebra** if its  $\mathcal{L}_{\text{lat}}$ -reduct is a lattice and if every element  $b$  has in  $L$  a **topological complement** relatively to every element, denoted  $a - b$ . By definition  $a - b$  is the least element  $c$  such that  $a \leq b \vee c$ . Equivalently  $P(a - b)$  is the topological closure of the relative complement  $P(a) \setminus P(b)$ , so the name of  $a - b$ . Reversing the order of a co-Heyting Heyting algebra  $L$  gives a Heyting algebra  $L^*$ , with  $b \rightarrow a$  in  $L^*$  corresponding to  $a - b$  in  $L$ , and every co-Heyting algebra is of this form. From the theory of Heyting algebras (see for example [Joh82]) we know that every co-Heyting algebra is distributive and that the class of all co-Heyting algebras is a variety (in the sense of universal algebra). Observe that in co-Heyting algebras the  $\ll$  relation is quantifier-free definable since

$$b \ll a \iff b \leq a - b.$$

So it will be preserved by  $\mathcal{L}_{\text{TC}}$ -embeddings. On the other hand the dimension will not be preserved in general by  $\mathcal{L}_{\text{TC}}$ -embeddings.

We will use the following rules, the proof of which are elementary exercises (using either Stone’s duality or corresponding properties of Heyting algebras).

**TC<sub>1</sub>**:  $a = (a \wedge b) \vee (a - b)$ .

In particular if  $a$  is  $\vee$ -irreducible then  $b < a \implies b \ll a$ .

**TC<sub>2</sub>**:  $(a_1 \vee a_2) - b = (a_1 - b) \vee (a_2 - b)$ .

**TC<sub>3</sub>**:  $(a - b) - b = a - b$ .

In particular  $(a - b) \wedge b \ll a - b \leq a$ .

**TC<sub>4</sub>**: More generally  $a - (b_1 \vee b_2) = (a - b_1) - b_2$ .

So if  $a - b_1 = a$  then  $a - (b_1 \vee b_2) = a - b_2$ .

**Fact 2.4 (Theorem 3.8 in [DJ11])** *For every element  $a$  in a co-Heyting algebra  $L$ ,  $\dim_L a$  is the least upper bound of the set of positive integers  $n$  such that there exists  $a_0, \dots, a_n \in L$  such that*

$$\mathbf{0} \neq a_0 \ll a_1 \ll \dots \ll a_n \leq a.$$

There is a well-established duality between (co-)Heyting algebras and so-called Esakia spaces with p-morphisms, from which we will pick up Fact 2.5 below. We first need a notation. Given an element  $x$  in a poset  $I$  and a subset  $X$  of  $I$  let

$$x^\downarrow = \{y \in I \mid y \leq x\} \qquad X^\downarrow = \bigcup_{x \in I} x^\downarrow.$$

The dual notation  $x^\uparrow$  and  $X^\uparrow$  is defined accordingly. The family  $\mathcal{D}^\downarrow(I)$  of decreasing subsets of  $I$  (that is the sets  $X \subseteq I$  such that  $X^\downarrow = X$ ) are the closed sets of a topology on  $I$ , hence a co-Heyting algebra with respect to the following operations.

$$X \vee Y = X \cup Y \qquad X \wedge Y = X \cap Y \qquad X - Y = (X \setminus Y)^\downarrow$$

The  $\vee$ -irreducible elements of  $\mathcal{D}^\downarrow(I)$  are precisely the sets  $x^\downarrow$  for  $x \in I$ .

**Fact 2.5** *Let  $L$  be a finite co-Heyting algebra and  $\mathcal{I}$  an ordered set. Assume that there is a surjective increasing map  $\pi : \mathcal{I} \rightarrow \mathcal{I}(L)$  such that  $\pi(x^\uparrow) \subseteq \pi(x)^\uparrow$  for every  $x \in \mathcal{I}(L)$ . Then there exists an  $\mathcal{L}_{\text{TC}}$ -embedding  $\varphi$  of  $L$  into  $\mathcal{D}^\downarrow(\mathcal{I})$  such that<sup>6</sup>  $\pi(\varphi(a)) = a^\downarrow \cap \mathcal{I}(L)$  for every  $a \in L$ .*

## 2.6 (Sub)scaled lattices.

Recall that  $\mathcal{L}_{\text{SC}} = \mathcal{L}_{\text{lat}} \cup \{-, C^i\}_{i \in \mathbf{N}} = \mathcal{L}_{\text{TC}} \cup \{C^i\}_{i \in \mathbf{N}}$  where  $\{C^i\}_{i \in \mathbf{N}}$  is a family of new unary function symbols. With the examples of the introduction in mind, define the **sc-dimension** of a non-zero element  $a$  of an  $\mathcal{L}_{\text{SC}}$ -structure  $L$  as

$$\text{sc-dim } a = \min\{k \in \mathbf{N} \mid a = \bigvee_{0 \leq i \leq k} C^i(a)\}.$$

Of course this is defined only if  $\text{sc-dim } a = \bigvee_{0 \leq i \leq k} C^i(a)$  for some  $k$ . If it is not defined we let  $\text{sc-dim } a = +\infty$ , and by convention  $\text{sc-dim } \mathbf{0} = -\infty$ . The **sc-dimension of  $L$** , denoted  $\text{sc-dim}(L)$ , is the sc-dimension of  $\mathbf{1}_L$ . In general the dimension of an element in a co-Heyting algebra is not preserved by  $\mathcal{L}_{\text{TC}}$ -embeddings. At the contrary the sc-dimension of an element is obviously preserved by  $\mathcal{L}_{\text{SC}}$ -embeddings, and this is the “raison d’être” of this structure.

A  **$d$ -subscaled lattice** is an  $\mathcal{L}_{\text{SC}}$ -structure whose  $\mathcal{L}_{\text{TC}}$ -reduct is a co-Heyting algebra and which satisfies the following list of axioms:

$$\mathbf{SC}_1^d: \bigvee_{0 \leq i \leq d} C^i(a) = a \quad \text{and} \quad \forall i > d, C^i(a) = \mathbf{0}.$$

$$\mathbf{SC}_2: \forall I \subseteq \{0, \dots, d\}, \forall k:$$

$$C^k\left(\bigvee_{i \in I} C^i(a)\right) = \begin{cases} \mathbf{0} & \text{if } k \notin I \\ C^k(a) & \text{if } k \in I \end{cases}$$

$$\mathbf{SC}_3: \forall k \geq \max(\text{sc-dim}(a), \text{sc-dim}(b)), \quad C^k(a \vee b) = C^k(a) \vee C^k(b)$$

$$\mathbf{SC}_4: \forall i \neq j, \quad \text{sc-dim}(C^i(a) \wedge C^j(a)) < \min(i, j)$$

$$\mathbf{SC}_5: \forall k \geq \text{sc-dim}(b), \quad C^k(a) - b = C^k(a) - C^k(b).$$

$$\mathbf{SC}_6: \text{If } b \ll a \text{ then } \text{sc-dim } b < \text{sc-dim } a.$$

It is a  **$d$ -scaled lattice** if it satisfies in addition the following property:

$$\mathbf{SC}_0: \text{sc-dim } a = \dim a$$

All the geometric  $\mathcal{L}_{\text{SC}}$ -structures in  $\text{SC}_{\text{def}}(\mathcal{K}, d)$  or  $\text{SC}_{\text{Zar}}(K, d)$  (defined after Example 1.1) are  $d$ -scaled lattices (see Fact 2.2). However  $\text{SC}_0$  does not follow from the other axioms as the following example shows.

<sup>6</sup>Note that the compositum  $\pi \circ \varphi$  is not defined. In this proposition  $\varphi(a)$  is a decreasing subset of  $\mathcal{I}$  and  $\pi(\varphi(a)) = \{\pi(\xi) \mid \xi \in \varphi(a)\}$ .

**Example 2.7** Let  $L$  be an arbitrary noetherian lattice, and  $D: \mathcal{I}(L) \rightarrow \{0, \dots, d\}$  be a strictly increasing map. For every  $a, b \in L$ , if  $\mathcal{C}(a)$  denotes the set of all  $\vee$ -irreducible components of  $a$ , let

$$\begin{aligned} a - b &= \mathbb{W}\{c \in \mathcal{C}(a) / c \not\leq b\}, \\ (\forall k) \quad \mathbf{C}_D^k(a) &= \mathbb{W}\{c \in \mathcal{C}(a) / D(c) = k\}. \end{aligned}$$

This is a typical example of a  $d$ -subscaled lattice in which the sc-dimension does not coincide with the dimension, except of course if  $D(a) = \dim_L a$  for every  $a \in \mathcal{I}(L)$ . Conversely, every noetherian (in particular every finite)  $d$ -subscaled lattice is of that kind.

We call **(sub)scaled lattices** the  $\mathcal{L}_{\text{SC}}$ -structures whose  $\mathcal{L}_{\text{SC}}$ -reduct is a  $d$ -(sub)scaled for some  $d \in \mathbf{N}$ . Of course this is not an elementary class. At the contrary, for any fixed  $d \in \mathbf{N}$ ,  $\text{SC}_1^d$  to  $\text{SC}_5$  are expressible by equations,  $\text{SC}_6$  by a universal formula and  $\text{SC}_0$  by a first order formula in  $\mathcal{L}_{\text{SC}}$ , hence  $d$ -scaled (resp.  $d$ -subscaled) lattices form elementary class. As the terminology suggests, we will see that  $d$ -subscaled lattices are precisely the  $\mathcal{L}_{\text{SC}}$ -substructures of  $d$ -scaled lattices.

By analogy with our guiding geometric examples, we say that an element  $a$  in a  $d$ -subscaled lattice is  **$k$ -sc-pure** if

$$\forall b \in L \ (a - b \neq \mathbf{0} \Rightarrow \text{sc-dim}(a - b) = k).$$

We will see that  $a$  is  $k$ -sc-pure if and only if  $a = \mathbf{C}^k(a)$  (this is  $\text{SC}_{13}$  in Section 3). Then either  $a = \mathbf{0}$  or  $\text{sc-dim } a = k$ . In the latter case we say that  $a$  **has pure sc-dimension  $k$** . For any  $a$ , the element  $\mathbf{C}^k(a)$  is called the  **$k$ -sc-pure component** of  $a$ , or simply its  **$k$ -pure component** if  $L$  is a scaled lattice. By construction these notions coincide with their geometric counterparts in  $\text{SC}_{\text{def}}(\mathcal{K}, d)$  and  $\text{SC}_{\text{Zar}}(K, d)$ .

The following notation will be convenient in induction arguments. If  $\mathcal{L}$  is any of our languages  $\mathcal{L}_{\text{lat}}$ ,  $\mathcal{L}_{\text{TC}}$  or  $\mathcal{L}_{\text{SC}}$  we let  $\mathcal{L}^* = \mathcal{L} \setminus \{\mathbf{1}\}$ . Given an  $\mathcal{L}$ -structure  $L$  whose reduct to  $\mathcal{L}_{\text{lat}}$  is a lattice, for any  $a \in L$  we let

$$L(a) = \{b \in L / b \leq a\}.$$

$L(a)$  is a typical example of  $\mathcal{L}^*$ -substructure of  $L$ .

### 3 Basic properties and embeddings

The next properties follow easily from the axioms of  $d$ -subscaled lattices.

**SC<sub>7</sub>**:  $\text{sc-dim } a = \max\{k / \mathbf{C}^k(a) \neq \mathbf{0}\}$ .

In particular  $\forall k, \text{sc-dim } \mathbf{C}^k(a) = k \iff \mathbf{C}^k(a) \neq \mathbf{0}$ .

**SC<sub>8</sub>**:  $\forall k \geq \text{sc-dim}(a), \text{sc-dim}(b \wedge a) < k \implies \mathbf{C}^k(a) - b = \mathbf{C}^k(a)$ .

**SC<sub>9</sub>**:  $\text{sc-dim } a \vee b = \max(\text{sc-dim } a, \text{sc-dim } b)$ .

In particular  $b \leq a \implies \text{sc-dim } b \leq \text{sc-dim } a$ .

**SC<sub>10</sub>**:  $\forall k \geq \text{sc-dim}(a), \mathbf{C}^k(a)$  is the largest  $k$ -sc-pure element lower than  $a$ .

**SC<sub>11</sub>**:  $\dim a \leq \text{sc-dim } a$ .

**SC<sub>12</sub>**:  $\forall I \subseteq \{0, \dots, d\}, \quad a - \mathbb{W}_{i \in I} \mathbf{C}^i(a) = \mathbb{W}_{i \notin I} \mathbf{C}^i(a)$ .

In particular  $\text{sc-dim} \left( a - \mathbb{W}_{i \geq k} \mathbf{C}^i(a) \right) < k$ .



**SC<sub>13</sub>**:  $\forall k, C^k(a) = a \iff \forall b (a - b \neq \mathbf{0} \Rightarrow \text{sc-dim } a - b = k)$ .

That is  $a$  is  $k$ -sc-pure if and only if  $a = C^k(a)$ .

In particular if  $a$  is  $\vee$ -irreducible then  $a$  is sc-pure by SC<sub>1</sub><sup>d</sup>.

*Proof:* (Sketch) SC<sub>7</sub> follows from SC<sub>1</sub><sup>d</sup> and SC<sub>2</sub>; SC<sub>8</sub> from SC<sub>5</sub> and SC<sub>7</sub>; SC<sub>9</sub> from SC<sub>2</sub>, SC<sub>3</sub> and SC<sub>7</sub>; SC<sub>10</sub> from SC<sub>2</sub> and SC<sub>3</sub>; SC<sub>11</sub> from SC<sub>6</sub> by Fact 2.4. Only the two last properties require a little effort.

**SC<sub>12</sub>**: For every  $l \in I$ ,  $C^l(a) \leq \bigvee_{i \in I} C^i(a)$  hence  $C^l(a) - \bigvee_{i \in I} C^i(a) = \mathbf{0}$ . On the other hand for every  $l \notin I$  and every  $i \in I$ ,  $C^l(a) - C^i(a) = C^l(a)$  by SC<sub>4</sub> and SC<sub>5</sub>. So  $C^l(a) - \bigvee_{i \in I} C^i(a) = C^l(a)$  by TC<sub>4</sub>. Finally by SC<sub>1</sub><sup>d</sup> and TC<sub>2</sub>,

$$a - \bigvee_{i \in I} C^i(a) = \bigvee_{l \leq d} (C^l(a) - \bigvee_{i \in I} C^i(a)) = \bigvee_{l \notin I} C^l(a).$$

**SC<sub>13</sub>**: Assume that  $a = C^k(a)$  and  $a - b \neq \mathbf{0}$  for some  $b$ . Then  $\text{sc-dim } a = k$  so  $\text{sc-dim}(a - b) \leq k$  and  $\text{sc-dim}(a \wedge b) \leq k$  by SC<sub>9</sub>. Since  $a = (a - b) \vee (a \wedge b)$  it follows by SC<sub>3</sub> that

$$C^k(a) = C^k((a - b) \vee (a \wedge b)) = C^k(a - b) \vee C^k(a \wedge b).$$

By assumption  $C^k(a) = a$ , and  $C^k(a \wedge b) \leq a \wedge b$  by SC<sub>1</sub><sup>d</sup>. So  $a \leq C^k(a - b) \vee (a \wedge b) \leq C^k(a - b) \vee b$  which implies that  $a - b \leq C^k(a - b)$ . In particular  $C^k(a - b) \neq \mathbf{0}$ . Since  $\text{sc-dim}(a - b) \leq k$  it follows that  $\text{sc-dim}(a - b) = k$  by SC<sub>7</sub>.

Conversely assume that  $a \neq C^k(a)$  (hence  $a \neq \mathbf{0}$ ). For  $b = C^k(a)$  we then have  $a - b \neq \mathbf{0}$  on one hand and  $\text{sc-dim}(a - b) \neq k$  by SC<sub>7</sub> on the other hand, because  $C^k(a - b) = \mathbf{0}$  by SC<sub>12</sub> and SC<sub>2</sub>.

■

**Proposition 3.1** *The  $\mathcal{L}_{\text{SC}}$ -structure of a  $d$ -scaled lattice  $L$  is uniformly definable in the  $\mathcal{L}_{\text{lat}}$ -structure of  $L$ . In particular it is uniquely determined by this  $\mathcal{L}_{\text{lat}}$ -structure.*

*Proof:* Clearly the  $\mathcal{L}_{\text{TC}}$ -structure is an extension by definition of the lattice structure of  $L$ . For every positive integer  $k$  the class of  $k$ -pure elements is uniformly definable, using the definability of  $\ll$  and Fact 2.4. Then so is the function  $C^k$  for every  $k$ , by decreasing induction on  $k$ . Indeed by SC<sub>10</sub> and SC<sub>12</sub>,  $C^k(a)$  is the largest  $k$ -pure element  $c$  such that  $c \leq a - \bigvee_{i > k} C^i(a)$ .

■

We need a reasonably easy criterion for an  $\mathcal{L}_{\text{lat}}$ -embedding of subscaled lattices to be an  $\mathcal{L}_{\text{SC}}$ -embedding. In the special case of a noetherian<sup>7</sup> embedded lattice, it is given by Proposition 3.3 below, whose proof will use the next characterisation of sc-pure components.

**Proposition 3.2** *Let  $L$  be a subscaled lattice and  $a, a_0, \dots, a_d \in L$  be such that  $a = \bigvee_{i \leq d} a_i$ , each  $a_i$  is  $i$ -sc-pure and  $\text{sc-dim}(a_i \wedge a_j) < \min(i, j)$  for every  $i \neq j$ . Then  $C^i(a) = a_i$  for every  $i$ .*

*Proof:* Note first that  $C^k(a_k) = a_k$  for every  $k \leq d$  by SC<sub>13</sub>. Hence for every  $k \leq i \leq d$  we have by SC<sub>3</sub> and SC<sub>2</sub>

$$C^i\left(\bigvee_{k \leq i} a_k\right) = \bigvee_{k \leq i} C^i(a_k) = \bigvee_{k \leq i} C^i(C^k(a_k)) = C^i(a_i) = a_i. \quad (1)$$

<sup>7</sup>Although we won't use it, let us mention that in the general case of an  $\mathcal{L}_{\text{lat}}$ -embedding  $\varphi : L \rightarrow L'$  between arbitrary subscaled lattices, one may easily derive from Proposition 3.3, by means of the Local Finiteness Theorem 4.1 and the model-theoretic compactness theorem, that  $\varphi$  is an  $\mathcal{L}_{\text{SC}}$ -embedding if and only if it preserves sc-dimension and sc-purity, that is for every  $a \in L$  and every  $k \in \mathbf{N}$ ,  $C^k(a) = a \Rightarrow C^k(\varphi(a)) = \varphi(a)$ .

In particular  $C^d(a) = a_d$ . Now assume that for some  $i < d$  we have proved that  $C^j(a) = a_j$  for  $i < j \leq d$ . By SC<sub>12</sub> and SC<sub>2</sub> we then have

$$C^i(a - \bigvee_{j>i} a_j) = C^i(a - \bigvee_{j>i} C^j(a)) = C^i(\bigvee_{j \leq i} C^j(a)) = C^i(a). \quad (2)$$

On the other hand  $a - \bigvee_{j>i} a_j = \bigvee_{k \leq d} (a_k - \bigvee_{j>i} a_j)$  by TC<sub>2</sub>. For  $k > i$  obviously  $a_k - \bigvee_{j>i} a_j = \mathbf{0}$ . For  $k < i$ ,  $a_k = C^k(a)$  and  $a_j = C^j(a_j)$  imply that  $\text{sc-dim}(a_k \wedge a_j) < k$  for  $j > i$  by SC<sub>4</sub>. Hence  $a_k - a_j = a_k$  by SC<sub>8</sub> and finally  $a_k - \bigvee_{j>i} a_j = a_k$  by TC<sub>4</sub>, so

$$a - \bigvee_{j>i} a_j = \bigvee_{k \leq i} a_k. \quad (3)$$

By (1), (2), (3) we conclude that  $C^i(a) = a_i$ . The result follows for every  $i$  by decreasing induction.

■

**Proposition 3.3** *Let  $L_0$  be a noetherian subscaled lattice,  $L$  a subscaled lattice, and  $\varphi : L_0 \rightarrow L$  an  $\mathcal{L}_{\text{lat}}$ -embedding such that for every  $a \in \mathcal{I}(L_0)$ ,  $\varphi(a)$  is sc-pure and has the same sc-dimension as  $a$ . Then  $\varphi$  is an  $\mathcal{L}_{\text{SC}}$ -embedding.*

**Remark 3.4** Clearly the same statement remains true with  $\mathcal{L}_{\text{lat}}$  and  $\mathcal{L}_{\text{SC}}$  replaced respectively by  $\mathcal{L}_{\text{lat}}^*$  and  $\mathcal{L}_{\text{SC}}^*$ . We will freely use these variants.

*Proof:*  $L_0$  and  $L$  are  $d$ -subscaled lattices for some  $d \in \mathbf{N}$ . Given  $a \in L_0$  and  $k$  a non-negative integer, we first check that  $\varphi(C^k(a))$  is  $k$ -sc-pure. Note that every  $\vee$ -irreducible component  $c$  of  $C^k(a)$  in  $L_0$  has sc-pure dimension  $k$ . Indeed  $C^k(a)$  is  $k$ -sc-pure by SC<sub>13</sub>, and  $c = C^k(a) - b \neq \mathbf{0}$  where  $b$  is the join of all the other  $\vee$ -irreducible components of  $C^k(a)$ , hence  $\text{sc-dim}(c) = \text{sc-dim}(C^k(a) - b) = k$ . Moreover  $c$  is sc-pure because it is  $\vee$ -irreducible, hence  $c$  is  $k$ -pure. By our assumption on  $\varphi$  it follows that  $\varphi(c)$  is  $k$ -sc-pure. Every finite union of  $k$ -sc-pure elements being  $k$ -sc-pure by SC<sub>3</sub>, it follows that

$$\varphi(C^k(a)) \text{ is } k\text{-sc-pure.} \quad (4)$$

Now for every  $l \neq k$  we have  $\text{sc-dim}(C^k(a) \wedge C^l(a)) < \min(k, l)$  by SC<sub>4</sub>. It follows that each  $\vee$ -irreducible component  $c$  of  $C^k(a) \wedge C^l(a)$  has sc-dimension strictly less than  $\min(k, l)$ , hence so does  $\varphi(c)$  by assumption. By SC<sub>9</sub> we conclude that

$$\text{sc-dim}(C^k(a) \wedge C^l(a)) < \min(k, l) \quad (\forall l \neq k). \quad (5)$$

$\varphi(a) = \bigvee_{k \leq d} \varphi(C^k(a))$  by SC<sub>1</sub><sup>d</sup> and because  $\varphi$  is an  $\mathcal{L}_{\text{lat}}$ -embedding. By (4), (5) and Proposition 3.2 it follows that  $C^k(\varphi(a)) = \varphi(C^k(a))$  for every  $k \leq d$ . Since  $\varphi$  is injective, this implies by SC<sub>7</sub> that for every  $a \in L_0$

$$\text{sc-dim } a = \text{sc-dim } \varphi(a). \quad (6)$$

It only remains to check that  $\varphi(a - b) = \varphi(a) - \varphi(b)$  for every  $a, b \in L_0$ . By TC<sub>2</sub>, replacing if necessary  $a$  by its  $\vee$ -irreducible components, we may assume w.l.o.g. that  $a$  itself is  $\vee$ -irreducible in  $L_0$ . This implies that  $a = C^k(a)$  for some  $k$ , hence  $\varphi(a)$  is  $k$ -sc-pure by assumption on  $\varphi$ . It then remains two possibilities for  $a - b$ :

- If  $b \geq a$  then  $\varphi(b) \geq \varphi(a)$ , hence  $a - b = \mathbf{0}$  and  $\varphi(a) - \varphi(b) = \mathbf{0}$ , so  $\varphi(a - b) = \varphi(\mathbf{0}) = \mathbf{0} = \varphi(a) - \varphi(b)$ .

- Otherwise  $b \wedge a < a$  hence  $b \wedge a \ll a$  by  $\text{TC}_1$ , that is  $a - b = a$ . So we have to prove that  $\varphi(a) - \varphi(b) = \varphi(a)$ . By  $\text{SC}_6$   $\text{sc-dim } b \wedge a < \text{sc-dim } a$ , hence  $\text{sc-dim}(\varphi(b) \wedge \varphi(a)) < \text{sc-dim}(\varphi(a)) = k$  by (6). Since  $\varphi(a)$  is  $k$ -sc-pure it follows that  $\varphi(a) - \varphi(b) = \varphi(a)$  by  $\text{SC}_5$ .

■

**Corollary 3.5** *Let  $L_0$  be a noetherian lattice embedded in a subscaled lattice  $L$ . Assume that every  $b' < b \in \mathcal{I}(L_0)$  are sc-pure in  $L$  and  $\text{sc-dim } b' < \text{sc-dim } b$  in  $L$ . Then the restrictions to  $L_0$  of the  $\mathcal{L}_{\text{SC}}$ -operations “ $-$ ” and “ $C_i$ ” of  $L$  turn  $L_0$  into an  $\mathcal{L}_{\text{SC}}$ -substructure which is a subscaled lattice.*

*Proof:* The assumptions imply that the map  $D : a \mapsto \text{sc-dim } a$  is a strictly increasing map from  $\mathcal{I}(L_0)$  to  $\mathbf{N}$ . Endow  $L_0$  with the structure of subscaled lattice determined by  $D$  as in Example 2.7. By construction the inclusion map from  $L_0$  to  $L$  is an  $\mathcal{L}_{\text{lat}}$ -embedding which preserves the sc-purity and sc-dimension of every  $b \in \mathcal{I}(L_0)$ , hence is an  $\mathcal{L}_{\text{SC}}$ -embedding by Proposition 3.3.

■

## 4 Local finiteness

We prove in this section that every finitely generated subscaled lattice is finite. This result is far non-obvious, due to the lack of any known normal form for terms in  $\mathcal{L}_{\text{SC}}$ . It contrasts with the general situation in co-Heyting algebras, which can be both infinite and generated by a single element. Our main ingredient, which explains this difference, is the uniform bound given *a priori* for the sc-dimension of any element in a given  $d$ -subscaled lattice.

**Theorem 4.1** *Any  $d$ -subscaled lattice  $L$  generated by  $n$  elements is finite. More precisely, the cardinality of  $\mathcal{I}(L)$  is bounded by the function  $\mu(n, d)$  defined by*

$$\mu(n, d) = 2^n + \mu(2^{n+1}, d - 1).$$

for  $d \geq 0$ , and  $\mu(n, d) = 0$  for  $d < 0$ .

*Proof:* The only subscaled lattice of sc-dimension  $d < 0$  is the one-element lattice  $\{\mathbf{0}\}$ , so the result is trivial in this case. Assume that  $d \geq 0$  and that the result is proved for every  $d' < d$  and every non-negative integer  $n$ .

Let  $L$  be a subscaled lattice of sc-dimension  $d$  generated by elements  $x_1, \dots, x_n$ . Let  $\Omega_n$  be the family of all subsets of  $\{1, \dots, n\}$  (so  $\Omega_0 = \{\emptyset\}$ ). For every  $I \in \Omega_n$  let  $I^c = \Omega_n \setminus I$  and

$$y_I = \left( \bigwedge_{i \in I} x_i \right) - \left( \bigvee_{i \in I^c} x_i \right), \quad z_I = C^d(y_I).$$

The family of all  $\mathcal{Y}_I = \bigcap_{i \in I} P(x_i) \cap \bigcap_{i \in I^c} P(x_i)^c$  is a partition of  $\text{Spec}(L)$ . Indeed the  $\mathcal{Y}_i$ 's are the atoms of the boolean algebra generated in the power set  $\mathcal{P}(\text{Spec}(L))$  by the  $P(x_i)$ 's. Moreover each  $P(y_I)$  is the topological closure  $\overline{\mathcal{Y}}_I$  of  $\mathcal{Y}_I$  in  $\text{Spec}(L)$  hence for every  $x \in L$

$$P(x) = \bigcup_{I \in \Omega_n} P(x) \cap \mathcal{Y}_I \subseteq \bigcup_{I \in \Omega_n} P(x) \cap \overline{\mathcal{Y}}_I = P\left( \bigvee_{I \in \Omega_n} x \wedge y_I \right).$$

So  $x \leq \bigvee_{I \in \Omega_n} (x \wedge y_I)$  by Stone's duality. The reverse inequality being obvious we have proved that

$$\forall x \in L, \quad x = \bigvee_{I \in \Omega_n} (x \wedge y_I). \quad (7)$$

In particular  $\text{SC}_3$  also gives

$$C^d(\mathbf{1}) = C^d\left(\bigvee_{I \in \Omega_n} y_I\right) = \bigvee_{I \in \Omega_n} z_I. \quad (8)$$

For every  $I \neq J \in \Omega_n$ , if for example  $I \not\subseteq J$  choose any  $i \in I \setminus J$  and observe that  $y_I \leq x_i$  and  $y_J \leq \mathbf{1} - x_i$  so  $y_I \wedge y_J \ll \mathbf{1} - x_i$  by  $\text{TC}_3$ . By  $\text{SC}_6$  and the  $d$ -sc-purity of the  $z_I$ 's it follows that

$$\text{sc-dim } z_I \wedge z_J < d \quad \text{hence} \quad z_I - z_J = z_I. \quad (9)$$

It follows from  $\text{SC}_9$ ,  $\text{SC}_{12}$  and (9) above, that the element

$$a = (\mathbf{1} - C^d(\mathbf{1})) \vee \left(\bigvee_{I \in \Omega_n} (y_I - z_I)\right) \vee \left(\bigvee_{I \neq J \in \Omega_n} (z_I \wedge z_J)\right)$$

has sc-dimension strictly smaller than  $d$ . So the induction hypothesis applies to the  $\mathcal{L}_{\text{SC}}$ -substructure  $L_{\bar{0}}$  of  $L(a)$  generated by the  $(y_I - z_I)$ 's and the  $(z_I \wedge a)$ 's:  $L_{\bar{0}}$  is finite, with at most  $\mu(2|\Omega_n|, d-1)$   $\vee$ -irreducible elements. Note that  $L_{\bar{0}}$  is an  $\mathcal{L}_{\text{SC}}^*$ -substructure of  $L$  (recall that  $\mathcal{L}_{\text{SC}}^* = \mathcal{L}_{\text{SC}} \setminus \{\mathbf{1}\}$ ). Finally let  $L_1$  be the upper semi-lattice generated in  $L$  by  $L_{\bar{0}} \cup \{z_I\}_{I \in \Omega_n}$ . By construction  $L_1$  is finite and  $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_{\bar{0}}) \cup \{z_I\}_{I \in \Omega_n}$ , so  $|\mathcal{I}(L_1)| \leq 2^n + \mu(2^{n+1}, d-1) = \mu(n, d)$ . It is then sufficient to show that  $L_1 = L$ .

We first prove that  $L_1$  is a lattice. By (8),  $C^d(\mathbf{1}) \vee a = \bigvee_{I \in \Omega_n} z_I \vee a \in L_1$  hence  $\mathbf{1} = C^d(\mathbf{1}) \vee a \in L_1$ . For every  $I \in \Omega_n$  and every  $b \in L_{\bar{0}}$ ,  $z_I \wedge b = (z_I \wedge a) \wedge b \in L_{\bar{0}}$ . For every  $I \neq J \in \Omega_n$ ,  $z_I \wedge z_J = (z_I \wedge a) \wedge (z_J \wedge a) \in L_{\bar{0}}$ . So by the distributivity law,  $L_1$  is a sublattice of  $L$ .

In order to conclude that  $L_1$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $L$ , by Corollary 3.5 it only remains to check that for every  $b' < b \in \mathcal{I}(L_1)$ ,  $b$  is sc-pure in  $L$  and  $\text{sc-dim } b' < \text{sc-dim } b$  in  $L$ . Since  $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_{\bar{0}}) \cup \{z_I\}_{I \in \Omega_n}$  we can distinguish two cases.

*Case 1:*  $b \in \mathcal{I}(L_{\bar{0}})$ . Then  $b$  is sc-pure in  $L_{\bar{0}}$  by  $\text{SC}_{13}$  hence also in  $L$  since  $L_{\bar{0}}$  is an  $\mathcal{L}_{\text{SC}}^*$ -substructure of  $L$ . Similarly  $b' \ll b$  in  $L_{\bar{0}}$  by  $\text{TC}_1$  that is  $b - b' = b$  in  $L_{\bar{0}}$  hence also in  $L$ . Thus  $b' \ll b$  in  $L$  which implies that  $\text{sc-dim } b' < \text{sc-dim } b$  in  $L$  by  $\text{SC}_6$ .

*Case 2:*  $b = z_I$  for some  $I \in \Omega_n$ . Then  $b = C^d(y_I)$  is sc-pure in  $L$  and  $\text{sc-dim } b = d$ . If  $b' = z_J$  for some other  $J \in \Omega_n$  then on one hand  $\text{sc-dim}(b') = d$  and on the other hand  $I \neq J$  hence  $b' = b' \wedge b = z_I \wedge z_J$  has sc-dimension  $< d$  by (9), a contradiction. So necessarily  $b' \in \mathcal{I}(L_{\bar{0}})$ , in particular  $b' \leq a$  hence  $\text{sc-dim}(b') \leq \text{sc-dim}(a) < d$ .

So  $L_1$  is indeed an  $\mathcal{L}_{\text{SC}}$ -substructure of  $L$ . Finally every  $y_I = (y_I - z_I) \vee z_I \in L_1$  and (7) gives, for every  $i \leq n$ ,

$$x_i = \bigvee_{I \in \Omega_n} x_i \wedge y_I \leq \bigvee_{\substack{I \in \Omega_n \\ i \in I}} y_I \leq x_i.$$

So equality holds, hence each  $x_i \in L_1$ , which finally proves that  $L_1 = L$ .

■

**Corollary 4.2** *For every  $n, d$  there are finitely many non-isomorphic subscaled lattices of sc-dimension  $d$  generated by  $n$  elements.*

*Proof:* Any such subscaled lattice  $L$  is finite, with  $|\mathcal{I}(L)| \leq \mu(n, d)$  by Theorem 4.1. Clearly there are finitely many non-isomorphic lattices such that  $|\mathcal{I}(L)| \leq \mu(n, d)$  and each of them admits finitely many non-isomorphic  $\mathcal{L}_{\text{SC}}$ -structures of  $d$ -subscaled lattices.

■

## 5 Linear representation

In this section we prove that the theory of  $d$ -subscaled lattices is the universal theory of various natural classes of geometric  $d$ -scaled lattices, including  $\text{SC}_{\text{def}}(\mathcal{K}, d)$  in Example 1.1 as well as  $\text{SC}_{\text{Zar}}(K, d)$ . The argument is based on an elementary representation theorem for  $d$ -subscaled lattices, combined with the local finiteness result of Section 4.

Given an arbitrary field  $K$ , a non-empty linear variety  $X \subseteq K^m$  is determined by the data of an arbitrary point  $P \in X$  and the vector subspace  $\vec{X}$  of  $K^m$ , *via* the relation  $X = P + \vec{X}$  (the orbit of  $P$  under the action of  $\vec{X}$  by translation). We call  $X$  a **special linear variety** (resp. a **special linear set**) if  $X$  is a linear variety such that  $\vec{X}$  is generated by a subset of the canonical basis of  $K^m$  (resp. if  $X$  is a finite union of special linear varieties). The family  $\text{L}_{\text{lin}}(X)$  of all special linear subsets of  $X$  is the family of closed sets of a noetherian topology on  $X$ , hence a noetherian lattice. For every  $A \in \text{L}_{\text{lin}}(X)$  we let  $D(A)$  be the dimension of  $A$  in the sense of linear algebra. This endows  $\text{L}_{\text{lin}}(X)$  with a natural structure of scaled lattice as in Example 2.7.

**Remark 5.1** For every  $A \in \text{L}_{\text{lin}}(X)$ , if  $K$  is infinite then  $\text{sc-dim } A = \dim_{\text{L}_{\text{lin}}(X)} A =$  the dimension of  $A$  as defined in linear algebra. It coincides with the Krull dimension as well. If moreover  $A$  is  $\vee$ -irreducible in  $\text{L}_{\text{lin}}(X)$  then it is pure dimensional, hence it is sc-pure both in  $\text{L}_{\text{lin}}(X)$  and  $\text{L}_{\text{Zar}}(X)$ . By Proposition 3.3 it follows that  $\text{L}_{\text{lin}}(X)$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $\text{L}_{\text{Zar}}(X)$ . Similarly if  $\mathcal{K}$  is an  $\mathcal{o}$ -minimal,  $C$ -minimal or  $P$ -minimal expansion of  $K$ , then  $\text{L}_{\text{lin}}(X)$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $\text{L}_{\text{def}}(X)$ .

In what follows  $K^m$  is identified to  $K^m \times \{0\}^r \subseteq K^{m+r}$ . The very easy result below prepares the proof of Proposition 5.3.

**Proposition 5.2** *For every two special linear sets  $C \subseteq B \subseteq K^m$  and every non-negative integer  $n \geq \dim C$  there exists a special linear set  $A \subseteq K^{m+n}$  of pure dimension  $n$  such that  $A \cap B = C$ .*

*Proof:* The result being rather trivial if  $C$  is empty, we can assume w.l.o.g. that  $C \neq \emptyset$ . Let  $(e_1, \dots, e_{m+n})$  be the canonical basis of  $K^{m+n}$ . If  $I$  is a subset of  $\{1, \dots, m+n\}$  we let  $\vec{E}(I)$  denote the vector space generated in  $K^n$  by  $(e_i)_{i \in I}$ . Decompose  $C$  as a union of special linear varieties  $C_1, \dots, C_p$ , and write each  $C_i = P_i + \vec{E}(J_i)$  with  $|J_i| = \dim C_i \leq n$ . Let  $I_i = J_i \cup \{m+1, \dots, m+n-|J_i|\}$  and  $A_i = P_i + \vec{E}(I_i)$  for every  $i \leq p$ . Finally let  $A = A_1 \cup \dots \cup A_p$ . By construction each  $A_i$  has pure dimension  $|I_i| = n$ , hence  $A$  has pure dimension  $n$ . Clearly each  $A_i \cap K^m = C_i$ , hence  $A \cap K^m = C$  and *a fortiori*  $A \cap B = C$ .  
■

**Proposition 5.3 (Linear representation)** *Let  $K$  be an infinite field,  $d \geq 0$  an integer and  $L$  a finite  $d$ -subscaled lattice. Then there exists a special linear set  $X$  over  $K$  of dimension  $\leq d$  and an  $\mathcal{L}_{\text{SC}}$ -embedding  $\varphi : L \rightarrow \text{L}_{\text{lin}}(X)$ .*

*Proof:* By induction on the number  $r$  of  $\vee$ -irreducible elements of an arbitrary  $d$ -subscaled lattice  $L$ , we prove that there exists an  $\mathcal{L}_{\text{SC}}^*$ -embedding  $\varphi$  of  $L$  into  $\text{L}_{\text{lin}}(K^m)$  for some  $m$  depending on  $L$ . Taking  $X = \varphi(\mathbf{1}_L)$  then gives the conclusion. Indeed  $X$  is a special linear set over  $K$ ,  $\dim X = \text{sc-dim}(\mathbf{1}_L) \leq d$  because  $\varphi$  preserves the sc-dimension, and  $\varphi$  is obviously an  $\mathcal{L}_{\text{SC}}$ -embedding of  $L$  into  $\text{L}_{\text{lin}}(X)$ .

If  $r = 0$  then  $L$  is the one-element lattice  $\{\mathbf{0}\}$ , hence an  $\mathcal{L}_{\text{SC}}^*$ -substructure of  $\text{L}_{\text{lin}}(K)$ . So, given a fixed  $r \geq 1$ , we can assume by induction that the result is proved for  $r - 1$ . Let  $L$  be a  $d$ -subscaled lattice with  $\vee$ -irreducible elements  $a_1, \dots, a_r$ .

Renumbering if necessary we may assume that  $a_r$  is maximal among the  $a_i$ 's. Let  $a = a_r$ ,  $b = \mathbb{W}_{1 \leq i < r} a_i$ ,  $c = a \wedge b$  and  $\varphi$  an  $\mathcal{L}_{\text{SC}}^*$ -embedding of  $L(b)$  into some  $L_{\text{lin}}(K^m)$  given by the induction hypothesis. Since  $a$  is  $\vee$ -irreducible in  $L$  it is sc-pure. Moreover  $c \leq a$  by  $\text{TC}_1$ , hence  $a$  has pure sc-dimension  $n$  for some  $n \geq \text{sc-dim}(c)$  by  $\text{SC}_6$ . Let  $B, C$  be the respective images of  $b, c$  by  $\varphi$ . Proposition 5.2 gives a special linear set  $A \subseteq K^{m+n}$  of pure dimension  $n$  such that  $A \cap B = C$ . Identifying  $K^m$  with  $K^m \times \{0\}^n \subseteq K^{m+n}$  turns  $\varphi$  into an  $\mathcal{L}_{\text{SC}}^*$ -embedding of  $L(b)$  into  $L_{\text{lin}}(K^{m+n})$ .

Every  $x \in L$  writes uniquely  $x_a \vee x_b$  with  $x_a \in \{\mathbf{0}, a\}$  and  $x_b \in L(b)$  by grouping appropriately the  $\vee$ -irreducible components of  $x$ , using the maximality of  $a_n$ . So we can let

$$\bar{\varphi}(x) = \begin{cases} \varphi(x_b) & \text{if } x_a = \mathbf{0}, \\ A \cup \varphi(x_b) & \text{if } x_a = a. \end{cases}$$

This is a well-defined  $\mathcal{L}_{\text{lat}}^*$ -embedding of  $L$  into  $L_{\text{lin}}(K^{m+n})$ . Moreover  $\bar{\varphi}$  is an  $\mathcal{L}_{\text{SC}}^*$ -embedding by Proposition 3.3. This finishes the induction.

■

Given an infinite field  $K$  and a positive integer  $d$ , we let  $\text{SC}_{\text{lin}}(K, d)$  be the class of  $d$ -scaled lattices  $L_{\text{lin}}(X)$  with  $X$  ranging over the special linear sets over  $K$  of dimension at most  $d$ .

**Theorem 5.4** *The universal theories of  $\text{SC}_{\text{def}}(\mathcal{K}, d)$  (resp.  $\text{SC}_{\text{Zar}}(K, d)$ ,  $\text{SC}_{\text{lin}}(K, d)$ ) in the language  $\mathcal{L}_{\text{SC}}$  are the same for every fixed integer  $d \geq 0$  and every  $o$ -minimal,  $C$ -minimal or  $P$ -minimal expansion<sup>8</sup>  $\mathcal{K}$  of a field  $K$  (resp. for every infinite field  $K$ ). This is the theory of  $d$ -subscaled lattices.*

*Proof:* As already mentioned, for every such expansion  $\mathcal{K}$  of  $K$  the good properties of the dimension theory for definable sets  $X \subseteq K^m$  ensure that  $L_{\text{def}}(X)$  is a  $d$ -scaled lattice. Obviously the same holds true for  $L_{\text{lin}}(X)$  and  $L_{\text{Zar}}(X)$ . So the universal theory of any of these classes contains the theory of  $d$ -subscaled lattices. For the converse, thanks to Remark 5.1 it suffices to prove that every  $d$ -subscaled lattice  $L$  embeds into a model of the theory of  $\text{SC}_{\text{lin}}(K, d)$ . If  $L$  is finite this is Proposition 5.3. The general case then follows from the model-theoretic compactness theorem, because  $L$  is locally finite by Theorem 4.1.

■

## 6 Minimal extensions

Minimal proper extensions of any finite subscaled lattices are entirely determined by so-called ‘‘SC-signatures’’ (see below). Since this is a special case of minimal extensions of finite co-Heyting algebras, we first recall the main results of [DJ18] on this subject, and try to reduce to them as much as possible.

We need some specific notation and definitions. Given a finite lattice  $L_0$ , an  $\mathcal{L}_{\text{lat}}$ -extension  $L$ , elements  $a \in L_0$  and  $x \in L$  we write:

- $a^- = \mathbb{W}\{b \in L_0 / b < a\}$ .
- $g(x, L_0) = \mathbb{M}\{a \in L_0 / x \leq a\}$ .

Clearly  $a \in \mathcal{I}(L_0)$  if and only if  $a^-$  is the unique predecessor of  $a$  in  $L_0$  (otherwise  $a^- = a$ ).

Assume that  $L_0$  and  $L$  are co-Heyting algebras (or topologically complemented lattices, or TC-lattice for short). A **TC-signature** in  $L_0$  is a triple  $(g, H, r)$  where  $g \in \mathcal{I}(L_0)$ ,  $H$  is a set of one or two elements  $h_1, h_2 \in L_0$  and  $r \in \{1, 2\}$  are such that:

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<sup>8</sup>More generally this holds true for dp-minimal expansions as explained in Footnote 3.

- either  $r = 1$  and  $h_1 = h_2 < g$ ;
- or  $r = 2$  and  $h_1 \vee h_2 < g$ .

A couple  $(x_1, x_2)$  of elements of  $L$  is **TC-primitive over**  $L_0$  if there is  $g \in \mathcal{I}(L_0)$  such that

**P1**  $g^- \wedge x_1$  and  $g^- \wedge x_2$  belong to  $L_0$ .

**P2** One of the following happens:

1.  $x_1 = x_2$  and  $g^- \wedge x_1 \ll x_1 \ll g$ .
2.  $x_1 \neq x_2$ ,  $x_1 \wedge x_2 \in L_0$  and  $g - x_1 = x_2$ ,  $g - x_2 = x_1$ .

This implies that each  $x_i \notin L_0$ , that  $g = g(x_1, L_0) = g(x_2, L_0)$  and that the triple  $\sigma_{\text{TC}}(x_1, x_2) = (g, H, r)$  defined as follows is a signature in  $L_0$ , called the **signature of**  $(x_1, x_2)$  **in**  $L_0$ .

$$g = g(x_1, L_0) \quad H = \{g^- \wedge x_1, g^- \wedge x_2\} \quad r = \text{Card}\{x_1, x_2\}$$

Finally we say that  $L$  is a **TC-primitive extension** of  $L_0$  if it is  $\mathcal{L}_{\text{TC}}$ -generated over  $L_0$  by a primitive tuple. For the convenience of the reader we collect here all the properties of TC-signatures and TC-primitive extensions that we are going to use. All the references are taken from [DJ18].

**Proposition 6.1** *Let  $L_0$  be a finite co-Heyting algebra and  $L$  an  $\mathcal{L}_{\text{TC}}$ -extension.*

1. [Theorem 3.3] *If  $L$  is  $\mathcal{L}_{\text{TC}}$ -generated over  $L_0$  by a TC-primitive tuple  $(x_1, x_2)$ , then  $L$  is exactly the upper semi-lattice generated over  $L_0$  by  $x_1$  and  $x_2$ . It is a finite co-Heyting algebra and one of the following happens:*

- (a)  $x_1 = x_2$  and  $\mathcal{I}(L) = \mathcal{I}(L_0) \cup \{x_1\}$ .
- (b)  $x_1 \neq x_2$  and  $\mathcal{I}(L) = (\mathcal{I}(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$ .

*In particular  $(x_1, x_2)$  is (up to permutation) the only TC-primitive tuple over  $L_0$  in  $L$ . We call  $\sigma_{\text{TC}}(x_1, x_2)$  the **TC-signature of  $L$  in  $L_0$**  and denote it  $\sigma_{\text{TC}}(L)$ .*

2. [Remark 3.6] *TC-signatures in  $L_0$  and TC-primitive extensions of  $L_0$  are in one-to-one correspondence: every TC-signature in  $L_0$  is the TC-signature of a TC-primitive extension, and two TC-primitive extensions of  $L_0$  are  $\mathcal{L}_{\text{TC}}$ -isomorphic over  $L_0$  if and only if they have the same TC-signature in  $L_0$ .*

3. [Corollary 3.4] *If  $L$  is finite, the following are equivalent.*

- (a)  $L$  is a minimal proper extension of  $L_0$ .
- (b)  $L$  is a TC-primitive extension of  $L_0$ .
- (c)  $\text{Card}(\mathcal{I}(L)) = \text{Card}(\mathcal{I}(L_0)) + 1$ .

*As a consequence every finite  $\mathcal{L}_{\text{TC}}$ -extension  $L'$  of  $L_0$  is the union of a tower of TC-primitive extensions  $L_0 \subset L_1 \subset \dots \subset L_n = L'$  with  $n = \text{Card}(\mathcal{I}(L')) - \text{Card}(\mathcal{I}(L_0))$ .*

We will refer to the  $k$ -th item of the above proposition as to Proposition 6.1. $k$ .

Now let  $L_0$  be a finite subscaled lattice and  $L$  and  $\mathcal{L}_{\text{SC}}$ -extension. An **SC-signature** in  $L_0$  is a triple  $\sigma = (g, H, q)$  where  $g \in \mathcal{I}(L_0)$ ,  $H$  is a set of one or two elements  $h_1, h_2 \in L_0$  and  $q \in \mathbf{N}$  are such that:

- either  $\text{sc-dim } h_1 < q < \text{sc-dim } g$  and  $h_1 = h_2 < g$ ;
- or  $q = \text{sc-dim } g$  and  $h_1 \vee h_2 = g^-$ .

Let  $r_\sigma = 1$  if  $q < \text{sc-dim } g$ ,  $r_\sigma = 2$  if  $q = \text{sc-dim } g$ , and  $\sigma^{\text{TC}} = (g, H, r_\sigma)$ . By construction this is a TC-signature in  $L_0$ . Given an  $\mathcal{L}_{\text{SC}}$ -extension  $L$  of  $L_0$ , a tuple  $(x_1, x_2)$  of elements of  $L$  is **SC-primitive over**  $L_0$  if it is TC-primitive over  $L_0^{\text{TC}}$  and if in addition

**P3**  $x_1, x_2$  are sc-pure of the same sc-dimension.

Such an SC-primitive tuple determines the so-called **SC-signature of**  $(x_1, x_2)$  **in**  $L_0$ , denoted  $\sigma_{\text{SC}}(x_1, x_2) = (g, H, q)$  and defined as follows.

$$g = g(x_1, L_0) \quad H = \{g^- \wedge x_1, g^- \wedge x_2\} \quad q = \text{sc-dim } x_1$$

Note that, by condition **P2** of the definition of TC-signatures,  $x_1 = x_2$  if and only if  $x_1 \ll g$ , and otherwise  $\text{sc-dim } x_1 = \text{sc-dim } x_2 = \text{sc-dim } g$ . This ensures that  $\sigma_{\text{TC}}(x_1, x_2) = (\sigma_{\text{SC}}(x_1, x_2))^{\text{TC}}$ .

Let  $L_0^{\text{TC}}$  and  $L^{\text{TC}}$  denote the respective  $\mathcal{L}_{\text{TC}}$ -reducts of  $L_0$  and  $L$ . For every subset  $X_0$  of  $L$  we let:

- $L_0\langle X_0 \rangle$  = the  $\mathcal{L}_{\text{SC}}$ -structure generated by  $L_0 \cup X_0$  in  $L$ ;
- $L_0^{\text{TC}}\langle X_0 \rangle$  = the  $\mathcal{L}_{\text{TC}}$ -structure generated by  $L_0^{\text{TC}} \cup X_0$  in  $L^{\text{TC}}$ .

We say that  $L$  is an **SC-primitive extension** of  $L_0$ , if there exists a tuple  $(x_1, x_2)$  SC-primitive over  $L_0$  such that  $L = L_0\langle x_1, x_2 \rangle$  (then clearly  $L = L_0\langle x_1 \rangle = L_0\langle x_2 \rangle$ ). By Lemma 6.2 below and Proposition 6.1.1 such a tuple is necessarily unique.

**Lemma 6.2** *Let  $L_0$  be finite subscaled lattice, and  $L$  an  $\mathcal{L}_{\text{SC}}$ -extension generated over  $L_0$  by an SC-primitive tuple  $(x_1, x_2)$ . Then  $L^{\text{TC}} = L_0\langle x_1, x_2 \rangle$ ,  $(x_1, x_2)$  is TC-primitive over  $L_0^{\text{TC}}$  and  $\sigma_{\text{TC}}(x_1, x_2) = (\sigma_{\text{SC}}(x_1, x_2))^{\text{TC}}$ .*

*Proof:* That  $(x_1, x_2)$  is TC-primitive over  $L_0^{\text{TC}}$  and  $\sigma_{\text{TC}}(x_1, x_2) = (\sigma_{\text{SC}}(x_1, x_2))^{\text{TC}}$  is only a reminder: it follows directly from the definitions. Let  $L_1 = L_0\langle x_1, x_2 \rangle$ , in order to conclude that  $L_1 = L$  it only remains to prove that  $L_1$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $L$ . By Corollary 3.5 it suffices to check that for every  $b' < b \in \mathcal{I}(L_1)$ ,  $b$  is sc-pure in  $L$  and  $\text{sc-dim } b' < \text{sc-dim } b$  in  $L$ .

If  $b \in \mathcal{I}(L_0)$ , then  $b$  is sc-pure in  $L_0$  by SC<sub>13</sub>, hence also in  $L$  because  $L_0$  is an  $\mathcal{L}_{\text{SC}}$ -substructure of  $L$ . Otherwise  $b = x_i$  for some  $i \in \{1, 2\}$ . Then  $b$  is sc-pure in  $L$  by definition of SC-primitive tuples over  $L_0$ .

In both cases  $b' \ll b$  in  $L_1$  by TC<sub>1</sub>, that is  $b - b' = b$  in  $L_1$ , hence also in  $L$  because  $L_1$  is an  $\mathcal{L}_{\text{TC}}$ -substructure of  $L$ . So  $b' \ll b$  in  $L$  hence  $\text{sc-dim}(b') < \text{sc-dim}(b)$  in  $L$  by SC<sub>6</sub>.

■

**Lemma 6.3** *Let  $L_0$  be finite subscaled lattice,  $L_1$  an  $\mathcal{L}_{\text{TC}}$ -extension generated over  $L_0^{\text{TC}}$  by a TC-primitive tuple  $(x_1, x_2)$ , and  $\tau = (g, \{h_1, h_2\}, q)$  an SC-signature in  $L_0$  such that  $\tau^{\text{TC}} = \sigma_{\text{TC}}(x_1, x_2)$ . Then there exists a unique structure of subscaled lattice expanding  $L_1$  which makes it an  $\mathcal{L}_{\text{SC}}$ -extension of  $L_0$  such that  $(x_1, x_2)$  is SC-primitive over  $L_0$  and  $\sigma_{\text{SC}}(x_1, x_2) = \tau$ .*

*Proof:* By Proposition 6.1.1,  $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0) \cup \{x_1, x_2\}$ . For every  $x \in \mathcal{I}(L_0)$  let  $D(x) = \text{sc-dim } x$ , and let  $D(x_1) = D(x_2) = q$ . This defines by restriction a function from  $\mathcal{I}(L_1)$  to  $\mathbb{N}$ . Assume that  $D$  is strictly increasing. Then it determines as in Example 2.7 an  $\mathcal{L}_{\text{SC}}$ -structure on  $L_1$  expanding its  $\mathcal{L}_{\text{TC}}$ -structure. Let us denote it  $L$ , so that  $L^{\text{TC}} = L_1$ . Every  $\vee$ -irreducible element of  $L_0$  remains sc-pure in  $L$  with the same sc-dimension, hence by Proposition 3.3 the inclusion of  $L_0$



into  $L$  is an  $\mathcal{L}_{\text{SC}}$ -embedding. This is clearly the only possible  $\mathcal{L}_{\text{SC}}$ -structure on  $L_1$  which makes it an  $\mathcal{L}_{\text{SC}}$ -extension of  $L$  such that  $\text{sc-dim } x_1 = \text{sc-dim } x_2 = q$ . So it only remains to prove that  $D$  is strictly increasing.

Let  $b < a$  in  $\mathcal{I}(L)$ , if  $a, b \in \mathcal{I}(L_0)$  then  $D(b) < D(a)$  by 6. So we can assume that  $a$  or  $b$  does not belong to  $\mathcal{I}(L_0)$ . By Proposition 6.1.1 one of them must belong to  $\{x_1, x_2\}$  and the other one to  $\mathcal{I}(L_0)$ . Note that our assumption  $\sigma_{\text{TC}}(x_1, x_2) = \sigma^{\text{TC}}$  implies that (for  $i = 1, 2$ )  $g = g(x_i, L_0)$ , and  $h_i = x_i \wedge g^-$  (up to re-numbering) and: either  $x_1 = x_2$ ,  $h_1 = h_2 < g$  and  $\text{sc-dim } h_1 < q < \text{sc-dim } g$ ; or  $x_1 \neq x_2$ ,  $h_1 \vee h_2 = g^-$  and  $q = \text{sc-dim } g$ .

*Case 1:*  $b = x_1$  or  $b = x_2$ , hence  $D(b) = q$ . Then  $a \in \mathcal{I}(L_0)$ , in particular  $a \in L_0$ , hence  $g \leq a$  and so  $\text{sc-dim } g \leq \text{sc-dim } a$ . If  $x_1 = x_2$  then  $q < \text{sc-dim } g \leq \text{sc-dim } a$  hence  $D(b) < D(a)$ . If  $x_1 \neq x_2$  then  $q = \text{sc-dim } g$ , and  $g$  is not  $\vee$ -irreducible in  $L$  hence  $g \neq a$ . So  $g \ll a$  (because  $g < a$  and  $a$  is  $\vee$ -irreducible) hence  $\text{sc-dim } g < \text{sc-dim } a$  by 6, that is  $D(b) < D(a)$ .

*Case 2:*  $a = x_1$  or  $a = x_2$ , hence  $D(a) = q$ . Then again  $b \in L_0$ , and  $b < a \leq g$  hence  $b \leq g^-$ . If  $x_1 = x_2$ , since  $b \leq a \wedge g^- = h_1$  we get  $\text{sc-dim } b \leq \text{sc-dim } h_1 < q$ , hence  $D(b) < D(a)$ . If  $x_1 \neq x_2$  then  $\text{sc-dim } g = q$ . Since  $b < g$  we have  $b \ll g$  (because  $g$  is  $\vee$ -irreducible) hence  $\text{sc-dim } b < \text{sc-dim } g$  by 6. So  $\text{sc-dim } b < q$ , that is  $D(b) < D(a)$ .

■

We can now pack all this together.

**Proposition 6.4** *Let  $L_0$  be a finite subscaled lattice and  $L$  an  $\mathcal{L}_{\text{SC}}$ -extension.*

1. *If  $L$  is  $\mathcal{L}_{\text{SC}}$ -generated over  $L_0$  by an SC-primitive tuple  $(x_1, x_2)$ , then  $L$  is exactly the upper semi-lattice generated over  $L_0$  by  $x_1$  and  $x_2$ . It is a finite subscaled lattice and one of the following happens :*

- (a)  $x_1 = x_2$  and  $\mathcal{I}(L) = \mathcal{I}(L_0) \cup \{x_1\}$ .
- (b)  $x_1 \neq x_2$  and  $\mathcal{I}(L) = (\mathcal{I}(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$ .

*In particular  $(x_1, x_2)$  is (up to permutation) the only SC-primitive tuple over  $L_0$  in  $L$ . We call  $\sigma_{\text{SC}}(x_1, x_2)$  the **SC-signature of  $L$  in  $L_0$**  and denote it  $\sigma_{\text{SC}}(L)$ .*

2. *SC-signatures in  $L_0$  and SC-primitive extensions of  $L_0$  are in one-to-one correspondence: every SC-signature in  $L_0$  is the SC-signature of an SC-primitive extension, and two SC-primitive extensions of  $L_0$  are  $\mathcal{L}_{\text{SC}}$ -isomorphic over  $L_0$  if and only if they have the same SC-signature in  $L_0$ .*
3. *If  $L$  is finite, the following are equivalent.*

- (a)  $L$  is a minimal proper  $\mathcal{L}_{\text{SC}}$ -extension of  $L_0$ .
- (b)  $L$  is an SC-primitive extension of  $L_0$ .
- (c)  $\text{Card}(\mathcal{I}(L)) = \text{Card}(\mathcal{I}(L_0)) + 1$ .

*As a consequence every finite  $\mathcal{L}_{\text{SC}}$ -extension  $L'$  of  $L_0$  is the union of a tower of SC-primitive extensions  $L_0 \subset L_1 \subset \dots \subset L_n = L'$  with  $n = \text{Card}(\mathcal{I}(L')) - \text{Card}(\mathcal{I}(L_0))$ .*

We will refer to the  $k$ -th item of the above proposition as to Proposition 6.4. $k$ .

*Proof:* If  $L$  is  $\mathcal{L}_{\text{SC}}$ -generated over  $L_0$  by an SC-primitive tuple  $(x_1, x_2)$ , then by Lemma 6.2  $L^{\text{TC}}$  is also  $\mathcal{L}_{\text{TC}}$ -generated over  $L_0^{\text{TC}}$  by  $(x_1, x_2)$ , which is TC-primitive. The first item follows from Proposition 6.1.1.

Let  $\sigma$  be an SC-signature in  $L_0$ . Then  $\sigma^{\text{TC}}$  is a TC-signature in  $L_0$ . Proposition 6.1.2 gives a TC-primitive  $\mathcal{L}_{\text{TC}}$ -extension  $L_1$  of  $L_0^{\text{TC}}$  with TC-signature  $\sigma^{\text{TC}}$  in  $L_0$ . Lemma 6.3 then gives

a unique structure of subscaled lattice expanding  $L_1$  which makes it an SC-primitive extension of  $L_0$  with signature  $\sigma$  in  $L_0$ . Let us denote it  $L$ , so that  $L^{\text{TC}} = L_1$ . Now if  $L'$  is another SC-primitive extension with signature  $\sigma$  in  $L_0$ , by Proposition 6.1.2  $L'^{\text{TC}}$  is  $\mathcal{L}_{\text{TC}}$ -isomorphic to  $L^{\text{TC}}$  over  $L_0$ . The image of  $L'$  via this endomorphism defines an  $\mathcal{L}_{\text{SC}}$ -structure expanding  $L^{\text{TC}}$ , which makes it an SC-primitive extension of  $L_0$  with the same signature as  $L$ . By the uniqueness of such a structure, given by Lemma 6.3, it follows that this  $\mathcal{L}_{\text{TC}}$ -isomorphism from  $L'^{\text{TC}}$  to  $L^{\text{TC}}$  is actually an  $\mathcal{L}_{\text{SC}}$ -isomorphism, which proves the result.

We finely prove item (3) by cyclic implication. Note that (3b) $\Rightarrow$ (3c) follows from item 1) above.

(3c) $\Rightarrow$ (3a). Let  $L'$  be a proper  $\mathcal{L}_{\text{SC}}$ -extension of  $L_0$  contained in  $L$ . Then  $L'^{\text{TC}}$  is a proper  $\mathcal{L}_{\text{TC}}$ -extension of  $L_0^{\text{TC}}$  contained in  $L^{\text{TC}}$ . By Proposition 6.1.3, (3c) implies that  $L'^{\text{TC}}$  is a minimal proper  $\mathcal{L}_{\text{TC}}$ -extension of  $L_0^{\text{TC}}$ . So  $L'^{\text{TC}} = L^{\text{TC}}$ , thus necessarily  $L' = L$ , which proves that  $L$  is minimal.

(3a) $\Rightarrow$ (3b). Let  $x_1$  be a minimal element in  $\mathcal{I}(L) \setminus \mathcal{I}(L_0)$ . Let  $g = g(x_1, L_0)$ ; if  $x_1 \ll g$  let  $x_2 = x_1$ , otherwise let  $x_2 = g - x_1$ . The proof of Corollary 3.4 in [DJ18] shows that  $(x_1, x_2)$  is TC-primitive over  $L_0^{\text{TC}}$ . In particular  $x_1, x_2 \in \mathcal{I}(L)$  so they are sc-pure by SC13. The same holds true for  $g$ , hence if  $x_1 \neq x_2$  then  $x_1 = g - x_2$  and  $x_2 = g - x_1$  have the same dimension (the dimension of  $g$ , by definition of the sc-purity of  $g$ ). So  $(x_1, x_2)$  is actually SC-primitive. Since  $L_0^{\text{TC}}\langle x_1, x_2 \rangle = L^{\text{TC}}$ , *a fortiori*  $L_0\langle x_1, x_2 \rangle = L$  hence  $L$  is SC-primitive over  $L_0$ .  
■

## 7 Model-completion of scaled lattices

We call **super scaled lattice** every subscaled lattice  $L$  which satisfy the following additional properties, both of which are clearly axiomatizable by  $\forall\exists$ -formulas in  $\mathcal{L}_{\text{SC}}$ . If  $\text{sc-dim } L \leq d$  we call it a **super  $d$ -scaled lattice**.

**Catenarity** For every non-negative integers  $r \leq q \leq p$  and every elements  $c \leq a \neq \mathbf{0}$ , if  $c$  is  $r$ -sc-pure and  $a$  is  $p$ -sc-pure then there exists a non-zero  $q$ -sc-pure element  $b$  such that  $c \leq b \leq a$ .

If  $\text{Spec} L$  is noetherian this property is equivalent to the usual notion of catenarity, namely that any two maximal chains in  $\text{Spec} L$  having the same first and last elements have the same length. In particular every  $d$ -scaled lattice  $L$  of type  $L_{\text{Zar}}(X)$  or  $L_{\text{lin}}(X)$  satisfies this property. At the contrary none of them satisfy the next property, as it implies that  $L$  is atomless.

**Splitting** For every elements  $b_1, b_2, a$ , if  $b_1 \vee b_2 \ll a \neq \mathbf{0}$  then there exists non-zero elements  $a_1 \geq b_1$  and  $a_2 \geq b_2$  such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

We will then say  $a_1, a_2$  **split  $a$  along  $b_1, b_2$** .

**Remark 7.1** If  $r < p \leq q$  in the Catenarity axiom, the conclusion can be strengthened to  $c \ll b \leq a$ . Indeed  $b$  has pure sc-dimension  $q$  and  $c \wedge b = c$  has sc-dimension  $< q$  hence  $b - c = b$  by SC8. In particular *every subscaled lattice satisfying the Catenarity axiom is a scaled lattice*. Indeed, given any element  $a$  of sc-dimension  $d \geq 1$ , repeated applications of the Catenarity axiom to  $C^d(a)$ ,  $c = \mathbf{0}$  and each integer  $p$  from 0 to  $d$ , gives a chain of sc-pure elements  $a_0, \dots, a_d$  such that

$$\mathbf{0} \neq a_0 \ll a_1 \ll \dots \ll a_d \leq a.$$

By Fact 2.2 it follows that  $\dim a \geq d$ , and by SC<sub>11</sub> that  $\dim a = d$ .

**Lemma 7.2** *Let  $a, b_1, b_2$  be elements of a finite subscaled lattice  $L$ . If  $b_1 \vee b_2 \ll a \neq \mathbf{0}$  then  $L$  embeds in a finite subscaled lattice  $L'$  containing non-zero elements  $a_1, a_2$  which split  $a$  along  $b_1, b_2$ . Moreover, if  $C^0(a) \neq \mathbf{0}$  we can require<sup>9</sup> that all the atoms of  $L'$  belong to  $L_0$ .*

*Proof:* We are going to prove by induction on  $d = \text{sc-dim } a$  a slightly more precise result, namely that in addition  $x \leq a$  for every  $x \in \mathcal{I}(L') \setminus \mathcal{I}(L)$ . Let  $g_1, \dots, g_n$  be the  $\vee$ -irreducible components of  $a$  in  $L$ . ( $n \geq 1$  because  $a \neq \mathbf{0}$ ). If  $d = 0$  our assumption that  $b_1 \vee b_2 \ll a$  implies by SC<sub>6</sub> that  $b_1 = b_2 = \mathbf{0}$ . If  $n = 1$ , that is  $a = g_1$  is  $\vee$ -irreducible, then  $\sigma = (g, \{\mathbf{0}\}, 0)$  is a signature in  $L$ . Proposition 6.4.2 gives an SC-primitive tuple  $(a_1, a_2)$  generating an  $\mathcal{L}_{\text{SC}}$ -extension  $L_1$  over  $L$  with signature  $\sigma$ . This signature ensures that  $(a_1, a_2)$  splits  $a$  along  $(\mathbf{0}, \mathbf{0})$ . If  $n \geq 2$ ,  $a_1 = g_1$  and  $a_2 = a - a_1$  will do the job. So the result is proved for  $d = 0$ .

Now assume that  $d \geq 1$  and the result is valid until  $d - 1$ . Note that  $g_1^- \vee \dots \vee g_n^-$  is the greatest element  $c \in L$  such that  $c \ll a$ , in particular

$$b_1 \vee b_2 \leq g_1^- \vee \dots \vee g_n^-. \quad (10)$$

Let  $u = (\bigvee_{i \leq n} g_i^-) - (b_1 \vee b_2)$  and  $u^* = u - C^0(u)$ . Since  $u \ll a$  we have  $\text{sc-dim } u < d$  by SC<sub>6</sub>. We are claiming that  $L$  embeds in a finite subscaled lattice  $L$  without new atoms, in which all the  $g_i$ 's are still  $\vee$ -irreducible with the same predecessor as in  $L_0$ , and in which there are elements  $u_1^*, u_2^*$  which satisfy all the conditions to split  $u^*$  along  $b_1 \wedge u^*, b_2 \wedge u^*$ , except that  $u_1^*, u_2^*$  might be zero elements.

By TC<sub>3</sub>,  $(b_1 \vee b_2) \wedge u^* \ll u^*$  so if  $\text{sc-dim } u \leq 0$  we can simply take  $u_1^* = u_2^* = \mathbf{0}$  and  $L_0 = L$ . On the other hand, if  $\text{sc-dim } u > 0$  the induction hypothesis applies to  $u^*, b_1 \wedge u^*, b_2 \wedge u^*$ . It gives a finite subscaled lattice  $L_0$  containing  $L$  and elements  $u_1^*, u_2^* \in L_0$  which split  $u^*$  along  $b_1 \wedge u^*, b_2 \wedge u^*$ . Moreover we can require that  $L_0$  do not contain any new atom because  $C^0(u^*) = \mathbf{0}$ , and that  $x \leq u^*$  for every  $x \in \mathcal{I}(L) \setminus \mathcal{I}(L_0)$ . For every  $x \in \mathcal{I}(L)$  such that  $x < g_i$  for some  $i \leq n$ , if  $x \in L_0$  then  $x \leq g_i^-$  (where  $g_i^-$  still denotes the predecessor of  $g_i$  in  $L_0$ ). If  $x \notin L_0$  then  $x \leq u^*$  by construction hence  $x \leq g_i \wedge u^*$ . The latter belongs to  $L_0$  and is strictly lower than  $g_i$ , hence lower than  $g_i^-$ , so  $x < g_i$ . It follows that  $g_i$  has a unique predecessor in  $L$  which is  $g_i^-$ , in particular  $g_i$  remains  $\vee$ -irreducible in  $L$ . This proves our claim in both cases.

Now let  $u_1 = C^0(u) \vee u_1^*$  and  $u_2 = u_2^*$ . We have in particular

$$u_1 \vee u_2 = \bigvee_{i \leq n} g_i^- - (b_1 \vee b_2). \quad (11)$$

Since  $u - b_2 = u$  by TC<sub>3</sub> necessarily  $b_2 \wedge c \ll c$  for every  $\vee$ -irreducible component  $c$  of  $u$ , hence  $b_2 \wedge C^0(u) \ll C^0(u)$ . By SC<sub>6</sub> it follows that  $b_2 \wedge C^0(u) = \mathbf{0}$  hence  $b_2 \wedge u_1 = b_2 \wedge u_1^*$ . Similarly  $u^* \wedge C^0(u) = \mathbf{0}$  because  $u^* - C^0(u) = u^*$  by SC<sub>12</sub> and TC<sub>3</sub>. *A fortiori*  $u_2^* \wedge C^0(u) = \mathbf{0}$  hence  $u_2^* \wedge u_1 = u_2^* \wedge u_1^*$ . Note also that  $b_1 \wedge u_2^* = b_1 \wedge u^* \wedge u_2^* \leq u_1^* \wedge u_2^*$ , and symmetrically  $b_2 \wedge u_1^* \leq u_1^* \wedge u_2^*$ . Altogether, since  $u_2 = u_2^*$  and  $u_1^* \wedge u_2^* \leq b_1 \wedge b_2$  by construction, this gives

$$(b_1 \wedge u_2) \vee (b_2 \wedge u_1) \vee (u_1 \wedge u_2) \leq (b_1 \wedge b_2)$$

hence

$$(b_1 \vee u_1) \wedge (b_2 \vee u_2) = (b_1 \wedge b_2) \vee (b_1 \wedge u_2) \vee (b_2 \wedge u_1) \vee (u_1 \wedge u_2) = (b_1 \wedge b_2). \quad (12)$$

After this preparation, for each  $i$  let

$$h_{i,1} = g_i^- \wedge (b_1 \vee u_1), \quad h_{i,2} = g_i^- \wedge (b_2 \vee u_2), \quad \sigma_i = (g_i, \{h_{i,1}, h_{i,2}\}, \text{sc-dim } g_i)$$

<sup>9</sup>This additional condition if  $C^0(a) \neq \mathbf{0}$  will be used only later, in Section 9.

Using (11) we get

$$\begin{aligned}
h_{i,1} \vee h_{i,2} &= g_i^- \wedge (b_1 \vee u_1 \vee b_2 \vee u_2) \\
&= g_i^- \wedge \left[ b_1 \vee b_2 \vee \left( \bigvee_{j \leq n} g_j^- - (b_1 \vee b_2) \right) \right] \\
&= g_i^- \wedge \bigvee_{j \leq n} g_j^- = g_i^-.
\end{aligned}$$

So each  $\sigma_i$  is an SC-signature in  $L_0$ . In particular Proposition 6.4.2 gives an SC-primitive extension  $L_1 = L_0 \langle a_{1,1}, a_{1,2} \rangle$  with SC-signature  $\sigma_1$  in  $L_0$ . By Proposition 6.4.1,  $\mathcal{I}(L_1) = (\mathcal{I}(L_0) \setminus \{a_1\}) \cup \{a_{1,1}, a_{1,2}\}$ . In particular  $g_2 \in \mathcal{I}(L_1)$ , hence  $\sigma_2$  is still an SC-signature in  $L_1$ . Repeating the construction  $n$  times (note that  $a \neq \mathbf{0}$  ensures that  $n \geq 1$ ) gives a chain of  $\mathcal{L}_{\text{SC}}$ -extensions  $(L_i)_{i \leq n}$  and for each  $i > 0$ , an SC-primitive tuple  $(a_{i,1}, a_{i,2})$  generating  $L_i$  over  $L_{i-1}$  with signature  $\sigma_i$  in  $L_{i-1}$ . Each  $a_i = a_{i,1} \vee a_{i,2}$  and by Proposition 6.4.1

$$\mathcal{I}(L_n) = (\mathcal{I}(L_0) \setminus \{a_1, \dots, a_n\}) \cup \{a_{1,1}, a_{1,2}, \dots, a_{n,1}, a_{n,2}\} \quad (13)$$

so  $a_{1,1}, a_{1,2}, \dots, a_{n,1}, a_{n,2}$  are the  $\vee$ -irreducible components of  $a$  in  $L_n$ . Moreover every  $c \in \mathcal{I}(L)$  such that  $c < a_{i,k}$  for some  $i, k$  must belong to  $L_0$ , hence the predecessor of  $a_{i,k}$  is the same in every  $L_j$  and belongs to  $L_0$ . We can then denote it  $a_{i,k}^-$  without ambiguity, and by construction we have

$$a_{i,k}^- = a_{i,k} \wedge g(a_i, L_{i-1}) = a_{i,k} \wedge g_i = h_{i,k}. \quad (14)$$

Let  $a_1 = \bigvee_{i \leq n} a_{i,1}$ ,  $a_2 = \bigvee_{i \leq n} a_{i,2}$ ,  $h_1 = \bigvee_{i \leq n} h_{i,1}$  and  $h_2 = \bigvee_{i \leq n} h_{i,2}$ . We are going to check that  $a_1, a_2$  split  $a$  along  $b_1, b_2$ . Both of them are non-zero and since the  $a_{i,k}$ 's are the  $\vee$ -irreducible components of  $a$  we have  $a - a_1 = a_2$ ,  $a - a_2 = a_1$ . Each  $a_{i,1} \geq h_{i,1}$  by construction, hence  $a_1 \geq h_1$  and symmetrically  $a_2 \geq h_2$ . Moreover for  $k \in \{1, 2\}$

$$h_k = \bigvee_{i \leq n} h_{i,k} \geq \bigvee_{i \leq n} g_i^- \wedge b_k = b_k$$

where the last equality comes from (10), so  $a_k \geq b_k$ . It remains to check that  $a_1 \wedge a_2 = b_1 \wedge b_2$ .

For  $i \neq j$ ,  $a_{i,1}$  and  $a_{j,2}$  are mutually incomparable hence by (14)

$$a_{i,1} \wedge a_{j,2} = a_{i,1}^- \wedge a_{j,2}^- = h_{i,1} \wedge h_{j,2}.$$

On the other hand  $a_{i,1} \wedge a_{i,2} = h_{i,1} \wedge h_{i,2}$  by construction. The conclusion follows, using (12).

$$\begin{aligned}
a_1 \wedge a_2 &= \bigvee_{i,j} a_{i,1} \wedge a_{j,2} = \bigvee_{i,j} h_{i,1} \wedge h_{j,2} \\
&= \bigvee_{i,j} [g_i^- \wedge (b_1 \vee u_1)] \wedge [g_j^- \wedge (b_2 \vee u_2)] \\
&= \bigvee_{i,j} (g_i^- \wedge g_j^-) \wedge [(b_1 \vee u_1) \wedge (b_2 \vee u_2)] \\
&= \left( \bigvee_i g_i^- \right) \wedge \left( \bigvee_j g_j^- \right) \wedge [(b_1 \vee u_1) \wedge (b_2 \vee u_2)] \\
&= (b_1 \vee u_1) \wedge (b_2 \vee u_2) = b_1 \wedge b_2.
\end{aligned}$$

■

**Theorem 7.3** *The theory of super  $d$ -scaled lattices is the model-completion of the theory of  $d$ -subscaled lattices*

*Proof:* By standard model-theoretic arguments it suffices to prove that every existentially closed  $d$ -subscaled lattice is super  $d$ -scaled, and that for every super  $d$ -scaled lattice  $\hat{L}$ , every finitely generated  $d$ -subscaled lattice  $L$  and every common  $\mathcal{L}_{\text{SC}}$ -substructure  $L_0$ , there is an embedding of  $L$  into  $\hat{L}$  over  $L_0$ .

Let  $L$  be an existentially closed  $d$ -subscaled lattice, and  $L_0$  a finitely generated substructure. By Theorem 4.1,  $L_0$  is finite. By Proposition 5.3,  $L_0$   $\mathcal{L}_{\text{SC}}$ -embeds the  $d$ -scaled lattice  $L_{\text{def}}(X)$  of some special linear set  $X$ , which is in particular a Catenary lattice. By the model-theoretic compactness Theorem it follows that  $L$  is catenary. Similarly Theorem 4.1, Lemma 7.2 and the model-theoretic compactness Theorem prove that  $L$  has the Splitting property, hence  $L$  is super  $d$ -scaled.

Conversely assume that  $L$  is a super  $d$ -scaled lattice,  $L_0$  is a finitely generated  $d$ -subscaled lattice and  $L_0$  common  $\mathcal{L}_{\text{SC}}$ -substructure. By Theorem 4.1 and Proposition 6.4.3a, we are reduced to the case where  $L$  is a primitive extension of  $L_0$ . Let  $\sigma = (g, \{h_1, h_2\}, q)$  be its SC-signature. By Proposition 6.4.2 it suffices to find a  $x_1, x_2 \in \hat{L}$  such that  $(x_1, x_2)$  is SC-primitive over  $L_0$  and  $\sigma_{\text{SC}}(x_1, x_2) = \sigma$ . We distinguish two cases, and let  $g^-$  denotes the predecessor of  $g$  in  $L_0$ .

*Case 1:*  $\text{sc-dim } h_1 < q < \text{sc-dim } g$  and  $h_1 = h_2 < g$ . Let  $p = \text{sc-dim } g$  and  $r = \text{sc-dim } h_1$ . Let  $y_1, y_2 \in \hat{L}$  which split  $g$  along  $h_1, g^-$ . For  $0 \leq i \leq d$ , either  $i < q$  or  $C^i(h_1) = \mathbf{0}$  (because  $\text{sc-dim } h_1 = r < q$ ), hence  $\text{sc-dim } C^i(h_1) < q < \text{sc-dim } g$ . The Catenarity property then applies to  $C^i(h_1) \leq y_1$  and gives  $x_i \in \hat{L}$  such that  $C^i(h_1) \leq x_i \leq y_1$  and  $x_i$  has pure dimension  $q$ . Let  $x = \mathbb{W}_{0 \leq i \leq d} x_i$ , by construction  $h_1 = \mathbb{W}_{i \leq d} C^i(h_1) \leq x \leq y_1$  and  $x$  has pure dimension  $q$ . In particular

$$h_1 \leq x \wedge g^- \leq y_1 \wedge y_2 = h_1 \wedge g^- = h_1$$

hence  $x \wedge g^- = h_1 \in L_0$ . Moreover  $x \wedge g^- = h_1 \ll x$  because  $\text{dim } h_1 < q$  and  $x$  has pure dimension  $q$ . Finally  $x \ll g$  because  $\text{dim } x_i = q < p$  and  $g$  has pure dimension  $p$ . Altogether this proves that  $(x, x)$  is an SC-primitive tuple over  $L_0$  with SC-signature  $\sigma$ .

*Case 2:*  $q = \text{sc-dim } g$  and  $h_1 \vee h_2 = g^-$ . Let  $y_1, y_2 \in \hat{L}$  which split  $g$  along  $h_1, h_2$ . By construction  $y_1 \vee y_2 = g$ , and since  $g$  has pure SC-dimension  $q$  so does each  $y_i$ . In addition  $y_1 \wedge y_2 = h_1 \wedge h_2 \in L_0$ . Moreover

$$y_1 \wedge h_2 \leq y_1 \wedge y_2 = h_1 \wedge h_2$$

hence  $y_1 \wedge (h_1 \vee h_2) = h_1 \vee (y_1 \wedge h_2) = h_1$ . Since  $h_1 \vee h_2 = g^-$  it follows that  $y_1 \wedge g^- = h_1 \in L_0$ , and symmetrically  $y_2 \wedge g^- = h_2 \in L_0$ . So  $(y_1, y_2)$  is an SC-primitive tuple over  $L_0$  with SC-signature  $\sigma$ .

■

**Remark 7.4** The proof of Theorem 7.3 shows that if  $L_0$  be a finite  $\mathcal{L}_{\text{SC}}$ -substructure of a super scaled lattice  $\hat{L}$ , then every signature  $\sigma$  in  $L_0$  is the signature of an SC-primitive extension of  $L_0$  in  $\hat{L}$ .

The completions of the theory of super  $d$ -scaled lattices are easy to classify. Let us say that a  $d$ -subscaled lattice is **prime** if it does not contain any proper  $d$ -subscaled lattice, or equivalently if it is generated by the empty set. Every prime  $d$ -subscaled lattice is finite. By Corollary 4.2 there exists finitely many prime  $d$ -subscaled lattices up to isomorphism.

**Corollary 7.5** *The theory of super  $d$ -scaled lattices containing (a copy of) a given prime  $d$ -subscaled lattice is  $\aleph_0$ -categorical, hence complete. It is also recursively axiomatizable, hence decidable. Since every completion of the theory of super  $d$ -scaled lattices is of that kind, the theory of super  $d$ -scaled lattices is decidable.*

*Proof:* Let  $L, L'$  be any two countable super  $d$ -scaled lattices containing isomorphic prime  $d$ -subscaled lattices  $L_0$  and  $L'_0$ . By Remark 7.4 any partial isomorphism between  $L$  and  $L'$ , extending the given isomorphism between  $L_0$  and  $L'_0$ , can be extended by va-et-vient. This proves the first statement. The other ones are immediate consequences. ■

## 8 Atomic scaled lattices

Every super scaled lattice is atomless because of the Splitting Property, hence none of the geometric scaled lattice in  $\text{SC}_{\text{def}}(\mathcal{K}, d)$ ,  $\text{SC}_{\text{Zar}}(K, d)$ ,  $\text{SC}_{\text{lin}}(K, d)$  can be super scaled. In order to apply our study to some of them we introduce now a variant of subscaled lattices intended to protect atoms against splitting.

Let  $\mathcal{L}_{\text{ASC}} = \mathcal{L}_{\text{SC}} \cup \{\text{At}_k\}_{k \in \mathbf{N}^*}$  with each  $\text{At}_k$  a new unary predicate symbol. For any  $\mathcal{L}_{\text{ASC}}$ -structure  $L$  we denote by  $\text{At}_k(L)$  the set of elements  $a$  in  $L$  such that  $L \models \text{At}_k(a)$ , and we let  $\text{At}_0(L) = L \setminus \bigcup_{k>0} \text{At}_k(L)$ . We call  $L$  an **ASC-lattice** if its  $\mathcal{L}_{\text{SC}}$ -reduct is a scaled lattice and if satisfies the following condition.

**ASC<sub>0</sub>** :  $(\forall k > 0), a \in \text{At}_k(L)$  if and only if  $a$  is the join of exactly  $k$  atoms in  $L$ .

**Remark 8.1** This condition can be expressed by  $\forall\exists$  formulas in  $\mathcal{L}_{\text{ASC}}$  by saying first that  $\text{At}_1(L)$  is the set of atoms of  $L$ , and then that  $\text{At}_k(L)$  is the set of elements of  $L$  which are the join of exactly  $k$  elements of  $\text{At}_1(L)$ .

Every ASC-lattice obviously satisfies also the following schemes (for  $k, l > 0$ ) of universal axioms, whose intuitive meaning is explained afterwards.

**ASC<sub>1</sub>** :  $(\forall k \neq l), \forall a, \text{At}_k(a) \rightarrow \neg \text{At}_l(a)$

**ASC<sub>2</sub>** :  $(\forall k), \forall a, a_0, \dots, a_{2^k}, \text{At}_k(a) \rightarrow$   

$$\left[ \bigwedge_{0 \leq i < 2^k} (a_i \leq a) \rightarrow \bigvee_{0 \leq i < j < 2^k} (a_i = a_j) \right] \wedge \text{sc-dim } a = 0$$

**ASC<sub>3</sub>** :  $(\forall k), \forall a, a_1, a_2,$   

$$\left[ (a = a_1 \vee a_2) \wedge (a_1 \wedge a_2 = \mathbf{0}) \wedge (a_1 \neq \mathbf{0}) \wedge (a_2 \neq \mathbf{0}) \right]$$
  

$$\rightarrow \left[ \text{At}_k(a) \leftrightarrow \bigvee_{0 < l < k} (\text{At}_l(a_1) \wedge \text{At}_{k-l}(a_2)) \right]$$

We call **sub-ASC-lattices** the  $\mathcal{L}_{\text{ASC}}$ -structures  $L$  whose  $\mathcal{L}_{\text{SC}}$ -reduct is a subscaled lattice and which satisfy ASC<sub>1</sub> to ASC<sub>3</sub> (but not necessarily ASC<sub>0</sub>).

ASC<sub>1</sub> obviously means that  $(\text{At}_k(L))_{k \in \mathbf{N}}$  is a partition of  $L$ . For any  $a \in L$  we then define  $\text{asc}(a)$  as the unique  $k \in \mathbf{N}$  such that  $a \in \text{At}_k(L)$ .

ASC<sub>2</sub> says that if  $\text{asc}(a) = k > 0$  then  $L(a)$  has at most  $2^k$  element and  $\text{sc-dim}(a) = 0$ . Then  $\text{dim}(a) = 0$  by SC<sub>11</sub> hence  $L(a)$  is a co-Heyting algebra with dimension 0, that is a Boolean algebra. So ASC<sub>2</sub> actually says that  $\text{sc-dim } a = 0$  and  $L(a)$  is a Boolean algebra with  $n$  atoms for some non zero  $n \leq k$ . In particular every  $a \in \text{At}_1(L)$  is an atom of  $L$ .

ASC<sub>3</sub> says that if  $a$  is the join of two non-zero disjoint elements  $a_1, a_2$  then  $\text{asc}(a)$  is non-zero if and only if  $\text{asc}(a_1)$  and  $\text{asc}(a_2)$  are non-zero, in which case  $\text{asc}(a) = \text{asc}(a_1) + \text{asc}(a_2)$ . By a straightforward induction this extends to any decomposition of  $a$  as the join of finitely

many pairwise disjoint elements. In view of  $\text{ASC}_2$  it then says that  $\text{asc}(a) > 0$  if and only if  $a$  is the join of finitely many atoms  $a_1, \dots, a_n$  of  $L$  such that each  $\text{asc}(a_i) > 0$ , in which case  $\text{asc}(a) = \sum_{1 \leq i \leq n} \text{asc}(a_i)$ .

**Remark 8.2** It follows immediately that an  $\mathcal{L}_{\text{SC}}$ -embedding of sub-ASC-lattices  $\varphi: L \rightarrow L'$  is an  $\mathcal{L}_{\text{ASC}}$ -embedding if and only if  $\text{asc}(a) = \text{asc}(\varphi(a))$  for every atom  $a \in L$ .

**Remark 8.3** Obviously every finitely generated substructure of a sub-ASC-lattices is finite by the Local Finiteness Theorem 4.1 because  $\mathcal{L}_{\text{ASC}}$  expands  $\mathcal{L}_{\text{SC}}$  only by relational symbols.

Every scaled lattice  $L$  admits a unique structure of ASC-lattice which is an expansion by definition of its lattice structure. We denote by  $L^{\text{At}}$  this expansion of  $L$ .

**Proposition 8.4 (Linear representation)** *Let  $K$  be an infinite field and  $L_0$  be a finite sub-ASC-lattice. For every integer  $N \geq 0$  there exists a special linear set  $X_N$  over  $K$  and an  $\mathcal{L}_{\text{ASC}}^*$ -embedding  $\varphi_N: L_0 \rightarrow L_{\text{lin}}^{\text{At}}(K^m)$  such that for every atom  $a$  of  $L_0$ :*

- If  $\text{asc}(a) > 0$  then  $\text{asc}(\varphi_N(a)) = \text{asc}(a)$ .
- If  $\text{asc}(a) = 0$  then  $\text{asc}(\varphi_N(a)) \geq N$ .

*Proof:* By induction on lexicographically ordered tuples of integers  $(r, s)$  we prove that the result is true for every finite sub-ASC-lattice  $L_0$  having  $r$   $\vee$ -irreducible elements,  $s$  of which have the same sc-dimension as  $L_0$ .

If  $r = 0$  then  $s = 0$  and the unique embedding of  $L_0 = \{\mathbf{0}\}$  into  $L_{\text{lin}}^{\text{At}}(P)$ , for an arbitrary point  $P$  of  $K$ , has the required property. So let us assume that  $r \geq 1$  and that the result is proved for every  $(r', s') < (r, s)$ . Let  $d = \text{sc-dim } L$  and  $a_1, \dots, a_r$  be the elements of  $\mathcal{I}(L_0)$  ordered by increasing sc-dimension, so that  $\text{sc-dim } a_r = d \geq 0$ .

*Case 1:*  $d = 0$ . Then  $L$  is a boolean algebra and  $a_1, \dots, a_r$  are its atoms. Let  $A_1, \dots, A_r$  be pairwise disjoint subsets of  $K$  such that:

- If  $\text{asc}(a_i) > 0$  then  $A_i$  has  $\text{asc}(a_i)$  elements, so  $\text{asc}(A_i) = \text{asc}(a_i)$ .
- If  $\text{asc}(a_i) = 0$  then  $A_i$  has  $N$  elements, so  $\text{asc}(A_i) = N$ .

Let  $X$  be the union of all these  $A_i$ 's. Clearly the map  $\varphi$  which maps each  $a_i$  to  $A_i$  extends uniquely to an  $\mathcal{L}_{\text{SC}}$ -embedding of  $L_0$  into  $L_{\text{lin}}^{\text{At}}(X)$  which has the required properties.

*Case 2:*  $d > 0$ . The upper semi-lattice  $L_0^-$  generated by  $a_1, \dots, a_{r-1}$  is an  $\mathcal{L}_{\text{ASC}}^*$ -substructure of  $L_0$  to which the induction hypothesis applies. This gives for some integer  $m$  a special linear set  $B \subseteq K^m$  over  $K$  and an  $\mathcal{L}_{\text{SC}}$ -embedding  $\psi: L_0^- \rightarrow L_{\text{lin}}^{\text{At}}(B)$  having the required properties. Let  $C = \varphi(\mathbf{1}_{L_0^-} \wedge a_r)$  and  $n = \text{sc-dim } a_r$ . Proposition 5.2 gives a special linear set  $A \subseteq K^{m+n}$  such that  $A \cap B = C$ . One can extend  $\psi$  to an  $\mathcal{L}_{\text{SC}}$ -embedding  $\varphi$  of  $L_0$  into  $L_{\text{lin}}^{\text{At}}(A \cup B)$  exactly like in the proof of Proposition 5.3. Then  $\varphi$  inherits from  $\psi$  the required properties because all the atoms of  $L$  already belong to  $L_0^-$ .

■

Let  $\text{ASC}_{\text{Zar}}(K, d)$ ,  $\text{ASC}_{\text{lin}}(K, d)$ ,  $\text{ASC}_{\text{def}}(K, d)$  denote the class of all ASC-lattices  $L^{\text{At}}$  for  $L$  ranging over  $\text{SC}_{\text{Zar}}(K, d)$ ,  $\text{SC}_{\text{lin}}(K, d)$ ,  $\text{SC}_{\text{def}}(K, d)$  respectively.

**Corollary 8.5** *For every integer  $d \geq 0$ , the universal theories of  $\text{ASC}_{\text{def}}(\mathcal{K}, d)$  (resp. of  $\text{ASC}_{\text{Zar}}(K, d)$  or  $\text{ASC}_{\text{lin}}(K, d)$ ) is the same for every  $o$ -minimal or  $P$ -minimal expansion of a field  $K$  (resp. every infinite field  $K$ ). This is the theory of sub-ASC-lattices.*

*Proof:* Since  $\text{ASC}_{\text{lin}}(K, d)$  is contained in the other classes, all of which are contained in the class of ASC-lattices, it suffices to prove that conversely every sub-ASC-lattice  $\mathcal{L}_{\text{ASC}}$ -embeds into an ultraproduct of elements of  $\text{ASC}_{\text{lin}}(K, d)$ . By the model-theoretic compactness theorem, it suffices to prove it for any finitely generated sub-ASC-lattice  $L_0$ .

By Theorem 4.1,  $L_0$  is finite. For any integer  $N \geq 0$  let  $\varphi_N : L_0 \rightarrow L_{\text{lin}}^{\text{At}}(X_N)$  be an  $\mathcal{L}_{\text{SC}}$ -embedding given by Proposition 8.4. Let  $\mathcal{U}$  be a non principal ultrafilter in the Boolean algebra of subsets of  $\mathbf{N}$ , and consider the ultraproduct  $L = \prod_{N \in \mathbf{N}} L_{\text{lin}}^{\text{At}}(X_N) / \mathcal{U}$ . Then  $\varphi = \prod_{N \in \mathbf{N}} \varphi_N / \mathcal{U}$  is an  $\mathcal{L}_{\text{SC}}$ -embedding of  $L_0$  into the  $L$ . In order to prove that it is an  $\mathcal{L}_{\text{ASC}}$ -embedding, by Remark 8.2 it remains check that for every atom  $a$  of  $L_0$ ,  $\text{asc}(\varphi(a)) = \text{asc}(a)$ . So let  $a$  be an atom of  $L_0$  and  $k = \text{asc}(a)$ .

If  $k > 0$  then for every  $N \geq k$ ,  $L_{\text{lin}}^{\text{At}}(X_N) \models \text{At}_k(\varphi_N(a))$  by construction. So  $L \models \text{At}_k(\varphi(a))$ , that is  $\text{asc}(\varphi(a)) = k$ .

If  $k = 0$ , let  $l$  be any strictly positive integer. For every  $N \geq l$ ,  $L_{\text{lin}}^{\text{At}}(X_N) \models \text{At}_N(\varphi_N(a))$  by construction, hence  $L_{\text{lin}}^{\text{At}}(X_N) \not\models \text{At}_l(\varphi_N(a))$ . So  $L \not\models \text{At}_l(\varphi(a))$ , and this being true for every  $l > 0$  it follows that  $\text{asc}(\varphi(a)) = 0$ .

■

## 9 Model-completion of atomic scaled lattices

Let us call **super ASC-lattices** those ASC-lattices which satisfy the following axioms, all of which are axiomatizable by  $\forall\exists$ -formulas in  $\mathcal{L}_{\text{ASC}}$ . We are going to show that its theory is the model-completion of the theory of sub-ASC-lattices of dimension at most  $d$  (resp. exactly  $d$ ).

**Atomicity:** Every element  $x$  is the least upper bound of the set of atoms smaller than  $x$ .

**Catenarity:** For every non-negative integers  $r \leq q \leq p$  and every elements  $c \leq a \neq \mathbf{0}$ , if  $c$  is  $r$ -sc-pure and  $a$  is  $p$ -sc-pure then there exists a non-zero  $q$ -sc-pure element  $b$  such that  $c \leq b \leq a$ .

**ASC-Splitting:** For every  $b_1, b_2, a$ , if  $b_1 \vee b_2 \ll a \neq \mathbf{0}$  and  $C^0(a) = \mathbf{0}$  there exists non-zero elements  $a_1 \geq b_1$  and  $a_2 \geq b_2$  such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

**Remark 9.1** An immediate consequence of the atomicity axiom is that for every elements  $x, y$  in a super ASC-lattice  $L$  such that  $y < x$  and  $\text{sc-dim}(x - y) \geq 1$ , there are infinitely many atoms  $a \in L$  such that  $a \leq x$  and  $a \wedge y = \mathbf{0}$ . Indeed let  $A$  be the set of atoms  $a \in L$  such that  $a \leq x - y$ , and  $B$  the subset of those  $a$  such that  $a \wedge y \neq \mathbf{0}$ . Assume for a contradiction that  $B$  is finite and let  $b = \bigvee_{a \in B} a$ . Note that  $b \leq y$  and  $\text{sc-dim } b = \dim b = 0$ . Then by the Atomicity axiom

$$x - y = \bigvee_{a \in A} a \leq y \vee b, \text{ hence } x - y \leq (y \vee b) - y = b - y \leq b.$$

This implies that  $\text{sc-dim}(x - y) \leq \text{sc-dim } b = 0$ , a contradiction.

Primitive tuples and sc-primitive extensions are defined for sub-ASC-lattices exactly like for subscaled lattices.

We define **ASC-signatures** in a finite sub-ASC-lattice  $L_0$  as triples  $(g, H, q)$  with  $H$  a set of non-necessarily distinct pairs  $(h_1, k_1), (h_2, k_2)$  in  $L_0 \times \mathbf{N}$  such that  $(g, \{h_1, h_2\}), q$  is an SC-signature in the  $\mathcal{L}_{\text{SC}}$ -reduct of  $L_0$  and (see Example 9.2):



1. If  $q < \text{sc-dim } g$  then  $k_1 = k_2$ .  
Note that  $h_1 = h_2$  in this case because  $(g, \{h_1, h_2\}, q)$  is a signature.
2. If  $q \neq 0$  then  $k_1 = k_2 = 0$ .
3. If  $k_1 = 0$  or  $k_2 = 0$  then  $\text{asc}(g) = 0$ .
4. If  $k_1 \neq 0, k_2 \neq 0$  and  $\text{sc-dim } g = 0$  then  $\text{asc}(g) = k_1 + k_2$ .

**Example 9.2** Let  $L_0$  be a finite sub-ASC-lattice, and  $L$  an  $\mathcal{L}_{\text{SC}}$ -extension of  $L_0$  generated by a (necessarily unique) SC-primitive tuple  $(x_1, x_2)$ . Let  $(g, \{h_1, h_2\}, q)$  be the SC-signature of  $L$  in  $L_0$  and  $k_i = \text{asc}(x_i)$ . Then  $(g, \{(h_1, k_1), (h_2, k_2), q\})$  is easily an ASC-signature in  $L_0$ . We call it the **ASC-signature** of  $L$  and of  $(x_1, x_2)$  in  $L_0$ .

The same argument as in Proposition 6.4.2 shows (using Remark 8.2) that two SC-primitive extensions of a finite sub-ASC-lattice  $L_0$  are  $\mathcal{L}_{\text{ASC}}$ -isomorphic over  $L_0$  if and only if they have the same ASC-signature in  $L_0$ .

**Lemma 9.3** *Let  $L_0$  be a finite  $\mathcal{L}_{\text{ASC}}$ -substructure of a super ASC-lattice  $\hat{L}$ . Let  $\sigma_{\text{At}} = (g, q, \{(h_1, k_1), (h_2, k_2)\})$  be an ASC-signature in  $L_0$ . Assume that  $q \neq 0$  or  $k_1 k_2 \neq 0$ . Otherwise assume that  $\hat{L}$  is  $\aleph_0$ -saturated. Then there exists a primitive tuple  $(x_1, x_2) \in \hat{L}$  over  $L_0$  whose signature is  $\sigma_{\text{At}}$ .*

*Proof:* Let  $\sigma = (g, \{h_1, h_2\}, q)$ . This is an SC-signature in  $L_0$  (more precisely in its  $\mathcal{L}_{\text{SC}}$ -reduct).

*Case 1:*  $\text{sc-dim } g \geq 1$  and  $q \geq 1$ . Then  $C^0(g) = 0$  and by definition of ASC-signatures  $k_1 = k_2 = 0$ . By Remark 7.4 there is an SC-primitive tuple  $(x_1, x_2)$  in  $\hat{L}$  with signature  $\sigma$  in  $L_0$ . Moreover each  $\text{asc } x_i = 0$  (because  $\text{sc-dim } x_i = p \geq 1$ ) and each  $k_i = 0$ , so the ASC-signature of  $(x_1, x_2)$  is  $\sigma_{\text{At}}$ .

*Case 2:*  $\text{sc-dim } g \geq 1$  and  $q = 0$ . Then  $C^0(g) = 0$  again and since  $\text{sc-dim}(h_1 \vee h_2) < q = 0$  by definition of SC-signatures we get that  $h_1 = h_2 = \mathbf{0}$ . Finally  $k_1 = k_2$  by definition of ASC-signatures since  $q = 0 < \text{sc-dim } g$ . By Remark 9.1 there are infinitely many atoms  $z$  in  $\hat{L}$  such that  $z \leq g$  and  $z \wedge g^- = \mathbf{0}$ . If  $k_1 > 0$  let  $x$  be the join of  $k_1$  such atoms of  $\hat{L}$ . Otherwise  $\hat{L}$  is  $\aleph_0$ -saturated by assumption hence it contains an element  $x \leq g$  of dimension 0 such that  $x \wedge g^- = \mathbf{0}$  and  $\hat{L}(x)$  has infinitely many atoms. By the Atomicity Property  $\text{asc}(x) = 0$ . So in both cases  $(x, x)$  is an SC-primitive tuple over  $L_0$  with ASC-signature  $\sigma_{\text{At}}$ .

*Case 3:*  $\text{sc-dim } g = 0$ . Then  $q = 0$ ,  $g$  is an atom of  $L_0$  and  $h_1 = h_2 = \mathbf{0}$ . In each of the two remaining sub-cases, we build a tuple  $(x_1, x_2)$  and leave as an exercise to check that  $(x_1, x_2)$  is SC-primitive over  $L_0$  with ASC-signature  $\sigma_{\text{At}}$ .

If  $k_1$  and  $k_2$  are non-zero then  $\text{asc}(g) = k_1 + k_2$  hence  $\hat{L}(g)$  contains  $k_1 + k_2$  atoms. Let  $x_1$  be the join of  $k_1$  of them and  $x_2$  be the join of the others.

Otherwise, by symmetry we can assume that  $k_1 = 0$ . Then  $\text{asc}(g) = 0$  by definition of ASC-signatures so  $\hat{L}(g)$  contains infinitely many atoms. By  $\aleph_0$ -saturation it follows that  $\hat{L}$  contains an element  $x$  lower than  $g$  such that both  $L(x)$  and  $L(g - x)$  contain infinitely many atoms, hence  $\text{asc}(x) = \text{asc}(g - x) = 0$ . If  $k_2 = 0$  let  $(x_1, x_2) = (x, g - x)$ . Otherwise let  $x_2$  be the join of  $k_2$  atoms in  $\hat{L}(g)$  and let  $x_1 = g - x_2$ .

■

**Theorem 9.4** *The theory of super ASC-lattices of sc-dimension at most  $d$  (resp. exactly  $d$ ) is the model-completion of the theory of ASC-lattices of dimension at most  $d$  (resp. exactly  $d$ ). It admits  $\aleph_0$  completions, each of which is decidable, and it is decidable.*

*Proof:* We only sketch the proof of the first statement as it essentially the same as for Theorem 7.3.

On one hand, given a finite sub-ASC-lattice  $L_0$ , we can embed it in an extension satisfying the Atomicity and Catenary Property by Proposition 8.4, and the ASC-Splitting Property by means of Lemma 7.2 applied to any  $a, b_1, b_2 \in L_0$  such that  $b_1 \vee b_2 \ll a \neq \mathbf{0}$  and  $C^0(a) = \mathbf{0}$  (note that this last assumption ensures that the extension built in Lemma 7.2 is an  $\mathcal{L}_{\text{ASC}}$ -extension). That every existentially closed sub-ASC-lattice is a super ASC-lattice then follows, by the model-theoretic compactness theorem.

On the other hand, given an  $\aleph_0$ -saturated super ASC-lattice  $\hat{L}$ , a finite  $\mathcal{L}_{\text{ASC}}$ -substructure  $L_0$  and a finite extension  $L$  of  $L_0$ , we reduce to the case where  $L$  is SC-primitive and let  $\sigma$  be its ASC-signature in  $L_0$ . Lemma 9.3 gives an SC-primitive extension  $L_1$  of  $L_0$  in  $\hat{L}$  with the same signature in  $L_0$ , hence an embedding of  $L_1$  into  $\hat{L}$  over  $L_0$  (which maps  $L$  to  $L_1$ ).

This proves the first statement. The last one follows because there are finitely many subscaled lattices of dimension at most  $d$  (resp. exactly  $d$ ) and on each of them  $\aleph_0$  different structures of sub-ASC-lattices. So the completions of the theory of super ASC-lattices, which are determined by their prime model, can be recursively enumerated.

■

We say that a sub-ASC-lattice  $L$  is **standard** if every element of sc-dimension 0 belongs to some  $\text{At}_k(L)$  for some  $k > 0$ . The existence of standard super ASC-lattices (see Section 10) and non-standard super ASC-lattices (by the model theoretic compactness theorem) implies that the theory of super ASC-lattices containing a given prime sub-ASC-lattice is not  $\aleph_0$ -categoric, contrary to what happens for super scaled lattice. However we can recover  $\aleph_0$ -categoricity by restricting to standard models.

**Proposition 9.5** *Let  $L_1, L_2$  be two standard countable super ASC-lattices. Then every  $\mathcal{L}_{\text{ASC}}$ -isomorphism from a finite sub-ASC-lattice  $L_{1,0} \subset L_1$  to a sub-ASC-lattice  $L_{2,0} \subset L_2$  extends to an  $\mathcal{L}_{\text{ASC}}$ -isomorphism from  $L_1$  to  $L_2$ . In particular  $L_1$  and  $L_2$  are isomorphic if and only if their prime  $\mathcal{L}_{\text{ASC}}$ -substructures (those generated by the empty set) are isomorphic.*

*Proof:* Let  $\varphi$  be an  $\mathcal{L}_{\text{ASC}}$ -isomorphism from  $L_{1,0}$  to  $L_{2,0}$ . Pick any element  $x \in L_1 \setminus L_{1,0}$ . The subscaled lattice generated in  $L_1$  by  $L_{1,0} \cup \{x\}$  (more precisely their  $\mathcal{L}_{\text{SC}}$ -reducts) is finite hence by Proposition 6.4.3 there is a chain  $L_{1,0} \subset L_{1,1} \subset \dots \subset L_{1,r}$  of SC-primitive extensions of subscaled lattices such that  $L_{1,0} \cup \{x\} \subseteq L_{1,r}$ . Endow each  $L_{1,i}$  with the  $\mathcal{L}_{\text{ASC}}$ -structure induced by  $L_1$ . It suffices to prove that  $\varphi$  extends to an  $\mathcal{L}_{\text{ASC}}$ -embedding  $\varphi_1 : L_{1,1} \rightarrow L_2$ . Indeed, repeating the argument will give an  $\mathcal{L}_{\text{ASC}}$ -embedding  $\varphi_r : L_{1,r} \rightarrow L_2$  extending  $\varphi$ , and by symmetry the conclusion will then follow by Fraïssé's va-et-vient.

Identifying  $L_{1,0}$  with its image by  $\varphi$  we can replace  $L_{1,0}$  and  $L_{2,0}$  by a common  $\mathcal{L}_{\text{ASC}}$ -structure  $L_0$  of  $L_1$  and  $L_2$ . Now  $L_{1,1}$  is generated over  $L_0$  by an SC-primitive tuple  $(x_1, x_2)$  with signature  $\sigma_{\text{At}} = (g, \{(h_1, k_1), (h_2, k_2)\}, q)$ . In particular  $q = \text{sc-dim } x_i$  and  $k_i = \text{asc}(x_i)$  for  $i = 1, 2$ . If  $q = 0$  then for each  $i$ ,  $\text{sc-dim } x_i = 0$  hence  $k_i > 0$  because  $L_1$  is standard. With other words  $q \neq 0$  or  $k_1 k_2 \neq 0$  hence Lemma 9.3 gives an sc-primitive tuple  $(y_1, y_2)$  in  $L_2$  with signature  $\sigma_{\text{At}}$ . Let  $L_{2,1}$  be the asc-substructure of  $L_2$  generated by  $L_{1,1} \cup \{y_1, y_2\}$ . By Proposition 6.4.2  $\varphi$  extends to an  $\mathcal{L}_{\text{SC}}$ -isomorphism  $\varphi_1$  from  $L_{1,1}$  to  $L_{2,1}$  which maps each  $x_i$  to  $y_i$ . By construction  $\text{asc}(x_i) = \text{asc}(y_i)$ , and by Proposition 6.4.1  $\varphi_1$  is the identity map on  $L_0$ , so  $\text{asc}(\varphi_1(z)) = \text{asc}(z)$  for every  $z \in \mathcal{I}(L_{1,1})$ . Hence  $\varphi_1$  is an  $\mathcal{L}_{\text{ASC}}$ -isomorphism by Remark 8.2, which proves the result.

■

## 10 Applications to lattices of $p$ -adic semi-algebraic sets

In all this section  $K$  denotes a fixed  $p$ -adically closed field. For every semi-algebraic set  $X$  contained in  $K^m$  we let  $L(X)$  denote the lattice of semi-algebraic subsets of  $X$  closed in  $X$ , endowed with its natural structure of ASC-lattice. Note that every  $A \in L(X)$  of dimension 0 is finite, hence  $L(X)$  is *standard*.

As already mentioned in the introduction, the results of the previous section lead us to conjecture in [Dar06] and finally to prove in [Dar17] the following result.

**Theorem 10.1 (Theorem 3.4 in [Dar17])** *Let  $X$  be a non-empty semi-algebraic subset of  $K^m$  without isolated points. Assume that  $X$  is open in its topological closure  $\overline{X}$  and let  $Y_1, \dots, Y_s$  be a collection of closed semi-algebraic subsets of  $\partial X = \overline{X} \setminus X$  such that  $Y_1 \cup \dots \cup Y_s = \partial X$ . Then there is a partition of  $X$  in non-empty semi-algebraic sets  $X_1, \dots, X_s$  such that  $\partial X_i = Y_i$  for  $1 \leq i \leq s$ .*

We can now combine this theorem with the results of Section 10 in order to get the following applications.

**Theorem 10.2** *Let  $X$  be any semi-algebraic subset of  $K^m$ . Then  $L(X)$  is a super ASC-lattice. In particular its complete theory is decidable and eliminates the quantifiers in  $\mathcal{L}_{\text{ASC}}$ .*

*Proof:* By construction  $L(X)$  is an ASC-lattice satisfying the Atomicity property. The Catenarity Property is well known (it follows for example from Denef's Cell Decomposition) hence we only have to check the Splitting Property. So let  $A, B_1, B_2 \in L(X)$  such that  $B_1 \cup B_2 \ll A$  and  $A$  has no isolated point.

The same holds true for their closures in  $K^m$ , denoted  $\overline{A}, \overline{B}_1, \overline{B}_2$ . Apply Theorem 10.1 to  $X = \overline{A} \setminus (\overline{B}_1 \cup \overline{B}_2)$ ,  $Y_1 = \overline{B}_1$  and  $Y_2 = \overline{B}_2$ . It gives a partition of  $X$  in non-empty semi-algebraic sets  $X_1, X_2$  whose frontiers are respectively  $\overline{B}_1, \overline{B}_2$ . Then  $\overline{X}_1 \cup \overline{X}_2 = \overline{A}$ ,  $\overline{X}_1 \cap \overline{X}_2 = \overline{B}_1 \cap \overline{B}_2$  and each  $\overline{X}_i = X_i \cup \overline{B}_i$ . Let  $A_1 = \overline{X}_1 \cap A = (X_1 \cap A) \cup B_1$  and define  $A_2$  accordingly. We have to check that  $A_1, A_2$  split  $A$  along  $B_1, B_2$ .

$A$  is dense in  $\overline{A} = \overline{X}$  and  $X_1 = \overline{X} \setminus (X_2 \cup \overline{B}_1 \cup \overline{B}_2) = \overline{X} \setminus (\overline{X}_2 \cup \overline{B}_1)$  is open in  $\overline{X}$ , hence  $A \cap X_1$  is dense in  $X_1$ . In particular  $A \cap X_1 \neq \emptyset$ , and symmetrically  $A \cap X_2 \neq \emptyset$ . Clearly  $A_1 \cup A_2 = A$ ,  $A_1 \cap A_2 = B_1 \cap B_2$  and each  $A_i \supseteq B_i$  by construction. So it only remains to check that  $A - A_1 = A_2$  in  $L(X)$ , that is that the closure of  $A \setminus A_1$  in  $X$  (hence in  $A$ ) is  $A_2$ . Note that  $A \setminus A_1 = A \setminus \overline{X}_1$  and

$$\overline{A} \setminus \overline{X}_1 = (X_1 \cup X_2 \cup \overline{B}_1 \cup \overline{B}_2) \setminus (X_1 \cup \overline{B}_1) = X_2 \cup (\overline{B}_2 \setminus \overline{B}_1).$$

In particular  $A \setminus A_1 = A \setminus \overline{X}_1 = (\overline{A} \setminus \overline{X}_1) \cap A$  contains  $X_2 \cap A$  and is contained in  $(X_2 \cup \overline{B}_2) \cap A = \overline{X}_2 \cap A = A_2$ . The conclusion will follow, if we can prove that  $X_2 \cap A$  is dense in  $A_2$ . Since  $A_2 = (X_2 \cap A) \cup B_2$  it suffices to check that  $B_2 \subseteq \overline{X}_2 \cap A$ . But this is clear since  $X_2 \cap A$  is dense in  $X_2$ , hence in  $\overline{X}_2 = X_2 \cup \overline{B}_2$ .

■

**Corollary 10.3** *Let  $F$  be a  $q$ -adically closed field (for some prime  $q$  not necessarily equal to  $p$ ). Let  $X \subseteq K^m$  and  $Y \subseteq F^n$  be two semi-algebraic sets.*

1. *If  $m = n$ ,  $K \preceq F$  and  $X = Y \cap K^n$  then  $L(X) \preceq L(Y)$ .*
2.  *$L(X) \equiv L(Y) \iff$  their prime  $\mathcal{L}_{\text{ASC}}$ -substructures are isomorphic.  
In particular  $L(K^m) \equiv L(F^n)$  if and only if  $m = n$ .*

3. If  $K$  and  $F$  are countable then  $L(X) \equiv L(Y) \iff L(X) \simeq L(Y)$ .

*Proof:* The two first points follow immediately from Theorem 10.2. Note that  $L(K^m) \equiv L(F^m)$  is a special case because their prime sublattice is just the two-element lattice with the same  $\mathcal{L}_{\text{ASC}}$ -structure, because  $K^m$  and  $F^m$  both have pure dimension  $m$ . The last point follows from Proposition 9.5 since both  $L(X)$  and  $L(Y)$  are standard and countable.

■

Given a pair of semi-algebraic sets  $X \subseteq K^m$  and  $Y \subseteq F^n$ , we say that a homeomorphism  $\psi : X \rightarrow Y$  is **pre-algebraic** if for every semi-algebraic sets  $A \subseteq X$  and  $B \subseteq Y$  defined over  $K$  and  $F$  respectively,  $\varphi(A)$  and  $\varphi^{-1}(B)$  are still semi-algebraic sets defined over  $K$  and  $F$ . It is obviously sufficient to check this for semi-algebraic sets  $A, B$  closed in  $X, Y$  respectively. With other words, a bijection  $\psi : X \rightarrow Y$  is a pre-algebraic if and only if taking direct images by  $\psi$  defines an  $\mathcal{L}_{\text{ASC}}$ -isomorphism from  $L(X)$  to  $L(Y)$  (which also ensures that  $\psi$  is a homeomorphism). When  $K = F$ , semi-algebraic homeomorphisms are obviously pre-algebraic. The converse is false, as the following example shows.

**Example 10.4** Assume that the  $p$ -valuation of  $K$  has value group  $\mathbf{Z}$ , and let  $R$  be its valuation ring. Applying Theorem 10.5 below to  $X = K$  and  $Y = R$  gives a pre-algebraic homeomorphism  $\varphi : K \rightarrow R$ . Since its value group is  $\mathbf{Z}$ , the  $p$ -valuation defines a metric on  $K$  and its completion  $K'$  is known to be an elementary extension of  $K$ . If  $\varphi$  would be semi-algebraic, it would then uniquely extend to a semi-algebraic homeomorphism from  $K'$  to its  $p$ -valuation ring  $R'$ . But this is not possible because  $R'$  is compact and  $K'$  is not. Thus  $\varphi$  is not semi-algebraic.

**Theorem 10.5** *Let  $K, F$  be countable  $p$ -adically closed fields, and  $X \subseteq K^m, Y \subseteq F^n$  be two semi-algebraic sets. Let  $L^0(X)$  and  $L^0(Y)$  be the prime  $\mathcal{L}_{\text{ASC}}$ -substructures of  $L(X)$  and  $L(Y)$  respectively. Then  $X$  and  $Y$  are pre-algebraically homeomorphic if and only if  $L^0(X)$  and  $L^0(Y)$  are  $\mathcal{L}_{\text{ASC}}$ -isomorphic. In particular, any two semi-algebraic sets over  $K$  and  $F$  with the same pure dimension  $d \geq 1$  are pre-algebraically homeomorphic.*

*Proof:* One direction is obvious: every pre-algebraic homeomorphism  $\psi : X \rightarrow Y$  induces an  $\mathcal{L}_{\text{ASC}}$ -isomorphism from  $L(X)$  to  $L(Y)$ , which maps their respective prime  $\mathcal{L}_{\text{ASC}}$ -substructures one to each other. Conversely, assume that an  $\mathcal{L}_{\text{ASC}}$ -isomorphism is given from  $L^0(X)$  to  $L^0(Y)$ . By Proposition 9.5 it extends to an  $\mathcal{L}_{\text{ASC}}$ -isomorphism  $\varphi : L(X) \rightarrow L(Y)$ . For every  $t \in X$ ,  $\varphi$  maps  $\{t\}$  to an atom  $\{t'\}$  of  $L(Y)$ . Let  $\psi(t) = t'$ , this defines a bijection  $\psi : X \rightarrow Y$  such that  $\psi(A) = \varphi(A)$  for every  $A \in L(X)$ , hence  $\psi$  is a pre-algebraic homeomorphism. The last statement follows.

■

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