Scaled lattices of closed p-adic semi-algebraic sets

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Abstract

Let p be prime number, K be a p-adically closed field, $X \subseteq K^m$ a semi-algebraic set defined over K and $L(X)$ the lattice of semi-algebraic subsets of X which are closed in X. We prove that the complete theory of $L(X)$ eliminates the quantifiers in a certain language $\mathcal{L}_{\mathrm{ASC}}$, the $\mathcal{L}_{\mathrm{ASC}}$-structure on $L(X)$ being an extension by definition of the lattice structure. Moreover it is decidable, contrary to what happens over a real closed field. We classify these $\mathcal{L}_{\mathrm{ASC}}$-structures up to elementary equivalence, and get in particular that the complete theory of $L(K^m)$ only depends on m, not on K nor even on p. As an application we obtain a classification of semi-algebraic sets over countable p-adically closed fields up to so-called "pre-algebraic" homeomorphisms.

1 Introduction

Our knowledge of the geometry of semi-algebraic sets over p-adic fields, that is the field $\mathbf{Q}_p$ of p-adic numbers and its finite extensions, has grown intensively in the past decades. After Macintyre's quantifier elimination theorem [Mac76], Denef's cell decomposition theorem [Den84] lead to develop a good dimension theory for p-adic semi-algebraic sets [SvdD88], and to classify them up to semi-algebraic bijection [Clu01]. There has been recently a new step forward with the triangulation of p-adic semi-algebraic sets [Dar17]. All this has been generalised to p-adically closed fields, that is fields which are elementarily equivalent to a p-adic one. In the present paper we use a consequence of the p-adic triangulation to prove the next result (Theorem 10.2).

Theorem A *Let K be a p-adically closed field, $X \subseteq K^m$ a semi-algebraic set and $L(X)$ the lattice of all semi-algebraic subsets of X which are closed in X. Then the complete theory of $L(X)$ is decidable, and eliminates the quantifier in $\mathcal{L}_{\mathrm{ASC}}$ (a certain expansion by definition of the lattice language, see below).*

Remark 1.1 In particular $L(\mathbf{Q}_p^m)$ is decidable for every positive integer m. This contrasts with the known results on archimedian local fields: $L(\mathbf{R}^m)$ and $L(\mathbf{C}^m)$ are undecidable for every $m \geq 2$ by [Grz51] (see [Tre17] for more on this topic). Note that the decomposition of semi-algebraic sets over $\mathbf{R}$ in finitely many semi-algebraic connected components is crucial in the proof of Grzegorczyk's undecidability result, a property which disappears in the p-adic case.

The above result follows from a fine-grain study of the model-theory of various classes of geometric lattices such as the following ones.

*This paper is a major revision of [Dar06], with extended results.

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Example 1.2 Let $\mathcal{K}$ be an *o*-minimal or *P*-minimal expansion[1] of a field K. For every definable sets $A, B \subseteq X \subseteq K^m$ let $A - B = \overline{A \setminus B} \cap A$ where the overline stands for the topological closure w.r.t. the topology induced on K^m by the order or the p-valuation of K. For every $a \in A$, the **local dimension** of A at a is the maximum of the dimensions of the definable neighborhoods of a in A, and A is called **pure dimensional** if it has the same local dimension at every point. For every non-negative integer i let

$$\mathrm{C}^i(A) = \overline{\{a \in A \,/\, \dim(A, a) = i\}} \cap A.$$

This is a definable subset of A called the ***i*-pure component** of A. We call $\mathrm{L}_{\mathrm{def}}(X)$ the lattice of all the definable subsets of X which are closed in X, enriched with the above functions "$-$" and C^i for every i.

Let $\mathcal{L}_{\mathrm{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$ be the language of lattices, and $\mathcal{L}_{\mathrm{SC}} = \mathcal{L}_{\mathrm{lat}} \cup \{-, (C^i)_{i \geq 0}\}$ be its expansion the above function symbols. Finally let $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ be the class of these $\mathcal{L}_{\mathrm{SC}}$-structures $\mathrm{L}_{\mathrm{def}}(X)$ for all the sets X of dimension at most d definable over $\mathcal{K}$. A similar construction can be done over a pure field K, with the Zariski topology on K^m. We let $\mathrm{SC}_{\mathrm{Zar}}(K, d)$ denote the corresponding class of $\mathcal{L}_{\mathrm{SC}}$-structures.

Theorem B *Given any non-negative integer d, the universal theories of $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ (resp. $\mathrm{SC}_{\mathrm{Zar}}(K, d)$) in the language $\mathcal{L}_{\mathrm{SC}}$ are the same for every o-minimal or P-minimal expansion $\mathcal{K}$ of a field K (resp. for every infinite field K).*

In order to prove this we give in Section 2 an explicit list of universal axioms for a theory T_d in $\mathcal{L}_{\mathrm{SC}}$, the models of which we call d-subscaled lattices. All the examples given above are d-scaled lattices, a natural subclass of d-subscaled lattices (the class of d-scaled lattices is elementary but not universal). After some technical preliminaries in Section 3 we prove in Section 4 that every finitely generated d-subscaled lattice is finite. Combining this with a linear representation for finite d-subscaled lattices (Proposition 5.3) and with the model-theoretic compactness theorem, we then prove in Section 5 the theory of d-subscaled lattices is exactly the universal theory of $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ and of $\mathrm{SC}_{\mathrm{Zar}}(K, d)$ (Theorem 5.3).

A detailed study of the minimal finitely generated extensions of finite d-subscaled lattices, achieved in Section 6, leads us in Section 7 to our third main result (Theorem 7.3 and Corollary 7.5).

Theorem C *For every non-negative integer d, the theory of d-subscaled lattices admits a model-completion $\overline{T}_d$ which is finitely axiomatizable and $\aleph_0$-categorical. Moreover $\overline{T}_d$ has finitely many prime models, hence it is decidable as well as all its completions.*

Remark 1.3 Since 0-subscaled lattices are exactly non-trivial boolean algebras (in which the $\mathcal{L}_{\mathrm{SC}}$-structure and the boolean structure are quantifier-free bi-definable) the above model-completion result for subscaled lattices is a generalisation to arbitrary finite dimension d of the well known theorem on the model-completion of boolean algebras.

Remark 1.4 Our d-subscaled lattices form (after language reduction) an elementary class of co-Heyting algebras, closely related to the "slices" studied in [Hos67], [Ono71] and others (see [DJ11] for a more precise account of this relation). So this paper might also be considered as a contribution to the model-theory of co-Heyting algebras with at most d slices.

[1] The reader unfamiliar with *o*-minimality, and *P*-minimality may consult [vdD98], [HM97], or restrict to the special case where $K = \mathbf{R}$ or K is a finite extension of $\mathbf{Q}_p$, and consider that "definable" is a synonym for "semi-algebraic".

The axiomatization of $\bar{T}_d$ given in Section 7 consists in a pair of axioms expressing a "Catenarity" and a "Splitting" property which both have a natural topological and geometric meaning. In particular the Splitting Property expresses a very strong form of disconnectedness, which implies that the models of $\bar{T}_d$ are atomless. So we develop in Sections 8 and 9 a variant of this quantifier elimination result in a language $\mathcal{L}_{\mathrm{ASC}} = \mathcal{L}_{\mathrm{SC}} \cup \{\mathrm{At}_k\}_{k\geq 1}$, where each At_k is a unary predicate symbol, to be interpreted as the set of elements which are the join of exactly k atoms. The model-completion $\bar{T}_d^{At}$ that we obtain is axiomatized by the Catenarity property and a little restriction of the Splitting Property which preserves the atoms. This theory $\bar{T}_d^{At}$ has $\aleph_0$ prime models which can easily be classified in terms of the prime models of $\bar{T}_d$, from which it follows that it is decidable as well as all its completions (Theorem 9.4).

In the initial version of this paper [Dar04] we conjectured that $\mathrm{L}_{\mathrm{def}}(\mathbf{Q}_p^d)$ might be a natural model of $\bar{T}_d^{At}$. This intuition proved to be crucial in the proof of the triangulation of semi-algebraic sets over a p-adically closed field [Dar17]. Conversely it follows from this triangulation that $\mathrm{L}_{\mathrm{def}}(X)$ is indeed a model of $\bar{T}_d^{At}$, for every semi-algebraic[2] set $X \subseteq K^m$ of dimension $\leq d$, from which Theorem A follows in the last Section 10.

The prime $\mathcal{L}_{\mathrm{ASC}}$-substructure of $\mathrm{L}_{\mathrm{def}}(X)$ (which is generated by the empty set) is finite. We expect it to be an essential invariant for the classification of semi-algebraic sets over p-adically closed fields up to semi-algebraic homeomorphisms. Such a classification is far from being achieved, but the next result exemplifies the role that this invariant can play in a weaker classification, up to "pre-algebraic" homeomorphisms (this is defined in Section 10).

Theorem D *Let K, F be* countable *p-adically closed fields, and $X \subseteq K^m$, $Y \subseteq F^n$ be two semi-algebraic sets. Let $L^0(X)$ and $L^0(Y)$ be the prime $\mathcal{L}_{\mathrm{ASC}}$-substructures of $L(X)$ and $L(Y)$ respectively. Then X and Y are pre-algebraically homeomorphic if and only if $L^0(X)$ and $L^0(Y)$ are $\mathcal{L}_{\mathrm{ASC}}$-isomorphic. In particular, any two semi-algebraic sets over K and F with the same pure dimension $d \geq 1$ are pre-algebraically homeomorphic.*

2 Notation and definitions

$\mathbf{N}$ denotes the set of non-negative integers, and $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$. If $\mathcal{N}$ is an unbounded non-empty subset of $\mathbf{N}$ (resp. the empty subset) we set $\max \mathcal{N} = +\infty$ (resp. $\max \mathcal{N} = -\infty$). The symbols $\subseteq$ and $\subset$ denote respectively the inclusion and the strict inclusion. The logical connectives 'or', 'and' and their iterated forms will be denoted by $\bigvee$, $\bigwedge$, $\bigvee\!\!\!\bigvee$ and $\bigwedge\!\!\!\bigwedge$ respectively.

2.1 Lattices and dimension

In this paper a **lattice** is a partially ordered set in which every finite subset has a greatest lower element and a least greater element. This applies in particular to the empty subset, hence our lattices must have a least and a greatest element. $\mathcal{L}_{\mathrm{lat}} = \{\mathbf{0}, \mathbf{1}, \vee, \wedge\}$ is the language of lattices, each symbol having its obvious meaning. As usually $b \leq a$ is an abbreviation for $a \vee b = a$ and similarly for $b < a$, $b \geq a$ and $b > a$. Iterated $\vee$ and $\wedge$ operations are denoted by $\bigvee\!\!\!\bigvee_{i\in I} a_i$ and $\bigwedge\!\!\!\bigwedge_{i\in I} a_i$ respectively. If the index set I is empty then $\bigvee\!\!\!\bigvee_{i\in I} a_i = \mathbf{0}$ and $\bigwedge\!\!\!\bigwedge_{i\in I} a_i = \mathbf{1}$. Given a subset S of a lattice L, the **upper semi-lattice** generated in L by S is the set of finite joins of elements of L. We will make intensive use of the following definable relation:

$$b \ll a \iff \forall c\, (c < a \Rightarrow b \vee c < a)$$

This is a strict order on $L \setminus \{\mathbf{0}\}$ (but not on L because $\mathbf{0} \ll a$ for every a including $a = \mathbf{0}$).

[2] A generalization to definable sets over more general P-minimal fields, if possible, has still to be done.

The **spectrum** of a lattice L is the set $\mathrm{Spec}(L)$ of all prime filters of L, endowed with the so-called Zariski topology, defined by taking as a basis of closed sets all the sets

$$P(a) = \{\mathbf{p} \in \mathrm{Spec}(L) \,/\, a \in \mathbf{p}\}$$

for a ranging over L. Stone-Priestley's duality asserts that if L is distributive (which is always the case in this paper) the map $a \mapsto P(a)$ is an isomorphism between L and the lattice of closed subsets S of $\mathrm{Spec}(L)$ such that the complement of S in $\mathrm{Spec}(L)$ is compact.

We call a lattice **noetherian** if it is isomorphic to the lattice of closed sets of a noetherian topological space. By Stone-Priestley's duality a lattice L is noetherian if and only if its spectrum is a noetherian topological space. In such a lattice every filter is principal and every element a writes uniquely as the join of its (finitely many) $\vee$**-irreducible components**, which are the maximal elements in the set of $\vee$-irreducible[3] elements of L smaller than a. We denote by $\mathcal{I}(L)$ the set of all $\vee$-irreducible elements of L.

We define the **dimension of an element** a **in a lattice** L, denoted $\dim_L a$, as the least upper bound (in $\mathbf{N} \cup \{-\infty, +\infty\}$) of the set of non-negative integers n such that

$$\exists \mathbf{p}_0 \subset \cdots \subset \mathbf{p}_n \in P(a).$$

This is nothing but the ordinary topological or Krull dimension (defined by chains of irreducible closed subsets) of the spectral space $P(a)$. By construction $\dim_L \mathbf{0} = -\infty$ and $\dim_L a \vee b = \max(\dim_L a, \dim_L b)$. The index L is necessary since $\dim_L a$ is not preserved by $\mathcal{L}_{\mathrm{lat}}$-embeddings, but we will omit it whenever the ambient lattice is clear from the context. We let the **dimension of** L be the dimension of $\mathbf{1}_L$ in L.

In all the geometric examples given the introduction, a set A is pure dimensional if and only if $\dim U = \dim A$ for every non-empty definable subset U of A which is open in A. This motivates the next definition: given an integer k we say that an element a of a distributive lattice L is k**-pure in** L if and only if

$$\forall b \in L \ (a - b \neq \mathbf{0} \Rightarrow \dim_L a - b = k).$$

Then either $a = \mathbf{0}$ or $\dim_L a = k$. In the latter case we say that a **has pure dimension** k **in** L.

Fact 2.2 *If $L = L(X)$ is any of the lattices of the introduction, then for any $A \in L(X)$, $\dim_{L(X)} A$ is exactly the usual dimension of A as a semi-algebraic set over K, and A is pure dimensional in $L(X)$ if and only if it so in the geometric sense.*

The second assertion in Fact 2.2 follows from the first, which itself follows from Fact 2.4 below.

2.3 Co-Heyting algebras

We let $\mathcal{L}_{\mathrm{TC}} = \mathcal{L}_{\mathrm{lat}} \cup \{-\}$ with '$-$' a binary function symbol. An $\mathcal{L}_{\mathrm{TC}}$-structure L is a **co-Heyting algebra** if its $\mathcal{L}_{\mathrm{lat}}$-reduct is a lattice and if every element b has in L a **topological complement** relatively to every element, denoted $a - b$. By definition $a - b$ is the least element c such that $a \leq b \vee c$. Equivalently $P(a-b)$ is the topological closure of the relative complement $P(a) \setminus P(b)$, so the name of $a - b$. Reversing the order of a co-Heyting Heyting algebra L gives a Heyting algebra L^*, with $b \to a$ in L^* corresponding to $a - b$ in L, and every co-Heyting algebra is of this form. From the theory of Heyting algebras (see for example [Joh82]) we know that every

[3] An element c of a lattice L is $\vee$-irreducible if it is *non-zero* and if $a \vee b = x$ implies $a = x$ or $b = x$.

co-Heyting algebra is distributive and that the class of all co-Heyting algebras is a variety (in the sense of universal algebra). Observe that in co-Heyting algebras the $\ll$ relation is quantifier-free definable since

$$b \ll a \iff b \leq a - b.$$

So it will be preserved by $\mathcal{L}_{\mathrm{TC}}$-embeddings. On the other hand the dimension will not be preserved in general by $\mathcal{L}_{\mathrm{TC}}$-embeddings.

We will use the following rules, the proof of which are elementary exercises (using either Stone-Priestley's duality or corresponding properties of Heyting algebras).

TC$_1$: $a = (a \wedge b) \vee (a - b)$.
In particular if a is $\vee$-irreducible then $b < a \implies b \ll a$.

TC$_2$: $(a_1 \vee a_2) - b = (a_1 - b) \vee (a_2 - b)$.

TC$_3$: $(a - b) - b = a - b$.
In particular $(a - b) \wedge b \ll a - b \leq a$.

TC$_4$: More generally $a - (b_1 \vee b_2) = (a - b_1) - b_2$.
So if $a - b_1 = a$ then $a - (b_1 \vee b_2) = a - b_2$.

Fact 2.4 (Theorem 3.8 in [DJ11]) *For every element a in a co-Heyting algebra L, $\dim_L a$ is the least upper bound of the set of positive integers n such that there exists $a_0, \ldots, a_n \in L$ such that*

$$\mathbf{0} \neq a_0 \ll a_1 \ll \cdots \ll a_n \leq a.$$

There is a well-established duality between finite (co-)Heyting algebras and finite posets, from which we will pick up Fact 2.5 below. We first need a notation. Given an element x in a poset I and a subset X of I let

$$x^{\downarrow} = \{y \in I \,/\, y \leq x\} \qquad\qquad X{\downarrow} = \bigcup_{x \in I} x^{\downarrow}.$$

The dual notation $x^{\uparrow}$ and $X{\uparrow}$ is defined accordingly. The family $\mathcal{D}^{\downarrow}(I)$ of decreasing subsets of I (that is the sets $X \subseteq I$ such that $X{\downarrow} = X$) are the closed sets of a topology on I, hence a co-Heyting algebra with respect to the following operations.

$$X \vee Y = X \cup Y \qquad\qquad X \wedge Y = X \cap Y \qquad\qquad X - Y = (X \setminus Y){\downarrow}$$

The $\vee$-irreducible elements of $\mathcal{D}^{\downarrow}(I)$ are precisely the sets $x^{\downarrow}$ for $x \in I$.

Fact 2.5 (Proposition 2.7 in [DJ]) *Let L be a finite co-Heyting algebra and $\mathcal{I}$ an ordered set. Assume that there is a surjective increasing map $\pi : \mathcal{I} \to \mathcal{I}(L)$ such that $\pi(x^{\uparrow}) \subseteq \pi(x){\uparrow}$ for every $x \in \mathcal{I}(L)$. Then there exists an $\mathcal{L}_{\mathrm{TC}}$-embedding φ of L into $\mathcal{D}^{\downarrow}(\mathcal{I})$ such that[4] $\pi(\varphi(a)) = a^{\downarrow} \cap \mathcal{I}(L)$ for every $a \in L$.*

[4]Note that the compositum $\pi \circ \varphi$ is not defined. In this proposition $\varphi(a)$ is a decreasing subset of $\mathcal{I}$ and $\pi(\varphi(a)) = \{\pi(\xi) \,/\, \xi \in \varphi(a)\}$.

2.6 (Sub)scaled lattices.

Recall that $\mathcal{L}_{\mathrm{SC}} = \mathcal{L}_{\mathrm{lat}} \cup \{-, \mathrm{C}^i\}_{i\in\mathbf{N}} = \mathcal{L}_{\mathrm{TC}} \cup \{\mathrm{C}^i\}_{i\in\mathbf{N}}$ where $\{\mathrm{C}^i\}_{i\in\mathbf{N}}$ is a family of new unary function symbols. With the examples of the introduction in mind, define the **sc-dimension** of a non-zero element a of an $\mathcal{L}_{\mathrm{SC}}$-structure L as

$$\text{sc-dim}\, a = \min\big\{k \in \mathbf{N} \,/\, a = \bigvee_{0\le i\le k} \mathrm{C}^i(a)\big\}.$$

Of course this is defined only if $\text{sc-dim}\, a = \bigvee_{0\le i\le k} \mathrm{C}^i(a)$ for some k. If it is not defined we let $\text{sc-dim}\, a = +\infty$, and by convention $\text{sc-dim}\, \mathbf{0} = -\infty$. The **sc-dimension of** L, denoted $\text{sc-dim}(L)$, is the sc-dimension of $\mathbf{1}_L$. In general the dimension of an element in a co-Heyting algebra is not preserved by $\mathcal{L}_{\mathrm{TC}}$-embeddings. At the contrary the sc-dimension of an element is obviously preserved by $\mathcal{L}_{\mathrm{SC}}$-embeddings, and this is the "raison d'être" of this structure.

A *d*-**subscaled lattice** is an $\mathcal{L}_{\mathrm{SC}}$-structure whose $\mathcal{L}_{\mathrm{TC}}$-reduct is a co-Heyting algebra and which satisfies the following list of axioms:

$\mathbf{SC_1^d}$: $\bigvee_{0\le i\le d} \mathrm{C}^i(a) = a \quad$ and $\quad \forall i > d,\ C^i(a) = \mathbf{0}$.

$\mathbf{SC_2}$: $\forall I \subseteq \{0,\dots,d\},\ \forall k$:

$$\mathrm{C}^k\Big(\bigvee_{i\in I} \mathrm{C}^i(a)\Big) = \begin{cases} \mathbf{0} & \text{if } k \notin I \\ \mathrm{C}^k(a) & \text{if } k \in I \end{cases}$$

$\mathbf{SC_3}$: $\forall k \ge \max(\text{sc-dim}(a), \text{sc-dim}(b)),\quad \mathrm{C}^k(a \vee b) = \mathrm{C}^k(a) \vee \mathrm{C}^k(b)$

$\mathbf{SC_4}$: $\forall i \ne j,\quad \text{sc-dim}\,\big(\mathrm{C}^i(a) \wedge \mathrm{C}^j(a)\big) < \min(i,j)$

$\mathbf{SC_5}$: $\forall k \ge \text{sc-dim}(b),\quad \mathrm{C}^k(a) - b = \mathrm{C}^k(a) - \mathrm{C}^k(b)$.

$\mathbf{SC_6}$: If $b \ll a$ then $\text{sc-dim}\, b < \text{sc-dim}\, a$.

It is a *d*-**scaled lattice** if it satisfies in addition the following property:

$\mathbf{SC_0}$: $\text{sc-dim}\, a = \dim a$

All the geometric $\mathcal{L}_{\mathrm{SC}}$-structures in $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ or $\mathrm{SC}_{\mathrm{Zar}}(K, d)$ (defined after Example 1.2) are d-scaled lattices (see Fact 2.2). However SC_0 does not follow from the other axioms as the following example shows.

Example 2.7 Let L be an arbitrary noetherian lattice, and $D\colon \mathcal{I}(L) \to \{0,\dots,d\}$ be a strictly increasing map. For every $a, b \in L$, if $\mathcal{C}(a)$ denotes the set of all $\vee$-irreducible components of a, let

$$a - b = \bigvee\{c \in \mathcal{C}(a) \,/\, c \not\le b\},$$
$$(\forall k) \qquad \mathrm{C}^k_D(a) = \bigvee\{c \in \mathcal{C}(a) \,/\, D(c) = k\}.$$

This is a typical example of a d-subscaled lattice in which the sc-dimension does not coincide with the dimension, except of course if $D(a) = \dim_L a$ for every $a \in \mathcal{I}(L)$. Conversely, every noetherian (in particular every finite) d-subscaled lattice is of that kind.

We call **(sub)scaled lattices** the $\mathcal{L}_{\mathrm{SC}}$-structures whose $\mathcal{L}_{\mathrm{SC}}$-reduct is a d-(sub)scaled for some $d \in \mathbf{N}$. Of course this is not an elementary class. At the contrary, for any fixed $d \in \mathbf{N}$, $\mathrm{SC}_1^{\mathrm{d}}$ to SC_5 are expressible by equations, SC_6 by a universal formula and SC_0 by a first order formula in $\mathcal{L}_{\mathrm{SC}}$, hence d-scaled (resp. d-subscaled) lattices form elementary class. As the terminology

suggests, we will see that d-subscaled lattices are precisely the $\mathcal{L}_{\mathrm{SC}}$-substructures of d-scaled lattices.

By analogy with our guiding geometric examples, we say that an element a in a d-subscaled lattice is **k-sc-pure** if

$$\forall b \in L \ (a - b \neq \mathbf{0} \Rightarrow \text{sc-dim}(a - b) = k).$$

We will see that a is k-sc-pure if and only if $a = \mathrm{C}^k(a)$ (this is SC_{13} in Section 3). Then either $a = \mathbf{0}$ or $\text{sc-dim}\, a = k$. In the latter case we say that a **has pure sc-dimension** k. For any a, the element $\mathrm{C}^k(a)$ is called the **k-sc-pure component** of a, or simply its **k-pure component** if L is a scaled lattice. By construction these notions coincide with their geometric counterparts in $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ and $\mathrm{SC}_{\mathrm{Zar}}(K, d)$.

The following notation will be convenient in induction arguments. If $\mathcal{L}$ is any of our languages $\mathcal{L}_{\mathrm{lat}}$, $\mathcal{L}_{\mathrm{TC}}$ or $\mathcal{L}_{\mathrm{SC}}$ we let $\mathcal{L}^* = \mathcal{L} \setminus \{\mathbf{1}\}$. Given an $\mathcal{L}$-structure L whose reduct to $\mathcal{L}_{\mathrm{lat}}$ is a lattice, for any $a \in L$ we let

$$L(a) = \{b \in L \,/\, b \leq a\}.$$

$L(a)$ is a typical example of $\mathcal{L}^*$-substructure of L.

3 Basic properties and embeddings

The next properties follow easily from the axioms of d-subscaled lattices.

$\mathbf{SC_7}$ $\text{sc-dim}\, a = \max\{k \,/\, \mathrm{C}^k(a) \neq \mathbf{0}\}$.
In particular $\forall k,\ \text{sc-dim}\, \mathrm{C}^k(a) = k \iff \mathrm{C}^k(a) \neq \mathbf{0}$.

$\mathbf{SC_8}$ $\forall k \geq \text{sc-dim}(a), \text{sc-dim}(b \wedge a) < k \implies \mathrm{C}^k(a) - b = \mathrm{C}^k(a)$.

$\mathbf{SC_9}$ $\text{sc-dim}\, a \vee b = \max(\text{sc-dim}\, a, \text{sc-dim}\, b)$.
In particular $b \leq a \Rightarrow \text{sc-dim}\, b \leq \text{sc-dim}\, a$.

$\mathbf{SC_{10}}$ $\forall k \geq \text{sc-dim}(a)$, $\mathrm{C}^k(a)$ is the largest k-sc-pure element lower than a.

$\mathbf{SC_{11}}$ $\dim a \leq \text{sc-dim}\, a$.

$\mathbf{SC_{12}}$ $\forall I \subseteq \{0, \ldots, d\}, \quad a - \bigvee\limits_{i \in I} \mathrm{C}^i(a) = \bigvee\limits_{i \notin I} \mathrm{C}^i(a)$.
In particular $\text{sc-dim}\left(a - \bigvee\limits_{i \geq k} \mathrm{C}^i(a)\right) < k$.

$\mathbf{SC_{13}}$ $\forall k, \quad \mathrm{C}^k(a) = a \iff \forall b \ \big(a - b \neq \mathbf{0} \Rightarrow \text{sc-dim}\, a - b = k\big)$.
That is a is k-sc-pure if and only if $a = \mathrm{C}^k(a)$.
In particular if a is $\vee$-irreducible then a is sc-pure by $\mathrm{SC}_1^{\mathrm{d}}$.

Proof: (Sketch) SC_7 follows from $\mathrm{SC}_1^{\mathrm{d}}$ and SC_2; SC_8 from SC_5 and SC_7; SC_9 from SC_2, SC_3 and SC_7; SC_{10} from SC_2 and SC_3; SC_{11} from SC_6 by Fact 2.4. Only the two last properties require a little effort.

SC_{12}**:** For every $l \in I$, $\mathrm{C}^l(a) \leq \bigvee_{i \in I} \mathrm{C}^i(a)$ hence $\mathrm{C}^l(a) - \bigvee_{i \in I} \mathrm{C}^i(a) = \mathbf{0}$. On the other hand for every $l \notin I$ and every $i \in I$, $\mathrm{C}^l(a) - \mathrm{C}^i(a) = \mathrm{C}^l(a)$ by SC_4 and SC_5. So $\mathrm{C}^l(a) - \bigvee_{i \in I} \mathrm{C}^i(a) = \mathrm{C}^l(a)$ by TC_4. Finally by $\mathrm{SC}_1^{\mathrm{d}}$ and TC_2,

$$a - \bigvee_{i \in I} \mathrm{C}^i(a) = \bigvee_{l \leq d} \Big(\mathrm{C}^l(a) - \bigvee_{i \in I} \mathrm{C}^i(a)\Big) = \bigvee_{l \notin I} \mathrm{C}^l(a).$$

SC_{13}**:** Assume that $a = \mathrm{C}^k(a)$ and $a - b \neq \mathbf{0}$ for some b. Then sc-dim $a = k$ so sc-dim$(a-b) \leq k$ and sc-dim$(a \wedge b) \leq k$ by SC_9. Since $a = (a-b) \vee (a \wedge b)$ it follows by SC_3 that

$$\mathrm{C}^k(a) = \mathrm{C}^k\left((a-b) \vee (a \wedge b)\right) = \mathrm{C}^k(a-b) \vee \mathrm{C}^k(a \wedge b).$$

By assumption $\mathrm{C}^k(a) = a$, and $\mathrm{C}^k(a \wedge b) \leq a \wedge b$ by $\mathrm{SC}_1^{\mathrm{d}}$. So $a \leq \mathrm{C}^k(a-b) \vee (a \wedge b) \leq \mathrm{C}^k(a-b) \vee b$ which implies that $a - b \leq \mathrm{C}^k(a-b)$. In particular $\mathrm{C}^k(a-b) \neq \mathbf{0}$. Since sc-dim$(a-b) \leq k$ it follows that sc-dim$(a-b) = k$ by SC_7.

Conversely assume that $a \neq \mathrm{C}^k(a)$ (hence $a \neq \mathbf{0}$). For $b = \mathrm{C}^k(a)$ we then have $a - b \neq \mathbf{0}$ on one hand and sc-dim$(a-b) \neq k$ by SC_7 on the other hand, because $\mathrm{C}^k(a-b) = \mathbf{0}$ by SC_{12} and SC_2.

■

Proposition 3.1 *The $\mathcal{L}_{\mathrm{SC}}$-structure of a d-scaled lattice L is uniformly definable in the $\mathcal{L}_{\mathrm{lat}}$-structure of L. In particular it is uniquely determined by this $\mathcal{L}_{\mathrm{lat}}$-structure.*

Proof: Clearly the $\mathcal{L}_{\mathrm{TC}}$-structure is an extension by definition of the lattice structure of L. For every positive integer k the class of k-pure elements is uniformly definable, using the definability of $\ll$ and Fact 2.4. Then so is the function C^k for every k, by decreasing induction on k. Indeed by SC_{10} and SC_{12}, $\mathrm{C}^k(a)$ is the largest k-pure element c such that $c \leq a - \bigvee_{i>k} \mathrm{C}^i(a)$.

■

We need a reasonably easy criterion for an $\mathcal{L}_{\mathrm{lat}}$-embedding of subscaled lattices to be an $\mathcal{L}_{\mathrm{SC}}$-embedding. In the special case of a noetherian[5] embedded lattice, it is given by Proposition 3.3 below, whose proof will use the next characterisation of sc-pure components.

Proposition 3.2 *Let L be a subscaled lattice and $a, a_0, \ldots, a_d \in L$ be such that $a = \bigvee_{i \leq d} a_i$, each a_i is i-sc-pure and sc-dim$(a_i \wedge a_j) < \min(i,j)$ for every $i \neq j$. Then $\mathrm{C}^i(a) = a_i$ for every i.*

Proof: Note first that $\mathrm{C}^k(a_k) = a_k$ for every $k \leq d$ by SC_{13}. Hence for every $k \leq i \leq d$ we have by SC_3 and SC_2

$$\mathrm{C}^i\Big(\bigvee_{k \leq i} a_k\Big) = \bigvee_{k \leq i} \mathrm{C}^i(a_k) = \bigvee_{k \leq i} \mathrm{C}^i\left(\mathrm{C}^k(a_k)\right) = \mathrm{C}^i(a_i) = a_i. \tag{1}$$

In particular $\mathrm{C}^d(a) = a_d$. Now assume that for some $i < d$ we have proved that $\mathrm{C}^j(a) = a_j$ for $i < j \leq d$. By SC_{12} and SC_2 we then have

$$\mathrm{C}^i\left(a - \bigvee_{j>i} a_j\right) = \mathrm{C}^i\left(a - \bigvee_{j>i} \mathrm{C}^j(a)\right) = \mathrm{C}^i\Big(\bigvee_{j \leq i} \mathrm{C}^j(a)\Big) = \mathrm{C}^i(a). \tag{2}$$

On the other hand $a - \bigvee_{j>i} a_j = \bigvee_{k \leq d}(a_k - \bigvee_{j>i} a_j)$ by TC_2. For $k > i$ obviously $a_k - \bigvee_{j>i} a_j = \mathbf{0}$. For $k < i$, $a_k = \mathrm{C}^k(a)$ and $a_j = \mathrm{C}^j(a_j)$ imply that sc-dim$(a_k \wedge a_j) < k$ for $j > i$ by SC_4. Hence $a_k - a_j = a_k$ by SC_8 and finally $a_k - \bigvee_{j>i} a_j = a_k$ by TC_4, so

$$a - \bigvee_{j>i} a_j = \bigvee_{k \leq i} a_k. \tag{3}$$

By (1), (2), (3) we conclude that $\mathrm{C}^i(a) = a_i$. The result follows for every i by decreasing induction.

■

[5] Although we won't use it, let us mention that in the general case of an $\mathcal{L}_{\mathrm{lat}}$-embedding $\varphi : L \to L'$ between arbitrary subscaled lattices, one may easily derive from Proposition 3.3, by means of the Local Finiteness Theorem 4.1 and the model-theoretic compactness theorem, that φ is an $\mathcal{L}_{\mathrm{SC}}$-embedding if and only if it preserves sc-dimension and sc-purity, that is for every $a \in L$ and every $k \in \mathbf{N}$, $\mathrm{C}^k(a) = a \Rightarrow \mathrm{C}^k(\varphi(a)) = \varphi(a)$.

Proposition 3.3 *Let L_0 be a noetherian subscaled lattice, L a subscaled lattice, and $\varphi : L_0 \to L$ an $\mathcal{L}_{\mathrm{lat}}$-embedding such that for every $a \in \mathcal{I}(L_0)$, $\varphi(a)$ is sc-pure and has the same sc-dimension as a. Then φ is an $\mathcal{L}_{\mathrm{SC}}$-embedding.*

Remark 3.4 Clearly the same statement remains true with $\mathcal{L}_{\mathrm{lat}}$ and $\mathcal{L}_{\mathrm{SC}}$ replaced respectively by $\mathcal{L}^*_{\mathrm{lat}}$ and $\mathcal{L}^*_{\mathrm{SC}}$. We will freely use these variants.

Proof: L_0 and L are d-subscaled lattices for some $d \in \mathbf{N}$. Given $a \in L_0$ and k a non-negative integer, we first check that $\varphi(\mathrm{C}^k(a))$ is k-sc-pure. Note that every $\vee$-irreducible component c of $\mathrm{C}^k(a)$ in L_0 has sc-pure dimension k. Indeed $\mathrm{C}^k(a)$ is k-sc-pure by SC_{13}, and $c = \mathrm{C}^k(a) - b \neq \mathbf{0}$ where b is the join of all the other $\vee$-irreducible components of $\mathrm{C}^k(a)$, hence $\text{sc-dim}(c) = \text{sc-dim}(\mathrm{C}^k(a) - b) = k$. Moreover c is sc-pure because it is $\vee$-irreducible, hence c is k-pure. By our assumption on φ it follows that $\varphi(c)$ is k-sc-pure. Every finite union of k-sc-pure elements being k-sc-pure by SC_3, it follows that

$$\varphi\big(\mathrm{C}^k(a)\big) \text{ is } k\text{-sc-pure.} \tag{4}$$

Now for every $l \neq k$ we have $\text{sc-dim}(\mathrm{C}^k(a) \wedge \mathrm{C}^l(a)) < \min(k,l)$ by SC_4. It follows that each $\vee$-irreducible component c of $\mathrm{C}^k(a) \wedge \mathrm{C}^l(a)$ has sc-dimension strictly less than $\min(k,l)$, hence so does $\varphi(c)$ by assumption. By SC_9 we conclude that

$$\text{sc-dim}\,\big(\mathrm{C}^k(a) \wedge \mathrm{C}^l(a)\big) < \min(k,l) \quad (\forall l \neq k). \tag{5}$$

$\varphi(a) = \mathbb{W}_{k \leq d}\, \varphi(\mathrm{C}^k(a))$ by $\mathrm{SC}^{\mathrm{d}}_1$ and because φ is an $\mathcal{L}_{\mathrm{lat}}$-embedding. By (4), (5) and Proposition 3.2 it follows that $\mathrm{C}^k(\varphi(a)) = \varphi(\mathrm{C}^k(a))$ for every $k \leq d$. Since φ is injective, this implies by SC_7 that for every $a \in L_0$

$$\text{sc-dim}\, a = \text{sc-dim}\, \varphi(a). \tag{6}$$

It only remains to check that $\varphi(a-b) = \varphi(a) - \varphi(b)$ for every $a, b \in L_0$. By TC_2, replacing if necessary a by its $\vee$-irreducible components, we may assume w.l.o.g. that a itself is $\vee$-irreducible in L_0. This implies that $a = \mathrm{C}^k(a)$ for some k, hence $\varphi(a)$ is k-sc-pure by assumption on φ. It then remains two possibilities for $a - b$:

- If $b \geq a$ then $\varphi(b) \geq \varphi(a)$, hence $a - b = \mathbf{0}$ and $\varphi(a) - \varphi(b) = \mathbf{0}$, so $\varphi(a-b) = \varphi(\mathbf{0}) = \mathbf{0} = \varphi(a) - \varphi(b)$.
- Otherwise $b \wedge a < a$ hence $b \wedge a \ll a$ by TC_1, that is $a - b = a$. So we have to prove that $\varphi(a) - \varphi(b) = \varphi(a)$. By SC_6 $\text{sc-dim}\, b \wedge a < \text{sc-dim}\, a$, hence $\text{sc-dim}(\varphi(b) \wedge \varphi(a)) < \text{sc-dim}(\varphi(a)) = k$ by (6). Since $\varphi(a)$ is k-sc-pure it follows that $\varphi(a) - \varphi(b) = \varphi(a)$ by SC_5.

■

Corollary 3.5 *Let L_0 be a noetherian lattice embedded in a subscaled lattice L. Assume that every $b' < b \in \mathcal{I}(L_0)$ are sc-pure in L and sc-dim $b' <$ sc-dim b in L. Then the restrictions to L_0 of the $\mathcal{L}_{\mathrm{SC}}$-operations "$-$" and "$C_i$" of L turn L_0 into an $\mathcal{L}_{\mathrm{SC}}$-substructure which is a subscaled lattice.*

Proof: The assumptions imply that the map $D : a \mapsto \text{sc-dim}\, a$ is a strictly increasing map from $\mathcal{I}(L_0)$ to $\mathbf{N}$. Endow L_0 with the structure of subscaled lattice determined by D as in Example 2.7. By construction the inclusion map from L_0 to L is an $\mathcal{L}_{\mathrm{lat}}$-embedding which preserves the sc-purity and sc-dimension of every $b \in \mathcal{I}(L_0)$, hence is an $\mathcal{L}_{\mathrm{SC}}$-embedding by Proposition 3.3.

■

4 Local finiteness

We prove in this section that every finitely generated subscaled lattice is finite. This result is far non-obvious, due to the lack of any known normal form for terms in $\mathcal{L}_{\mathrm{SC}}$. It contrasts with the general situation in co-Heyting algebras, which can be both infinite and generated by a single element. Our main ingredient, which explains this difference, is the uniform bound given *a priori* for the sc-dimension of any element in a given d-subscaled lattice.

Theorem 4.1 *Any d-subscaled lattice L generated by n elements is finite. More precisely, the cardinality of $\mathcal{I}(L)$ is bounded by the function $\mu(n,d)$ defined by*

$$\mu(n,d) = 2^n + \mu(2^{n+1}, d-1).$$

for $d \geq 0$, and $\mu(n,d) = 0$ for $d < 0$.

Proof: The only subscaled lattice of sc-dimension $d < 0$ is the one-element lattice $\{\mathbf{0}\}$, so the result is trivial in this case. Assume that $d \geq 0$ and that the result is proved for every $d' < d$ and every non-negative integer n.

Let L be a subscaled lattice of sc-dimension d generated by elements $x_1, \ldots, x_n$. Let Ω_n be the family of all subsets of $\{1, \ldots, n\}$ (so $\Omega_0 = \{\emptyset\}$). For every $I \in \Omega_n$ let $I^c = \Omega_n \setminus I$ and

$$y_I = \Big(\bigwedge_{i\in I} x_i\Big) - \Big(\bigvee_{i\in I^c} x_i\Big), \qquad z_I = \mathrm{C}^d(y_I).$$

The family of all $\mathcal{Y}_I = \bigcap_{i\in I} P(x_i) \cap \bigcap_{i\in I^c} P(x_i)^c$ is a partition of $\mathrm{Spec}(L)$. Indeed the $\mathcal{Y}_i$'s are the atoms of the boolean algebra generated in the power set $\mathcal{P}(\mathrm{Spec}(L))$ by the $P(x_i)$'s. Moreover each $P(y_I)$ is the topological closure $\overline{\mathcal{Y}}_I$ of $\mathcal{Y}_I$ in $\mathrm{Spec}(L)$ hence for every $x \in L$

$$P(x) = \bigcup_{I\in\Omega_n} P(x) \cap \mathcal{Y}_I \subseteq \bigcup_{I\in\Omega_n} P(x) \cap \overline{\mathcal{Y}}_I = P\Big(\bigvee_{I\in\Omega_n} x \wedge y_I\Big).$$

So $x \leq \bigvee_{I\in\Omega_n} (x \wedge y_I)$ by Stone-Priestley's duality. The reverse inequality being obvious we have proved that

$$\forall x \in L, \quad x = \bigvee_{I\in\Omega_n} (x \wedge y_I). \tag{7}$$

In particular SC_3 also gives

$$\mathrm{C}^d(\mathbf{1}) = \mathrm{C}^d\Big(\bigvee_{I\in\Omega_n} y_I\Big) = \bigvee_{I\in\Omega_n} z_I. \tag{8}$$

For every $I \neq J \in \Omega_n$, if for example $I \not\subseteq J$ choose any $i \in I \setminus J$ and observe that $y_I \leq x_i$ and $y_J \leq \mathbf{1} - x_i$ so $y_I \wedge y_J \ll \mathbf{1} - x_i$ by TC_3. By SC_6 and the d-sc-purity of the z_I's it follows that

$$\text{sc-dim}\, z_I \wedge z_J < d \quad \text{hence} \quad z_I - z_J = z_I. \tag{9}$$

It follows from SC_9, SC_{12} and (9) above, that the element

$$a = \big(\mathbf{1} - \mathrm{C}^d(\mathbf{1})\big) \vee \Big(\bigvee_{I\in\Omega_n} (y_I - z_I)\Big) \vee \Big(\bigvee_{I\neq J\in\Omega_n} (z_I \wedge z_J)\Big)$$

has sc-dimension strictly smaller than d. So the induction hypothesis applies to the $\mathcal{L}_{\mathrm{SC}}$-substructure L_0^- of $L(a)$ generated by the $(y_I - z_I)$'s and the $(z_I \wedge a)$'s: L_0^- is finite, with at most $\mu(2|\Omega_n|, d-1)$ $\vee$-irreducible elements. Note that L_0^- is an $\mathcal{L}^*_{\mathrm{SC}}$-substructure of L (recall that

$\mathcal{L}^*_{\mathrm{SC}} = \mathcal{L}_{\mathrm{SC}} \setminus \{\mathbf{1}\}$). Finally let L_1 be the upper semi-lattice generated in L by $L_0^- \cup \{z_I\}_{I\in\Omega_n}$. By construction L_1 is finite and $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0^-) \cup \{z_I\}_{I\in\Omega_n}$, so $|\mathcal{I}(L_1)| \leq 2^n + \mu(2^{n+1}, d-1) = \mu(n,d)$. It is then sufficient to show that $L_1 = L$.

We first prove that L_1 is a lattice. By (8), $\mathrm{C}^d(\mathbf{1}) \vee a = \bigvee_{I\in\Omega_n} z_I \vee a \in L_1$ hence $\mathbf{1} = \mathrm{C}^d(\mathbf{1}) \vee a \in L_1$. For every $I \in \Omega_n$ and every $b \in L_0^-$, $z_I \wedge b = (z_I \wedge a) \wedge b \in L_0^-$. For every $I \neq J \in \Omega_n$, $z_I \wedge z_J = (z_I \wedge a) \wedge (z_J \wedge a) \in L_0^-$. So by the distributivity law, L_1 is a sublattice of L.

In order to conclude that L_1 is an $\mathcal{L}_{\mathrm{SC}}$-substructure of L, by Corollary 3.5 it only remains to check that for every $b' < b \in \mathcal{I}(L_1)$, b is sc-pure in L and $\text{sc-dim}\, b' < \text{sc-dim}\, b$ in L. Since $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0^-) \cup \{z_I\}_{I\in\Omega_n}$ we can distinguish two cases.

Case 1: $b \in \mathcal{I}(L_0^-)$. Then b is sc-pure in L_0^- by SC_{13} hence also in L since L_0^- is an $\mathcal{L}^*_{\mathrm{SC}}$-substructure of L. Similarly $b' \ll b$ in L_0^- by TC_1 that is $b - b' = b$ in L_0^- hence also in L. Thus $b' \ll b$ in L which implies that $\text{sc-dim}\, b' < \text{sc-dim}\, b$ in L by SC_6.

Case 2: $b = z_I$ for some $I \in \Omega_n$. Then $b = \mathrm{C}^d(y_I)$ is sc-pure in L and $\text{sc-dim}\, b = d$. If $b' = z_J$ for some other $J \in \Omega_n$ then on one hand $\text{sc-dim}(b') = d$ and on the other hand $I \neq J$ hence $b' = b' \wedge b = z_I \wedge z_J$ has sc-dimension $< d$ by (9), a contradiction. So necessarily $b' \in \mathcal{I}(L_0^-)$, in particular $b' \leq a$ hence $\text{sc-dim}(b') \leq \text{sc-dim}(a) < d$

So L_1 is indeed an $\mathcal{L}_{\mathrm{SC}}$-substructure of L. Finally every $y_I = (y_I - z_I) \vee z_I \in L_1$ and (7) gives, for every $i \leq n$,

$$x_i = \bigvee_{I\in\Omega_n} x_i \wedge y_I \leq \bigvee_{\substack{I\in\Omega_n \\ i\in I}} y_I \leq x_i.$$

So equality holds, hence each $x_i \in L_1$, which finally proves that $L_1 = L$.

■

Corollary 4.2 *For every n, d there are finitely many non-isomorphic subscaled lattices of sc-dimension d generated by n elements.*

Proof: Any such subscaled lattice L is finite, with $|\mathcal{I}(L)| \leq \mu(n,d)$ by Theorem 4.1. Clearly there are finitely many non-isomorphic lattices such that $|\mathcal{I}(L)| \leq \mu(n,d)$ and each of them admits finitely many non-isomorphic $\mathcal{L}_{\mathrm{SC}}$-structures of d-subscaled lattices.

■

5 Linear representation

In this section we prove that the theory of d-subscaled lattices is the universal theory of various natural classes of $\mathcal{L}_{\mathrm{SC}}$-structures, including $\mathrm{SC}_{\mathrm{Zar}}(K,d)$. The argument is based on an elementary representation theorem for d-subscaled lattices, combined with the local finiteness result of Section 4.

Given an arbitrary field K, a non-empty linear variety $X \subseteq K^m$ is determined by the data of an arbitrary point $P \in X$ and the vector subspace $\overrightarrow{X}$ of K^m, *via* the relation $X = P + \overrightarrow{X}$ (the orbit of P under the action of $\overrightarrow{X}$ by translation). We call X a **special linear variety** (resp. a **special linear set**) if X is a linear variety such that $\overrightarrow{X}$ is generated by a subset of the canonical basis of K^m (resp. if X is a finite union of special linear varieties). The family $\mathrm{L}_{\mathrm{lin}}(X)$ of all special linear subsets of X is the family of closed sets of a noetherian topology on X, hence a noetherian lattice. For every $A \in \mathrm{L}_{\mathrm{lin}}(X)$ we let $D(A)$ be the dimension of A in the sense of linear algebra. This endows $\mathrm{L}_{\mathrm{lin}}(X)$ with a natural structure of scaled lattice as in Example 2.7.

Remark 5.1 For every $A \in \mathrm{L}_{\mathrm{lin}}(X)$, if K is infinite then sc-dim $A = \dim_{\mathrm{L}_{\mathrm{lin}}(X)} A =$ the dimension of A as defined in linear algebra. It coincides with the Krull dimension as well. If moreover A is $\vee$-irreducible in $\mathrm{L}_{\mathrm{lin}}(X)$ then it is pure dimensional, hence it is sc-pure both in $\mathrm{L}_{\mathrm{lin}}(X)$ and $\mathrm{L}_{\mathrm{Zar}}(X)$. By Proposition 3.3 it follows that $\mathrm{L}_{\mathrm{lin}}(X)$ is an $\mathcal{L}_{\mathrm{SC}}$-substructure of $\mathrm{L}_{\mathrm{Zar}}(X)$. Similarly if K is a real closed field or p-adically closed field then $\mathrm{L}_{\mathrm{lin}}(X)$ is an $\mathcal{L}_{\mathrm{SC}}$-substructure of $\mathrm{L}_{\mathrm{def}}(X)$.

In what follows K^m is identified to $K^m \times \{0\}^r \subseteq K^{m+r}$. The very easy result below prepares the proof of Proposition 5.3.

Proposition 5.2 *For every two special linear sets $C \subseteq B \subseteq K^m$ and every non-negative integer $n \geq \dim C$ there exists a special linear set $A \subseteq K^{m+n}$ of pure dimension n such that $A \cap B = C$.*

Proof: The result being rather trivial if C is empty, we can assume w.l.o.g. that $C \neq \emptyset$. Let $(e_1, \ldots, e_{m+n})$ be the canonical basis of K^{m+n}. If I is a subset of $\{1, \ldots, m+n\}$ we let $\overrightarrow{E}(I)$ denote the vector space generated in K^n by $(e_i)_{i \in I}$. Decompose C as a union of special linear varieties $C_1, \ldots, C_p$, and write each $C_i = P_i + \overrightarrow{E}(J_i)$ with $|J_i| = \dim C_i \leq n$. Let $I_i = J_i \cup \{m+1, \ldots, m+n-|J_i|\}$ and $A_i = P_i + \overrightarrow{E}(I_i)$ for every $i \leq p$. Finally let $A = A_1 \cup \cdots \cup A_p$. By construction each A_i has pure dimension $|I_i| = n$, hence A has pure dimension n. Clearly each $A_i \cap K^m = C_i$, hence $A \cap K^m = C$ and *a fortiori* $A \cap B = C$.

■

Proposition 5.3 (Linear representation) *Let K be an infinite field, $d \geq 0$ an integer and L a finite d-subscaled lattice. Then there exists a special linear set X over K of dimension $\leq d$ and an $\mathcal{L}_{\mathrm{SC}}$-embedding $\varphi : L \to \mathrm{L}_{\mathrm{lin}}(X)$.*

Proof: By induction on the number r of $\vee$-irreducible elements of an arbitrary d-subscaled lattice L, we prove that there exists an $\mathcal{L}^*_{\mathrm{SC}}$-embedding φ of L into $\mathrm{L}_{\mathrm{lin}}(K^m)$ for some m depending on L. Taking $X = \varphi(\mathbf{1}_L)$ then gives the conclusion. Indeed X is a special linear set over K, $\dim X = \text{sc-dim}(\mathbf{1}_L) \leq d$ because φ preserves the sc-dimension, and φ is obviously an $\mathcal{L}_{\mathrm{SC}}$-embedding of L into $\mathrm{L}_{\mathrm{lin}}(X)$.

If $r = 0$ then L is the one-element lattice $\{\mathbf{0}\}$, hence an $\mathcal{L}^*_{\mathrm{SC}}$-substructure of $\mathrm{L}_{\mathrm{lin}}(K)$. So, given a fixed $r \geq 1$, we can assume by induction that the result is proved for $r-1$. Let L be a d-subscaled lattice with $\vee$-irreducible elements $a_1, \ldots, a_r$.

Renumbering if necessary we may assume that a_r is maximal among the a_i's. Let $a = a_r$, $b = \mathbb{V}_{1 \leq i < r} a_i$, $c = a \wedge b$ and φ an $\mathcal{L}^*_{\mathrm{SC}}$-embedding of $L(b)$ into some $\mathrm{L}_{\mathrm{lin}}(K^m)$ given by the induction hypothesis. Since a is $\vee$-irreducible in L it is sc-pure. Moreover $c \leq a$ by TC_1, hence a has pure sc-dimension n for some $n \geq \text{sc-dim}(c)$ by SC_6. Let B, C be the respective images of b, c by φ. Proposition 5.2 gives a special linear set $A \subseteq K^{m+n}$ of pure dimension n such that $A \cap B = C$. Identifying K^m with $K^m \times \{0\}^n \subseteq K^{m+n}$ turns φ into an $\mathcal{L}^*_{\mathrm{SC}}$-embedding of $L(b)$ into $\mathrm{L}_{\mathrm{lin}}(K^{m+n})$.

Every $x \in L$ writes uniquely $x_a \vee x_b$ with $x_a \in \{\mathbf{0}, a\}$ and $x_b \in L(b)$ by grouping appropriately the $\vee$-irreducible components of x, using the maximality of a_n. So we can let

$$\bar{\varphi}(x) = \begin{cases} \varphi(x_b) & \text{if } x_a = \mathbf{0}, \\ A \cup \varphi(x_b) & \text{if } x_a = a. \end{cases}$$

This is a well-defined $\mathcal{L}^*_{\mathrm{lat}}$-embedding of L into $\mathrm{L}_{\mathrm{lin}}(K^{m+n})$. Moreover $\bar{\varphi}$ is an $\mathcal{L}^*_{\mathrm{SC}}$-embedding by Proposition 3.3. This finishes the induction.

■

For any infinite field K and positive integer d let $\mathrm{SC}_{\mathrm{lin}}(K, d)$ be the class of d-scaled lattices $\mathrm{L}_{\mathrm{lin}}(X)$ with X ranging over the special linear sets over K of dimension at most d.

Theorem 5.4 *Given any non-negative integer d, the universal theories of $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$ (resp. $\mathrm{SC}_{\mathrm{Zar}}(K, d)$, $\mathrm{SC}_{\mathrm{lin}}(K, d)$) in the language $\mathcal{L}_{\mathrm{SC}}$ are the same for every o-minimal or P-minimal expansion $\mathcal{K}$ of a field K (resp. for every infinite field K). This is the theory of d-subscaled lattices.*

Proof: The universal theory of any of these classes obviously contains the theory of d-subscaled lattices. For the converse, thanks to Remark 5.1 it suffices to prove that every d-subscaled lattice L embeds into a model of the theory of $\mathrm{SC}_{\mathrm{lin}}(K, d)$. If L is finite this is Proposition 5.3. The general case then follows from the model-theoretic compactness theorem, because L is locally finite by Theorem 4.1.

■

6 Minimal extensions

Minimal proper extensions of any finite subscaled lattices are entirely determined by so-called "SC-signatures" (see below). Since this is a special case of minimal extensions of finite co-Heyting algebras, we first recall what the main results of [DJ] on this subject, and try to reduce to them as much as possible.

We need some specific notation and definitions. Given a finite lattice L_0, an $\mathcal{L}_{\mathrm{lat}}$-extension L, elements $a \in L_0$ and $x \in L$ we write:

- $a^- = \curlyvee\{b \in L_0 \,/\, b < a\}$.
- $g(x, L_0) = \curlywedge\{a \in L_0 \,/\, x \leq a\}$.

Clearly $a \in \mathcal{I}(L_0)$ if and only if a^- is the unique predecessor of a in L_0 (otherwise $a^- = a$).

Assume that L_0 and L are co-Heyting algebras (or topologically complemented lattices, or TC-lattice for short). A **TC-signature** in L_0 is a triple (g, H, r) where $g \in \mathcal{I}(L_0)$, H is a set of one or two elements $h_1, h_2 \in L_0$ and $r \in \{1, 2\}$ are such that:

- either $r = 1$ and $h_1 = h_2 < g$;
- or $r = 2$ and $h_1 \vee h_2 < g$.

A couple (x_1, x_2) of elements of L is **TC-primitive over** L_0 if there is $g \in \mathcal{I}(L_0)$ such that

P1 $g^- \wedge x_1$ and $g^- \wedge x_2$ belong to L_0.

P2 One of the following happens:

1. $x_1 = x_2$ and $g^- \wedge x_1 \ll x_1 \ll g$.
2. $x_1 \neq x_2$, $x_1 \wedge x_2 \in L_0$ and $g - x_1 = x_2$, $g - x_2 = x_1$.

This implies that each $x_i \notin L_0$, that $g = g(x_1, L_0) = g(x_2, L_0)$ and that the triple $\sigma_{\mathrm{TC}}(x_1, x_2) = (g, H, r)$ defined as follows is a signature in L_0, called the **signature of** (x_1, x_2) **in** L_0.

$$g = g(x_1, L_0) \qquad H = \{g^- \wedge x_1, g^- \wedge x_2\} \qquad r = \mathrm{Card}\{x_1, x_2\}$$

Finally we say that L is a **TC-primitive extension** of L_0 if it is $\mathcal{L}_{\mathrm{TC}}$-generated over L_0 by a primitive tuple. For the convenience of the reader we collect here all the properties of TC-signatures and TC-primitive extensions that we are going to use. All the references are taken from [DJ].

Proposition 6.1 *Let L_0 be a finite co-Heyting algebra and L an $\mathcal{L}_{\mathrm{TC}}$-extension.*

1. *[Theorem 3.3] If L is $\mathcal{L}_{\mathrm{TC}}$-generated over L_0 by a TC-primitive tuple (x_1, x_2), then L is exactly the upper semi-lattice generated over L_0 by x_1 and x_2. It is a finite co-Heyting algebra and one of the following happens :*

   (a) $x_1 = x_2$ and $\mathcal{I}(L) = \mathcal{I}(L_0) \cup \{x_1\}$.

   (b) $x_1 \neq x_2$ and $\mathcal{I}(L) = (\mathcal{I}(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$.

   In particular (x_1, x_2) is (up to permutation) the only TC-primitive tuple over L_0 in L. We call $\sigma_{\mathrm{TC}}(x_1, x_2)$ the **TC-signature of L in L_0** *and denote it $\sigma_{\mathrm{TC}}(L)$.*

2. *[Remark 3.6] TC-signatures in L_0 and TC-primitive extensions of L_0 are in one-to-one correspondence: every TC-signature in L_0 is the TC-signature of a TC-primitive extension, and two TC-primitive extensions of L_0 are $\mathcal{L}_{\mathrm{TC}}$-isomorphic over L_0 if and only if they have the same TC-signature in L_0.*

3. *[Corollary 3.4] If L is finite, the following are equivalent.*

   (a) L is a minimal proper extension of L_0.

   (b) L is a TC-primitive extension of L_0.

   (c) $\mathrm{Card}(\mathcal{I}(L)) = \mathrm{Card}(\mathcal{I}(L_0)) + 1$.

   As a consequence every finite $\mathcal{L}_{\mathrm{TC}}$-extension L' of L_0 is the union of a tower of TC-primitive extensions $L_0 \subset L_1 \subset \cdots \subset L_n = L'$ with $n = \mathrm{Card}(\mathcal{I}(L')) - \mathrm{Card}(\mathcal{I}(L_0))$.

We will refer to the k-th item of the above proposition as to Proposition 6.1.k.

Now let L_0 be a finite subscaled lattice and L and $\mathcal{L}_{\mathrm{SC}}$-extension. An **SC-signature** in L_0 is a triple $\sigma = (g, H, q)$ where $g \in \mathcal{I}(L_0)$, H is a set of one or two elements $h_1, h_2 \in L_0$ and $q \in \mathbf{N}$ are such that:

- either $\text{sc-dim}\, h_1 < q < \text{sc-dim}\, g$ and $h_1 = h_2 < g$;
- or $q = \text{sc-dim}\, g$ and $h_1 \vee h_2 = g^-$.

Let $r_\sigma = 1$ if $q < \text{sc-dim}\, g$, $r_\sigma = 2$ if $q = \text{sc-dim}\, g$, and $\sigma^{\mathrm{TC}} = (g, H, r_\sigma)$. By construction this is a TC-signature in L_0. Given an $\mathcal{L}_{\mathrm{SC}}$-extension L of L_0, a tuple (x_1, x_2) of elements of L is **SC-primitive over** L_0 if it is TC-primitive over L_0^{TC} and if in addition

P3 x_1, x_2 are sc-pure of the same sc-dimension.

Such an SC-primitive tuple determines the so-called **SC-signature of (x_1, x_2) in L_0**, denoted $\sigma_{\mathrm{SC}}(x_1, x_2) = (g, H, q)$ and defined as follows.

$$g = g(x_1, L_0) \qquad H = \{g^- \wedge x_1, g^- \wedge x_2\} \qquad q = \text{sc-dim}\, x_1$$

Note that, by condition **P2** of the definition of TC-signatures, $x_1 = x_2$ if and only if $x_1 \ll g$, and otherwise $\text{sc-dim}\, x_1 = \text{sc-dim}\, x_2 = \text{sc-dim}\, g$. This ensures that $\sigma_{\mathrm{TC}}(x_1, x_2) = (\sigma_{\mathrm{SC}}(x_1, x_2))^{\mathrm{TC}}$.

Let L_0^{TC} and L^{TC} denote the respective $\mathcal{L}_{\mathrm{TC}}$-reducts of L_0 and L. For every subset X_0 of L we let:

- $L_0\langle X_0\rangle$ = the $\mathcal{L}_{\mathrm{SC}}$-structure generated by $L_0 \cup X_0$ in L;

- $L_0^{\mathrm{TC}}\langle X_0\rangle$ = the $\mathcal{L}_{\mathrm{TC}}$-structure generated by $L_0^{\mathrm{TC}} \cup X_0$ in L^{TC}.

We say that L is an **SC-primitive extension** of L_0, if there exists a tuple (x_1, x_2) SC-primitive over L_0 such that $L = L_0\langle x_1, x_2\rangle$ (then clearly $L = L_0\langle x_1\rangle = L_0\langle x_2\rangle$). By Lemma 6.2 below and Proposition 6.1.1 such a tuple is necessarily unique.

Lemma 6.2 *Let L_0 be finite subscaled lattice, and L an $\mathcal{L}_{\mathrm{SC}}$-extension generated over L_0 by an SC-primitive tuple (x_1, x_2). Then $L^{\mathrm{TC}} = L_0\langle x_1, x_2\rangle$, (x_1, x_2) is TC-primitive over L_0^{TC} and $\sigma_{\mathrm{TC}}(x_1, x_2) = (\sigma_{\mathrm{SC}}(x_1, x_2))^{\mathrm{TC}}$.*

Proof: That (x_1, x_2) is TC-primitive over L_0^{TC} and $\sigma_{\mathrm{TC}}(x_1, x_2) = (\sigma_{\mathrm{SC}}(x_1, x_2))^{\mathrm{TC}}$ is only a reminder: it follows directly from the definitions. Let $L_1 = L_0\langle x_1, x_2\rangle$, in order to conclude that $L_1 = L$ it only remains to prove that L_1 is an $\mathcal{L}_{\mathrm{SC}}$-substructure of L. By Corollary 3.5 it suffices to check that for every $b' < b \in \mathcal{I}(L_1)$, b is sc-pure in L and $\text{sc-dim}\, b' < \text{sc-dim}\, b$ in L.

If $b \in \mathcal{I}(L_0)$, then b is sc-pure in L_0 by SC_{13}, hence also in L because L_0 is an $\mathcal{L}_{\mathrm{SC}}$-substructure of L. Otherwise $b = x_i$ for some $i \in \{1, 2\}$. Then b is sc-pure in L by definition of SC-primitive tuples over L_0.

In both cases $b' \ll b$ in L_1 by TC_1, that is $b - b' = b$ in L_1, hence also in L because L_1 is an $\mathcal{L}_{\mathrm{TC}}$-substructure of L. So $b' \ll b$ in L hence $\text{sc-dim}(b') < \text{sc-dim}(b)$ in L by SC_6.
■

Lemma 6.3 *Let L_0 be finite subscaled lattice, L_1 an $\mathcal{L}_{\mathrm{TC}}$-extension generated over L_0^{TC} by a TC-primitive tuple (x_1, x_2), and $\tau = (g, \{h_1, h_2\}, q)$ an SC-signature in L_0 such that $\tau^{\mathrm{TC}} = \sigma_{\mathrm{TC}}(x_1, x_2)$. Then there exists a unique structure of subscaled lattice expanding L_1 which makes it an $\mathcal{L}_{\mathrm{SC}}$-extension of L_0 such that (x_1, x_2) is SC-primitive over L_0 and $\sigma_{\mathrm{SC}}(x_1, x_2) = \tau$.*

Proof: By Proposition 6.1.1, $\mathcal{I}(L_1) \subseteq \mathcal{I}(L_0) \cup \{x_1, x_2\}$. For every $x \in \mathcal{I}(L_0)$ let $D(x) = \text{sc-dim}\, x$, and let $D(x_1) = D(x_2) = q$. This defines by restriction a function from $\mathcal{I}(L_1)$ to $\mathbf{N}$. Assume that D is strictly increasing. Then it determines as in Example 2.7 an $\mathcal{L}_{\mathrm{SC}}$-structure on L_1 expanding its $\mathcal{L}_{\mathrm{TC}}$-structure. Let us denote it L, so that $L^{\mathrm{TC}} = L_1$. Every $\vee$-irreducible element of L_0 remains sc-pure in L with the same sc-dimension, hence by Proposition 3.3 the inclusion of L_0 into L is an $\mathcal{L}_{\mathrm{SC}}$-embedding. This is clearly the only possible $\mathcal{L}_{\mathrm{SC}}$-structure on L_1 which makes it an $\mathcal{L}_{\mathrm{SC}}$-extension of L such that $\text{sc-dim}\, x_1 = \text{sc-dim}\, x_2 = q$. So it only remains to prove that D is strictly increasing.

Let $b < a$ in $\mathcal{I}(L)$, if $a, b \in \mathcal{I}(L_0)$ then $D(b) < D(a)$ by 6. So we can assume that a or b does not belong to $\mathcal{I}(L_0)$. By Proposition 6.1.1 one of them must belong to $\{x_1, x_2\}$ and the other one to $\mathcal{I}(L_0)$. Note that our assumption $\sigma_{\mathrm{TC}}(x_1, x_2) = \sigma^{\mathrm{TC}}$ implies that (for $i = 1, 2$) $g = g(x_i, L_0)$, and $h_i = x_i \wedge g^-$ (up to re-numbering) and: either $x_1 = x_2$, $h_1 = h_2 < g$ and $\text{sc-dim}\, h_1 < q < \text{sc-dim}\, g$; or $x_1 \neq x_2$, $h_1 \vee h_2 = g^-$ and $q = \text{sc-dim}\, g$.

Case 1: $b = x_1$ or $b = x_2$, hence $D(b) = q$. Then $a \in \mathcal{I}(L_0)$, in particular $a \in L_0$, hence $g \leq a$ and so $\text{sc-dim}\, g \leq \text{sc-dim}\, a$. If $x_1 = x_2$ then $q < \text{sc-dim}\, g \leq \text{sc-dim}\, a$ hence $D(b) < D(a)$. If $x_1 \neq x_2$ then $q = \text{sc-dim}\, g$, and g is not $\vee$-irreducible in L hence $g \neq a$. So $g \ll a$ (because $g < a$ and a is $\vee$-irreducible) hence $\text{sc-dim}\, g < \text{sc-dim}\, a$ by 6, that is $D(b) < D(a)$.

Case 2: $a = x_1$ or $a = x_2$, hence $D(a) = q$. Then again $b \in L_0$, and $b < a \leq g$ hence $b \leq g^-$. If $x_1 = x_2$, since $b \leq a \wedge g^- = h_1$ we get $\text{sc-dim}\, b \leq \text{sc-dim}\, h_1 < q$, hence $D(b) < D(a)$. If $x_1 \neq x_2$ then $\text{sc-dim}\, g = q$. Since $b < g$ we have $b \ll g$ (because g is $\vee$-irreducible) hence $\text{sc-dim}\, b < \text{sc-dim}\, g$ by 6. So $\text{sc-dim}\, b < q$, that is $D(b) < D(a)$.
■

We can now pack all this together.

Proposition 6.4 *Let L_0 be a finite subscaled lattice and L an $\mathcal{L}_{\mathrm{SC}}$-extension.*

1. *If L is $\mathcal{L}_{\mathrm{SC}}$-generated over L_0 by an SC-primitive tuple (x_1, x_2), then L is exactly the upper semi-lattice generated over L_0 by x_1 and x_2. It is a finite subscaled latice and one of the following happens :*

   (a) $x_1 = x_2$ *and* $\mathcal{I}(L) = \mathcal{I}(L_0) \cup \{x_1\}$.

   (b) $x_1 \neq x_2$ *and* $\mathcal{I}(L) = (\mathcal{I}(L_0) \setminus \{g\}) \cup \{x_1, x_2\}$.

   In particular (x_1, x_2) is (up to permutation) the only SC-primitive tuple over L_0 in L. We call $\sigma_{\mathrm{SC}}(x_1, x_2)$ the **SC-signature of L in L_0** *and denote it $\sigma_{\mathrm{SC}}(L)$.*

2. *SC-signatures in L_0 and SC-primitive extensions of L_0 are in one-to-one correspondence: every SC-signature in L_0 is the SC-signature of an SC-primitive extension, and two SC-primitive extensions of L_0 are $\mathcal{L}_{\mathrm{SC}}$-isomorphic over L_0 if and only if they have the same SC-signature in L_0.*

3. *If L is finite, the following are equivalent.*

   (a) *L is a minimal proper $\mathcal{L}_{\mathrm{SC}}$-extension of L_0.*

   (b) *L is an SC-primitive extension of L_0.*

   (c) $\mathrm{Card}(\mathcal{I}(L)) = \mathrm{Card}(\mathcal{I}(L_0)) + 1$.

   As a consequence every finite $\mathcal{L}_{\mathrm{SC}}$-extension L' of L_0 is the union of a tower of SC-primitive extensions $L_0 \subset L_1 \subset \cdots \subset L_n = L'$ with $n = \mathrm{Card}(\mathcal{I}(L')) - \mathrm{Card}(\mathcal{I}(L_0))$.

We will refer to the k-th item of the above proposition as to Proposition 6.4.k.

Proof: If L is $\mathcal{L}_{\mathrm{SC}}$-generated over L_0 by an SC-primitive tuple (x_1, x_2), then by Lemma 6.2 L^{TC} is also $\mathcal{L}_{\mathrm{TC}}$-generated over L_0^{TC} by (x_1, x_2), which is TC-primitive. The first item the follows from Proposition 6.1.1.

Let σ be an SC-signature in L_0. Then σ^{TC} is a TC-signature in L_0. Proposition 6.1.2 gives a TC-primitive $\mathcal{L}_{\mathrm{TC}}$-extension L_1 of L_0^{TC} with TC-signature σ^{TC} in L_0. Lemma 6.3 then gives a unique structure of subscaled lattice expanding L_1 which makes it an SC-primitive extension of L_0 with signature σ in L_0. Let us denote it L, so that $L^{\mathrm{TC}} = L_1$. Now if L' is another SC-primitive extension with signature σ in L_0, by Proposition 6.1.2 L'^{TC} is $\mathcal{L}_{\mathrm{TC}}$-isomorphic to L^{TC} over L_0. The image of L' *via* this endomorphism defines an $\mathcal{L}_{\mathrm{SC}}$-structure expanding L^{TC}, which makes it an SC-primitive extension of L_0 with the same signature as L. By the uniqueness of such a structure, given by Lemma 6.3, it follows that this $\mathcal{L}_{\mathrm{TC}}$-isomorphism from L'^{TC} to L^{TC} is actually an $\mathcal{L}_{\mathrm{SC}}$-isomorphism, which proves the result.

We finely prove item (3 by cyclic implication. Note that (3b)⇒(3c) follows from item 1) above.

(3c)⇒(3a). Let L' be a proper $\mathcal{L}_{\mathrm{SC}}$-extension of L_0 contained in L. Then L'^{TC} is a proper $\mathcal{L}_{\mathrm{TC}}$-extension of L_0^{TC} contained in L^{TC}. By Proposition 6.1.3, (3c) implies that L^{TC} is a minimal proper $\mathcal{L}_{\mathrm{TC}}$-extension of L_0^{TC}. So $L'^{\mathrm{TC}} = L^{\mathrm{TC}}$, thus necessarily $L' = L$, which proves that L is minimal.

(3a)⇒(3b). Let x_1 be a minimal element in $\mathcal{I}(L) \setminus \mathcal{I}(L_0)$. Let $g = g(x_1, L_0)$,; if $x_1 \ll g$ let $x_2 = x_1$, otherwise let $x_2 = g - x_1$. The proof of Corollary 3.4 in [DJ] shows that (x_1, x_2) is TC-primitive over L_0^{TC}. In particular $x_1, x_2 \in \mathcal{I}(L)$ so they are sc-pure by SC_{13}. The same holds true for g, hence if $x_1 \neq x_2$ then $x_1 = g - x_2$ and $x_2 = g - x_1$ have the same dimension (the dimension of g, by definition of the sc-purity of g). So (x_1, x_2) is actually SC-primitive. Since $L_0^{\mathrm{TC}}\langle x_1, x_2\rangle = L^{\mathrm{TC}}$, *a fortiori* $L_0\langle x_1, x_2\rangle = L$ hence L is SC-primitive over L_0.

■

7 Model-completion of scaled lattices

We call **super scaled lattice** every subscaled lattice L which satisfy the following additional properties, both of which are clearly axiomatizable by $\forall\exists$-formulas in $\mathcal{L}_{\mathrm{SC}}$. If $\operatorname{sc-dim} L \leq d$ we call it a **super d-scaled lattice**.

Catenarity For every non-negative integers $r \leq q \leq p$ and every elements $c \leq a \neq \mathbf{0}$, if c is r-sc-pure and a is p-sc-pure then there exists a non-zero q-sc-pure element b such that $c \leq b \leq a$.

If $\mathrm{Spec}L$ is noetherian this property is equivalent to the usual notion of catenarity, namely that any two maximal chains in $\mathrm{Spec}L$ having the same first and last elements have the same length. In particular every d-scaled lattice L of type $\mathrm{L}_{\mathrm{Zar}}(X)$ or $\mathrm{L}_{\mathrm{lin}}(X)$ satisfies this property. At the contrary none of them satisfy the next property, as it implies that L is atomless.

Splitting: For every elements b_1, b_2, a, if $b_1 \vee b_2 \ll a \neq \mathbf{0}$ then there exists non-zero elements $a_1 \geq b_1$ and $a_2 \geq b_2$ such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

We will then say a_1, a_2 **split** a **along** b_1, b_2.

Remark 7.1 If $r < p \leq q$ in the Catenarity axiom, the conclusion can be strengthen to $c \ll b \leq a$. Indeed b has pure sc-dimension q and $c \wedge b = c$ has sc-dimension $< q$ hence $b - c = b$ by SC_8. In particular *every subscaled lattice satisfying the Catenarity axiom is a scaled lattice.* Indeed, given any element a of sc-dimension $d \geq 1$, repeated applications of the Catenarity axiom to $\mathrm{C}^d(a)$, $c = \mathbf{0}$ and each integer p from 0 to d, gives a chain of sc-pure elements $a_0, \ldots, a_d$ such that

$$\mathbf{0} \neq a_0 \ll a_1 \ll \cdots \ll a_d \leq a.$$

By Fact 2.2 it follows that $\dim a \geq d$, and by SC_{11} that $\dim a = d$.

Lemma 7.2 *Let a, b_1, b_2 be elements of a finite subscaled lattice L. If $b_1 \vee b_2 \ll a \neq \mathbf{0}$ then L embeds in a finite subscaled lattice L' containing non-zero elements a_1, a_2 which split a along b_1, b_2. Moreover, if $\mathrm{C}^0(a) \neq \mathbf{0}$ we can require[6] that all the atoms of L' belong to L_0.*

Proof: We are going to prove by induction on $d = \operatorname{sc-dim} a$ a slightly more precise result, namely that in addition $x \leq a$ for every $x \in \mathcal{I}(L') \setminus \mathcal{I}(L)$. Let $g_1, \ldots, g_n$ be the $\vee$-irreducible components of a in L. ($n \geq 1$ because $a \neq \mathbf{0}$). If $d = 0$ our assumption that $b_1 \vee b_2 \ll a$ implies by SC_6 that $b_1 = b_2 = \mathbf{0}$. If $n = 1$, that is $a = g_1$ is $\vee$-irreducible, then $\sigma = (g, \{\mathbf{0}\}, 0)$ is a signature in L. Proposition 6.4.2 gives an SC-primitive tuple (a_1, a_2) generating an $\mathcal{L}_{\mathrm{SC}}$-extension L_1 over L with signature σ. This signature ensures that (a_1, a_2) splits a along $(\mathbf{0}, \mathbf{0})$. If $n \geq 2$, $a_1 = g_1$ and $a_2 = a - a_1$ will do the job. So the result is proved for $d = 0$.

Now assume that $d \geq 1$ and the result is valid until $d - 1$. Note that $g_1^- \vee \cdots \vee g_n^-$ is the greatest element $c \in L$ such that $c \ll a$, in particular

$$b_1 \vee b_2 \leq g_1^- \vee \cdots \vee g_n^-. \tag{10}$$

[6]This additional condition if $\mathrm{C}^0(a) \neq \mathbf{0}$ will be used only later, in Section 9.

Let $u = (\bigvee_{i\leq n} g_i^-) - (b_1 \vee b_2)$ and $u^* = u - \mathrm{C}^0(u)$. Since $u \ll a$ we have $\text{sc-dim}\, u < d$ by SC_6. We are claiming that L embeds in a finite subscaled lattice L without new atoms, in which all the g_i's are still $\vee$-irreducible with the same predecessor as in L_0, and in which there are elements u_1^*, u_2^* which satisfy all the conditions to split u^* along $b_1 \wedge u^*$, $b_2 \wedge u^*$, except that u_1^*, u_2^* might be zero elements.

By TC_3, $(b_1 \vee b_2) \wedge u^* \ll u^*$ so if $\text{sc-dim}\, u \leq 0$ we can simply take $u_1^* = u_2^* = \mathbf{0}$ and $L_0 = L$. On the other hand, if $\text{sc-dim}\, u > 0$ the induction hypothesis applies to u^*, $b_1 \wedge u^*$, $b_2 \wedge u^*$. It gives a finite subscaled lattice L_0 containing L and elements $u_1^*, u_2^* \in L_0$ which split u^* along $b_1 \wedge u^*$, $b_2 \wedge u^*$. Moreover we can require that L_0 do not contain any new atom because $\mathrm{C}^0(u^*) = \mathbf{0}$, and that $x \leq u^*$ for every $x \in \mathcal{I}(L) \setminus \mathcal{I}(L_0)$. For every $x \in \mathcal{I}(L)$ such that $x < g_i$ for some $i \leq n$, if $x \in L_0$ then $x \leq g_i^-$ (where g_i^- still denotes the predecessor of g_i in L_0). If $x \notin L_0$ then $x \leq u^*$ by construction hence $x \leq g_i \wedge u^*$. The latter belongs to L_0 and is strictly lower than g_i, hence lower than g_i^-, so $x < g_i$. It follows that g_i has a unique predecessor in L which is g_i^-, in particular g_i remains $\vee$-irreducible in L. This proves our claim in both cases.

Now let $u_1 = \mathrm{C}^0(u) \vee u_1^*$ and $u_2 = u_2^*$. We have in particular

$$u_1 \vee u_2 = \bigvee_{i\leq n} g_i^- - (b_1 \vee b_2). \tag{11}$$

Since $u - b_2 = u$ by TC_3 necessarily $b_2 \wedge c \ll c$ for every $\vee$-irreducible component c of u, hence $b_2 \wedge \mathrm{C}^0(u) \ll \mathrm{C}^0(u)$. By SC_6 it follows that $b_2 \wedge \mathrm{C}^0(u) = \mathbf{0}$ hence $b_2 \wedge u_1 = b_2 \wedge u_1^*$. Similarly $u^* \wedge \mathrm{C}^0(u) = 0$ because $u^* - \mathrm{C}^0(u) = u^*$ by SC_{12} and TC_3. *A fortiori* $u_2^* \wedge \mathrm{C}^0(u) = \mathbf{0}$ hence $u_2^* \wedge u_1 = u_2^* \wedge u_1^*$. Note also that $b_1 \wedge u_2^* = b_1 \wedge u^* \wedge u_2^* \leq u_1^* \wedge u_2^*$, and symmetrically $b_2 \wedge u_1^* \leq u_1^* \wedge u_2^*$. Altogether, since $u_2 = u_2^*$ and $u_1^* \wedge u_2^* \leq b_1 \wedge b_2$ by construction, this gives

$$(b_1 \wedge u_2) \vee (b_2 \wedge u_1) \vee (u_1 \wedge u_2) \leq (b_1 \wedge b_2)$$

hence

$$(b_1 \vee u_1) \wedge (b_2 \vee u_2) = (b_1 \wedge b_2) \vee (b_1 \wedge u_2) \vee (b_2 \wedge u_1) \vee (u_1 \wedge u_2) = (b_1 \wedge b_2). \tag{12}$$

After this preparation, for each i let

$$h_{i,1} = g_i^- \wedge (b_1 \vee u_1), \qquad h_{i,2} = g_i^- \wedge (b_2 \vee u_2), \qquad \sigma_i = \big(g_i, \{h_{i,1}, h_{i,2}\}, \text{sc-dim}\, g_i\big)$$

Using (11) we get

$$\begin{aligned} h_{i,1} \vee h_{i,2} &= g_i^- \wedge (b_1 \vee u_1 \vee b_2 \vee u_2) \\ &= g_i^- \wedge \left[b_1 \vee b_2 \vee \left(\bigvee_{j\leq n} g_i^- - (b_1 \vee b_2) \right) \right] \\ &= g_i^- \wedge \bigvee_{j\leq n} g_j^- = g_i^-. \end{aligned}$$

So each σ_i is an SC-signature in L_0. In particular Proposition 6.4.2 gives an SC-primitive extension $L_1 = L_0\langle a_{1,1}, a_{1,2}\rangle$ with SC-signature σ_1 in L_0. By Proposition 6.4.1, $\mathcal{I}(L_1) = (\mathcal{I}(L_0) \setminus \{a_1\}) \cup \{a_{1,1}, a_{1,2}\}$. In particular $g_2 \in \mathcal{I}(L_1)$, hence σ_2 is still an SC-signature in L_1. Repeating the construction n times (note that $a \neq \mathbf{0}$ ensures that $n \geq 1$) gives a chain of $\mathcal{L}_{\mathrm{SC}}$-extensions $(L_i)_{i\leq n}$ and for each $i > 0$, an SC-primitive tuple $(a_{i,1}, a_{i,2})$ generating L_i over L_{i-1} with signature σ_i in L_{i-1}. Each $a_i = a_{i,1} \vee a_{i,2}$ and by Proposition 6.4.1

$$\mathcal{I}(L_n) = \big(\mathcal{I}(L_0) \setminus \{a_1, \ldots, a_n\}\big) \cup \{a_{1,1}, a_{1,2}, \ldots, a_{n,1}, a_{n,2}\} \tag{13}$$

so $a_{1,1}, a_{1,2}, \ldots, a_{n,1}, a_{n,2}$ are the $\vee$-irreducible components of a in L_n. Moreover every $c \in \mathcal{I}(L)$ such that $c < a_{i,k}$ for some i, k must belong to L_0, hence the predecessor of $a_{i,k}$ is the same in

every L_j and belongs to L_0. We can then denote it $a^-_{i,k}$ without ambiguity, and by construction we have

$$a^-_{i,k} = a_{i,k} \wedge g(a_i, L_{i-1}) = a_{i,k} \wedge g_i = h_{i,k}. \tag{14}$$

Let $a_1 = \bigvee_{i\leq n} a_{i,1}$, $a_2 = \bigvee_{i\leq n} a_{i,2}$, $h_1 = \bigvee_{i\leq n} h_{i,1}$ and $h_2 = \bigvee_{i\leq n} h_{i,2}$. We are going to check that a_1, a_2 split a along b_1, b_2. Both of them are non-zero and since the $a_{i,k}$'s are the $\vee$-irreducible components of a we have $a - a_1 = a_2$, $a - a_2 = a_1$. Each $a_{i,1} \geq h_{i,1}$ by construction, hence $a_1 \geq h_1$ and symmetrically $a_2 \geq h_2$. Moreover for $k \in \{1, 2\}$

$$h_k = \bigvee_{i\leq n} h_{i,k} \geq \bigvee_{i\leq n} g_i^- \wedge b_k = b_k$$

where the last equality comes from (10), so $a_k \geq b_k$. It remains to check that $a_1 \wedge a_2 = b_1 \wedge b_2$.

For $i \neq j$, $a_{i,1}$ and $a_{j,2}$ are mutually incomparable hence by (14)

$$a_{i,1} \wedge a_{j,2} = a^-_{i,1} \wedge a^-_{i,2} = h_{i,1} \wedge h_{j,2}.$$

On the other hand $a_{i,1} \wedge a_{i,2} = h_{i,1} \wedge h_{i,2}$ by construction. The conclusion follows, using (12).

$$\begin{aligned}
a_1 \wedge a_2 &= \bigvee_{i,j} a_{i,1} \wedge a_{j,2} = \bigvee_{i,j} h_{i,1} \wedge h_{j,2} \\
&= \bigvee_{i,j} \left[g_i^- \wedge (b_1 \vee u_1)\right] \wedge \left[g_j^- \wedge (b_2 \vee u_2)\right] \\
&= \bigvee_{i,j} (g_i^- \wedge g_j^-) \wedge \left[(b_1 \vee u_1) \wedge (b_2 \vee u_2)\right] \\
&= \left(\bigvee_i g_i^-\right) \wedge \left(\bigvee_j g_j^-\right) \wedge \left[(b_1 \vee u_1) \wedge (b_2 \vee u_2)\right] \\
&= (b_1 \vee u_1) \wedge (b_2 \vee u_2) = b_1 \wedge b_2.
\end{aligned}$$

■

Theorem 7.3 *The theory of super d-scaled lattices is the model-completion of the theory of d-subscaled lattices*

Proof: By standard model-theoretic arguments it suffices to prove that every existentially closed d-subscaled lattice is super d-scaled, and that for every super d-scaled lattice $\hat{L}$, every finitely generated d-subscaled lattice L and every common $\mathcal{L}_{\mathrm{SC}}$-substructure L_0, there is an embedding of L into $\hat{L}$ over L_0.

Let L be an existentially closed d-subscaled lattice, and L_0 a finitely generated substructure. By Theorem 4.1, L_0 is finite. By Proposition 5.3, L_0 $\mathcal{L}_{\mathrm{SC}}$-embeds the d-scaled lattice $\mathrm{L}_{\mathrm{def}}(X)$ of some special linear set X, which is in particular a Catenary lattice. By the model-theoretic compactness Theorem it it follows that L is catenary. Similarly Theorem 4.1, Lemma 7.2 and the model-theoretic compactness Theorem prove that L has the Splitting property, hence L is super d-scaled.

Conversely assume that L is a super d-scaled lattice, L is a finitely generated d-subscaled lattice L and L_0 common $\mathcal{L}_{\mathrm{SC}}$-substructure. By Theorem 4.1 and Proposition 6.4.3a, we are reduced to the case where L is a primitive extension of L_0. Let $\sigma = (g, \{h_1, h_2\}, q)$ be its SC-signature. By Proposition 6.4.2 it suffices to find a $x_1, x_2 \in \hat{L}$ such that (x_1, x_2) is SC-primitive over L_0 and $\sigma_{\mathrm{SC}}(x_1, x_2) = \sigma$. We distinguish two cases, and let g^- denotes the predecessor of g in L_0.

Case 1: $\text{sc-dim}\, h_1 < q < \text{sc-dim}\, g$ and $h_1 = h_2 < g$. Let $p = \text{sc-dim}\, g$ and $r = \text{sc-dim}\, h_1$. Let $y_1, y_2 \in \hat{L}$ which split g along h_1, g^-. For $0 \leq i \leq d$, either $i < q$ or $\mathrm{C}^i(h_1) = \mathbf{0}$ (because $\text{sc-dim}\, h_1 = r < q$), hence $\text{sc-dim}\, \mathrm{C}^i(h_1) < q < \text{sc-dim}\, g$. The Catenarity property then applies to $\mathrm{C}^i(h_1) \leq y_1$ and gives $x_i \in \hat{L}$ such that $\mathrm{C}^i(h_1) \leq x_i \leq y_1$ and x_i has pure dimension q. Let

$x = \mathbb{W}_{0\leq i\leq d}\, x_i$, by construction $h_1 = \mathbb{W}_{i\leq d}\, \mathrm{C}^i(h_1) \leq x \leq y_1$ and x has pure dimension q. In particular

$$h_1 \leq x \wedge g^- \leq y_1 \wedge y_2 = h_1 \wedge g^- = h_1$$

hence $x \wedge g^- = h_1 \in L_0$. Moreover $x \wedge g^- = h_1 \ll x$ because $\dim h_1 < q$ and x has pure dimension q. Finally $x \ll g$ because $\dim xi = q < p$ and g has pure dimension p. Altogether this proves that (x, x) is an SC-primitive tuple over L_0 with SC-signature σ.

Case 2: $q = \text{sc-dim}\, g$ and $h_1 \vee h_2 = g^-$. Let $y_1, y_2 \in \hat{L}$ which split g along h_1, h_2. By construction $y_1 \vee y_2 = g$, and since g has pure SC-dimension q so does each y_i. In addition $y_1 \wedge y_2 = h_1 \wedge h_2 \in L_0$. Moreover

$$y_1 \wedge h_2 \leq y_1 \wedge y_2 = h_1 \wedge h_2$$

hence $y_1 \wedge (h_1 \vee h_2) = h_1 \vee (y_1 \wedge h_2) = h_1$. Since $h_1 \vee h_2 = g^-$ it follows that $y_1 \wedge g^- = h_1 \in L_0$, and symmetrically $y_2 \wedge g^- = h_2 \in L_0$. So (y_1, y_2) is an SC-primitive tuple over L_0 with SC-signature σ.

■

Remark 7.4 The proof of Theorem 7.3 shows that if L_0 be a finite $\mathcal{L}_{\mathrm{SC}}$-substructure of a super scaled lattice $\hat{L}$, then every signature σ in L_0 is the signature of an SC-primitive extension of L_0 in $\hat{L}$.

The completions of the theory of super d-scaled lattices are easy to classify. Let us say that a d-subscaled lattice is **prime** if it does not contain any proper d-subscaled lattice, or equivalently if it is generated by the empty set. Every prime d-subscaled lattice is finite. By Corollary 4.2 there exists finitely many prime d-subscaled lattices up to isomorphism.

Corollary 7.5 *The theory of super d-scaled lattices containing (a copy of) a given prime d-subscaled lattice is $\aleph_0$-categorical, hence complete. It is also recursively axiomatizable, hence decidable. Since every completion of the theory of super d-scaled lattices is of that kind, the theory of super d-scaled lattices is decidable.*

Proof: Let L, L' be any two countable super d-scaled lattices containing isomorphic prime d-subscaled lattices L_0 and L'_0. By Remark 7.4 any partial isomorphism between L and L', extending the given isomorphism between L_0 and L'_0, can be extended by va-et-vient. This proves the first statement. The other ones are immediate consequences.

■

8 Atomic scaled lattices

Every super scaled lattice is atomless because of the Splitting Property, hence none of the geometric scaled lattice in $\mathrm{SC}_{\mathrm{def}}(\mathcal{K}, d)$, $\mathrm{SC}_{\mathrm{Zar}}(K, d)$, $\mathrm{SC}_{\mathrm{lin}}(K, d)$ can be super scaled. In order to apply our study to some of them we introduce now a variant of subscaled lattices intended to protect atoms against splitting.

Let $\mathcal{L}_{\mathrm{ASC}} = \mathcal{L}_{\mathrm{SC}} \cup \{\mathrm{At}_k\}_{k\in\mathbf{N}^*}$ with each At_k a new unary predicate symbol. For any $\mathcal{L}_{\mathrm{ASC}}$-structure L we denote by $\mathrm{At}_k(L)$ the set of elements a in L such that $L \models \mathrm{At}_k(a)$, and we let $\mathrm{At}_0(L) = L \setminus \bigcup_{k>0} \mathrm{At}_k(L)$. We call L an **ASC-lattice** if its $\mathcal{L}_{\mathrm{SC}}$-reduct is a scaled lattice and if satisfies the following condition.

$\mathbf{ASC_0}$: $(\forall k > 0)$, $a \in \mathrm{At}_k(L)$ if and only if a is the join of exactly k atoms in L.

Remark 8.1 This condition can be expressed by $\forall\exists$ formulas in $\mathcal{L}_{\mathrm{ASC}}$ by saying first that $\mathrm{At}_1(L)$ is the set of atoms of L, and then that $\mathrm{At}_k(L)$ is the set of elements of L which are the join of exactly k elements of $\mathrm{At}_1(L)$.

Every ASC-lattice obviously satisfies also the following schemes (for $k, l > 0$) of universal axioms:

$\mathbf{ASC_1}$: $(\forall k \neq l)$, $\quad \forall a,\ \mathrm{At}_k(a) \to \neg\, \mathrm{At}_l(a)$

$\mathbf{ASC_2}$: $(\forall k)$, $\quad \forall a, a_0, \ldots, a_{2^k}$, $\quad \mathrm{At}_k(a) \longrightarrow$

$$\left[\bigwedge_{0\leq i\leq 2^k} (a_i \leq a) \longrightarrow \bigvee_{0\leq i<j\leq 2^k} (a_i = a_j)\right] \bigwedge \text{sc-dim}\, a = 0$$

$\mathbf{ASC_3}$: $(\forall k)$, $\quad \forall a, a_1, a_2,$

$$\Big[(a = a_1 \vee a_2) \bigwedge (a_1 \wedge a_2 = \mathbf{0}) \bigwedge (a_1 \neq \mathbf{0}) \bigwedge (a_2 \neq \mathbf{0})\Big]$$

$$\longrightarrow \left[\mathrm{At}_k(a) \longleftrightarrow \bigvee_{0<l<k} \big(\,\mathrm{At}_l(a_1) \bigwedge \mathrm{At}_{k-l}(a_2)\big)\right]$$

We call **sub-ASC-lattices** the $\mathcal{L}_{\mathrm{ASC}}$-structures L whose $\mathcal{L}_{\mathrm{SC}}$-reduct is a subscaled lattice and which satisfy ASC_1 to ASC_3 (but not necessarily ASC_0).

ASC_1 obviously means that $(\mathrm{At}_k(L))_{k\in\mathbf{N}}$ is a partition of L. For any $a \in L$ we then define $\mathrm{asc}(a)$ as the unique $k \in \mathbf{N}$ such that $a \in \mathrm{At}_k(L)$.

ASC_2 says that if $\mathrm{asc}(a) = k > 0$ then $L(a)$ has at most 2^k element and $\text{sc-dim}(a) = 0$. Then $\dim(a) = 0$ by SC_{11} hence $L(a)$ is a co-Heyting algebra with dimension 0, that is a Boolean algebra. So ASC_2 actually says that $\text{sc-dim}\, a = 0$ and $L(a)$ is a Boolean algebra with n atoms for some non zero $n \leq k$. In particular every $a \in \mathrm{At}_1(L)$ is an atom of L.

ASC_3 says that if a is the join of two non-zero disjoint elements a_1, a_2 then $\mathrm{asc}(a)$ is non-zero if and only if $\mathrm{asc}(a_1)$ and $\mathrm{asc}(a_2)$ are non-zero, in which case $\mathrm{asc}(a) = \mathrm{asc}(a_1) + \mathrm{asc}(a_2)$. By a straightforward induction this extends to any decomposition of a as the join of finitely many pairwise disjoint elements. In view of ASC_2 it then says that $\mathrm{asc}(a) > 0$ if and only if a is the join of finitely many atoms $a_1, \ldots, a_n$ of L such that each $\mathrm{asc}(a_i) > 0$, in which case $\mathrm{asc}(a) = \sum_{1\leq i\leq n} \mathrm{asc}(a_i)$.

Remark 8.2 It follows immediately that an $\mathcal{L}_{\mathrm{SC}}$-embedding of sub-ASC-lattices $\varphi\colon L \to L'$ is an $\mathcal{L}_{\mathrm{ASC}}$-embedding if and only if $\mathrm{asc}(a) = \mathrm{asc}(\varphi(a))$ for every *atom* $a \in L$.

Remark 8.3 Obviously every finitely generated substructure of a sub-ASC-lattices is finite by the Local Finiteness Theorem 4.1 because $\mathcal{L}_{\mathrm{ASC}}$ expands $\mathcal{L}_{\mathrm{SC}}$ only by relational symbols.

Every scaled lattice L admits a unique structure of ASC-lattice which is an expansion by definition of its lattice structure. We denote by L^{At} this expansion of L.

Proposition 8.4 (Linear representation) *Let K be an infinite field and L_0 be a finite sub-ASC-lattice. For every integer $N \geq 0$ there exists a special linear set X_N over K and an $\mathcal{L}^*_{\mathrm{ASC}}$-embedding $\varphi_N\colon L_0 \to \mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(K^m)$ such that for every atom a of L_0:*

- *If $asc(a) > 0$ then $asc(\varphi_N(a)) = asc(a)$.*
- *If $asc(a) = 0$ then $asc(\varphi_N(a)) \geq N$.*

Proof: By induction on lexicographically ordered tuples of integers (r, s) we prove that the result is true for every finite sub-ASC-lattice L_0 having r $\vee$-irreducible elements, s of which have the same sc-dimension as L_0.

If $r = 0$ then $s = 0$ and the unique embedding of $L_0 = \{\mathbf{0}\}$ into $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(P)$, for an arbitrary point P of K, has the required property. So let us assume that $r \geq 1$ and that the result is proved for every $(r', s') < (r, s)$. Let $d = \text{sc-dim}\, L$ and $a_1, \ldots, a_r$ be the elements of $\mathcal{I}(L_0)$ ordered by increasing sc-dimension, so that $\text{sc-dim}\, a_r = d \geq 0$.

Case 1: $d = 0$. Then L is a boolean algebra and $a_1, \ldots, a_r$ are its atoms. Let $A_1, \ldots, A_r$ be pairwise disjoint subsets of K such that:

- If $\mathrm{asc}(a_i) > 0$ then A_i has $\mathrm{asc}(a_i)$ elements, so $\mathrm{asc}(A_i) = \mathrm{asc}(a_i)$.
- If $\mathrm{asc}(a_i) = 0$ then A_i has N elements, so $\mathrm{asc}(A_i) = N$.

Let X be the union of all these A_i's. Clearly the map φ which maps each a_i to A_i extends uniquely to an $\mathcal{L}_{\mathrm{SC}}$-embedding of L_0 into $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X)$ which has the required properties.

Case 2: $d > 0$. The upper semi-lattice L_0^- generated by $a_1, \ldots, a_{r-1}$ is an $\mathcal{L}^*_{\mathrm{ASC}}$-substructure of L_0 to which the induction hypothesis applies. This gives for some integer m a special linear set $B \subseteq K^m$ over K and an $\mathcal{L}_{\mathrm{SC}}$-embedding $\psi\colon L_0^- \to \mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(B)$ having the required properties. Let $C = \varphi(\mathbf{1}_{L_0^-} \wedge a_r)$ and $n = \text{sc-dim}\, a_r$. Proposition 5.2 gives a special linear set $A \subseteq K^{m+n}$ such that $A \cap B = C$. One can extend ψ to an $\mathcal{L}_{\mathrm{SC}}$-embedding φ of L_0 into $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(A \cup B)$ exactly like in the proof of Proposition 5.3. Then φ inherits from ψ the required properties because all the atoms of L already belong to L_0^-.

■

Let $\mathrm{ASC}_{\mathrm{Zar}}(K, d)$, $\mathrm{ASC}_{\mathrm{lin}}(K, d)$, $\mathrm{ASC}_{\mathrm{def}}(K, d)$ denote the class of all ASC-lattices L^{At} for L ranging over $\mathrm{SC}_{\mathrm{Zar}}(K, d)$, $\mathrm{SC}_{\mathrm{lin}}(K, d)$, $\mathrm{SC}_{\mathrm{def}}(K, d)$ respectively.

Corollary 8.5 *For every integer $d \geq 0$, the universal theories of $\mathrm{ASC}_{\mathrm{def}}(\mathcal{K}, d)$ (resp. of $\mathrm{ASC}_{\mathrm{Zar}}(K, d)$ or $\mathrm{ASC}_{\mathrm{lin}}(K, d)$) is the same for every o-minimal or P-minimal expansion of a field K (resp. every infinite field K). This is the theory of sub-ASC-lattices.*

Proof: Since $\mathrm{ASC}_{\mathrm{lin}}(K, d)$ is contained in the other classes, all of which are contained in the class of ASC-lattices, it suffices to prove that conversely every sub-ASC-lattice $\mathcal{L}_{\mathrm{ASC}}$-embeds into an ultraproduct of elements of $\mathrm{ASC}_{\mathrm{lin}}(K, d)$. By the model-theoretic compactness theorem, it suffices to prove it for any finitely generated sub-ASC-lattice L_0.

By Theorem 4.1, L_0 is finite. For any integer $N \geq 0$ let $\varphi_N : L_0 \to \mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N)$ be an $\mathcal{L}_{\mathrm{SC}}$-embedding given by Proposition 8.4. Let $\mathcal{U}$ be a non principal ultrafilter in the Boolean algebra of subsets of $\mathbf{N}$, and consider the ultraproduct $L = \prod_{N \in \mathbf{N}} \mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N)/\mathcal{U}$. Then $\varphi = \prod_{N \in \mathbf{N}} \varphi_N/\mathcal{U}$ is an $\mathcal{L}_{\mathrm{SC}}$-embedding of L_0 into the L. In order to prove that it is an $\mathcal{L}_{\mathrm{ASC}}$-embedding, by Remark 8.2 it remains check that for every atom a of L_0, $\mathrm{asc}(\varphi(a)) = \mathrm{asc}(a)$. So let a be an atom of L_0 and $k = \mathrm{asc}(a)$.

If $k > 0$ then for every $N \geq k$, $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N) \models \mathrm{At}_k(\varphi_N(a))$ by construction. So $L \models \mathrm{At}_k(\varphi(a))$, that is $\mathrm{asc}(\varphi(a)) = k$.

If $k = 0$, let l be any strictly positive integer. For every $N \geq l$, $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N) \models \mathrm{At}_N(\varphi_N(a))$ by construction, hence $\mathrm{L}^{\mathrm{At}}_{\mathrm{lin}}(X_N) \not\models \mathrm{At}_l(\varphi_N(a))$. So $L \not\models \mathrm{At}_l(\varphi(a))$, and this being true for every $l > 0$ it follows that $\mathrm{asc}(\varphi(a)) = 0$.

■

9 Model-completion of atomic scaled lattices

Let us call **super ASC-lattices** those ASC-lattices which satisfy the following axioms, all of which are axiomatizable by $\forall\exists$-formulas in $\mathcal{L}_{\mathrm{ASC}}$. We are going to show that its theory is the model-completion of the theory of sub-ASC-lattices of dimension at most d (resp. exactly d).

Atomicity: Every element x is the least upper bound of the set of atoms smaller than x.

Catenarity: For every non-negative integers $r \leq q \leq p$ and every elements $c \leq a \neq \mathbf{0}$, if c is r-sc-pure and a is p-sc-pure then there exists a non-zero q-sc-pure element b such that $c \leq b \leq a$.

ASC-Splitting: For every b_1, b_2, a, if $b_1 \vee b_2 \ll a \neq \mathbf{0}$ and $\mathrm{C}^0(a) = \mathbf{0}$ there exists non-zero elements $a_1 \geq b_1$ and $a_2 \geq b_2$ such that:

$$\begin{cases} a_1 = a - a_2 \\ a_2 = a - a_1 \\ a_1 \wedge a_2 = b_1 \wedge b_2 \end{cases}$$

Remark 9.1 An immediate consequence of the atomicity axiom is that for every elements x, y in a super ASC-lattice L such that $y < x$ and $\text{sc-dim}(x-y) \geq 1$, there are infinitely many atoms $a \in L$ such that $a \leq x$ and $a \wedge y = \mathbf{0}$. Indeed let A be the set of atoms $a \in L$ such that $a \leq x - y$, and B the subset of those a such that $a \wedge y \neq \mathbf{0}$. Assume for a contradiction that B is finite and let $b = \bigvee_{a \in B} a$. Note that $b \leq y$ and $\text{sc-dim}\, b = \dim b = 0$. Then by the Atomicity axiom

$$x - y = \bigvee_{a \in A} a \leq y \vee b, \text{ hence } x - y \leq (y \vee b) - y = b - y \leq b.$$

This implies that $\text{sc-dim}(x-y) \leq \text{sc-dim}\, b = 0$, a contradiction.

Primitive tuples and sc-primitive extensions are defined for sub-ASC-lattices exactly like for subscaled lattices.

We define **ASC-signatures** in a finite sub-ASC-lattice L_0 as triples (g, H, q) with H a set of non-necessarily distinct pairs (h_1, k_1), (h_2, k_2) in $L_0 \times \mathbf{N}$ such that $(g, \{h_1, h_2\}), q$ is an SC-signature in the $\mathcal{L}_{\mathrm{SC}}$-reduct of L_0 and (see Example 9.2):

1. If $q < \text{sc-dim}\, g$ then $k_1 = k_2$.
   Note that $h_1 = h_2$ in this case because $(g, \{h_1, h_2\}, q)$ is a signature.
2. If $q \neq 0$ then $k_1 = k_2 = 0$.
3. If $k_1 = 0$ or $k_2 = 0$ then $\mathrm{asc}(g) = 0$.
4. If $k_1 \neq 0$, $k_2 \neq 0$ and $\text{sc-dim}\, g = 0$ then $\mathrm{asc}(g) = k_1 + k_2$.

Example 9.2 Let L_0 be a finite sub-ASC-lattice, and L an $\mathcal{L}_{\mathrm{SC}}$-extension of L_0 generated by a (neccessarily unique) SC-primitive tuple (x_1, x_2). Let $(g, \{h_1, h_2\}, q)$ be the SC-signature of L in L_0 and $k_i = \mathrm{asc}(x_i)$. Then $\big(g, \{(h_1, k_1), (h_2, k_2), q\}\big)$ is easily an ASC-signature in L_0. We call it the **ASC-signature** of L and of (x_1, x_2) in L_0.

The same argument as in Proposition 6.4.2 shows (using Remark 8.2) that two SC-primitive extensions of a finite sub-ASC-lattice L_0 are $\mathcal{L}_{\mathrm{ASC}}$-isomorphic over L_0 if and only if they have the same ASC-signature in L_0.

Lemma 9.3 *Let L_0 be a finite $\mathcal{L}_{\mathrm{ASC}}$-substructure of a super ASC-lattice $\hat{L}$. Let $\sigma_{\mathrm{At}} = (g, q, \{(h_1, k_1), (h_2, k_2)\})$ be an ASC-signature in L_0. Assume that $q \neq 0$ or $k_1 k_2 \neq 0$. Otherwise assume that $\hat{L}$ is $\aleph_0$-saturated. Then there exists a primitive tuple $(x_1, x_2) \in \hat{L}$ over L_0 whose signature is σ_{At}.*

Proof: Let $\sigma = (g, \{h_1, h_2\}, q)$. This is an SC-signature in L_0 (more precisely in its $\mathcal{L}_{\mathrm{SC}}$-reduct).

Case 1: sc-dim $g \geq 1$ and $q \geq 1$. Then $\mathrm{C}^0(g) = 0$ and by definition of ASC-signatures $k_1 = k_2 = 0$. By Remark 7.4 there is an SC--primitive tuple (x_1, x_2) in $\hat{L}$ with signature σ in L_0. Moreover each asc $x_i = 0$ (because sc-dim $x_i = p \geq 1$) and each $k_i = 0$, so the ASC-signature of (x_1, x_2) is σ_{At}.

Case 2: sc-dim $g \geq 1$ and $q = 0$. Then $\mathrm{C}^0(g) = 0$ again and since sc-dim$(h_1 \vee h_2) < q = 0$ by definition of SC-signatures we get that $h_1 = h_2 = \mathbf{0}$. Finally $k_1 = k_2$ by definition of ASC-signatures since $q = 0 <$ sc-dim g. By Remark 9.1 there are infinitely many atoms z in $\hat{L}$ such that $z \leq g$ and $z \wedge g^- = \mathbf{0}$. If $k_1 > 0$ let x be the join of k_1 such atoms of $\hat{L}$. Otherwise $\hat{L}$ is $\aleph_0$-saturated by assumption hence it contains an element $x \leq g$ of dimension 0 such that $x \wedge g^- = \mathbf{0}$ and $\hat{L}(x)$ has infinitely many atoms. By the Atomicity Property asc$(x) = 0$. So in both cases (x, x) is an SC-primitive tuple over L_0 with ASC-signature σ_{At}.

Case 3: sc-dim $g = 0$. Then $q = 0$, g is an atom of L_0 and $h_1 = h_2 = \mathbf{0}$. In each of the two remaining sub-cases, we build a tuple (x_1, x_2) and leave as an exercise to check that (x_1, x_2) is SC-primitive over L_0 with ASC-signature σ_{At}.

If k_1 and k_2 are non-zero then asc$(g) = k_1 + k_2$ hence $\hat{L}(g)$ contains $k_1 + k_2$ atoms. Let x_1 be the join of k_1 of them and x_2 be the join of the others.

Otherwise, by symmetry we can assume that $k_1 = 0$. Then asc$(g) = 0$ by definition of ASC-signatures so $\hat{L}(g)$ contains infinitely many atoms. By $\aleph_0$-saturation it follows that $\hat{L}$ contains an element x lower than g such that both $L(x)$ and $L(g - x)$ contain infinitely many atoms, hence asc$(x) =$ asc$(g - x) = 0$. If $k_2 = 0$ let $(x_1, x_2) = (x, g - x)$. Otherwise let x_2 be the join of k_2 atoms in $\hat{L}(g)$ and let $x_1 = g - x_2$.

■

Theorem 9.4 *The theory of super ASC-lattices of sc-dimension at most d (resp. exactly d) is the model-completion of the theory of ASC-lattices of dimension at most d (resp. exactly d). It admits $\aleph_0$ completions, each of which is decidable, and it is decidable.*

Proof: We only sketch the proof of the first statement as it essentially the same as for Theorem 7.3.

On one hand, given a finite sub-ASC-lattice L_0, we can embed it in an extension satisfying the Atomicity and Catenary Property by Proposision 8.4, and the ASC-Splitting Property by means of Lemma 7.2 applied to any $a, b_1, b_2 \in L_0$ such that $b_1 \vee b_2 \ll a \neq \mathbf{0}$ and $\mathrm{C}^0(a) = \mathbf{0}$ (note that this last assumption ensures that the extension built in Lemma 7.2 is an $\mathcal{L}_{\mathrm{ASC}}$-extension). That every existentially closed sub-ASC-lattice is a super ASC-lattice then follows, by the model-theoretic compactness theorem.

On the other hand, given an $\aleph_0$-saturated super ASC-lattice $\hat{L}$, a finite $\mathcal{L}_{\mathrm{ASC}}$-substructure L_0 and a finite extension L of L_0, we reduce to the case where L is SC-primitive and let σ be its ASC-signature in L_0. Lemma 9.3 gives an SC-primitive extension L_1 of L_0 in $\hat{L}$ with the same signature in L_0, hence an embedding of L_1 into $\hat{L}$ over L_0 (which maps L to L_1).

This proves the first statement. The last one follows because there are finitely many subscaled lattices of dimension at most d (resp. exactly d) and on each of them $\aleph_0$ different structures of

sub-ASC-lattices. So the completions of the theory of super ASC-lattices, which are determined by their prime model, can be recursively enumerated.

■

We say that a sub-ASC-lattice L is **standard** if every element of sc-dimension 0 belongs to some $\mathrm{At}_k(L)$ for some $k > 0$. The existence of standard super ASC-lattices (see Section 10) and non-standard super ASC-lattices (by the model theoretic compactness theorem) implies that the theory of super ASC-lattices containing a given prime sub-ASC-lattice is not $\aleph_0$-categoric, contrary to what happens for super scaled lattice. However we can recover $\aleph_0$-categoricity by restricting to standard models.

Proposition 9.5 *Let L_1, L_2 be two* standard *countable super ASC-lattices. Then every $\mathcal{L}_{\mathrm{ASC}}$-isomorphism from a finite sub-ASC-lattice $L_{1,0} \subset L_1$ to a sub-ASC-lattice $L_{2,0} \subset L_2$ extends to an $\mathcal{L}_{\mathrm{ASC}}$-isomorphism from L_1 to L_2. In particular L_1 and L_2 are isomorphic if and only if their prime $\mathcal{L}_{\mathrm{ASC}}$-substructures (those generated by the empty set) are isomorphic.*

Proof: Let φ be an $\mathcal{L}_{\mathrm{ASC}}$-isomorphism from $L_{1,0}$ to $L_{2,0}$. Pick any element $x \in L_1 \setminus L_{1,0}$. The subscaled lattice generated in L_1 by $L_{1,0} \cup \{x\}$ (more precisely their $\mathcal{L}_{\mathrm{SC}}$-reducts) is finite hence by Proposition 6.4.3 there is a chain $L_{1,0} \subset L_{1,1} \subset \cdots \subset L_{1,r}$ of SC-primitive extensions of subscaled lattices such that $L_{1,0} \cup \{x\} \subseteq L_{1,r}$. Endow each $L_{1,i}$ with the $\mathcal{L}_{\mathrm{ASC}}$-structure induced by L_1. It suffices to prove that φ extends to an $\mathcal{L}_{\mathrm{ASC}}$-embedding $\varphi_1 : L_{1,1} \to L_2$. Indeed, repeating the argument will give an $\mathcal{L}_{\mathrm{ASC}}$-embedding $\varphi_r : L_{1,r} \to L_2$ extending φ, and by symmetry the conclusion will then follow by Fraissé's va-et-vient.

Identifying $L_{1,0}$ with its image by φ we can replace $L_{1,0}$ and $L_{2,0}$ by a common $\mathcal{L}_{\mathrm{ASC}}$-structure L_0 of L_1 and L_2. Now $L_{1,1}$ is generated over L_0 by an SC-primitive tuple (x_1, x_2) with signature $\sigma_{\mathrm{At}} = (g, \{(h_1, k_1), (h_2, k_2)\}, q)$. In particular $q = \text{sc-dim}\, x_i$ and $k_i = \mathrm{asc}(x_i)$ for $i = 1, 2$. If $q = 0$ then for each i, $\text{sc-dim}\, x_i = 0$ hence $k_i > 0$ because L_1 is standard. With other words $q \neq 0$ or $k_1 k_2 \neq 0$ hence Lemma 9.3 gives an sc-primitive tuple (y_1, y_2) in L_2 with signature σ_{At}. Let $L_{2,1}$ be the asc-substructure of L_2 generated by $L_{1,1} \cup \{y_1, y_2\}$. By Proposition 6.4.2 φ extends to an $\mathcal{L}_{\mathrm{SC}}$-isomorphism φ_1 from $L_{1,1}$ to $L_{2,1}$ which maps each x_i to y_i. By construction $\mathrm{asc}(x_i) = \mathrm{asc}(y_i)$, and by Proposition 6.4.1 φ_1 is the identity map on L_0, so $\mathrm{asc}(\varphi_1(z)) = \mathrm{asc}(z)$ for every $z \in \mathcal{I}(L_{1,1})$. Hence φ_1 is an $\mathcal{L}_{\mathrm{ASC}}$-isomorphism by Remark 8.2, which proves the result.

■

10 Applications to lattices of p-adic semi-algebraic sets

In all this section K denotes a fixed p-adically closed field. For every semi-algebraic set X contained in K^m we let $L(X)$ denote the lattice of semi-algebraic subsets of X closed in X, endowed with its natural structure of ASC-lattice. Note that every $A \in L(X)$ of dimension 0 is finite, hence $L(X)$ is *standard*.

As already mentioned in the introduction, the results of the previous section lead us to conjecture in [Dar04] and finally to prove in [Dar17] the following result.

Theorem 10.1 (Theorem 3.4 in [Dar17]) *Let X be a non-empty semi-algebraic subset of K^m without isolated points. Assume that X is open in its topological closure $\overline{X}$ and let $Y_1, \ldots, Y_s$ be a collection of closed semi-algebraic subsets of $\partial X = \overline{X} \setminus X$ such that $Y_1 \cup \cdots \cup Y_s = \partial X$. Then there is a partition of X in non-empty semi-algebraic sets $X_1, \ldots, X_s$ such that $\partial X_i = Y_i$ for $1 \leq i \leq s$.*

We can now combine this theorem with the results of Section 10 in order to get the following applications.

Theorem 10.2 *Let X be any semi-algebraic subset of K^m. Then $L(X)$ is a super ASC-lattice. In particular its complete theory is decidable and eliminates the quantifiers in $\mathcal{L}_{\mathrm{ASC}}$.*

Proof: By construction $L(X)$ is an ASC-lattice satisfying the Atomicity property. The Catenarity Property is well known (it follows for example from Denef's Cell Decomposition) hence we only have to check the Splitting Property. So let $A, B_1, B_2 \in L(X)$ such that $B_1 \cup B_2 \ll A$ and A has no isolated point.

The same holds true for their closures in K^m, denoted $\overline{A}$, $\overline{B}_1$, $\overline{B}_2$. Apply Theorem 10.1 to $X = \overline{A} \setminus (\overline{B}_1 \cup \overline{B}_2)$, $Y_1 = \overline{B}_1$ and $Y_2 = \overline{B}_2$. It gives a partition of X in non-empty semi-algebraic sets X_1, X_2 whose frontiers are respectively $\overline{B}_1$, $\overline{B}_2$. Then $\overline{X}_1 \cup \overline{X}_2 = \overline{A}$, $\overline{X}_1 \cap \overline{X}_2 = \overline{B}_1 \cap \overline{B}_2$ and each $\overline{X}_i = X_i \cup \overline{B}_i$. Let $A_1 = \overline{X}_1 \cap A = (X_1 \cap A) \cup B_1$ and define A_2 accordingly. We have to check that A_1, A_2 split A along B_1, B_2.

A is dense in $\overline{A} = \overline{X}$ and $X_1 = \overline{X} \setminus (X_2 \cup \overline{B}_1 \cup \overline{B}_2) = \overline{X} \setminus (\overline{X}_2 \cup \overline{B}_1)$ is open in $\overline{X}$, hence $A \cap X_1$ is dense in X_1. In particular $A \cap X_1 \neq \emptyset$, and symmetrically $A \cap X_2 \neq \emptyset$. Clearly $A_1 \cup A_2 = A$, $A_1 \cap A_2 = B_1 \cap B_2$ and each $A_i \supseteq B_i$ by construction. So it only remains to check that $A - A_1 = A_2$ in $L(X)$, that is that the closure of $A \setminus A_1$ in X (hence in A) is A_2. Note that $A \setminus A_1 = A \setminus \overline{X}_1$ and

$$\overline{A} \setminus \overline{X}_1 = \big(X_1 \cup X_2 \cup \overline{B}_1 \cup \overline{B}_2\big) \setminus (X_1 \cup \overline{B}_1) = X_2 \cup \big(\overline{B}_2 \setminus \overline{B}_1\big).$$

In particular $A \setminus A_1 = A \setminus \overline{X}_1 = (\overline{A} \setminus \overline{X}_1) \cap A$ contains $X_2 \cap A$ and is contained in $(X_2 \cup \overline{B}_2) \cap A = \overline{X}_2 \cap A = A_2$. The conclusion will follow, if we can prove that $X_2 \cap A$ is dense in A_2. Since $A_2 = (X_2 \cap A) \cup B_2$ it suffices to check that $B_2 \subseteq \overline{X_2 \cap A}$. But this is clear since $X_2 \cap A$ is dense in X_2, hence in $\overline{X}_2 = X_2 \cup \overline{B}_2$.

■

Corollary 10.3 *Let F be a q-adically closed field (for some prime q not necessarily equal to p). Let $X \subseteq K^m$ and $Y \subseteq F^n$ be two semi-algebraic sets.*

1. *If $m = n$, $K \preccurlyeq F$ and $X = Y \cap K^n$ then $L(X) \preccurlyeq L(Y)$.*
2. *$L(X) \equiv L(Y) \iff$ their prime $\mathcal{L}_{\mathrm{ASC}}$-substructures are isomorphic. In particular $L(K^m) \equiv L(F^n)$ if and only if $m = n$.*
3. *If K and F are countable then $L(X) \equiv L(Y) \iff L(X) \simeq L(Y)$.*

Proof: The two first points follow immediately from Theorem 10.2. Note that $L(K^m) \equiv L(F^m)$ is a special case because their prime sublattice is just the two-element lattice with the same $\mathcal{L}_{\mathrm{ASC}}$-structure, because K^m and F^m both have pure dimension m. The last point follows from Proposition 9.5 since both $L(X)$ and $L(Y)$ are standard and countable.

■

Given a pair of semi-algebraic sets $X \subseteq K^m$ and $Y \subseteq F^n$, we say that a homeomorphism $\psi : X \to Y$ is **pre-algebraic** if for every semi-algebraic sets $A \subseteq X$ and $B \subseteq Y$ defined over K and F respectively, $\varphi(A)$ and $\varphi^{-1}(B)$ are still semi-algebraic sets defined over K and F. It is obviously sufficient to check this for semi-algebraic sets A, B closed in X, Y respectively. With other words, a bijection $\psi : X \to Y$ is a pre-algebraic if and only if taking direct images by ψ defines an $\mathcal{L}_{\mathrm{ASC}}$-isomorphism from $L(X)$ to $L(Y)$ (which also ensures that ψ is a homeomorphism).

Remark 10.4 When $K = F$, semi-algebraic homeomorphisms are obviously pre-algebraic but the converse is false. Indeed, applying the next proposition to $X = K$ and $Y = R$, the p-valuation of K, gives a pre-algebraic homeomorphism $\varphi : K \to R$. If the value group of K is $\mathbf{Z}$, the p-valuation defines a metric on K and its completion K' is known to be an elementary extension of K. Since its valuation ring R' is compact and K' is not, they cannot be homeomorphic. Hence φ do not extend to a homeomorphism from K' to R', in particular φ is not semi-algebraic.

Theorem 10.5 *Let K, F be* countable *p-adically closed fields, and $X \subseteq K^m$, $Y \subseteq F^n$ be two semi-algebraic sets. Let $L^0(X)$ and $L^0(Y)$ be the prime $\mathcal{L}_{\mathrm{ASC}}$-substructures of $L(X)$ and $L(Y)$ respectively. Then X and Y are pre-algebraically homeomorphic if and only if $L^0(X)$ and $L^0(Y)$ are $\mathcal{L}_{\mathrm{ASC}}$-isomorphic. In particular, any two semi-algebraic sets over K and F with the same pure dimension $d \geq 1$ are pre-algebraically homeomorphic.*

Proof: One direction is obvious: every pre-algebraic homeomorphism $\psi : X \to Y$ induces an $\mathcal{L}_{\mathrm{ASC}}$-isomorphism from $L(X)$ to $L(Y)$, which maps their respective prime $\mathcal{L}_{\mathrm{ASC}}$-substructures one to each other. Conversely, assume that an $\mathcal{L}_{\mathrm{ASC}}$-isomorphism is given from $L^0(X)$ to $L^0(Y)$. By Proposition 9.5 it extends to an $\mathcal{L}_{\mathrm{ASC}}$-isomorphism $\varphi : L(X) \to L(Y)$. For every $t \in X$, φ maps $\{t\}$ to an atom $\{t'\}$ of $L(Y)$. Let $\psi(t) = t'$, this defines a bijection $\psi : X \to Y$ such that $\psi(A) = \varphi(A)$ for every $A \in L(X)$, hence ψ is a pre-algebraic homeomorphism. The last statement follows.

■

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