NONDEGENERATE CRITICAL POINTS, MORSE-DARBOUX LEMMA AND PROPAGATORS IN BV FORMALISM

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Abstract. The notion of nondegenerate critical point in the BV formalism is studied. The analogs of the Morse and Darboux theorems in the BV formalism are proven. The theorem on the normal form of an arbitrary quadratic function on odd symplectic space is proven. This can be viewed as an analog of Jordan type decomposition for a pair of a symmetric pairing on vector space and an anti-symmetric pairing on the dual space. The last result was used for the construction of propagators in the BV formalism in the equivariant setting, see [2, 3, 4]

1. Morse and Darboux Theorems .

1.1. **Nondegenerate critical point of action functional.** Let M be a smooth real supervariety with an odd symplectic structure. Let $\{\cdot,\cdot\}$ denotes the corresponding odd bracket on functions on M. Let f be an even function on M, which satisfies the classical BV master equation

$$\{f,f\}=0. \tag{1.1}$$

Locally the algebra of functions on M with the odd bracket is isomorphic to the multi-vector algebra on $\mathbb{R}^n$ equipped with Schouten's bracket $\mathbb{R}\left[x^i,\xi_j=\frac{\partial}{\partial x^j}\right]$, $\left\{x^i,\xi_j\right\}=\delta^i_j$, see [1, 5]. The function f has a critical point at p if its development near p in the local coordinates x^i , ξ_j has no linear terms

$$f=const+\sum_{i_1i_2}f_{i_1i_2}x^{i_1}x^{i_2}+\sum_{j_1j_2}f^{j_1j_2}\xi_{j_1}\xi_{j_2}+\sum_{|I|,|J|>2}f^J_I x^{i_1}\dots x^{i_k}\xi_{j_1}\dots\xi_{j_l}.$$

Then the quadratic term $f_{(2)}=\sum_{i_1i_2}f_{i_1i_2}x^{i_1}x^{i_2}+\sum_{j_1j_2}f^{j_1j_2}\xi_{j_1}\xi_{j_2}$satisfies also $\left\{f_{(2)},f_{(2)}\right\}=0$. And therefore defines the differential $d_{lin}=\{f_{(2)},\cdot\}$ acting on the cotangent space at p.

Definition 1. The critical point of the solution f to (1.1) is *non-degenerate* if the cohomology of the differential $d_{lin}=\{f_{(2)},\cdot\}$ acting on the cotangents space at this point are trivial:

$$\ker\{f_{(2)},\cdot\}=\mathrm{im}\{f_{(2)},\cdot\}$$

Example 2. An example of such a situation is a function $f=f(x)$ on $\mathbb{R}^n$ considered as a multi-vector. The equation $\{f,f\}=0$ is satisfied because in this case f does not depend on ξ_j's. A critical point p of such solution is nondegenerate if and only if the matrix $\frac{\partial^2 f}{\partial x^i\partial x^j}$ is nondegenerate at this point, which corresponds to a Morse critical point.

Example 3. Another example of a solution is a Poisson bivector

$$\gamma = \sum_{j_1 j_2} \gamma^{j_1 j_2}(x)\xi_{j_1}\xi_{j_2}.$$

A critical point of this solution is nondegenerate if and only if the matrix $\gamma^{j_1 j_2}$ is nondegenerate, i.e. γ defines locally a symplectic structure.

1.2. **Canonical form.**

Theorem 4. *For any solution (1.1) with a non-degenerate critical point, there exists a change of local coordinates x^i, ξ_j on M, preserving the odd symplectic structure, after which f is quadratic in the new coordinates.*

$$f = const + \sum_{i_1 i_2} f_{i_1 i_2} x^{i_1} x^{i_2} + \sum_{j_1 j_2} f^{j_1 j_2}\xi_{j_1}\xi_{j_2}.$$

Proof. Let us construct step-by-step a local change of coordinates, preserving the odd symplectic structure. Suppose that f has an expansion

$$f = const + f_{(2)} + \sum_{|I|+|J|=k} f_I^J x^{i_1}\dots x^{i_k}\xi_{j_1}\dots\xi_{j_l} + \sum_{|I|+|J|>k} f_I^J x^{i_1}\dots x^{i_k}\xi_{j_1}\dots\xi_{j_l}$$

and $k > 2$. Then $\{f_{(2)}, f_{(k)}\} = 0$ where $f_{(k)} = \sum_{|I|+|J|=k} f_I^J x^{i_1}\dots x^{i_k}\xi_{j_1}\dots\xi_{j_l}$ is the term of the degree$k$. The cohomology of the differential $\{f_{(2)}, \cdot\}$ are trivial on polynomials of degree k, since they are trivial on linear functions. Then there exists a polynomial h_k of degree k, such that $\{f_{(2)}, h_k\} = f_{(k)}$. Let

$$v_k = \sum_i \frac{\partial h_k}{\partial x^i}\frac{\partial}{\partial \xi_i} - \frac{\partial h_k}{\partial \xi_i}\frac{\partial}{\partial x^i}$$

be the hamiltonian vector field of the hamiltonian h_k, and let $\varphi_k^t : x \to x + tv_k(x) + \dots$ be the one-parameter group of diffeomorphisms generated by the vector field $v_k(x)$. Then the change of variables in the form $x = \varphi_k^1(x)$ preserves the odd symplectic structure and reduces the solution f to

$$f = const + f_{(2)} + \sum_{|I|+|J|>k} \tilde{f}_I^J x^{i_1}\dots x^{i_k}\xi_{j_1}\dots\xi_{j_l}$$

□

In the two previous examples this result is equivalent to the Morse's theorem and to the Darboux's theorem correspondingly. This result is interesting because it makes possible to see the analogy in certain results between the Morse theory and the symplectic geometry. Also it is possible to develop analogs of some theorems from the Morse theory and from the symplectic geometry in a new situation of the other case of solutions with nondenerated critical points $f = h(x) + \gamma$ where γ is a Poisson bivector and $h(x)$ is a Casimir function of γ, nondegenerated in transverse directions to the symplectic leaves of the Poisson bivector γ.

2. Propagators in BV formalism.

Let g denotes an odd non-degenerate symmetric pairing $g : V^{\otimes 2} \to \Pi k$ on some $\mathbb{Z}/2\mathbb{Z}$-graded vector space V. Notice that this is the same as the odd symplectic structure on the $\mathbb{Z}/2\mathbb{Z}$-graded vector space ΠV.

Theorem 5. *Any odd anti-self-adjoint operator I over algebraically closed field k, char $(k) \neq 2$,*

$$g(Ix, y) + (-1)^{\overline{x}} g(x, Iy) = 0$$

can be reduced by linear transformation preserving the odd scalar product g to a sum consisting of the following blocks:

1_k- $\mathbb{Z}/2\mathbb{Z}-$*graded subspace spanned by* $2k$ *elements* $v, Iv, \ldots, I^{2k-1}v$ *with* $\overline{v} =$ *k mod* 2, $I^{2k}v = 0$,

$$g(I^m v, I^{m'} v) = \delta_{m+m',2k-1}(-1)^{m\overline{v}+\frac{1}{2}m(m+1)},$$

2_k- $\mathbb{Z}/2\mathbb{Z}-$*graded subspace spanned by* $2k$ *elements* $v, u, Iv, Iu \ldots, I^{k-1}v, I^{k-1}u$ *with* $\overline{v} + \bar{u} = k$ *mod* 2, $I^k u = I^k v = 0$, $g(I^m v, I^{m'} v) = 0$, $g(I^m u, I^{m'} u) = 0$,

$$g(I^m v, I^{m'} u) = \delta_{m+m',k-1}(-1)^{m\overline{v}+\frac{1}{2}m(m+1)},$$

3 - $\mathbb{Z}/2\mathbb{Z}-$*graded subspace on which I is invertible.*

Proof. For any $n \leq n'$ $\ker I^n \subseteq \ker I^{n'}$. There exists $n_{\max}$ such that for any $n > n_{\max}$ $\ker I^n = \ker I^{n_{\max}}$, since $\dim V < \infty$. The subspace $U = \ker I^{n_{\max}}$ is $I-$invariant and I is invertible on the quotient V/U. Therefore for any $v \in V$ there exists v' such that $I^{n_{\max}} v' = v \bmod U$. If the restriction $g|_U$ is degenerate, so that there exists $u \in U$ such that $g(u, u') = 0$ for any $u' \in U$, then, since g is symmetric,

$$g(u, I^{n_{\max}} v') = g(I^{n_{\max}} u, v') = 0$$

for any $v' \in V$. This implies that $g(u, v) = 0$ for any $v \in V$. Therefore such u does not exist and the restriction $g|_U$ is nondegenerate. Therefore $U^\perp \cap U = \{0\}$ and V is a direct sum of the $I-$invariant subspaces $V = U \oplus U^\perp$. In particular, $U^\perp \simeq V/U$ and I is invertible on $U^\perp$. This is the subspace of type 3.

Let n be the smallest number, such that there exists an element $v \in \ker I^n$ which is not orthogonal to $\ker I$: $g(v, \cdot)|_{\ker I} \neq 0$. It can be assumed without loss of generality that v is of certain parity $\bar{v}$. For any $k \leq n$, $I^k v \in \ker I^{n-k}$, $I^k v \notin \ker I^{n-k-1}$ and the elements $I^k v$ $k = 0, \ldots, n-1$ are linearly independent.

Consider first the case $g(v, I^{n-1}v) \neq 0$. Since k is algebraically closed, v can be normalized so that

$$g(v, I^{n-1}v) = 1. \tag{2.1}$$

It follows from the proposition 6 that if v is even then $n = 4k'$, and if v is odd then $n = 4k' - 2$, $k' \in \mathbb{N}$. For all $m + m' \geq n$: $g(I^m v, I^{m'} v) = g(I^{m+m'} v, v) = 0$. Also $g(I^m v, I^{m'} v) = 0$ when $m + m'$is even, for the parity reason. Assume that v is even, then $n = 4k'$,$I^{2l}v$ is even for any $l \in \mathbb{N}$ and $g(v, I^{4l+1}v) = 0$ for any l by the loc.cit. Let $l < k'$ be the biggest number such that $g(v, I^{4l-1}v) = a$, $a \neq 0$. Then $\widetilde{v} = v - \frac{1}{2}aI^{4k'-4l}v$ still has $g(\widetilde{v}, I^{4l'-1}\widetilde{v}) = 0$ for any $l < l' < k'$, $g(\widetilde{v}, I^{4k'-1}\widetilde{v}) = 1$ and in addition satisfies

$$g(\widetilde{v}, I^{4l-1}\widetilde{v}) = g(v - \frac{1}{2}aI^{4k'-4l}v, I^{4l-1}v - \frac{1}{2}aI^{4k'-1}v) = 0.$$

In this way v can be chosen so that $g(v, I^m v) = 0$ for all m unless $m = n-1$. Similarly, if v is odd, then $n = 4k' - 2$, $g(v, I^{4k'-3}v) = 1$,$I^{2l-1}v$ is even and $g(v, I^{4l-1}v) = 0$ for any $l \in \mathbb{N}$. Similarly, let $l < k'$ be the biggest number such that $g(v, I^{4l-3}v) = a$, $a \neq 0$. Then $\widetilde{v} = v - \frac{1}{2}aI^{4k'-4l}v$ still has $g(\widetilde{v}, I^{4l'-3}\widetilde{v}) = 0$ for any $l < l' < k'$, $g(\widetilde{v}, I^{4k'-3}\widetilde{v}) = 1$ and satisfies in addition $g(\widetilde{v}, I^{4l-3}\widetilde{v}) = 0$. In this way v can again be chosen so that $g(v, I^m v) = 0$ unless $m = n-1$. In particular, we

see that the restriction of g to the subspace $\tilde{U}$ spanned by $I^k v$, $k = 0, \dots, n-1$ is nondegenerate in the case (2.1). It follows, in particular, that U is the direct sum of the $I-$invariant subspaces $U = \tilde{U} \oplus \tilde{U}^{\perp}$.

Consider next the case $g(v, I^{n-1}v) = 0$, and let us denote $u' \in \ker I$ the element of parity $\overline{v}+1$ such that $g(v,u') = 1$. Then since u' is orthogonal to all elements from $\ker I^{n-1}$, therefore u' is in the image of the self-adjoint operator I^{n-1}, and there exists u such that $I^{n-1}u = u'$. Then

$$(-1)^{\overline{v}k+\frac{1}{2}k(k+1)} g(I^k v, I^{n-1-k}u) = g(v, I^{n-1}u) = 1. \tag{2.2}$$

We can assume also that $g(u, I^{n-1}u) = 0$, since otherwise this reduces to the previous case. I claim that v and u can be chosen so that except (2.2) all other pairings $g(I^k v, I^m u)$, $g(I^k v, I^m v)$ and $g(I^k u, I^m u)$ are zero. Let $g(v, I^{l'}u) = 0$, $g(v, I^{l'}v) = 0$, $g(u, I^{l'}u) = 0$ holds for all$l' \in [l+1, \dots, n-1]$. Let $g(v, I^l u) = a$, $g(v, I^l v) = b$, $g(u, I^l u) = c$. Put $\widetilde{v} = v - \frac{1}{2}bI^{n-l-1}u$, $\tilde{u} = u - \frac{1}{2}cI^{n-l-1}v - aI^{n-l-1}u$. Then, since $n-1 > l$, and $u, v \in \ker I^n$:

$$g(v - \frac{1}{2}bI^{n-l-1}u, I^l u - \frac{1}{2}cI^{n-1}v - aI^{n-1}u) = g(v, I^l u) - ag(v, I^{n-1}u) = 0.$$

Also if $b \neq 0$ then by the proposition 6, $g(I^{n-l-1}u, I^l v) = g(I^{n-1}u, v) = 1$, and therefore $g(\tilde{v}, I^l \tilde{v}) = 0$,since

$$g(v - \frac{1}{2}bI^{n-l-1}u, I^l(v - \frac{1}{2}bI^{n-l-1}u)) = g(v, I^l v) - \frac{1}{2}bg(v, I^{n-1}u) - \frac{1}{2}bg(I^{n-l-1}u, I^l v) = 0.$$

Similarly, $g(\tilde{u}, I^l \tilde{u}) = 0$. It follows that v and u can be chosen in this way so that all pairings between $I^k u$, $I^{k'}v$, except (2.2), are zero. Therefore the elements $v, u, Iv, Iu \dots, I^{k-1}v, I^{k-1}u$ span the subspace W of type 2_k, the restricion of g to W is nondegenerate, and U is again the direct sum of the $I-$invariant subspaces $U = W \oplus W^{\perp}$where $W^{\perp}$ is the othogonal complement to U'.

Continuing in this way one gets the decomposition of the space U into direct sum of subspaces of type 1_k and 2_k, and the subspace $U^{\perp}$ is the subspace of type 3. □

Proposition 6. *Let $g(v, I^l v) \neq 0$ where I is an odd anti-selfadjoint operator and v is an element of parity $\bar{v}$. Then if $\bar{v} = 0$, then $l = 4\tilde{k} - 1$, and if$\bar{v} = 1$, then $l = 4\tilde{k} - 3$, $\tilde{k} \in \mathbb{N}$. It follows, in particular, that in both cases$g(u, I^l v) = g(I^l u, v)$ for any u.*

Proof. If α is even then

$$g(I\alpha, \alpha) = (-1)^{\overline{\alpha}+1} g(\alpha, I\alpha) = -g(\alpha, I\alpha) = -g(I\alpha, \alpha) = 0$$

vanishes. Since $g(v, I^l v) \neq 0$, $l = 2k-1$, $k \in \mathbb{N}$. If $g(v, I^{2k-1}v) \neq 0$ then $g(I^k v, I^{k-1}v) \neq 0$, and therefore the element $I^{k-1}v$ is odd. It follows that if v is even then $k = 2\tilde{k}$, $l = 4\tilde{k} - 1$, and if v is odd then $l = 4\tilde{k} - 3$, $\tilde{k} \in \mathbb{N}$. For any u, v: $g(I^2 u, v) = -g(u, I^2 v)$. Therefore for v even

$$g(u, I^{4k'+3}v) = -g(I^{4k'+2}u, Iv) = g(I^{4k'+3}u, v).$$

Similarly, for v odd

$$g(u, I^{4k'+1}v) = g(I^{4k'}u, Iv) = (-1)^{\bar{v}+1} g(I^{4k'+1}u, v)$$

□

Let I be an odd anti-selfadjoint operator I in the canonical form of the direct sum of blocks of types 1_k, 2_k and 3.

Proposition 7. *There exists an odd operator $\tilde{I}$, such that*

$$\left[I, \tilde{I}\right] = 1 \tag{2.3}$$

if and only if the decomposition of the operator I into blocks by the theorem 5 has no blocks of type 2_{2n-1}, $n \in \mathbb{N}$. In this case the operator $\tilde{I}$ can be chosen to be self-adjoint $g(\tilde{I}x, y) = (-1)^{\overline{x}} g(x, \tilde{I}y)$.

Proof. Let us put for the elements of blocks of type 1_k:

$$\tilde{I}(I^{2l+1}v) = I^{2l}v,\ \tilde{I}(I^{2l}v) = 0,\ l = 0, \dots, k-1,$$

for the elements of blocks of type 2_{2k}:

$$\tilde{I}(I^{2l-1}v) = I^{2l}v,\ \tilde{I}(I^{2l-1}u) = I^{2l}u,\ \tilde{I}(I^{2l}v) = \tilde{I}(I^{2l}u) = 0,\ l = 0, \dots, k-1,$$

define also $\tilde{I}$ on block of type 3:

$$\tilde{I} = \frac{1}{2} I^{-1}.$$

Then $\left[\tilde{I}, I\right] = 1$ and also $g(\tilde{I}x, y) = (-1)^{\overline{x}} g(x, \tilde{I}y)$. □

Assume now that such operator $\tilde{I}$ exists on a sum $V = W \oplus W'$ of two $I-$invariant subspaces, where W is a block of type 2_{2k-1}. Let $\tilde{I}_1$ be the block-diagonal component of $\tilde{I}$ acting on W, $\tilde{I} = \begin{pmatrix} \tilde{I}_1 & \tilde{I}_3 \\ \tilde{I}_2 & \tilde{I}_4 \end{pmatrix}$. Then $\left[\tilde{I}_1, I\right] = 1$on W, so it is enough to consider the case $V = W$. Since W is again the sum of two $I-$invariant subspaces, it is enough to check that the operator satisfying the (2.3) cannot exist on the $I-$invariant subspace spanned by $v, Iv, \dots, I^{2k-2}v$. Since $I\left(I^{2k-2}v\right) = 0$, then $\tilde{I}\left(I^{2k-2}v\right) = I^{2k-3}v$, and therefore $\tilde{I}\left(I^{2k-3}v\right) = a_0 I^{2k-2}v$. It follows that $\tilde{I}\left(I^{2k-4}v\right) = I^{2k-5}v - a_0 I^{2k-3}v$ and therefore $\tilde{I}\left(I^{2k-5}v\right) = a_0 I^{2k-4}v + a_1 I^{2k-2}v$, etc. $\tilde{I}(Iv) = a_0 I^2 v + \sum_{j\geq 1} a_j I^{2(j+1)}v$. It follows that $\left[I, \tilde{I}\right] v \in \mathrm{Image}\,(I)$ and therefore $\left[I, \tilde{I}\right] v \neq v$. Such an odd anti-self-adjoint operator I satisfying (2.3) is called homotopy invertible.

This notion extends the notion of "contractible differential" to the case when $I^2 \neq 0$.

2.1. **Jourdan blocks decomposition for the pair of quadratic forms.** The odd non-degenerate symmetric pairing g is uniquely determined by the linear isomorphism $g : V_0^\vee \simeq \Pi V_1$. The components of the anti-self-adjoint operator I are then the linear maps $I_{10} \in \mathrm{Hom}_k(V_0, V_0^\vee)$, $I_{01} \in \mathrm{Hom}_k(V_0^\vee, V_0)$, I_{10} is an anti-symmetric pairing on V_0 and I_{01} is a symmetric pairing on $V_0^\vee$.

Proposition 8. *The anti-self-adjoint operators I on V are in one-to-one correspondence with pairs (h, h') where h is an anti-symmetric quadratic form on V_0 and h' is a symmetric quadratic form on $V_0^\vee$. The decomposition of I from the theorem 5 is equivalent to an analog of the "Jordan blocks" decomposition for such pairs.*

Proof. Follows from the theorem5, see also [6] for similar results. □

2.2. **Nondegeneracy of $g(Ix,x)$ on $(\Pi V)_0 \Rightarrow I$ is homotopy invertible.** The odd anti-self-adjoint operator I, or the pair (h,h') where h is an anti-symmetric quadratic form on V_0 and h' is a symmetric quadratic form on $V_0^\vee$, is the same as an even quadratic function on ΠV. The correspondence is given by

$$I \leftrightarrow g(Ix,x),\ x \in \Pi V.$$

Proposition 9. *Let the restriction of the quadratic function $g(Ix,x)$ on $(\Pi V)_0$ is a nondegenerate quadratic form. Then the decomposition of I has no blocks of type 2_{2n-1} and the operator $\widetilde{I}$ defined in the proposition 7 satisfies.*

$$\left[I,\tilde{I}\right] = 1.$$

Proof. Notice that $\bar{v}+\bar{u}=1$ for the block of type 2_{2n-1}, and therefore one of the elements $I^{k-1}v, I^{k-1}u$ from the kernel of I is alwas odd and then the quadratic function $g(Ix,x)$ on $(\Pi V)_0$ is degenerate. □

The propositions 7, 9 allow to construct the propagators, in particular in the equivariant setting when the quadratic term of the action functional does not necessarily satisfy $\left\{S_{(2)},S_{(2)}\right\}=0$. This is used for the construction of cohomology classes in the compactified modulis spaces of curves in [4].

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