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Stationary reaction-diffusion systems in L^1

El Haj Laamri*, Michel Pierre†,

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Abstract

We prove existence of solutions to stationary $m \times m$ reaction-diffusion systems where the data are in L^1 or in $L\text{Log}L$. We first give an abstract result where the “diffusions” are nonlinear m -accretive operators in L^1 and the reactive terms are assumed to satisfy m structural inequalities. It implies that the situation is controlled by an associated cross-diffusion system and provides L^1 -estimates on the reactive terms. Next we prove existence for specific systems modeling chemical reactions and which naturally satisfies less than m structural (in)equalities. The main difficulty is also to obtain L^1 -estimates on the nonlinear reactive terms.

Keywords: reaction-diffusion systems, nonlinear diffusion, cross-diffusion, global existence, porous media equation, weak solutions.

2010 MSC: 35K10, 35K40, 35K57

1 Introduction and main results

Our goal is to analyze the existence of solutions to stationary reaction-diffusion systems in L^1 -spaces. We first give a general abstract result for nonlinearities satisfying as many structural inequalities as the number of equations. This is done in the framework of abstract m -accretive operators in L^1 spaces for the diffusion part and the full system is controlled by a somehow “good” *associated cross-diffusion system*. We give several examples where this abstract result applies. Then, we provide a general result for more specific systems associated with general chemical reactions for which less structure holds for the nonlinearities.

We denote by (Ω, μ) a measured space where μ is a nonnegative measure on the set Ω with $\mu(\Omega) < +\infty$. We consider systems of the type

$$(S) \begin{cases} \forall i = 1, \dots, m, \\ u_i \in D(A_i) \cap L^1(\Omega, d\mu)^+, h_i(\cdot, u) \in L^1(\Omega, d\mu), \\ u_i + A_i u_i = h_i(\cdot, u_1, \dots, u_m) + f_i(\cdot) \in L^1(\Omega, d\mu), \end{cases} \quad (1)$$

where for all $i = 1, \dots, m$,

– A_i is a (possibly) nonlinear operator in $L^1(\Omega, d\mu)$ (generally a diffusion operator in applications), defined on $D(A_i) \subset L^1(\Omega, d\mu)$,

– $h_i : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}$ is a nonlinear “reactive” term,

– $f_i \in L^1(\Omega, d\mu)^+ [:= \{g \in L^1(\Omega, d\mu) ; g \geq 0, \mu - a.e.\}]$.

We are interested in nonnegative solutions $u = (u_1, \dots, u_m) \in L^1(\Omega, d\mu)^{+m}$.

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These systems are naturally associated with the *evolution reaction-diffusion systems*

$$\partial_t u_i(t) + A_i u_i(t) = h_i(\cdot, u(t)), \quad t \in [0, +\infty), \quad i = 1, \dots, m. \quad (2)$$

When approximating these evolution systems by an implicit time discretization scheme, we are led to solving the following set of equations, for all small time interval Δt

$$\begin{cases} \forall i = 1, \dots, m, \quad n = 0, 1, \dots, \quad u_i^{n+1} \in D(A_i), \\ \frac{u_i^{n+1} - u_i^n}{\Delta t} + A_i u_i^{n+1} = h_i(\cdot, u^{n+1}) \text{ or } u_i^{n+1} + (\Delta t)A_i u_i^{n+1} = (\Delta t)h_i(\cdot, u_i^{n+1}) + u_i^n. \end{cases}$$

This is exactly the system (S) with unknown $u = u^{n+1}$, up to trivially changing $(\Delta t)A_i$ into A_i and $(x, u) \mapsto [(\Delta t)h_i(x, u) + u_i^n(x)]_{1 \leq i \leq m}$ into $(x, u) \mapsto [h_i(x, u) + f_i(x)]_{1 \leq i \leq m}$.

The asymptotic steady states of the evolution system (2) are also the solutions $u = (u_1, \dots, u_m)$ to the system $\{A_i u_i = h_i(\cdot, u), i = 1, \dots, m, \}$ which is also essentially included in the system (S) up to a slight change of the operators A_i .

The kind of operators A_i we have in mind are :

- The Laplacian $u \mapsto -\Delta u$ on a bounded open set Ω of \mathbb{R}^N with various boundary conditions on $\partial\Omega$.
- More general elliptic operators $u \mapsto -\sum_{i=1}^N \partial_{x_i} \left(\sum_{j=1}^N a_{ij}(x) \partial_{x_j} u + b_i u \right)$ on the same Ω , with various boundary conditions as well.
- Nonlinear operators of porous media type like $u \mapsto -\Delta \varphi(u)$ where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function, again with different boundary conditions on $\partial\Omega$.
- Nonlinear operators of p -Laplacian type like $u \mapsto -\Delta_p u := -\nabla \cdot (|\nabla u|^{p-2} \nabla u)$ where $p \in (1, +\infty)$ and $|\cdot|$ is the euclidian norm in \mathbb{R}^N .
- Classical perturbations and various associations of all of those.

All these operators will satisfy the assumption (A) below which means that each A_i is an m-accretive operator in $L^1(\Omega, d\mu)$ whose resolvents are compact and preserve positivity, namely the following, where I denote the identity :

$$(A) \begin{cases} \forall i = 1, \dots, m, \\ A_i : D(A_i) \subset L^1(\Omega, d\mu) \rightarrow L^1(\Omega, d\mu), \quad \text{and for all } \lambda \in (0, +\infty), \\ (A1) \text{ m - accretivity : } I + \lambda A_i \text{ is onto and } (I + \lambda A_i)^{-1} \text{ is nonexpansive,} \\ (A2) \text{ positivity : } \int_{\Omega} \text{sign}^-(u_i) A u_i \geq 0, \forall u_i \in D(A_i), \\ (A3) \text{ compactness : } (I + \lambda A_i)^{-1} : L^1(\Omega, d\mu) \rightarrow L^1(\Omega, d\mu) \text{ is compact.} \end{cases} \quad (3)$$

Here I denotes the identity on $L^1(\Omega, d\mu)$ and in (A2), we define $\text{sign}^-(s) := -\text{sign}^+(-s)$, $\forall s \in \mathbb{R}$ where

$$\text{sign}^+(s) := 1, \quad \forall s \in [0, +\infty), \quad \text{sign}^+(s) = 0, \quad \forall s \in (-\infty, 0).$$

This implies immediately that $(I + \lambda A_i)^{-1} (L^1(\Omega, d\mu)^+) \subset L^1(\Omega, d\mu)^+$ since then, for $g \in L^1(\Omega, d\mu)^+$ and $u_i := (I + \lambda A_i)^{-1} g$

$$0 \geq \int_{\Omega} \text{sign}^-(u_i) g \, d\mu = \int_{\Omega} \text{sign}^-(u_i) [u_i + \lambda A_i u_i] \, d\mu \geq \int_{\Omega} u_i^- \, d\mu \Rightarrow u_i^- = 0 \quad \mu - a.e.$$

For simplicity, we consider only single-valued operators A_i , but everything would also work for multivalued m-accretive operators A_i .

The nonlinear reactive terms h_i will preserve positivity as well. We assume that, for all $i = 1, \dots, m$:

$$(H1) \begin{cases} (1) \quad h_i : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R} \text{ measurable ;} \\ (2) \quad \tilde{h}_i(\cdot, R) := \sup\{|h_i(\cdot, r)|; |r| \leq R\} \in L^1(\Omega, d\mu)^+, \quad \forall R \in [0, +\infty) ; \\ (3) \quad r \in \mathbb{R}^m \mapsto h_i(\cdot, r) \text{ is continuous ;} \\ (4) \quad \text{quasipositivity : } h_i(\cdot, r_1, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_m) \geq 0, \quad \forall r \in [0, +\infty)^m. \end{cases} \quad (4)$$

Moreover, the nonlinear reactive terms h_i we are considering are those for which mass conservation or, more generally, mass dissipation or at least mass control holds for the associated evolution system (2). It is the case when the h_i 's satisfy a relation like $\sum_{i=1}^m h_i(\cdot, r) \leq 0$ for all $r \in [0, +\infty)^m$ or more generally

$$(H2) \quad \sum_{i=1}^m a_i h_i(\cdot, r) \leq \sum_{i=1}^m b_i r_i + \omega(\cdot), \quad \omega \in L^1(\Omega, d\mu)^+, \text{ for some } 0 \leq b_i < a_i, i = 1, \dots, m, r \in [0, +\infty)^m. \quad (5)$$

As we will see, this structure naturally implies an a priori L^1 -estimate on the solutions u_i of the system (S) (see Proposition 2.2), together with “standard operators” A_i by which we mean

$$(A_{in f}) \quad a_\infty := \inf \left\{ \int_{\Omega} A_i u_i d\mu, u_i \in D(A_i) \cap L^1(\Omega, d\mu)^+ \right\} > -\infty \text{ for each } i = 1, \dots, m. \quad (6)$$

This assumption essentially holds when $0 \in D(A_i), i = 1, \dots, m$. It also holds with most diffusion operators with non homogeneous boundary conditions (see Section 3) except a few ones : this is discussed in Remark 3.4.

Actually, the main point will be to get a priori L^1 -estimates even on the nonlinearities $h_i(\cdot, u)$ where u is solution of System (S). This will be satisfied if more structure is required on the nonlinear functions h_i , namely (using the natural order in \mathbb{R}^m) :

$$(M) \quad \left\{ \begin{array}{l} \text{There exist two } m \times m \text{ matrices } M_0, M_1 \text{ where} \\ M_0 \text{ is invertible, with nonnegative entries and such that} \\ M_0 h(\cdot, r) \leq M_1 r + \Theta(\cdot), \forall r \in [0, +\infty)^m, \Theta \in L^1(\Omega, d\mu)^{+m}. \end{array} \right. \quad (7)$$

Applying the matrix M_0 to the system (S) leads to the following set of m inequalities for an associated *cross-diffusion system* :

$$M_0 u + M_0 A u = M_0 h(\cdot, u) + M_0 f \leq M_1 u + \Theta + M_0 f,$$

where we denote $Au := (A_i u_i)_{1 \leq i \leq m}$, $h(\cdot, u) := (h_i(\cdot, u))_{1 \leq i \leq m}$ and $f := (f_i)_{1 \leq i \leq m}$.

As proved in Proposition 2.3, this will imply an a priori estimate of $h_i(\cdot, u), i = 1, \dots, m$ in $L^1(\Omega, d\mu)$. More precisely, we will consider an approximate problem where the nonlinearities h_i are replaced by truncated versions h_i^n . Then (M) will imply that $h_i^n(\cdot, u^n)$ is bounded in $L^1(\Omega, d\mu)$ for the approximate solutions u^n . The compactness of u^n in $L^1(\Omega, d\mu)$ will then follow. Thus up to a subsequence, u^n will converge in $L^1(\Omega, d\mu)^m$ and μ -a.e. to some u and $h_i^n(\cdot, u^n)$ will also converge μ -a.e. to $h_i(\cdot, u)$ for all i . But this is not sufficient yet to pass to the limit in the system since one essentially needs the convergence of $h_i^n(\cdot, u^n)$ in $L^1(\Omega, d\mu)$. It is the case if $h_i^n(\cdot, u^n)$ is uniformly integrable, by Vitali's lemma. This will hold if we add the following technical assumption which is some kind of compatibility condition between the various operators A_i . It will be satisfied in the examples of Section 3.

$$(\Phi) \quad \left\{ \begin{array}{l} \text{There exist } \varphi : [0, +\infty)^m \rightarrow [0, +\infty) \text{ continuous with } \lim_{|r| \rightarrow +\infty} \varphi(r) = +\infty \text{ and } b \in (0, +\infty)^m \text{ such that} \\ \int_{\Omega} \text{sign}^+(\varphi(u) - k) b \cdot M_0 A u d\mu \geq 0, \forall u \in D(A) \cap L^1(\Omega, d\mu)^{+m}, \forall k \in [0, +\infty), D(A) := \prod_{i=1}^m D(A_i). \end{array} \right. \quad (8)$$

We now state our abstract result. It will be proved in Section 2 and applied to several explicit examples in Section 3. We refer to [11] where this kind of results and examples were already widely discussed and analyzed.

Theorem 1.1 Assume that $(A), (H1), (H2), (A_{inf}), (M), (\Phi)$ hold. Then the system (S) has a solution for all $f \in L^1(\Omega, d\mu)^{+m}$.

As already explained, the assumption (M) means that m independent inequalities hold between the m nonlinear functions h_i . But many systems come with less than m such relations. The following 2×2 system satisfies only one relation of type $(H2)$ and not (M) :

$$\begin{cases} u_1 - d_1 \Delta u_1 = (\beta_1 - \alpha_1)[u_1^{\alpha_1} u_2^{\alpha_2} - u_1^{\beta_1} u_2^{\beta_2}] + f_1 =: h_1(u_1, u_2) + f_1, \\ u_2 - d_2 \Delta u_2 = (\beta_2 - \alpha_2)[u_1^{\beta_1} u_2^{\beta_2} - u_1^{\alpha_1} u_2^{\alpha_2}] + f_2 =: h_2(u_1, u_2) + f_2, \\ \partial_\nu u_1 = 0 = \partial_\nu u_2 \text{ on } \partial\Omega, \end{cases} \quad (9)$$

where for $i = 1, 2$, $d_i \in (0, +\infty)$, $\alpha_i, \beta_i \in \{0\} \cup [1, +\infty)$, $f_i \in L^1(\Omega)^+$, $\Omega \subset \mathbb{R}^N$ equipped with the Lebesgue measure. This kind of nonlinearity appears in reversible chemical reactions with two species, namely

$$\alpha_1 A_1 + \alpha_2 A_2 \rightleftharpoons \beta_1 A_1 + \beta_2 A_2. \quad (10)$$

Here $(\beta_1 - \alpha_1)(\beta_2 - \alpha_2) < 0$. The nonlinearity h_1 and h_2 are quasipositive but satisfy the only one relation $\gamma_2 h_1 + \gamma_1 h_2 = 0$, $\gamma_i := |\beta_i - \alpha_i|$. It turns out that existence indeed holds for system (9) when $f_i \in L^1(\Omega)^+$ for all i , as we prove it here. Actually we prove existence for a general class of such ‘‘chemical systems’’ when $f_i \in L^1(\Omega)^+$ and also $f_i \log f_i \in L^1(\Omega)$. This is the second main result of this paper. Let us consider the system

$$(CHS) \begin{cases} \text{For all } i = 1, \dots, m, \\ u_i - d_i \Delta u_i = (\beta_i - \alpha_i) \left(k_1 \prod_{k=1}^m u_k^{\alpha_k} - k_2 \prod_{k=1}^m u_k^{\beta_k} \right) + f_i, \\ \partial_\nu u_i = 0 \text{ on } \partial\Omega, \end{cases} \quad (11)$$

where $k_1, k_2 \in (0, +\infty)$ and, for all $i = 1, \dots, m$, $d_i \in (0, +\infty)$, $\alpha_i, \beta_i \in \{0\} \cup [1, +\infty)$ and $f_i \in L^1(\Omega)^+$ where Ω is bounded regular open subset of \mathbb{R}^N and where

$$\begin{cases} I := \{i \in \{1, \dots, m\}; \alpha_i - \beta_i > 0\}, J := \{j \in \{1, \dots, m\}; \beta_j - \alpha_j > 0\} \\ \text{satisfy:} \\ I \neq \emptyset, J \neq \emptyset, I \cup J = \{1, \dots, m\}. \end{cases} \quad (12)$$

We denote by $|I|$ (resp. $|J|$) the number of elements of I (resp. J).

Theorem 1.2 Assume that $f_i \in L^1(\Omega)^+$, $f_i \log f_i \in L^1(\Omega)$ for all $i = 1, \dots, m$. Assume also that $|I| \leq 2$ (or $|J| \leq 2$). Then there exists a nonnegative solution $u \in W^{2,1}(\Omega)^{+m}$ of (CHS) with $h_i(u) \in L^1(\Omega)$ for all $i = 1, \dots, m$. If moreover $m = 2$, then the same result holds if $f_i \in L^1(\Omega)^+$, $i = 1, 2$ only.

Remark 1.3 System (11) arises when modeling a general reversible chemical reaction with m species, according to the mass action law and to Fick’s linear diffusion. In fact, it also contains systems written more generally as

$$v_i - d_i \Delta v_i = \lambda_i \left(K_1 \prod_{k=1}^m v_k^{\alpha_k} - K_2 \prod_{k=1}^m v_k^{\beta_k} \right) + \tilde{f}_i,$$

where $\lambda_i \in \mathbb{R}$, $\lambda_i(\beta_i - \alpha_i) > 0$, $i = 1, \dots, m$. Indeed, we may go back to the exact writing of (11) by setting

$$u_i := (\beta_i - \alpha_i)v_i/\lambda_i, \quad k_1 := K_1 \prod_k [\lambda_k / (\beta_k - \alpha_k)]^{\alpha_k}, \quad k_2 := K_2 \prod_k [\lambda_k / (\beta_k - \alpha_k)]^{\beta_k}, \quad f_i := (\beta_i - \alpha_i)\tilde{f}_i/\lambda_i.$$

On the other hand, it does not include systems like

$$\begin{cases} u_1 - d_1 \Delta u_1 = +[u_1^3 u_2^2 - u_1^2 u_2^3] + f_1 (= u_1^2 u_2^2 [u_1 - u_2] + f_1), \\ u_2 - d_2 \Delta u_2 = -[u_1^3 u_2^2 - u_1^2 u_2^3] + f_2. \end{cases}$$

Here the condition $\lambda_i(\beta_i - \alpha_i) > 0$ is not satisfied.

Remark 1.4 Note that besides the 2×2 system (9), Theorem 1.2 contains as particular cases some favorite systems of the literature like

$$h_i(u) = (-1)^i [u_1^{\alpha_1} u_3^{\alpha_3} - u_2^{\beta_2}], \quad i = 1, 2, 3, \quad \alpha_1, \alpha_3, \beta_2 \in [1, +\infty),$$

$$h_i(u) = (-1)^i [u_1 u_3 - u_2 u_4], \quad i = 1, 2, 3, 4.$$

We may use Remark 1.3 to write them as in (11). Analysis of systems of this kind may be found for instance in [12], [7], [18], [13], [9], [5], [6], etc.

In fact Theorem 1.2 applies to quite general systems like, for instance, those obtained by multiplying the above 3×3 (resp. 4×4) example by $u_1^{\sigma_1} u_2^{\sigma_2} u_3^{\sigma_3}$ (resp. $u_1^{\sigma_1} u_2^{\sigma_2} u_3^{\sigma_3} u_4^{\sigma_4}$) where $\sigma_i \in [0, +\infty)$. They lead to the following nonlinearities

$$h_i(u) = \lambda_i [\Pi_{k=1}^m u_k^{\alpha_k} - \Pi_{k=1}^m u_k^{\beta_k}],$$

where $m = 3$ (resp. $m = 4$) and $\lambda_i(\beta_i - \alpha_i) > 0$. Actually, Theorem 1.2 applies also to these same nonlinearities when $m = 5$. Indeed, since $I \cup J = \{1, 2, 3, 4, 5\}$, it follows that I or J contains at most two elements.

We believe that the result of Theorem 1.2 holds even if $|I|$ and $|J| > 2$. But it is not clear how to extend our main Lemma 4.2 to the general case. We leave this as an open problem.

Remark 1.5 Theorem 1.2 provides a solution u such that $h_i(u) \in L^1(\Omega)$. It is easy to write down explicit examples where the nonlinear terms $\Pi_k u_k^{\alpha_k}, \Pi_k u_k^{\beta_k}$ are not separately in $L^1(\Omega)$. For instance, let $N > 4$ and let us introduce the following function σ where $r := |x|, x \in \Omega := B(0, 1) \subset \mathbb{R}^N, b \in (N/2, N - 2), c \in (0, +\infty)$:

$$\sigma(r) := r^{-b} + br + c, \quad \forall 0 < r \leq 1,$$

$$\Rightarrow \sigma - \Delta \sigma = f, \quad f(r) := c + r^{-b} + br + b(N - 2 - b)r^{-(b+2)} - b(N - 1)r^{-1}, \quad \sigma'(1) = 0,$$

where c is chosen large enough so that $f \geq 0$. Note that $f \in L^p(\Omega), p \in [1, N/(b + 2))$. Let us now consider the solution (which exists by Theorem 1.2) of the system

$$\begin{cases} u_1, u_2 \in W^{2,1}(\Omega), \quad u_2^2 - u_1^2 \in L^1(\Omega), \\ u_1 - \Delta u_1 = u_2^2 - u_1^2 + f/2, \\ u_2 - \Delta u_2 = -[u_2^2 - u_1^2] + f/2, \\ \partial_\nu u_1 = \partial_\nu u_2 = 0 \text{ on } \partial\Omega. \end{cases}$$

Then $(u_1 + u_2) - \Delta(u_1 + u_2) = f, \partial_\nu(u_1 + u_2) = 0$ on $\partial\Omega$. Thus $u_1 + u_2 = \sigma$. But u_1, u_2 cannot be in $L^2(\Omega)$ since σ is not in $L^2(\Omega)$ by the choice of $b > N/2$.

Remark 1.6 It is classical that an entropy structure holds in System (11) since

$$\sum_{i=1}^m [\log u_i + \mu_i] h_i(u) = -[\log(k_1 \Pi_{k=1}^m u_k^{\alpha_k}) - \log(k_2 \Pi_{k=1}^m u_k^{\beta_k})] [k_1 \Pi_{k=1}^m u_k^{\alpha_k} - k_2 \Pi_{k=1}^m u_k^{\beta_k}] \leq 0,$$

when choosing $\mu_i := [\log k_1 - \log k_2] / [m(\alpha_i - \beta_i)]$. Under the assumptions of Theorem 1.2, and in particular the $L \text{Log} L$ assumption on the f_i , we have the estimate

$$\int_{\Omega} \sum_{i=1}^m [\log u_i + \mu_i] u_i + d_i \frac{|\nabla u_i|^2}{u_i} \leq \int_{\Omega} \sum_{i=1}^m [\log u_i + \mu_i] f_i \leq \int_{\Omega} \sum_{i=1}^m f_i \log f_i + (\mu_i - 1) f_i + u_i,$$

where, for the last inequality, we use the Young's inequality (75) with $r = f_i, s = \log u_i$.

But we will not use here this structure in the proof of Theorem 1.2. Our strategy will consist in proving that the nonlinearity $h_i(u)$ is a priori bounded in $L^1(\Omega)$. Then adequate compactness arguments allow us to pass to the limit in the approximate system. Note that the entropy inequality provides the extra information that $\nabla \sqrt{u_i} \in L^2(\Omega)$ for the solutions obtained in Theorem 1.2 when $f_i \log f_i \in L^1(\Omega), i = 1, \dots, m$.

Theorem 1.1 will be proved in Section 2, examples of applications are given in Section 3 and the proof of Theorem 1.2 is given in Section 4.

2 Proof of Theorem 1.1

Since μ is fixed, we will most of the time more simply write $L^1(\Omega)$, $L^1(\Omega)^+$ instead of $L^1(\Omega, d\mu)$, $L^1(\Omega, d\mu)^+$. We will use the natural order in \mathbb{R}^m namely $[r \leq \hat{r} \Leftrightarrow r_i \leq \hat{r}_i, i = 1, \dots, m]$ and we also denote $r^+ := (r_1^+, \dots, r_m^+)$.

Lemma 2.1 *Let $f = (f_1, \dots, f_m) \in L^1(\Omega)^{+m}$. We set*

$$h_i^n(x, r) := \frac{h_i(x, r)}{1 + n^{-1} \sum_{j=1}^m |h_j(x, r)|}, \quad \forall i = 1, \dots, m, \quad r \in \mathbb{R}^m, \quad x \in \Omega. \quad (13)$$

Assume that (A) and (H1) hold. Then the following approximate system

$$(S^n) \begin{cases} \forall i = 1, \dots, m, \quad u_i^n \in D(A_i) \cap L^1(\Omega)^+, \\ u_i^n + A_i u_i^n = h_i^n(\cdot, u^n) + f_i(\cdot) \quad \text{in } L^1(\Omega, d\mu), \end{cases} \quad (14)$$

has a nonnegative solution $u^n = (u_1^n, \dots, u_m^n)$.

Proof. We consider the mapping $\mathcal{T} : v \in L^1(\Omega)^m \mapsto w \in L^1(\Omega)^m$ where $w = (w_1, \dots, w_m)$ is the solution of

$$\begin{cases} \forall i = 1, \dots, m, \quad w_i \in D(A_i), \\ w_i + A_i w_i = h_i^n(\cdot, v^+) + f_i(\cdot) \quad \text{in } L^1(\Omega, d\mu), \end{cases} \quad (15)$$

where $v^+ := (v_1^+, \dots, v_m^+)$. The mapping \mathcal{T} is well defined since $|h_i^n(x, r)| \leq n$ for all $(x, r) \in \Omega \times \mathbb{R}^m$ and $f \in L^1(\Omega)^m$ so that the solution is given by

$$w_i = (I + A_i)^{-1} (h_i^n(\cdot, v^+) + f_i), \quad i = 1, \dots, m.$$

Moreover $\|h_i^n(\cdot, v) + f_i\|_{L^1(\Omega)} \leq n \mu(\Omega) + \|f_i\|_{L^1(\Omega)} =: M_i$. Let

$$K_i := (I + A_i)^{-1} [\{g \in L^1(\Omega) ; \|g\|_{L^1(\Omega)} \leq M_i\}], \quad K := K_1 \times \dots \times K_m \subset L^1(\Omega)^m.$$

By assumption (A) and in particular (A3), \bar{K} is a compact set of $L^1(\Omega)^m$ and $\bar{K} \supset \mathcal{T}(L^1(\Omega)^m)$. On the other hand, we easily check that \mathcal{T} is continuous. Thus \mathcal{T} is a compact operator from $L^1(\Omega)^m$ into the compact set $\bar{K} \subset L^1(\Omega)^m$. By Schauder's fixed point theorem (see e.g. [21]), \mathcal{T} has a fixed point u^n . This means that u^n is solution of (S^n) .

To prove the nonnegativity property of u^n , we multiply the i -th equation by $sign^-(u_i^n)$ and integrate to obtain, using $\int_{\Omega} sign^-(u_i) A_i u_i \geq 0$ (see (A2)),

$$\int_{\Omega} (u_i^n)^- d\mu \leq \int_{\Omega} [sign^-(u_i^n) h_i^n(\cdot, (u^n)^+) + sign^-(u_i^n) f_i] d\mu.$$

By the nonnegativity of f_i and the quasipositivity of h_i assumed in (H1), the integral on the right is nonpositive so that $\int_{\Omega} (u_i^n)^- d\mu \leq 0$ or $u_i^n \geq 0 \mu - a.e..$ \square

We will now progressively obtain estimates on u^n independently of n . We start with the control of the total mass of u^n .

Proposition 2.2 *Assume that (A), (H1), (H2), (A_{inf}) hold. Then there exists $C \in (0, +\infty)$ independent of n such that the solution u^n of the approximate system (S^n) satisfies*

$$\max_{1 \leq i \leq m} \|u_i^n\|_{L^1(\Omega)} \leq C.$$

Proof. According to (H2), we multiply each equation by a_i and we add them to obtain

$$\sum_{i=1}^m a_i(u_i^n + A_i u_i^n) = \sum_{i=1}^m a_i[h_i^n(\cdot, u^n) + f_i] \leq \sum_{i=1}^m [b_i u_i^n + a_i f_i].$$

We use (A_{inf}) to obtain

$$\sum_{i=1}^n (a_i - b_i) \int_{\Omega} u_i^n d\mu \leq -a_{\infty} \sum_{i=1}^m a_i + \sum_{i=1}^m a_i \int_{\Omega} f_i d\mu.$$

Using $a_i > b_i$ for all i yields the estimate of Proposition 2.2. \square

We now prove that the nonlinearities are bounded in L^1 independently of n .

Proposition 2.3 *Assume that (A), (H1), (H2), (A_{inf}), (M) hold. Then there exists $C \in (0, +\infty)$ independent of n such that the solution u^n of the approximate system (S^n) satisfies*

$$\max_{1 \leq i \leq m} \|h_i^n(\cdot, u^n)\|_{L^1(\Omega)} \leq C.$$

Proof. According to the assumption (M), let us multiply the system (S^n) by the matrix M_0 . As before, we denote

$$A u^n := (A_1 u^n, \dots, A_m u^n), \quad h^n(\cdot, u^n) := (h_1^n(\cdot, u^n), \dots, h_m^n(\cdot, u^n)), \quad f := (f_1, \dots, f_m).$$

Then

$$M_0 u^n + M_0 A u^n = M_0 h^n(\cdot, u^n) + M_0 f \leq M_1 u^n + \Theta + M_0 f. \quad (16)$$

Since the entries of M_0 are nonnegative, and thanks to (A_{inf}) and the nonnegativity of u^n , there exists $C_{\infty} \in [0, +\infty)^m$ such that

$$\int_{\Omega} [M_0 u^n + M_0 A u^n] d\mu \geq -C_{\infty} \quad \text{which implies} \quad \int_{\Omega} [M_0 h^n(\cdot, u^n) + M_0 f] d\mu \geq -C_{\infty}.$$

We deduce

$$\int_{\Omega} [M_1 u^n + \Theta - M_0 h^n(\cdot, u^n)] d\mu \leq C_{\infty} + \int_{\Omega} [M_1 u^n + \Theta + M_0 f] d\mu \leq D_{\infty} \in (0, +\infty)^m,$$

the last inequality using also Proposition 2.2. As the function $M_1 u^n + \Theta - M_0 h^n(\cdot, u^n)$ is nonnegative, these inequalities provide a bound for its $L^1(\Omega)^m$ -norm. It follows that $M_0 h^n(\cdot, u^n)$ is also bounded in $L^1(\Omega)^m$ independently of n . Since M_0 is invertible, we deduce that $h^n(\cdot, u^n)$ is itself bounded in $L^1(\Omega)^m$: indeed if $|\cdot|$ denotes the euclidian norm in \mathbb{R}^m and $\|\cdot\|$ the induced norm on the $m \times m$ matrices, we may write

$$|h^n(\cdot, u^n)| = |M_0^{-1} M_0 h^n(\cdot, u^n)| \leq \|M_0^{-1}\| |M_0 h^n(\cdot, u^n)|.$$

Whence Proposition 2.3. \square

Proposition 2.4 *Assume that (A), (H1), (H2), (A_{inf}), (M), (Φ) hold. Then, if u^n is the solution of the approximate system (S^n) , $\{h^n(\cdot, u^n)\}_n$ is uniformly integrable in Ω which means that, for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all measurable set $E \subset \Omega$*

$$\mu(E) < \delta \Rightarrow \int_E \sum_{i=1}^n |h_i^n(\cdot, u^n)| d\mu \leq \varepsilon \text{ for all } n.$$

Proof. By Proposition 2.3, we already know that $h^n(\cdot, u^n)$ is bounded in $L^1(\Omega)^m$. Let us prove that the extra condition (Φ) implies that it is even uniformly integrable.

First, since $u_i^n = (I + A_i)^{-1}(h^n(\cdot, u^n) + f_i)$, the compactness condition (A3) in (3) implies that, for all $i = 1, \dots, m$, u_i^n belongs to a compact set of $L^1(\Omega)$.

Let us show that $M_0 h^n(\cdot, u^n)$ is uniformly integrable. Since M_0 is invertible, this will imply that $h^n(\cdot, u^n)$ is itself uniformly integrable and end the proof of Proposition 2.4.

We will successively prove that $(M_0 h^n(\cdot, u^n))^+$ and $(M_0 h^n(\cdot, u^n))^-$ are uniformly integrable.

Note that by the definition (13), $h^n(\cdot, u^n) \leq h(\cdot, u^n)$. We may write (recall that the entries of M_0 are nonnegative)

$$M_0 h^n(\cdot, u^n) \leq M_0 h(\cdot, u^n) \leq M_1 u^n + \Theta \Rightarrow (M_0 h^n(\cdot, u^n))^+ \leq (M_1 u^n)^+ + \Theta. \quad (17)$$

Since u^n is in a compact set of $L^1(\Omega)^m$, so is $(M_1 u^n)^+$ and it is in particular uniformly integrable. We then deduce from the last inequality that $(M_0 h^n(\cdot, u^n))^+$ is uniformly integrable.

To control $(M_0 h^n(\cdot, u^n))^-$, we go back to (16) and, using (17), we rewrite it as follows

$$M_0 u^n + M_0 A u^n + (M_0 h^n(\cdot, u^n))^- = (M_0 h^n(\cdot, u^n))^+ + M_0 f \leq (M_1 u^n)^+ + \Theta + M_0 f. \quad (18)$$

According to (Φ), we multiply this inequality by $\text{sign}^+(\varphi(u^n) - k)b$. By (Φ) on one hand and by the nonnegativity of u^n, b and of the entries of M_0 on the other hand, we have

$$\int_{\Omega} \text{sign}^+(\varphi(u^n) - k)b \cdot M_0 A u^n d\mu \geq 0, \quad \int_{\Omega} \text{sign}^+(\varphi(u^n) - k)b \cdot M_0 u^n d\mu \geq 0, \quad \forall k \in [0, +\infty).$$

Combining with (18), we deduce

$$\int_{\Omega} \text{sign}^+(\varphi(u^n) - k)b \cdot (M_0 h^n(\cdot, u^n))^- d\mu \leq \int_{\Omega} \text{sign}^+(\varphi(u^n) - k)b \cdot [(M_1 u^n)^+ + \Theta + M_0 f] d\mu. \quad (19)$$

Since $\lim_{|r| \rightarrow +\infty} \varphi(r) = +\infty$, for all $k \in [0, +\infty)$, there exists $R(k)$ such that $\{|r| \geq R(k)\} \subset \{\varphi(r) \geq k\}$ and therefore

$$\{|u^n| \geq R(k)\} \subset \{\varphi(u^n) \geq k\}, \mu - a.e.. \quad (20)$$

Let $E \subset \Omega$ be a measurable set. Then

$$\int_E b \cdot (M_0 h^n(\cdot, u^n))^- d\mu \leq \int_{E \cap \{|u^n| \leq R(k)\}} b \cdot (M_0 h^n(\cdot, u^n))^- d\mu + \int_{E \cap \{|u^n| \geq R(k)\}} b \cdot (M_0 h^n(\cdot, u^n))^- d\mu =: I_-^n + I_+^n.$$

If we denote $\varphi_R := \max\{\varphi(r); |r| \leq R\}$, then $R \in [0, +\infty) \mapsto \varphi_R \in [0, +\infty)$ is a nondecreasing function such that $\{\varphi(r) \geq k\} \subset \{|r| \geq \psi(k)\}$ where $\psi(k) := \inf_{R \in [0, +\infty)} \varphi_R^{-1}([k, +\infty))$. By (20) and (19) and $\{\varphi(u^n) \geq k\} \subset \{|u^n| \geq \psi(k)\}$, we have

$$I_+^n \leq \int_{\{\varphi(u^n) \geq k\}} b \cdot [(M_1 u^n)^+ + \Theta + M_0 f] d\mu \leq \int_{\{|u^n| \geq \psi(k)\}} b \cdot [(M_1 u^n)^+ + \Theta + M_0 f] d\mu. \quad (21)$$

Since u^n lies in a compact set of $L^1(\Omega)^m$ and $\lim_{k \rightarrow +\infty} \psi(k) = +\infty$, then $\lim_{k \rightarrow +\infty} \mu(|u^n| \geq \psi(k)) = 0$ uniformly in n . Thus, given $\varepsilon \in (0, 1)$, there exists $k = k_\varepsilon$ large enough so that

$$I_+^n \leq \varepsilon/2 \text{ for all } n. \quad (22)$$

We can also control I_-^n as follows. We remark that (see assumption (2) in (H1))

$$|u^n| \leq R(k_\varepsilon) \Rightarrow |h_i^n(\cdot, u^n)| \leq |h_i(\cdot, u^n)| \leq \tilde{h}_i(\cdot, R(k_\varepsilon)).$$

This implies that for some $B \in (0, +\infty)$,

$$I_-^n = \int_{E \cap \{|u^n| \leq R(k)\}} b \cdot (M_0 h^n(\cdot, u^n))^- d\mu \leq \int_E B \sum_{i=1}^m \tilde{h}_i(\cdot, R(k_\varepsilon)) \quad (23)$$

We may choose δ small enough (independent of n) such that

$$\mu(E) \leq \delta \Rightarrow \int_E \sum_{i=1}^m \tilde{h}_i(\cdot, R(k_\varepsilon)) < \varepsilon/2B.$$

Combining with $I_+^n \leq \varepsilon/2$ proved above (see (22)), we deduce that $b \cdot (M_0 h^n(\cdot, u^n))^-$ is uniformly integrable. Since $b \in (0, +\infty)^m$, this implies that $(M_0 h^n(\cdot, u^n))^-$ is itself uniformly integrable. We already know that $(M_0 h^n(\cdot, u^n))^+$ is uniformly integrable. Thus so is $M_0 h^n(\cdot, u^n)$ and this ends the proof of Proposition 2.4. \square

Proof of Theorem 1.1. We consider the solution u^n of the approximate system (S^n) built in Lemma 2.1. By Propositions 2.2, 2.3, 2.4, up to a subsequence as $n \rightarrow +\infty$, we may assume that u^n converges in $L^1(\Omega)^m$ and μ -a.e. to some $u \in L^1(\Omega)^{+m}$. Moreover, by definition of h_i^n (see (13)) and the continuity property of h_i assumed in (H1), $h_i^n(\cdot, u^n)$ converges μ -a.e. to $h(\cdot, u)$. Moreover $h^n(\cdot, u^n)$ is uniformly integrable. By Vitali's Lemma (see e.g. [20] or [8]), $h^n(\cdot, u^n)$ converges also in $L^1(\Omega)^m$ to $h(\cdot, u)$. Since $u_i^n = (I + A_i)^{-1}(h^n(\cdot, u^n) + f_i)$ this implies that $u_i = (I + A_i)^{-1}(h(\cdot, u) + f_i)$ which means that u is solution of the limit system (S) and this ends the proof of Theorem 1.1. \square

3 Examples

In all examples below, Ω is a bounded open subset of \mathbb{R}^N with regular boundary and equipped with the Lebesgue measure.

3.1 Examples with linear diffusions and homogeneous boundary conditions

We start with a simple example associated with the Laplacian operator with homogeneous boundary conditions.

Corollary 3.1 *Assume the nonlinearity $h = (h_1, \dots, h_m)$ satisfies (H1), (H2), (M). Let $d_i \in (0, +\infty)$, $f_i \in L^1(\Omega)^+$, $i = 1, \dots, m$. Then the following system has a solution*

$$\begin{cases} \text{for all } i = 1, \dots, m; \\ u_i \in W_0^{1,1}(\Omega)^+, h_i(\cdot, u) \in L^1(\Omega), \\ u_i - d_i \Delta u_i = h_i(\cdot, u) + f_i. \end{cases} \quad (24)$$

Proof. We consider the operators

$$\begin{cases} D(A_i) := \{u \in W_0^{1,1}(\Omega) ; \Delta u \in L^1(\Omega)\}, \\ A_i u := d_i \Delta u. \end{cases} \quad (25)$$

It is classical that these operators A_i satisfy the three conditions of (A) in (3) (see e.g. [4]). Since $0 \in D(A_i)$, as already noticed just after (6), (A_{inf}) is also satisfied. And for (Φ) , if we denote $M_0 = [m_{ij}]_{1 \leq i, j \leq m}$, we choose $\varphi(r) := \sum_{j=1}^m (\sum_{i=1}^m m_{ij}) d_j r_j$, $b = (1, \dots, 1)^t \in (0, +\infty)^m$. Note that $\sum_{i=1}^m m_{ij} > 0$ for all $j = 1, \dots, m$ since $m_{ij} \geq 0$ and M_0 is invertible. In particular, $\lim_{|r| \rightarrow +\infty} \varphi(r) = +\infty$. Then, if $u \in D(A)$, $k \in (0, +\infty)$,

$$\int_{\Omega} \text{sign}^+(\varphi(u) - k) b \cdot M_0 A u = - \int_{\Omega} \text{sign}^+ \left(\sum_{i,j=1}^m m_{ij} d_j u_j - k \right) \Delta \left(\sum_{i,j=1}^m m_{ij} d_j u_j \right) \geq 0. \quad (26)$$

Then we may apply Theorem 1.1. □

Remark 3.2 As examples of functions h , we may for instance choose

$$\begin{cases} m = 2, \alpha_i, \beta_i \in \{0\} \cup [0, +\infty), i = 1, 2, \lambda \in (0, 1) \\ h_1(\cdot, u_1, u_2) = \lambda u_1^{\alpha_1} u_2^{\alpha_2} - u_1^{\beta_1} u_2^{\beta_2}, \\ h_2(\cdot, u_1, u_2) = -u_1^{\alpha_1} u_2^{\alpha_2} + u_1^{\beta_1} u_2^{\beta_2}. \end{cases}$$

We easily check that (H1) holds and that (H2) is satisfied with $a_i = 1, b_i = 0$ for all i and $\omega = 0$ (in other words $h_1 + h_2 \leq 0$). Moreover (M) is satisfied with $M_1 = 0, \Theta = (0, 0)^t$ and

$$M_0 = \begin{pmatrix} 1 & 1 \\ 1 & \lambda \end{pmatrix}.$$

Note that for $\lambda = 1$, M_0 is not invertible so that only one relation $h_1 + h_2 \leq 0$ holds and Theorem 1.1 does not apply. This kind of systems is considered in Theorem 1.2. □

• Here is another example of a nonlinearity $h = (h_1, h_2, h_3)$ which satisfies the corollary.

$$\begin{cases} m = 3, \alpha_i \in [1, +\infty), i = 1, 2, \\ h_1 = u_3 - u_1^{\alpha_1} u_2^{\alpha_2} = h_2 = -h_3. \end{cases}$$

The corresponding evolution problem is studied in [19] for $N \leq 5$, [15] for any dimension N with $\alpha_1 = \alpha_2 = 1$, and in [12] where $(\alpha_1, \alpha_2) \in [1, +\infty)^2$.

Here (H1) is obviously satisfied and so is (H2) with $a_i = 1, i = 1, 2, a_3 = 2, b_i = 0, i = 1, 2, 3, \omega = 0$. Then (M) is satisfied with $\Theta = (0, 0, 0)^t$ and

$$M_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A main point here is that the dependence in u_3 is linear. When it is superlinear, the system does not fit any more into the scope of Theorem 1.1. It is however analyzed in Theorem 1.2. □

3.2 About more general linear diffusions

Let us now make some comments on the following example where diffusions are more general than in Corollary 3.1.

$$\left\{ \begin{array}{l} u_i \in W_0^{1,1}(\Omega), h_i(u_1, u_2) \in L^1(\Omega), i = 1, 2, \\ u_1 - \sum_{i,j=1}^N a_{ij} \partial_{x_i x_j} u_1 = h_1(u_1, u_2) + f_1, \\ u_2 - \sum_{i,j=1}^N b_{ij} \partial_{x_i x_j} u_2 = h_2(u_1, u_2) + f_2, \end{array} \right. \quad (27)$$

where

$$a_{ij}, b_{ij} \in \mathbb{R}, \quad \sum_{i,j=1}^N a_{ij} \xi_i \xi_j, \sum_{i,j=1}^N b_{ij} \xi_i \xi_j \geq \alpha |\xi|^2, \alpha \in (0, +\infty), \quad \forall \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N. \quad (28)$$

We consider the operators

$$\left\{ \begin{array}{l} D(A_1) [\text{resp. } D(A_2)] := \left\{ v \in W_0^{1,1}(\Omega), \sum_{i,j=1}^N a_{ij} \partial_{x_i x_j} v \text{ [resp. } \sum_{i,j=1}^N b_{ij} \partial_{x_i x_j} v] \in L^1(\Omega) \right\}, \\ A_1 v := \sum_{i,j=1}^N a_{ij} \partial_{x_i x_j} v, \quad A_2 v := \sum_{i,j=1}^N b_{ij} \partial_{x_i x_j} v. \end{array} \right.$$

It is easy to see that the assumptions (A) , (A_{inf}) are satisfied. Thus if $(H1)$, $(H2)$, (M) are satisfied like in Corollary 3.1, then for the approximate solutions u^n of this system, as defined in the proof of Theorem 1.1, it follows that $u^n, h^n(\cdot, u^n)$ are bounded in $L^1(\Omega)^2$ and u^n lies in a compact set of $L^1(\Omega)^2$. However, the extra condition (Φ) is not satisfied in general so that it is not clear whether $h^n(\cdot, u^n)$ is uniformly integrable. Actually, we have to *choose here a different strategy* which does not seem to be generalized to the abstract setting of Section 2. It consists in looking at the equation satisfied by $T_k(u_1^n + \eta u_2^n)$ where T_k is a cut-off function. It is then easy to pass to the limit in the nonlinear terms of the truncated approximate system since they are multiplied by $T_k'(u_1^n + \eta u_2^n)$ which vanishes for u_i^n large for all i . Therefore a.e. convergence is sufficient to pass to the limit. The difficulty is then to prove precise estimates independent of n , in terms of η , in order to control the other terms as it done in the parabolic case (see [16], [17], [13]). This approach through cut-off functions T_k is precisely developed in Section 4 to prove Theorem 1.2 and we refer the reader to this other approach without giving more details here. \square

3.3 Examples with linear diffusions and Robin-type boundary conditions

Now we analyze what happens for systems like (24) when the boundary conditions are different. If we replace the homogeneous boundary conditions of (24) by homogeneous Neumann boundary conditions, then the result is exactly the same. On the other hand, the situation is quite more complicated if the boundary conditions are of different type for each of the u_i 's. This is actually connected with the content of the assumption (Φ) . For simplicity, we do it only for a 2×2 -system. Given a nonlinearity h satisfying $(H1)$, $(H2)$, (M) , we consider the following system with general Robin-type boundary conditions.

$$\left\{ \begin{array}{l} i = 1, 2, \\ u_i \in W^{1,1}(\Omega)^+, h_i(u_1, u_2) \in L^1(\Omega), \\ u_i - d_i \Delta u_i = h_i(u_1, u_2) + f_i, \\ \lambda_i u_i + (1 - \lambda_i) \partial_\nu u_i = \psi_i \text{ on } \partial\Omega, \\ \lambda_i \in [0, 1], d_i \in (0, +\infty), \psi_i \in [0, +\infty), f_i \in L^1(\Omega)^+. \end{array} \right. \quad (29)$$

Corollary 3.3 Let $(f_1, f_2) \in L^1(\Omega)^+ \times L^1(\Omega)^+$. Assume the nonlinearity h satisfies (H1), (H2), (M). Assume moreover $[0 \leq \lambda_1, \lambda_2 < 1]$ or $[\lambda_1 = \lambda_2 = 1 \text{ and } \psi_1 = \psi_2 = 0]$. Then the system (29) has a solution.

Remark 3.4 It is known that the case when one of the λ_i is equal to 1 is different (see the analysis in [14]). For instance, finite time blow up may occur for the associated evolution problem when the boundary conditions are $u_1 = 1, \partial_\nu u_2 = 0$ (see [2], [3]). Then the operator A_1 does not satisfy (A_{inf}) as easily seen by considering (as in [14]) the following simple example: $u_\sigma(x) := \cosh(\sigma x) / \cosh(\sigma)$ on $\Omega := (-1, 1)$ with $\sigma \in (0, +\infty)$. Then $u_\sigma \geq 0$ and $u_\sigma = 1$ on $\partial\Omega$. But $-\int_\Omega u_\sigma'' = u_\sigma'(-1) - u_\sigma'(1) \rightarrow -\infty$ as $\sigma \rightarrow +\infty$. Here we consider only the cases that directly fall into the scope of Theorem 1.1. A few other cases could be treated directly like $\lambda_1 = \lambda_2$ and positive data ψ_1, ψ_2 or also $\lambda_1 = 0 \neq \lambda_2$ and $\psi_1 = 0$.

Proof of Corollary 3.3. Here we define

$$\begin{cases} D(B_i) = \{u \in W^{2,1}(\Omega); \lambda_i u_i + (1 - \lambda_i) \partial_\nu u_i = \psi_i \text{ on } \partial\Omega\}, \\ B_i u := -d_i \Delta u_i. \end{cases}$$

Then the closure A_i of B_i in $L^1(\Omega)$ satisfies the assumption (A) (see [4]).

• Let us first assume $0 \leq \lambda_i < 1, i = 1, 2$. Then, for $u_i \in D(B_i)$

$$\int_\Omega B_i u_i = - \int_\Omega d_i \Delta u_i = - \int_\Omega d_i \partial_\nu u_i = d_i \int_\Omega (1 - \lambda_i)^{-1} (\lambda_i u_i - \psi_i) \geq -d_i (1 - \lambda_i)^{-1} \int_\Omega \psi_i.$$

This remains valid for A_i by closure. Thus the assumption (A_{inf}) is satisfied. For the condition (Φ) , we come back to (26) with the same φ, b . Let $p_q(r)$ be a standard approximation of $sign^+(r - k)$ like

$$p_q(r) = 0, \forall r \in (-\infty, k - 1/q]; p_q(r) = q(r - k) + 1, \forall r \in [k - 1/q, k]; p_q(r) = 1, \forall r \in [k, +\infty). \quad (30)$$

Then, for $u_j \in D(B_j), j = 1, 2$ and for $V := \sum_{i,j=1}^2 m_{ij} d_j u_j$

$$\begin{aligned} \int_\Omega p_q(\varphi(u) - k) b \cdot M_0 B u &= \int_\Omega p_q'(V) |\nabla V|^2 - \int_{\partial\Omega} p_q(V) \partial_\nu V \geq - \int_{\partial\Omega} p_q(V) \partial_\nu V \\ &- \int_{\partial\Omega} p_q(V) \partial_\nu V = \int_{\partial\Omega} p_q(V) \sum_{i,j=1}^2 m_{ij} d_j (1 - \lambda_j)^{-1} (\lambda_j u_j - \psi_j). \end{aligned}$$

If $\psi_j = 0$ for all $j = 1, \dots, m$, then $-\int_{\partial\Omega} p_q(V) \partial_\nu V \geq 0$ and letting $q \rightarrow +\infty$, we obtain that B and therefore A satisfies the condition (Φ) . When $\psi_j \neq 0$ for some j , we only obtain

$$\int_\Omega sign^+(\varphi(u) - k) b \cdot M_0 B u \geq - \int_{\partial\Omega} sign^+(V - k) \sum_{i,j=1}^m m_{ij} d_j (1 - \lambda_j)^{-1} \psi_j =: -\eta(V, k).$$

The point is that we can slightly modify the proof of Theorem 1.1 to get the same conclusion with this weaker estimate from below. Indeed, in the present case, we have to add the term $\eta(V_n, k)$ in the inequality (19) where $V_n = \sum_{i,j=1}^2 m_{ij} d_j u_j^n$. Since $\eta(V_n, k)$ tends to 0 as $k \rightarrow +\infty$ uniformly in n , the rest of the proof remains unchanged if we choose k_ε large enough so that $\eta(V, k_\varepsilon) < \varepsilon$ also.

• Assume now that $\lambda_1 = \lambda_2 = 1$. Then we make the same choice as in (26) and we see that

$$- \int_\Omega sign^+(\varphi(u) - k) b \cdot M_0 A u \geq - \int_{\partial\Omega} sign^+(\sum_{i,j=1}^2 m_{ij} d_j \psi_j - k) \partial_\nu (\sum_{i,j=1}^2 m_{ij} d_j u_j) \geq 0,$$

since $u_j \geq 0, u_j = 0$ on $\partial\Omega$ imply that $\partial_\nu u_j \leq 0$ on $\partial\Omega$.

For the same reason, (A_{inf}) holds since $-\int_\Omega \Delta u_j = - \int_{\partial\Omega} \partial_\nu u_j \geq 0$. □

3.4 Examples with nonlinear diffusions

Let us now consider *nonlinear diffusions*.

- We start with porous media type equations.

$$\begin{cases} \text{For } i = 1, \dots, m, \\ u_i \in L^1(\Omega)^+, \varphi_i(u_i) \in W_0^{1,1}(\Omega), h_i(u) \in L^1(\Omega), \\ u_i - \Delta \varphi_i(u_i) = h_i(u) + f_i, \end{cases} \quad (31)$$

where $\varphi_i : [0, +\infty) \rightarrow [0, +\infty)$ is continuous, increasing with $\varphi_i(0) = 0$, $\lim_{s \rightarrow +\infty} s^{-(N-2)^+/N} \varphi_i(s) = +\infty$.

Corollary 3.5 *Let $(f_1, \dots, f_m) \in (L^1(\Omega)^+)^m$. Assume the nonlinearity h satisfies (H1), (H2), (M). Then the system (31) has a solution.*

Proof. We naturally define

$$\begin{cases} D(A_i) := \{u_i \in L^1(\Omega) ; \varphi_i(u_i) \in W_0^{1,1}(\Omega), \Delta \varphi_i(u_i) \in L^1(\Omega)\}, \\ A_i u_i := -\Delta \varphi_i(u_i). \end{cases}$$

It is classical that the operators A_i satisfy the assumptions (A), (A_{inf}) (see [4], [1]). To check that (Φ) is satisfied, we consider $r \in [0, +\infty)^m \mapsto \varphi(r) := \sum_{i,j=1}^m m_{ij} \varphi_j(r_j)$ and $b := (1, \dots, 1)^t$. Then

$$\int_{\Omega} \text{sign}^+(\varphi(u) - k)b \cdot M_0 A u = - \int_{\Omega} \text{sign}^+\left(\sum_{i,j=1}^m m_{ij} \varphi_j(u_j) - k\right) \Delta \left(\sum_{i,j=1}^m m_{ij} \varphi_j(u_j)\right) \geq 0.$$

We then apply Theorem 1.1. □

- A second example with nonlinear diffusions is the following where $p \in (1, +\infty)$:

$$\begin{cases} \text{For } i = 1, 2, \\ u_i \in W_0^{1,p-1}(\Omega)^+, f_i \in L^1(\Omega), d_i \in (0, +\infty), \\ u_i - d_i \Delta_p u_i = h_i(u_1, u_2) + f_i, \end{cases} \quad (32)$$

where for $v \in W^{1,p-1}(\Omega)$, $\Delta_p v := \nabla \cdot (|\nabla v|^{p-2} \nabla v)$.

Corollary 3.6 *Let $(f_1, f_2) \in L^1(\Omega)^+ \times L^1(\Omega)^+$. Assume the nonlinearity h satisfies (H1), (H2), (M). Then the system (32) has a solution.*

Proof. Here we define for $i = 1, 2$

$$\begin{cases} D(B_i) := \{v \in W_0^{1,p}(\Omega) \cap L^2(\Omega) ; \Delta_p v \in L^2(\Omega)\}, \\ B_i(v) := -d_i \Delta_p v. \end{cases}$$

And the operators A_i are defined as the closure of B_i in $L^1(\Omega)$. Then the assumptions (A), (A_{inf}) are satisfied (see e.g. [10]). Let us prove that (Φ) holds. For p_q defined as in (30) and $\hat{u}_i \in D(B_i)$, $i = 1, 2$, we have

$$- \int_{\Omega} p_q(\hat{u}_1 + \hat{u}_2) (\Delta_p \hat{u}_1 + \Delta_p \hat{u}_2) = \int_{\Omega} (\nabla \hat{u}_1 + \nabla \hat{u}_2) (|\nabla \hat{u}_1|^{p-2} \nabla \hat{u}_1 + |\nabla \hat{u}_2|^{p-2} \nabla \hat{u}_2) p'_q(\hat{u}_1 + \hat{u}_2).$$

The mapping $r \in \mathbb{R}^N \mapsto |r|^p$ is convex. Therefore its gradient $r \in \mathbb{R}^N \mapsto p|r|^{p-2}r \in \mathbb{R}^N$ is monotone, which means

$$(r_1 - r_2) \cdot (|r_1|^{p-2}r_1 - |r_2|^{p-2}r_2) \geq 0, \quad \forall r_1, r_2 \in \mathbb{R}^N.$$

We apply this with $r_1 := \nabla \hat{u}_1(x), r_2 := -\nabla \hat{u}_2(x), x \in \Omega$ to deduce after integration on Ω :

$$-\int_{\Omega} p_q(\hat{u}_1 + \hat{u}_2)(\Delta_p \hat{u}_1 + \Delta_p \hat{u}_2) \geq 0.$$

And letting $q \rightarrow +\infty$ gives $-\int_{\Omega} \text{sign}^+(\hat{u}_1 + \hat{u}_2 - k)(\Delta_p \hat{u}_1 + \Delta_p \hat{u}_2) \geq 0$ and by closure

$$\int_{\Omega} \text{sign}^+(\hat{u}_1 + \hat{u}_2 - k)(A_1 \hat{u}_1 + A_2 \hat{u}_2) \geq 0 \quad \forall u_1 \in D(A_1), \forall u_2 \in D(A_2). \quad (33)$$

For the condition (M), we choose

$$b := (1, 1)^t, \quad \varphi(r_1, r_2) := c_1 r_1 + c_2 r_2, \quad c_1 := \{(m_{11} + m_{21})d_1\}^{1/(p-1)}, \quad c_2 := \{(m_{21} + m_{22})d_2\}^{1/(p-1)}$$

where $M_0 := (m_{ij})_{1 \leq i, j \leq 2}$. Then for $u_1 \in D(A_1), u_2 \in D(A_2), u = (u_1, u_2)$,

$$\begin{aligned} \int_{\Omega} \text{sign}^+(\varphi(u) - k)b \cdot M_0 A u &= \int_{\Omega} \text{sign}^+(c_1 u_1 + c_2 u_2 - k) \sum_{i,j=1}^2 m_{ij} d_j A_j u_j \\ &= \int_{\Omega} \text{sign}^+(c_1 u_1 + c_2 u_2 - k) [A_1(c_1 u_1) + A_2(c_2 u_2)] \\ &\geq 0, \end{aligned}$$

the last inequality coming from (33) applied with $\hat{u}_1 := c_1 u_1, \hat{u}_2 := c_2 u_2$. This ends the proof of Corollary 3.6. □

• To end this section, let us comment on the following system which is a model of situations where the operators A_i are *very different* from each other :

$$\begin{cases} p \neq 2 \\ u_1 \in W_0^{1,p-1}(\Omega)^+, u_2 \in W_0^{1,1}(\Omega)^+, \\ h_1, h_2, f_1, f_2 \in L^1(\Omega), \\ u_1 - \Delta_p u_1 = h_1(u_1, u_2) + f_1, \\ u_2 - \Delta u_2 = h_2(u_1, u_2) + f_2. \end{cases} \quad (34)$$

As a consequence, the compatibility condition (Φ) is not generally satisfied. However, if the nonlinearity h satisfies (H1), (H2), (M), then for the approximate solution u^n defined in Section 2, $h^n(\cdot, u^n)$ is bounded in $L^1(\Omega) \times L^1(\Omega)$ and therefore u^n lies in a compact set of $L^1(\Omega) \times L^1(\Omega)$. Thus we may assume that, up to a subsequence, u^n converges to some u in $L^1(\Omega) \times L^1(\Omega)$ and a.e. so that $h^n(\cdot, u^n)$ converges a.e. to $h(\cdot, u)$. Unfortunately we do not know whether we can pass to the limit in the approximate version of system (34). And it is not clear how the use of cut-off function T_k as in the next section can help. We leave this as an open problem. □

4 Proof of Theorem 1.2

For each $k = 1, \dots, m$, we define $\sigma_k := \min\{\alpha_k, \beta_k\}, \gamma_k := |\alpha_k - \beta_k|$ so that (see (12) for the definition of I, J):

$$h_k(u) = (\beta_k - \alpha_k) \left(\prod_{\ell=1}^m u_{\ell}^{\sigma_{\ell}} \right) B(u), \quad B(u) := k_1 \prod_{i \in I} u_i^{\gamma_i} - k_2 \prod_{j \in J} u_j^{\gamma_j}. \quad (35)$$

We first solve the approximate system with the bounded data $f_i^n := \inf\{f_i, n\}, n \in \mathbb{N}$.

Lemma 4.1 *There exists a nonnegative solution $u^n \in \cap_{p \in [1, \infty)} W^{2,p}(\Omega)^{+m}$ of*

$$\begin{cases} \text{For } i = 1, \dots, m, \\ u_i^n - d_i \Delta u_i^n = (\beta_i - \alpha_i) \prod_{k=1}^m (u_k^n)^{\sigma_k} B(u^n) + f_i^n \text{ in } \Omega, \\ \partial_\nu u_i = 0 \text{ on } \partial\Omega. \end{cases} \quad (36)$$

Moreover, we have

$$\gamma_j u_i^n + \gamma_i u_j^n - \Delta(\gamma_j d_i u_i^n + \gamma_i d_j u_j^n) = \gamma_j f_i^n + \gamma_i f_j^n, \quad \forall i \in I, j \in J, \quad (37)$$

and u^n is bounded in $L^{1+\eta}(\Omega)^m$ for some $\eta > 0$.

Proof. By the abstract Lemma 2.1, for $\varepsilon \in (0, 1)$, there exists a regular nonnegative solution u^ε of

$$\begin{cases} \text{For } i = 1, \dots, m, \\ u_i^\varepsilon - d_i \Delta u_i^\varepsilon = (\beta_i - \alpha_i) \frac{\prod_{k=1}^m (u_k^\varepsilon)^{\sigma_k} B(u^\varepsilon)}{1 + \varepsilon [\prod_{k=1}^m (u_k^\varepsilon)^{\alpha_k} + \prod_{k=1}^m (u_k^\varepsilon)^{\beta_k}]} + f_i^n \text{ in } \Omega, \\ \partial_\nu u_i^\varepsilon = 0 \text{ on } \partial\Omega. \end{cases} \quad (38)$$

Indeed the nonlinearity is here quasi-positive and uniformly bounded by $\max_i \gamma_i / \varepsilon$. Then by multiplying the equations in $i \in I, j \in J$ respectively by γ_j, γ_i , we have

$$(\gamma_j u_i^\varepsilon + \gamma_i u_j^\varepsilon) - \Delta(\gamma_j d_i u_i^\varepsilon + \gamma_i d_j u_j^\varepsilon) = \gamma_j f_i^n + \gamma_i f_j^n. \quad (39)$$

If $\bar{d} := \max_{1 \leq k \leq m} d_k$, this implies (since $u_i^\varepsilon, u_j^\varepsilon \geq 0$)

$$(\bar{d})^{-1} [\gamma_j d_i u_i^\varepsilon + \gamma_i d_j u_j^\varepsilon] - \Delta(\gamma_j d_i u_i^\varepsilon + \gamma_i d_j u_j^\varepsilon) \leq \gamma_j f_i^n + \gamma_i f_j^n, \quad \partial_\nu(\gamma_j d_i u_i^\varepsilon + \gamma_i d_j u_j^\varepsilon) = 0 \text{ on } \partial\Omega.$$

We deduce that

$$\gamma_j d_i u_i^\varepsilon + \gamma_i d_j u_j^\varepsilon \leq [(\bar{d})^{-1} I - \Delta]^{-1} (\gamma_j f_i^n + \gamma_i f_j^n). \quad (40)$$

Since the f_i^n are bounded by n (fixed), using the nonnegativity of u^ε , we obtain that the u_i^ε are bounded in $L^\infty(\Omega)$ independently of ε . By standard elliptic regularity results applied to the equation (38) in u_i^ε , they are also bounded in $\cap_{p \in [1, \infty)} W^{2,p}(\Omega)$. They are therefore included in a compact set of $L^\infty(\Omega)$ as $\varepsilon \rightarrow 0$. We can then easily pass to the limit as $\varepsilon \rightarrow 0$ and obtain the convergence of a subsequence of u^ε toward a solution u^n of (36).

Then the identities (37) follow by passing to the limit in (39). But (40) remains also valid at the limit and implies that $u_i^n, i = 1, \dots, m$ are bounded in $L^1(\Omega)$ independently of n . Next we may rewrite (37), using $\underline{d} := \min_{1 \leq k \leq m} d_k$:

$$\underline{d}^{-1} [\gamma_j d_i u_i^n + \gamma_i d_j u_j^n] - \Delta(\gamma_j d_i u_i^n + \gamma_i d_j u_j^n) = \gamma_j [(\underline{d}^{-1} d_i - 1) u_i^n + f_i^n] + \gamma_i [(\underline{d}^{-1} d_j - 1) u_j^n + f_j^n], \quad \forall i \in I, j \in J, \quad (41)$$

together with $\partial_\nu(\gamma_j d_i u_i^n + \gamma_i d_j u_j^n) = 0$ on $\partial\Omega$. Since the right-hand side is bounded in $L^1(\Omega)$, this implies that $\gamma_j d_i u_i^n + \gamma_i d_j u_j^n$ is bounded in $L^p(\Omega)$ for all $p \in [1, N/(N-2)^+)$ (see e.g. [4]). This ends the proof of Lemma 4.1. □

In order to rewrite the relations (41), we denote

$$g_k^n := (\underline{d}^{-1} d_k - 1) u_k^n + f_k^n \geq 0, \quad \forall k = 1, \dots, m. \quad (42)$$

When $f_k \log f_k$ is assumed to be in $L^1(\Omega)$ for all $k = 1, \dots, m$, then

$$\sup_{n \in \mathbb{N}} \left\{ \max_{1 \leq k \leq m} \int_{\Omega} g_k^n + |g_k^n \log g_k^n| \right\} < +\infty. \quad (43)$$

This is due to the “**L LOG L**” assumption on the f_i 's and on the $L^{1+\eta}$ -bound on the u_i^n stated in Lemma 4.1. Next we introduce the solutions G_k^n of

$$\underline{d}^{-1}G_k^n - \Delta G_k^n = g_k^n \text{ in } \Omega, \quad \partial_{\nu}G_k^n = 0 \text{ on } \partial\Omega, \quad G_k^n \geq 0, \quad k = 1, \dots, m, \quad (44)$$

so that the relations (37), (41) may be rewritten

$$\gamma_j d_i u_i^n + \gamma_i d_j u_j^n = \gamma_j G_i^n + \gamma_i G_j^n, \quad \forall i \in I, j \in J. \quad (45)$$

Our goal is to prove that the nonlinearity of the system (36) is bounded in $L^1(\Omega)^m$ independently of n . It will imply enough compactness on u^n to pass to the limit. The following lemma will provide the key estimate.

Lemma 4.2 *Under the assumptions of Theorem 1.2, there exists $\theta^n \in W^{2,1}(\Omega)^{+m}$ (unique) such that*

$$\begin{cases} \gamma_j d_i \theta_i^n + \gamma_i d_j \theta_j^n = \gamma_j G_i^n + \gamma_i G_j^n, \quad \forall i \in I, j \in J, \\ k_1 \prod_{i \in I} (\theta_i^n)^{\gamma_i} = k_2 \prod_{j \in J} (\theta_j^n)^{\gamma_j} \quad (\text{or } B(\theta^n) = 0), \\ \partial_{\nu} \theta_k^n = 0 \text{ on } \partial\Omega, \quad \forall k = 1, \dots, m. \end{cases} \quad (46)$$

Moreover

$$\sup_n \left\{ \max_{1 \leq k \leq m} \|\Delta \theta_k^n\|_{L^1(\Omega)} \right\} < +\infty. \quad (47)$$

We postpone the proof of this lemma and **we end the proof of Theorem 1.2.**

Without loss of generality, we assume that $1 \in I$. Using the equation in u_1^n given in the approximate system (36), and the fact that $B(\theta^n) = 0$, we have

$$u_1^n - \theta_1^n - d_1 \Delta(u_1^n - \theta_1^n) + \gamma_1 \prod_{k=1}^m (u_k^n)^{\sigma_k} [B(u^n) - B(\theta^n)] = \rho^n, \quad \rho^n := f_1^n - \theta_1^n + d_1 \Delta \theta_1^n. \quad (48)$$

Now the main point is that $B(u^n), B(\theta^n)$ have the property that

$$B(u^n) = b^n(\cdot, u_1^n), \quad B(\theta^n) = b^n(\cdot, \theta_1^n), \quad (49)$$

where $r \mapsto b^n(\cdot, r)$ is a *increasing function*. More precisely, we deduce from the relations (45), (46) that

$$\gamma_i d_j \theta_j^n = \gamma_j G_i^n + \gamma_i G_j^n - \gamma_j d_1 \theta_1^n, \quad \forall j \in J, \quad \gamma_1 d_i \theta_i^n = \gamma_i d_1 \theta_1^n + \gamma_1 G_i^n - \gamma_i G_1^n, \quad \forall i \in I.$$

$$\gamma_i d_j u_j^n = \gamma_j G_i^n + \gamma_i G_j^n - \gamma_j d_1 u_1^n, \quad \forall j \in J, \quad \gamma_1 d_i u_i^n = \gamma_i d_1 u_1^n + \gamma_1 G_i^n - \gamma_i G_1^n, \quad \forall i \in I.$$

Plugging this into $B(u^n), B(\theta^n)$, recalling that $B(v) = k_1 \prod_{i \in I} v_i^{\gamma_i} - k_2 \prod_{j \in J} v_j^{\gamma_j}$, we indeed obtain (49) by setting

$$b^n(x, r) := k_1 \prod_{i \in I} (\delta_i d_1 r + F_i^n(x))^{\gamma_i} - k_2 \prod_{j \in J} (F_j^n(x) - \delta_j d_1 r)^{\gamma_j}, \quad \forall r \in [r_-^n, r_+^n], \quad (50)$$

where we define

$$\begin{cases} \delta_k := \gamma_k / (\gamma_1 d_k), \quad \forall k = 1, \dots, m, \\ F_i^n := d_i^{-1} G_i^n - \delta_i G_1^n, \quad \forall i \in I, \quad F_j^n := \delta_j G_1^n + d_j^{-1} G_j^n, \quad \forall j \in J, \\ r_-^n(x) := \max_{i \in I} [F_i^n(x)]^- / (\delta_i d_1), \quad r_+^n(x) := \min_{j \in J} F_j^n(x) / (\delta_j d_1). \end{cases} \quad (51)$$

We multiply the equation (48) by $\text{sign}(u_1^n - \theta_1^n)$ and we integrate. Then we use the two following properties

$$\int_{\Omega} \text{sign}(u_1^n - \theta_1^n)[u_1^n - \theta_1^n - d_1 \Delta(u_1^n - \theta_1^n)] \geq 0,$$

$$\text{sign}(u_1^n - \theta_1^n)[B(u^n) - B(\theta^n)] = \text{sign}(u_1^n - \theta_1^n)[b^n(\cdot, u_1^n) - b^n(\cdot, \theta_1^n)] = |b^n(\cdot, u_1^n) - b^n(\cdot, \theta_1^n)|,$$

to obtain

$$\gamma_1 \int_{\Omega} \prod_{k=1}^m (u_k^n)^{\sigma_k} |b^n(\cdot, u_1^n) - b^n(\cdot, \theta_1^n)| \leq \int_{\Omega} |\rho^n|.$$

Since ρ^n is bounded in $L^1(\Omega)$ by Lemma 4.2, and since $b^n(\cdot, \theta_1^n) \equiv 0$, this implies that the nonlinearity of the equation in u_1^n

$$u_1^n - d_1 \Delta u_1^n + \gamma_1 \prod_{k=1}^m (u_k^n)^{\sigma_k} b^n(\cdot, u_1^n) = f_i^n \text{ in } \Omega,$$

is bounded in $L^1(\Omega)$ independently of n . This implies that Δu_1^n is bounded in $L^1(\Omega)$. Going back to (45), it follows that Δu_k^n is bounded in $L^1(\Omega)^m$ for all $k = 1, \dots, m$ as well. Up to a subsequence, we may deduce that u_k^n converges for all k to some u_k in $L^1(\Omega)$ and a.e. and that ∇u_k^n converges in $L^1(\Omega)^N$ to ∇u_k . Note also that by Fatou's lemma, $\prod_k (u_k)^{\sigma_k} B(u) \in L^1(\Omega)$.

Now to end the passing to the limit, we look at the equation satisfied by $T_R(v_i^n)$ where $v_i^n := u_i^n + \varepsilon \sum_{j \neq i} u_j^n$ and $\varepsilon \in (0, 1)$ and the T_R are C^2 -cut-off functions satisfying

$$\begin{cases} T_R(s) = s \text{ if } s \in [0, k-1], T'_R(s) = 0 \text{ if } s \geq k, T_R(s) \leq k, \forall s \in [0, +\infty), \\ 0 \leq T'_R(s) \leq 1, T''_R(s) \leq 0 \text{ for all } s \geq 0. \end{cases} \quad (52)$$

$$-\Delta T_R(v_i^n) = -T'_R(v_i^n) \Delta v_i^n - T''_R(v_i^n) |\nabla v_i^n|^2 \geq -T'_R(v_i^n) \Delta(u_i^n + \varepsilon \sum_{j \neq i} u_j^n).$$

Coming back to the system (36), we have that for all $\psi \in C^\infty(\bar{\Omega})^+$,

$$\int_{\Omega} \nabla \psi \nabla T_R(v_i^n) \geq \int_{\Omega} \psi T'_R(v_i^n) \left\{ \prod_k (u_k^n)^{\sigma_k} B(u^n) \left[\frac{\beta_i - \alpha_i}{d_i} + \varepsilon \sum_{j \neq i} \frac{\beta_j - \alpha_j}{d_j} \right] + \frac{f_i^n - u_i^n}{d_i} + \varepsilon \sum_{j \neq i} \frac{f_j^n - u_j^n}{d_j} \right\}.$$

We know that, up to a subsequence, u^n converges in $L^1(\Omega)^m$ and a.e. to some $u \in W^{1,1}(\Omega)^m$. We may pass to the limit along this subsequence in the above inequality. Indeed the nonlinear terms on the right converge a.e. and $T'_R(v_i^n) \prod_k (u_k^n)^{\sigma_k} B(u^n)$ is uniformly bounded, while the $f_k^n, u_k^n, k = 1, \dots, m$ converge in $L^1(\Omega)$. Moreover, $\nabla T_R(v_i^n) = T'_R(v_i^n) \nabla v_i^n$ converges in $L^1(\Omega)$ to $\nabla T_R(v_i), v_i := u_i + \varepsilon \sum_{j \neq i} u_j$. At the limit, we obtain the same inequality without the superscript n . Then, we let ε go to 0 to obtain

$$\int_{\Omega} \nabla \psi \nabla T_R(u_i) \geq \int_{\Omega} \psi T'_R(u_i) \left\{ \prod_k (u_k)^{\sigma_k} B(u) \frac{\beta_i - \alpha_i}{d_i} + \frac{f_i - u_i}{d_i} \right\}.$$

And now we let $R \rightarrow +\infty$ to obtain

$$\int_{\Omega} \nabla \psi \nabla u_i \geq \int_{\Omega} \psi \left\{ \prod_{k=1}^m (u_k)^{\sigma_k} B(u) \frac{\beta_i - \alpha_i}{d_i} + \frac{f_i - u_i}{d_i} \right\},$$

or equivalently

$$\forall i = 1, \dots, m, \int_{\Omega} \psi u_i + d_i \nabla \psi \nabla u_i \geq \int_{\Omega} \psi \{ \prod_k (u_k)^{\sigma_k} B(u) (\beta_i - \alpha_i) + f_i \}. \quad (53)$$

But, on the other hand, by passing directly to the limit as $n \rightarrow +\infty$ in (37), we have that for $i \in I, j \in J$

$$\int_{\Omega} \psi(\gamma_j u_i + \gamma_i u_j) + \nabla \psi \nabla [\gamma_j d_i u_i + \gamma_i d_j u_j] = \int_{\Omega} \psi(\gamma_j f_i + \gamma_i f_j).$$

This implies that all inequalities in (53) are actually equalities so that, for all $i = 1, \dots, m$,

$$u_i \in W^{2,1}(\Omega), \quad u_i - d_i \Delta u_i = h_i(u) + f_i \text{ in } \Omega, \quad \partial_{\nu} u_i = 0 \text{ on } \partial\Omega.$$

This ends the proof of Theorem 1.2. □

Proof of Lemma 4.2.

STEP 0 : The case $m = 2$. For simplicity, we drop the index 'n'. We assume $I = \{1\}, J = \{2\}$. Then the system (46) is equivalent to

$$\begin{cases} \theta_2 = G - \delta \theta_1, \quad G := [\gamma_2 G_1 + \gamma_1 G_2][\gamma_1 d_2]^{-1}, \quad \delta := \gamma_2 d_1 [\gamma_1 d_2]^{-1}, \\ k_1 \theta_1^{\gamma_1} = k_2 \theta_2^{\gamma_2} \Leftrightarrow \rho \theta_1^{\gamma} = G - \delta \theta_1, \quad \rho := [k_1/k_2]^{\gamma_2^{-1}} \quad \gamma := \gamma_1/\gamma_2, \\ \partial_{\nu} \theta_1 = 0 \text{ on } \partial\Omega. \end{cases}$$

It is easily seen that the equation in θ_1 has a unique regular solution (see **STEP 2** for more details) and differentiating this equation gives

$$\begin{aligned} \rho[\gamma \theta_1^{\gamma-1} + \delta] \nabla \theta_1 &= \nabla G, \quad \rho[\gamma \theta_1^{\gamma-1} + \delta] \Delta \theta_1 + \rho \gamma (\gamma - 1) \theta_1^{\gamma-2} |\nabla \theta_1|^2 = \Delta G, \\ \Rightarrow \Delta \theta_1 + \frac{\gamma(\gamma - 1) \theta_1^{\gamma-2} |\nabla \theta_1|^2}{\gamma \theta_1^{\gamma-1} + \delta} &= \frac{\Delta G}{\rho[\gamma \theta_1^{\gamma-1} + \delta]} \end{aligned} \quad (54)$$

If $\gamma = 1$, we immediately have

$$|\Delta \theta_1| = |\Delta G| [\rho(\gamma \theta_1^{\gamma-1} + \delta)]^{-1} \leq |\Delta G| [\rho \delta]^{-1}.$$

By (42) and (44), we know that ΔG is bounded in $L^1(\Omega)$ independently of n as soon as the f_k are (only) in $L^1(\Omega)^+$ for all k . Therefore so is $\Delta \theta_1$.

If $\gamma \neq 1$, then integrating (54) and using $\partial_{\nu} \theta_1 = 0$, we obtain

$$0 + \int_{\Omega} \frac{\gamma(\gamma - 1) \theta_1^{\gamma-2} |\nabla \theta_1|^2}{\gamma \theta_1^{\gamma-1} + \delta} = \int_{\Omega} \frac{\Delta G}{\rho[\gamma \theta_1^{\gamma-1} + \delta]}.$$

Again, the last integral is bounded independently of n if the f_k are (only) assumed to be in $L^1(\Omega)$. Therefore so is the first integral. But by positivity, this implies that $\theta_1^{\gamma-2} |\nabla \theta_1|^2 / [\gamma \theta_1^{\gamma-1} + \delta]$ is bounded in $L^1(\Omega)$ independently of n . So is $\Delta \theta_1$ by going back to (54). And finally the same holds for $\Delta \theta_2 = \Delta(G - \delta \theta_1)$.

We now come back to the general situation $m \geq 2$ and with the $L \text{Log} L$ assumptions on the f_k .

STEP 1 : Let us first treat the **trivial case** when there exists $(i_0, j_0) \in I \times J$ such that $\gamma_{j_0} G_{i_0}^n + \gamma_{i_0} G_{j_0}^n \equiv 0$ (i.e. $G_{i_0}^n \equiv 0 \equiv G_{j_0}^n$). Then, by the first line of (46), $\theta_{i_0} \equiv 0 \equiv \theta_{j_0}$. Using again the first line of (46), we deduce that $\theta_k^n = G_k^n$ for all $k = 1, \dots, n$. Thus, the conclusion of the lemma is obvious in this case.

Throughout the rest of the proof, we will assume that $\gamma_j G_i^n + \gamma_i G_j^n \neq 0$. By maximum principle applied to the equation (44) defining the G_k^n , $a_{ij}^n := \inf(\gamma_j G_i^n + \gamma_i G_j^n) > 0$. Also for the rest of the proof, choosing n_0 large enough, we fix $c \in (0, +\infty)$ such that

$$0 < c < a_{ij}^n / [\gamma_j d_i + \gamma_i d_j], \quad \forall (i, j) \in I \times J, \quad \forall n \geq n_0. \quad (55)$$

This definition of c will be used only in **STEPS 6 and 7** of the proof.

For simplicity, we now drop the superscript ' n ' in the rest of the proof.

STEP 2: Existence of θ^n satisfying (46). Again we assume (without loss of generality) that $1 \in I$ so that we may use (50), (51). Thus, for all $x \in \Omega$, the function $r \in [r_-(x), r_+(x)] \rightarrow b(x, r)$ is increasing where r_-, r_+ are defined in (51). Moreover

$$\begin{cases} b(x, r_-(x)) = -k_2 \prod_{j \in J} (F_j(x) - d_1 \delta_j r_-(x))^{\gamma_j} \leq 0, \\ b(x, r_+(x)) = k_1 \prod_{i \in I} (d_1 \delta_i r_+(x) + F_i(x))^{\gamma_i} \geq 0. \end{cases}$$

Thus there exists a unique $\theta_1(x) \in [r_-(x), r_+(x)]$ such that $b(x, \theta_1(x)) = 0$. Since the function $(x, r) \rightarrow b(x, r)$ is regular, by the implicit function theorem, so is $x \rightarrow \theta_1(x)$. The function θ_k are then uniquely determined from the first line of (46) which we rewrite as:

$$\theta_i := d_1 \delta_i \theta_1 + F_i, \quad \forall i \in I; \quad \theta_j := F_j - d_1 \delta_j \theta_1, \quad \forall j \in J. \quad (56)$$

It remains to prove that $\partial_\nu \theta_k = 0$ on $\partial\Omega$ for all $k = 1, \dots, m$. This will be a consequence of the following computation (see (60)).

STEP 3 : Differentiating $B(\theta) = 0$. The condition $B(\theta) = 0$ means

$$k_1 \prod_{i \in I} \theta_i^{\gamma_i} = k_2 \prod_{j \in J} \theta_j^{\gamma_j}, \quad \text{which implies} \quad \log k_1 + \sum_{i \in I} \gamma_i \log \theta_i = \log k_2 + \sum_{j \in J} \gamma_j \log \theta_j. \quad (57)$$

Differentiating this leads to

$$\sum_{i \in I} \gamma_i \frac{\nabla \theta_i}{\theta_i} = \sum_{j \in J} \gamma_j \frac{\nabla \theta_j}{\theta_j}. \quad (58)$$

Inserting (56) in this formula, we obtain $\nabla \theta_1$ in terms of the θ_k , namely

$$d_1 \nabla \theta_1 A = \sum_{j \in J} \frac{\gamma_j \nabla F_j}{\theta_j} - \sum_{i \in I} \frac{\gamma_i \nabla F_i}{\theta_i}, \quad A := \sum_{k=1}^m \frac{\gamma_k \delta_k}{\theta_k}. \quad (59)$$

Since $\nabla F_k \cdot \nu = 0$ on $\partial\Omega$ for all $k = 1, \dots, m$, it follows from this identity that $\nabla \theta_1 \cdot \nu = 0$ on $\partial\Omega$ as well. And by (56) it also follows that

$$\nabla \theta_k \cdot \nu = 0 \text{ on } \partial\Omega, \quad \forall k = 1, \dots, m. \quad (60)$$

Differentiating once more (58) gives

$$\sum_{i \in I} \frac{\gamma_i \Delta \theta_i}{\theta_i} - \frac{\gamma_i |\nabla \theta_i|^2}{\theta_i^2} = \sum_{j \in J} \frac{\gamma_j \Delta \theta_j}{\theta_j} - \frac{\gamma_j |\nabla \theta_j|^2}{\theta_j^2},$$

or also, using again (56) and the definition of A in (59),

$$d_1 A \Delta \theta_1 = \sum_{i \in I} -\frac{\gamma_i \Delta F_i}{\theta_i} + \frac{\gamma_i |\nabla \theta_i|^2}{\theta_i^2} + \sum_{j \in J} \frac{\gamma_j \Delta F_j}{\theta_j} - \frac{\gamma_j |\nabla \theta_j|^2}{\theta_j^2}. \quad (61)$$

Our goal is to estimate the L^1 -norm of $\Delta \theta_1$. We remark that, if we denote $\alpha_k := \gamma_k \delta_k / A \theta_k, k = 1, \dots, m$, then

$$0 \leq \alpha_k \leq 1, \quad \sum_{k=1}^m \alpha_k = 1.$$

But the relation (61) may be rewritten

$$d_1 \Delta \theta_1 = \sum_{i \in I} \alpha_i \left[-\frac{\Delta F_i}{\delta_i} + \frac{|\nabla \theta_i|^2}{\delta_i \theta_i} \right] + \sum_{j \in J} \alpha_j \left[\frac{\Delta F_j}{\delta_j} - \frac{|\nabla \theta_j|^2}{\delta_j \theta_j} \right]. \quad (62)$$

According to the definition of F_k in (51) and to the definition of G_k (or more precisely of G_k^n) in (44) and (43), we know that $\|\Delta F_k\|_{L^1(\Omega)}, k = 1, \dots, m$ is bounded in terms of the data (independently of n). Thus

$$\left\| \sum_{i \in I} -\alpha_i \frac{\Delta F_i}{\delta_i} + \sum_{j \in J} \alpha_j \frac{\Delta F_j}{\delta_j} \right\|_{L^1(\Omega)} \leq C, \quad (63)$$

where C is independent of n .

We also have that $\int_{\Omega} \Delta \theta_1 = \int_{\partial \Omega} \partial_{\nu} \theta_1 = 0$. Inserting this into (62) and (63) gives

$$\int_{\Omega} \sum_{j \in J} \alpha_j \frac{|\nabla \theta_j|^2}{\delta_j \theta_j} \leq \int_{\Omega} \sum_{i \in I} \alpha_i \frac{|\nabla \theta_i|^2}{\delta_i \theta_i} + C, \quad (64)$$

where again C does not depend on n . Therefore, it is sufficient to bound the right-hand side of (64) to obtain a bound on $\|\Delta \theta_1\|_{L^1(\Omega)}$ and *this will end the proof of Lemma 4.2* (since an L^1 -bound on $\Delta \theta_1$ implies an L^1 -bound on $\Delta \theta_k$ for all $k = 1, \dots, m$).

STEP 4 : A bound from below on the θ_k . The previous step indicates that one has to bound $|\nabla \theta_i|^2 / \theta_i$ in $L^1(\Omega)$ for all $i \in I$. The identity (59) says that

$$d_1 \frac{\nabla \theta_1}{\sqrt{\theta_1}} = \sum_{j \in J} \frac{\gamma_j \nabla F_j}{\theta_j \sqrt{\theta_1} A} - \sum_{i \in I} \frac{\gamma_i \nabla F_i}{\theta_i \sqrt{\theta_1} A}, \quad A := \sum_{k=1}^m \frac{\gamma_k \delta_k}{\theta_k}. \quad (65)$$

Thus we need to control $\theta_k \sqrt{\theta_1} A$ from below. We note that for all $k = 1, \dots, m$:

$$\theta_k \sqrt{\theta_1} A \geq \frac{\theta_k \gamma_1 \delta_1}{\sqrt{\theta_1}} + \sqrt{\theta_1} \gamma_k \delta_k.$$

Obviously, this implies $\theta_k \sqrt{\theta_1} A \geq \gamma_k \delta_k \sqrt{\theta_1}$. Since the minimum of $x \in (0, \infty) \rightarrow \frac{a}{\sqrt{x}} + b\sqrt{x}$ is equal to $2\sqrt{ab}$, for $a, b \in (0, +\infty)$, we also deduce that $\theta_k \sqrt{\theta_1} A$ is bounded from below by a factor of $\sqrt{\theta_k}$. Therefore there exists $c_k \in (0, +\infty)$ such that

$$\theta_k \sqrt{\theta_1} A \geq c_k \sup\{\sqrt{d_1 \delta_k \theta_1}, \sqrt{\theta_k}\}. \quad (66)$$

Remark 4.3 For the analysis below, we will use the classical identity

$$\max\{a, b\} = \frac{a + b + |a - b|}{2}, \quad \forall a, b \in [0, +\infty). \quad (67)$$

□

STEP 5 : Estimating the terms indexed by $j \in J$ in (65). The inequality (66) allows us to control the terms in $j \in J$ in (65). Using it together with (67), we may write, for all $j \in J$,

$$\theta_j^2 \theta_1 A^2 \geq c_j^2 \sup\{d_1 \delta_j \theta_1, \theta_j\} \geq \frac{c_j^2}{2} [d_1 \delta_j \theta_1 + \theta_j] = \frac{c_j^2}{2} F_j, \quad (68)$$

the last equality coming from (56). Therefore, this inequality, together with the second estimate of the technical Lemma 4.5 (stated below after the end of the present proof) applied to F_j , gives

$$\int_{\Omega} \frac{|\nabla F_j|^2}{\theta_j^2 \theta_1 A^2} \leq \frac{2}{c_j^2} \int_{\Omega} \frac{|\nabla F_j|^2}{F_j} \leq \int_{\Omega} \tilde{f}_j \log^+ \tilde{f}_j + (\underline{d} - 1) \tilde{f}_j + (\underline{de})^{-1},$$

where, by (51) and (44),

$$\tilde{f}_j = \underline{d}^{-1} F_j - \Delta F_j = \delta_j [\underline{d}^{-1} G_1 - \Delta G_1] + [\underline{d}^{-1} G_j - \Delta G_j] / d_j = \delta_j g_1 + g_j / d_j \geq 0.$$

We conclude from this and the estimate (43) that

$$\left\| \frac{\nabla F_j}{\theta_j \sqrt{\theta_1} A} \right\|_{L^2(\Omega)} \leq C, \quad \forall j \in J, \quad \text{where } C \text{ is independent of } n. \quad (69)$$

Remark 4.4 In **STEP 5**, we strongly used the fact that $F_j \geq 0$ and even $\underline{d}^{-1} F_j - \Delta F_j \geq 0$. Therefore we could apply the second estimate of Lemma 4.5. Unfortunately, this is different for the terms in $i \in I$ in (65) since

$$\underline{d}^{-1} F_i - \Delta F_i = [\underline{d}^{-1} G_i - \Delta G_i] / d_i - \delta_i [\underline{d}^{-1} G_1 - \Delta G_1] = g_i / d_i - \delta_i g_1. \quad (70)$$

As in (68), and using also (67), we deduce that, for all $i \in I$,

$$\theta_i^2 \theta_1 A^2 \geq c_i^2 \sup\{d_1 \delta_i \theta_1, \theta_i\} \geq \frac{c_i^2}{2} |\theta_i - d_1 \delta_i \theta_1| = \frac{c_i^2}{2} |F_i|, \quad (71)$$

the last equality coming from (56). This implies that the terms indexed by $i \in I$ in (65) may be estimated as follows

$$\int_{\Omega} \frac{|\nabla F_i|^2}{\theta_i^2 \theta_1 A^2} \leq \frac{2}{c_i^2} \int_{\Omega} \frac{|\nabla F_i|^2}{|F_i|}.$$

But since the sign of F_i is not constant on Ω , it is not clear how to bound this last integral.

Nevertheless, *assume we are able to prove* that, for some $i_0 \in I$

$$\sup\{d_1 \delta_{i_0} \theta_1, \theta_{i_0}\} \geq c_0 > 0, \quad \text{with } c_0 \text{ independent of } n. \quad (72)$$

Then using also (71), we can write

$$\theta_{i_0}^2 \theta_1 A^2 \geq c_{i_0}^2 \sup\{c_0, |F_{i_0}|/2\} \geq \frac{c_{i_0}^2}{4} [a + |F_{i_0}|], \quad a = 2c_0.$$

Then by the first estimate of the technical Lemma 4.5 below, we deduce

$$\int_{\Omega} \frac{|\nabla F_{i_0}|^2}{\theta_{i_0}^2 \theta_1 A^2} \leq \frac{4}{c_{i_0}^2} \int_{\Omega} \frac{|\nabla F_{i_0}|^2}{a + |F_{i_0}|} \leq \int_{\Omega} |\tilde{f}_{i_0}| \log |\tilde{f}_{i_0}| + |\tilde{f}_{i_0}| [a^{-1} \underline{d} - 1] + 1,$$

where $\tilde{f}_{i_0} = \underline{d}^{-1} F_{i_0} - \Delta F_{i_0} = g_{i_0}/d_1 - \delta_{i_0} g_1$ as indicated in (70). We conclude from the estimate (43) that

$$\left\| \frac{\nabla F_{i_0}}{\theta_{i_0} \sqrt{\theta_1} A} \right\|_{L^2(\Omega)} \leq C \quad \text{where } C \text{ is independent of } n. \quad (73)$$

□

STEP 6: More bounds from below on the θ_k . We prove here the two following facts, where the real number c was defined in (55):

$$\begin{cases} \Omega = \Omega_I \cup \Omega_J, \quad \Omega_I := \{x \in \Omega ; \theta_i(x) > c, \forall i \in I\}, \quad \Omega_J := \{x \in \Omega ; \theta_j(x) > c, \forall j \in J\}, \\ \sup_{i \in I} \theta_i \geq c_1 > 0, \quad \text{for some } c_1 \in (0, \infty) \text{ independent of } n. \end{cases} \quad (74)$$

For the first part of (74), assume by contradiction that there exists $x \in \Omega \setminus (\Omega_I \cup \Omega_J)$. Then there exists $(i, j) \in I \times J$ such that, for this x :

$$\theta_i(x) \leq c \text{ and } \theta_j(x) \leq c, \text{ or equivalently, } \gamma_j d_i \theta_i(x) \leq \gamma_j d_i c, \quad \gamma_i d_j \theta_j(x) \leq \gamma_i d_j c.$$

Let us add these last two inequalities. Using the first line of (46) and the definition of c in (55), we have

$$\gamma_j G_i(x) + \gamma_i G_j(x) = \gamma_j d_i \theta_i(x) + \gamma_i d_j \theta_j(x) \leq [\gamma_j d_i + \gamma_i d_j] c < \inf\{\gamma_j G_i + \gamma_i G_j\}.$$

And this is a contradiction. Whence the first statement of (74).

For the second one, let us first note that

$$\sup_{x \in \Omega_I} \theta_i(x) \geq \inf_{x \in \Omega_I} \theta_i(x) \geq c, \quad \forall i \in I.$$

Now let $x \in \Omega_J$ so that, by the previous statement, $\theta_j(x) \geq c$ for all $j \in J$. We then use the second line of (46) to obtain

$$k_2 c^{\sum_{j \in J} \gamma_j} \leq k_2 \prod_{j \in J} \theta_j(x)^{\gamma_j} = k_1 \prod_{i \in I} \theta_i(x)^{\gamma_i} \leq k_1 [\sup_{i \in I} \theta_i]^{\sum_{i \in I} \gamma_i}.$$

This implies that

$$\sup_{i \in I} \theta_i \geq c_1 := \min\left\{c, \left[k_2 c^{\sum_{j \in J} \gamma_j} / k_1 \right]^{[\sum_{i \in I} \gamma_i]^{-1}} \right\}.$$

Whence the second statement of (74).

STEP 7: End of the proof of Lemma 4.2. This is where we use that I (for instance) has at most two elements. Indeed, let us go back to the expression of $\nabla \theta_1 / \sqrt{\theta_1}$ in (65). We already know by **STEP 5** that all terms indexed by $j \in J$ are bounded in $L^2(\Omega)$. Since $F_1 \equiv 0$, there is at most one term indexed by $i \in I$, namely none if $I = \{1\}$, and only $\nabla F_{i_0} / (\theta_{i_0} \sqrt{\theta_1} A)$ if $I = \{1, i_0\}$.

If $I = \{1\}$, it immediately follows that $\nabla \theta_1 / \sqrt{\theta_1}$ is bounded in $L^2(\Omega)$ independently of n .

If $I = \{1, i_0\}$, then $\sup_{i \in I} \theta_i = \sup\{\theta_1, \theta_{i_0}\}$. It follows from the second line of (74) in **STEP 6** that (72) holds. Consequently, as proved in Remark 4.4, $\nabla F_{i_0} / (\theta_{i_0} \sqrt{\theta_1} A)$ is bounded in $L^2(\Omega)$ independently of n . Having controlled all terms in (65), we can conclude that $\nabla \theta_1 / \sqrt{\theta_1}$ is itself also bounded in $L^2(\Omega)$ independently of n . By symmetry, this also holds for $\nabla \theta_{i_0} / \sqrt{\theta_{i_0}}$. This implies that the right-hand side of (64) in **STEP 4** is bounded independently of n and *ends the proof of Lemma 4.2.*

□

Let us now state the following technical lemma which was used in two places in the previous proof.

Lemma 4.5 Let $F \in W^{2,1}(\Omega)$ such that, for some $d \in (0, +\infty)$

$$dF - \Delta F = f, \quad \partial_\nu F = 0 \text{ on } \partial\Omega \text{ with } f, f \log |f| \in L^1(\Omega).$$

For $a \in (0, +\infty)$, we have

$$\int_{\Omega} \frac{|\nabla F|^2}{a + |F|} \leq \int_{\Omega} |f| \log |f| + |f|[(ad)^{-1} - 1] + 1.$$

If moreover $f \geq 0$, then

$$\int_{\Omega} \frac{|\nabla F|^2}{F} \leq \int_{\Omega} f \log^+ f + (d^{-1} - 1)f + de^{-1}.$$

Proof. For the first inequality of the lemma, let us first remark that, if we set $F := a\tilde{F}$, then by homogeneity,

$$\int_{\Omega} \frac{|\nabla F|^2}{a + |F|} = a \int_{\Omega} \frac{|\nabla \tilde{F}|^2}{1 + |\tilde{F}|}.$$

Let us now multiply the equation $d\tilde{F} - \Delta \tilde{F} = f/a$ by $\text{sign}(\tilde{F}) \log(|\tilde{F}| + 1)$. We obtain

$$\int_{\Omega} |\tilde{F}| \log(|\tilde{F}| + 1) + \frac{|\nabla \tilde{F}|^2}{1 + |\tilde{F}|} = \frac{1}{a} \int_{\Omega} f \text{sign}(\tilde{F}) \log(|\tilde{F}| + 1) \leq \frac{1}{a} \int_{\Omega} |f| \log(|\tilde{F}| + 1).$$

Now we use the Young's convexity inequality

$$\forall r \in [0, +\infty), \forall s \in \mathbb{R}, \quad rs \leq (r \log r - r) + e^s. \quad (75)$$

We apply it with $r := |f(x)|, s := \log(|\tilde{F}(x)| + 1)$ to deduce

$$\int_{\Omega} |f| \log(|\tilde{F}| + 1) \leq \int_{\Omega} |f|[\log |f| - 1] + |\tilde{F}| + 1.$$

From the equation in \tilde{F} , we also derive $d \int_{\Omega} |\tilde{F}| \leq \int_{\Omega} |f|/a$. The first inequality of Lemma 4.5 follows.

If $f \geq 0$, then $F \geq 0$ by maximum principle. We multiply the equation in F by $\log(F + \varepsilon)$ and we integrate by parts. Then,

$$\int_{\Omega} dF \log(F + \varepsilon) + \frac{|\nabla F|^2}{F + \varepsilon} = \int_{\Omega} f \log(F + \varepsilon). \quad (76)$$

We apply the Young's inequality (75) with $r := f(x), s := \log(F(x) + \varepsilon)$. Then

$$f \log(F + \varepsilon) \leq (f \log f - f) + F + \varepsilon,$$

so that, using (76), we deduce

$$\int_{[F \geq 1]} dF \log(F + \varepsilon) + \int_{\Omega} \frac{|\nabla F|^2}{F + \varepsilon} \leq \int_{\Omega} f \log f - f + F + \varepsilon - d \int_{[F \leq 1]} F \log(F + \varepsilon).$$

Now we let $\varepsilon \rightarrow 0$ and we use

$$\int_{[F \geq 1]} F \log F \geq 0, \quad f \log f \leq f \log^+ f, \quad \int_{\Omega} dF = \int_{\Omega} f, \quad x \log x \geq -e^{-1}, \quad \forall x \in (0, 1),$$

to deduce the second estimate of Lemma 4.5. □

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