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SCALING LIMIT OF THE VRJP IN DIMENSION ONE AND BASS-BURDZY FLOW

TITUS LUPU, CHRISTOPHE SABOT, AND PIERRE TARRÈS

ABSTRACT. We introduce a continuous space limit of the Vertex Reinforced Jump Process (VRJP) in dimension one, which we call Linearly Reinforced Motion (LRM) on \mathbb{R} . It is constructed out of a convergent Bass-Burdzy flow. The proof goes through the representation of the VRJP as a mixture of Markov jump processes. As a by-product this gives a representation in terms of a mixture of diffusions of the LRM and of the Bass-Burdzy flow itself. We also show that our continuous space limit can be obtained out of the Edge Reinforced Random Walk (ERRW), since the ERRW and the VRJP are known to be closely related.

1. INTRODUCTION AND PRESENTATION OF RESULTS

Let $G = (V, E, \sim)$ be an electrical network with positive conductances $(C_e)_{e \in E}$, and let $(\theta_i)_{i \in V}$ be positive weights on the vertices V . The Vertex-Reinforced Jump Process (VRJP) is a continuous-time process $(\zeta_t)_{t \geq 0}$ taking values in V which, conditionally on the past at time t , jumps from a vertex $i \in V$ to $j \sim i$ at rate

$$(1.1) \quad C_{ij} L_j(t),$$

where

$$(1.2) \quad L_j(t) = \theta_j + \int_0^t \mathbf{1}_{\{\zeta_s = j\}} ds$$

is the local time at vertex j at time t , with the convention that the initial local time at j is θ_j .

The VRJP was introduced by Davis and Volkov [DV02, DV04] and is closely related to the Edge-Reinforced Random Walk (ERRW) introduced by Coppersmith and Diaconis in 1986 [CD86], and to supersymmetric hyperbolic model in quantum field theory, see [ST15, DSZ10]; see [DD10, BS12] for more references on the VRJP.

Our aim is to introduce a scaling limit of the VRJP on the one-dimensional lattice $2^{-n}\mathbb{Z}$ when n tends to infinity.

We start with a function $L_0 : \mathbb{R} \rightarrow (0, +\infty)$, which will correspond to initial local times of the scaling limit, such that

$$(1.3) \quad \int_0^{+\infty} L_0(x)^{-2} dx = \int_{-\infty}^0 L_0(x)^{-2} dx = +\infty.$$

As we will see further, (1.3) is a condition for non-explosion to infinity.

We define $(X_t^{(n)})_{t \geq 0}$ as the continuous-time VRJP started from 0 on the network $2^{-n}\mathbb{Z}$, with $C_e = C = 2^{2n-1}$ and $\theta_{i2^{-n}} = 2^{-n} L_0(i2^{-n})$. We define its local time as

$$\ell_t^{(n)}(x) = 2^n \int_0^t \mathbf{1}_{X_s^{(n)} = x} ds, \quad x \in 2^{-n}\mathbb{Z}.$$

The factor 2^n is the inverse of the size of a cell around a vertex. The jump rates at time t from x to $x + \sigma 2^{-n}$, $\sigma \in \{-1, 1\}$, are

$$(1.4) \quad 2^{2n-1} L_t^{(n)}(x + \sigma 2^{-n}),$$

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with

$$L_t^{(n)} = L_0^{(n)} + \ell_t^{(n)},$$

where $L_0^{(n)}$ is the restriction to $2^{-n}\mathbb{Z}$ of the initial occupation profile L_0 . The process is defined up to a time $t_{\max}^{(n)} \in (0, +\infty]$, as it might reach $-\infty$ or $+\infty$ in finite time.

We are interested in the limit in law of $((X_t^{(n)})_{0 \leq t \leq t_{\max}^{(n)}}, (L_t^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}, 0 \leq t \leq t_{\max}^{(n)}})$ as $n \rightarrow +\infty$.

The order in the conductances C and initial local times θ yields, up to a linear change of time, the only interesting limit, i.e. which is not Brownian motion or a constant process.

We will denote the limit process on \mathbb{R} by $((X_t)_{t \geq 0}, (L_t(x))_{x \in \mathbb{R}, t \geq 0})$. One can construct it out of the flow of solutions of the Bass-Burdzy equation:

$$(1.5) \quad \frac{dY_u}{du} = \begin{cases} -1 & \text{if } Y_u > B_u, \\ 1 & \text{if } Y_u < B_u, \end{cases}$$

where $(B_u)_{u \geq 0}$ is the standard Brownian motion on \mathbb{R} started from 0. Bass and Burdzy showed in [BB99] that (1.5) has a.s., for a given initial condition, a unique solution which is Lipschitz continuous. Let us explain how this equation naturally appears in our context.

Assume first that there is no reinforcement, that is to say $L_t^{(n)}$ is replaced by L_0 in the jump rates of (1.4). Then the processes would converge to a Markov diffusion with infinitesimal generator

$$\frac{1}{2}L_0(x)\frac{d^2}{dx^2} + L_0(x)\left(\frac{d}{dx}(\log(L_0(x)))\right)\frac{d}{dx}.$$

So if one does a change of scale

$$dy = L_0(x)^{-2}dx$$

(by the way, this is where the condition (1.3) comes from), and a change of time

$$du = L_0(X_t)^{-3}dt,$$

where X_t is the position of the particle at time t , one gets a Brownian motion (see [IM74],[Bre92], Chapter 16 or [RY99], Sections VII.2 and VII.3).

Now assume that we do have a reinforcement and that there is some limit process $(X_t)_{t \geq 0}$, with occupation densities $L_t - L_0$. Then one would like to have a dynamical change of scale

$$dS_t(x) = L_t(x)^{-2}dx,$$

such that $(S_t(X_t))_{t \geq 0}$ is a martingale (which corresponds to choosing $S_t^{-1}(0)$ in an appropriate way), and such that after a change of time

$$(1.6) \quad du = L_t(X_t)^{-3}dt,$$

this martingale becomes a Brownian motion $B_u = S_u(X_u)$. This corresponds to the idea that after time t , $X_{t+\Delta t}$, behaves, for $\Delta t \ll 1$, almost like a diffusion with infinitesimal generator

$$\frac{1}{2}L_t(x)\frac{d^2}{dx^2} + L_t(x)\frac{d}{dx}(\log(L_t(x)))\frac{d}{dx}.$$

Given $x_1 < x_2 \in \mathbb{R}$ fixed, in the time scale (1.6), we have that

$$\begin{aligned} \frac{d}{du}(S_u(x_2) - S_u(x_1)) &= \frac{dt}{du} \frac{d}{dt}(S_t(x_2) - S_t(x_1)) = L_t(X_t)^3 \frac{d}{dt} \int_{x_1}^{x_2} L_t(x)^{-2} dx \\ &= -2L_t(X_t)^3 \int_{x_1}^{x_2} L_t(x)^{-3} d_t L_t(x) dx = -2 \int_{x_1}^{x_2} d_t L_t(x) dx \\ &= -2\mathbf{1}_{x_1 < X_t < x_2} = -2\mathbf{1}_{S_u(x_1) < B_u < S_u(x_2)}. \end{aligned}$$

If we moreover take into account that after time t , X_{t+dt} should spend infinitesimally the same amount of time left and right from X_t , we get the equation

$$\frac{d}{du}(S_u(x)) = -\mathbf{1}_{S_u(x) > B_u} + \mathbf{1}_{S_u(x) < B_u},$$

which is exactly that of (1.5).

We will "reverse-engineer" the above construction. Let $(\Psi_u^B(y))_{y \in \mathbb{R}, u \geq 0}$ be the flow of solutions to (1.5). $u \mapsto \Psi_u^B(y)$ is the Lipschitz solution to (1.5) with initial condition $Y_0 = y$. We call Ψ^B the **convergent Bass-Burdzy** flow. It is a flow of diffeomorphisms of \mathbb{R} [BB99]. Let be

$$\xi_u = (\Psi_u^B)^{-1}(B_u).$$

The process $(\xi_u)_{u \geq 0}$ has a time-space continuous family of local times $(\Lambda_u(y))_{y \in \mathbb{R}, u \geq 0}$ [HW00], such that for all $f : \mathbb{R} \rightarrow \mathbb{R}$ bounded, Borel measurable, and all $u \geq 0$,

$$\int_0^u f(\xi_v) dv = \int_{\mathbb{R}} f(y) \Lambda_u(y) dy.$$

Moreover, $\Lambda_u(y) \leq 1/2$.

Definition 1.1. Let $L_0 : \mathbb{R} \rightarrow (0, +\infty)$ be a continuous function. Moreover, we assume that

$$\int_0^{+\infty} L_0(x)^{-2} dx = \int_{-\infty}^0 L_0(x)^{-2} dx = +\infty.$$

Let $x_0 \in \mathbb{R}$. Denote, for $x \in \mathbb{R}$,

$$(1.7) \quad S_0(x) = \int_{x_0}^x L_0(r)^{-2} dr.$$

Perform the change of time

$$(1.8) \quad dt = L_0(S_0^{-1}(\xi_u))^3 (1 - 2\Lambda_u(\xi_u))^{-\frac{3}{2}} du.$$

The process $(S_0^{-1}(\xi_{u(t)}))_{t \geq 0}$, where $u(t)$ is the inverse time change of (1.8), is called the **Linearly Reinforced Motion** starting from x_0 , with **initial occupation profile** L_0 . We call $(\xi_u)_{u \geq 0}$ the corresponding **reduced process** and $(B_u)_{u \geq 0}$ the corresponding **driving Brownian motion**. Set

$$(1.9) \quad L_t(x) = L_0(x) (1 - 2\Lambda_{u(t)}(S_0(x)))^{-\frac{1}{2}}.$$

$(L_t(x))_{x \in \mathbb{R}}$ is the **occupation profile at time t** .

Remark 1.2. The time change (1.8) is *a posteriori*

$$dt = L_t(X_t)^3 du.$$

Theorem 1.3. The VRJP process and its occupation profiles $(X_t^{(n)}, L_t^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}, 0 \leq t \leq t_{\max}^{(n)}}$ converge in law as $n \rightarrow +\infty$ to a Linearly Reinforced Motion started from 0 and its occupation profiles $(X_t, L_t(x))_{x \in \mathbb{R}, t \geq 0}$. The topology of the convergence is that of uniform convergence on compact subsets. In particular $t_{\max}^{(n)}$ converges in probability to $+\infty$.

Remark 1.4. Previously, a different Bass-Burdzy flow appeared in the study of continuous self-interacting processes. In [War05] it was shown that the flow of solutions to

$$\frac{dY_u}{du} = 2\mathbf{1}_{Y_u > B_u}$$

was related to the Brownian first passage bridge conditioned by its family of local times and to the Brownian burglar [WY98].

It was shown in [ST15] that on any electrical network, the VRJP has same law as a time-change of a mixture of Markov (non-reinforced) jump processes. In our setting, the random environment related to the VRJP converges. This gives us in the limit a description of the LRM as a time-changed diffusion in random environment.

Let S_0 be the change of scale defined by (1.7), with $x_0 = 0$. Let $(W(y))_{y \geq 0}$ and $(W(-y))_{y \geq 0}$ be two independent standard Brownian motions, started from 0, where y is seen as a space variable. We see $(W(y))_{y \in \mathbb{R}}$ as a Brownian motion parametrized by \mathbb{R} . Define

$$(1.10) \quad \mathcal{U}(x) = \sqrt{2}W \circ S_0(x) + |S_0(x)|.$$

Consider $(Z_q)_{q \geq 0}$ the diffusion in random potential $2\mathcal{U} - 2\log(L_0)$. Conditional on $(\mathcal{U}(x))_{x \in \mathbb{R}}$, it is a Markov diffusion on \mathbb{R} , started from $Z_0 = 0$, with infinitesimal generator

$$(1.11) \quad \frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{d}{dx} (\log(L_0(x)) - \mathcal{U}(x)) \right) \frac{d}{dx}.$$

We will denote by $(\lambda_q(x))_{x \in \mathbb{R}, y \geq 0}$ the family of local times of $(Z_q)_{q \geq 0}$.

Although the function $x \mapsto \log(L_0(x)) - \mathcal{U}(x)$ is in general not differentiable, the diffusion $(Z_q)_{q \geq 0}$ is well defined. For that, consider the natural scale function

$$(1.12) \quad \mathcal{S}(x) = \int_0^x L_0(r)^{-2} e^{2\mathcal{U}(r)} dr.$$

The condition (1.3) and the fact that \mathcal{U} is a.s. bounded from below imply that

$$\text{a.s. } \mathcal{S}(-\infty) = -\infty, \quad \mathcal{S}(+\infty) = +\infty.$$

$(\mathcal{S}(Z_q))_{q \geq 0}$ is a local martingale and a Markov diffusion with infinitesimal generator

$$\frac{1}{2} (\mathcal{S}' \circ \mathcal{S}^{-1}(\zeta))^2 \frac{d^2}{d\zeta^2}.$$

It is a time-changed Brownian motion, and in particular, it is defined up to $q = +\infty$. In the particular case $L_0 \equiv 1$, the generator (1.11) is equal to

$$\frac{1}{2} \frac{d^2}{dy^2} - \sqrt{2} \left(\frac{d}{dy} W(y) \right) \frac{d}{dy} - \text{sgn}(y) \frac{d}{dy},$$

$(\frac{d}{dy} W(y))_{y \in \mathbb{R}}$ being the white noise. For some background on diffusions in random Wiener potential, we refer to [Sch85, Bro86, Tan95] and the references therein.

Theorem 1.5. *The Linearly Reinforced Motion $(X_t)_{t \geq 0}$, started from 0, with initial occupation profile L_0 , has same law as a time-change of the mixture of diffusions $(Z_q)_{q \geq 0}$, where the time-change is given by*

$$(1.13) \quad dt = (L_0(Z_q)^2 + 2\lambda_q(Z_q))^{-\frac{1}{2}} dq.$$

Remark 1.6. The mixture of diffusions $(Z_q)_{q \geq 0}$ is itself a reinforced process. Informally, one can imagine it as having a time-dependent infinitesimal generator

$$\frac{1}{2} \frac{d^2}{dx^2} + \frac{1}{2} \frac{d}{dx} (\log(L_0(x)^2 + 2\lambda_q(x))) \frac{d}{dx}.$$

We will prove Theorem 1.5 by constructing out of the VRJP a discrete analogue of the convergent Bass-Burdzy flow.

Theorem 1.5 has an immediate implication on the reduced process $(\xi_u)_{u \geq 0}$.

Corollary 1.7. *Let $\xi_u = (\Psi_u^B)^{-1}(B_u)$ be the reduced process obtained out of the Bass-Burdzy flow $(\Psi_u^B)_{u \geq 0}$. Let $(\bar{Z}_{\bar{q}})_{\bar{q} \geq 0}$ be a process, that conditional on $(W(y))_{y \in \mathbb{R}}$ is a Markov diffusion with generator*

$$\frac{1}{2} \frac{d^2}{dy^2} - \sqrt{2} \left(\frac{d}{dy} W(y) \right) \frac{d}{dy} - \text{sgn}(y) \frac{d}{dy},$$

and $(\bar{\lambda}_{\bar{q}}(y))_{y \in \mathbb{R}, \bar{q} \geq 0}$ its family of local times. Let be the time change

$$du = (1 + 2\bar{\lambda}_{\bar{q}}(\bar{Z}_{\bar{q}}))^{-2} d\bar{q}.$$

Then the time changed process $(\bar{Z}_{\bar{q}(u)})_{u \geq 0}$ has same law as $(\xi_u)_{u \geq 0}$. Moreover, in this construction of $(\xi_u)_{u \geq 0}$, we have the following relation between the local times:

$$\Lambda_u(y) = \frac{\bar{\lambda}_{\bar{q}}(y)}{1 + 2\bar{\lambda}_{\bar{q}}(y)}.$$

Next table sums up the correspondences between different processes, an LRM with initial occupation profile (i.o.p.) L_0 , the LRM with initial occupation profile 1 denoted $(\chi_\tau)_{\tau \geq 0}$ (which is, as we will show, equivalent up to a change of scale and time to an LRM with different initial occupation profile), the reduced process $(\xi_u)_{u \geq 0}$, and the diffusion in random environment $(\bar{Z}_{\bar{q}})_{\bar{q} \geq 0}$. On the lines marked by "corr." (for "correspondence"), all the quantities are equal.

Process	X_t	χ_τ	ξ_u	$\bar{Z}_{\bar{q}}$
Description	LRM, i.o.p. L_0	LRM, i.o.p. 1	red. proc.	diffusion in rand. envir.
Space variable	x	y	y	y
Time variable	t	τ	u	\bar{q}
Local time	$L_t(x) - L_0(x)$	$L_\tau^\chi(y) - 1$	$\Lambda_u(y)$	$\bar{\lambda}_{\bar{q}}(y)$
Space corr.	$L_0(x)^{-2} dx$	dy	dy	dy
Time corr.	$L_t(X_t)^{-3} dt$	$L_\tau(\chi_\tau)^{-3} d\tau$	du	$(1 + 2\bar{\lambda}_{\bar{q}}(\bar{Z}_{\bar{q}}))^{-2} d\bar{q}$
Local time corr.	$\frac{1}{2} \left(1 - \frac{L_0(x)^2}{L_t(x)^2}\right)$	$\frac{1}{2} (1 - L_\tau^\chi(y)^{-2})$	$\Lambda_u(y)$	$\frac{\bar{\lambda}_{\bar{q}}(y)}{1 + 2\bar{\lambda}_{\bar{q}}(y)}$

The convergence of the VRJP to a continuous space process has a version for the Edge Reinforced Random Walk. For references on the ERRW see [Dia88, KR00, DR06, Rol06, MR07, ACK14]. It was shown in [ST15] that an ERRW has same distribution as the discrete-time process of a VRJP in a network with random conductances, hence it is a mixture of Markovian random walks.

In our context, we consider a discrete time reinforced walk $(\widehat{Z}_k^{(n)})_{k \geq 0}$ on $2^{-n}\mathbb{Z}$, started at 0. The weight of an edge $w_k^{(n)}(x, x + 2^{-n})$ at time k will be

$$\begin{aligned} w_k^{(n)}(x - 2^{-n}, x) &= w_k^{(n)}(x, x - 2^{-n}) \\ &= w_0^{(n)}(x - 2^{-n}, x) + \text{Card}\{j \in \{1, \dots, k\} \mid \{\widehat{Z}_{j-1}^{(n)}, \widehat{Z}_j^{(n)}\} = \{x - 2^{-n}, x\}\}, \end{aligned}$$

where $\{\cdot, \cdot\}$ stands for the undirected edge and $w_0(x - 2^{-n}, x) \in (0, +\infty)$. The transition probabilities are:

$$\mathbb{P}(\widehat{Z}_{k+1}^{(n)} = x \pm 2^{-n} \mid \widehat{Z}_k^{(n)} = x, (\widehat{Z}_j^{(n)})_{0 \leq j \leq k}) = \frac{w_k^{(n)}(x, x \pm 2^{-n})}{w_k^{(n)}(x, x - 2^{-n}) + w_k^{(n)}(x, x + 2^{-n})}.$$

For initial weights we will take

$$w_0^{(n)}(x - 2^{-n}, x) = 2^{n-1} L_0(x - 2^{-n}) L_0(x).$$

Proposition 1.8. *The family of processes*

$$(\widehat{Z}_{\lfloor 4^n q \rfloor}^{(n)}, 2^{-n} w_{\lfloor 4^n q \rfloor}^{(n)}(x - 2^{-n}, x))_{x \in 2^{-n}\mathbb{Z}, q \geq 0}$$

jointly convergences in law as $n \rightarrow +\infty$ towards

$$(Z_q, L_0(x)^2/2 + \lambda_q(x))_{x \in \mathbb{R}, q \geq 0},$$

where $(Z_q)_{q \geq 0}$ is the mixture of diffusion of Theorem 1.5. The spatial processes are considered to be interpolated linearly outside $2^{-n}\mathbb{Z}$.

Remark 1.9. The fact that the ERRW has a scaling limit which is a diffusion in random potential is reminiscent of the Sinai’s random walk [Sin82] scaling to a Brox diffusion [Bro86, Sei00, Pac16]. In the Brox diffusion however the random potential contains only a Wiener term and no drift as in our case. See also [Dav96] for the once-reinforced random walk scaling to Carmona-Petit-Yor process [CPY98].

Our paper is organized as follows. In Section 2 we will recall some properties of Bass-Burdzy flows and see what it implies for the Linearly Reinforced Motion. In Section 3 we will show the convergence of the random environment related to the VRJP, and as a consequence the convergence of the VRJP to a mixture of time-changed diffusions. We will also recall the random environment associated to the ERRW and deduce the convergence of the ERRW. In Section 4, we will show that this mixture of time-changed diffusions coincides with the Linearly Reinforced Motion, and deduce a couple of consequences of this, such as the long-time behaviour of the LRM. In our paper we will use different time scales, t, τ, u, q, \bar{q} , etc., and the notations like $q(t)$ will denote the change of time that transform one time scale into an other.

Next are some simulations of the Linearly Reinforced Motion at different scales, obtained by running VRJP-s on a fine lattice.

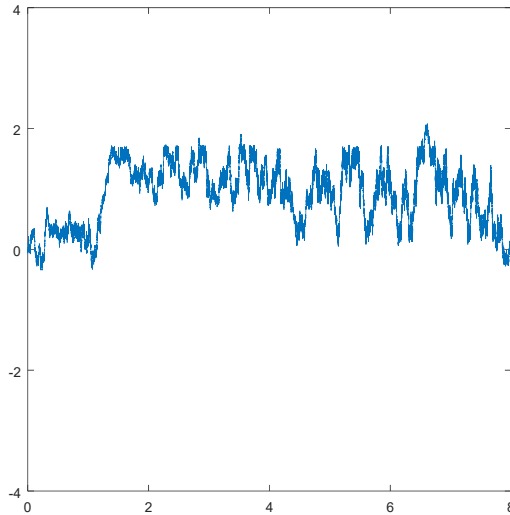


FIGURE 1. LRM with $L_0 \equiv 1$ on time-interval $[0, 8]$.

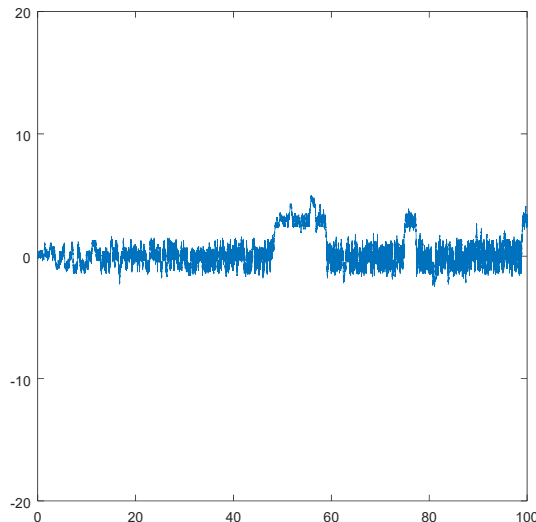


FIGURE 2. LRM with $L_0 \equiv 1$ on time-interval $[0, 100]$.

2. CONVERGENT BASS-BURDZY FLOW AND LINEARLY REINFORCED MOTION

Let $(B_u)_{u \geq 0}$ be a standard Brownian motion on \mathbb{R} , starting at 0, and let $(\mathcal{F}_u^B)_{u \geq 0}$ be the associated filtration. For $u_0 \geq 0$, we will denote

$$(B \circ \theta_{u_0})_u = B_{u_0+u} - B_{u_0}.$$

We consider the differential equation (1.5):

$$\frac{dY_u}{du} = \begin{cases} -1 & \text{if } Y_u > B_u, \\ 1 & \text{if } Y_u < B_u, \end{cases}$$

with some initial condition $Y_0 = y \in \mathbb{R}$. $(B_u - \Psi_u^B(y))_{y \in \mathbb{R}, u \geq 0}$ is a stochastic flow, solution to the SDE

$$(2.1) \quad d\zeta_u = dB_u - (-\mathbf{1}_{\zeta_u > 0} + \mathbf{1}_{\zeta_u < 0})du.$$

The equation(2.1) falls in the class studied in [Att10] (bounded variation drift). Next we list the main results on solutions to (1.5).

Proposition 2.1 (Bass-Burdzy [BB99], Hu-Warren [HW00], Attanasio [Att10]). *For every initial condition $Y_0 = y \in \mathbb{R}$, there is a.s. a unique solution to (1.5) which is Lipschitz continuous. We denote it $(\Psi_u^B(y))_{u \geq 0}$. For any $y_1, \dots, y_k \in \mathbb{R}$, the joint law of $(B_u, \Psi_u^B(y_1), \dots, \Psi_u^B(y_k))_{u \geq 0}$ is uniquely determined. One can construct $(\Psi_u^B(y))_{y \in \mathbb{R}, u \geq 0}$ simultaneously for all $y \in \mathbb{R}$ such that $(y, u) \mapsto \Psi_u^B(y)$ is continuous on $\mathbb{R} \times [0, +\infty)$. Moreover, we have the following properties:*

- (1) *The flow $(\Psi_u^B(y))_{y \in \mathbb{R}, u \geq 0}$ is adapted to the filtration $(\mathcal{F}_u^B)_{u \geq 0}$.*
- (2) *(Strong Markov property). For any U_0 stopping time for $(\mathcal{F}_u^B)_{u \geq 0}$,*

$$\Psi_{U_0+u}^B(y) = \Psi_u^{B \circ \theta_{U_0}}(y - B_{U_0}) + B_{U_0}.$$

- (3) *A.s., for any $\alpha \in (0, 1/2)$ and for all $u \geq 0$, $y \mapsto \Psi_u^B(y)$ is a $\mathcal{C}^{1,\alpha}$ -diffeomorphism of \mathbb{R} . That is to say, $y \mapsto \Psi_u^B(y)$ is an increasing bijection, $\frac{\partial}{\partial y} \Psi_u^B(y)$ is positive on \mathbb{R} , and both the functions $y \mapsto \frac{\partial}{\partial y} \Psi_u^B(y)$ and $y \mapsto \frac{\partial}{\partial y} (\Psi_u^B)^{-1}(y)$ are locally α -Hölder continuous.*
- (4) *The process $(B_u - \Psi_u^B(y))_{u \geq 0}$ admits semi-martingale local times at level 0, $(\mathcal{L}_u(y))_{u \geq 0}$,*

$$\mathcal{L}_u(y) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} \int_0^u \mathbf{1}_{|B_v - \Psi_v^B(y)| < \varepsilon} dv,$$

such that the map $(y, u) \mapsto \mathcal{L}_u(y)$ is continuous.

- (5) *For the space derivative of the flow, one has*

$$(2.2) \quad \frac{\partial}{\partial y} \Psi_u^B(y) = \exp(-2\mathcal{L}_u(y)).$$

- (6) *The process $((\Psi_u^B)^{-1}(B_u))_{u \geq 0} = (\xi_u)_{u \geq 0}$ admits occupation densities (local times) $(\Lambda_u(y))_{y \in \mathbb{R}, u \geq 0}$, continuous in (y, u) . Moreover, the following identity holds:*

$$(2.3) \quad \Lambda_u(y) = \frac{1}{2}(1 - \exp(-2\mathcal{L}_u(y))).$$

In particular, $\Lambda_u(y) \leq 1/2$.

- (7) *The process $(\xi_u)_{u \geq 0}$ is recurrent, that is to say, for all $u_0 \geq 0$, the process will visit a.s. all points after u_0 .*

Next we show some elementary properties of $(\xi_u)_{u \geq 0}$ which we did not found as such in our references [BB99, HW00, Att10].

Proposition 2.2. $(\xi_u)_{u \geq 0}$ *satisfies:*

- (1) *A.s., for any $\alpha \in (0, 1/2)$, the process $(\xi_u)_{u \geq 0}$ is locally α -Hölder continuous.*

(2) Let $u > 0$ and consider $(u_{i,j})_{0 \leq j \leq N_i, i \geq 0}$ a deterministic family such that

$$0 = u_{i,0} < u_{i,1} < \cdots < u_{i,N_i-1} < u_{N_i} = u$$

and

$$\lim_{i \rightarrow +\infty} \max_{1 \leq j \leq N_i} (u_{i,j} - u_{i,j-1}) = 0.$$

Then,

$$(2.4) \quad \lim_{i \rightarrow +\infty} \sum_{j=1}^{N_i} (\xi_{u_{i,j}} - \xi_{u_{i,j-1}})^2 = \int_0^u (1 - 2\Lambda_v(\xi_v))^{-2} dv$$

in probability.

(3) Let $(\rho_u)_{u \geq 0}$ be the process

$$\rho_u = \xi_u - \int_0^u (1 - 2\Lambda_v(\xi_v))^{-1} dB_v.$$

For a family $(u_{i,j})$ as above,

$$\lim_{i \rightarrow +\infty} \sum_{j=1}^{N_i} (\rho_{u_{i,j}} - \rho_{u_{i,j-1}})^2 = 0$$

in probability.

Proof. First note that for any $y \in \mathbb{R}$,

$$|(\Psi_u^B)^{-1}(y) - y| = |\Psi_u^B \circ (\Psi_u^B)^{-1}(y) - (\Psi_u^B)^{-1}(y)| \leq 2u,$$

and

$$|(\Psi_u^B)^{-1}(y_2) - (\Psi_u^B)^{-1}(y_1)| \leq \exp\left(2 \sup_{y \in \mathbb{R}} \mathcal{L}_u(y)\right) |y_2 - y_1| = (1 - 2 \sup_{y \in \mathbb{R}} \Lambda_u(y))^{-1} |y_2 - y_1|,$$

where for the second inequality we used that

$$\frac{\partial}{\partial y} (\Psi_u^B)^{-1}(y) = \exp(2\mathcal{L}_u((\Psi_u^B)^{-1}(y))) = (1 - 2\Lambda_u((\Psi_u^B)^{-1}(y)))^{-1}.$$

Then write

$$\xi_{u_2} = (\Psi_{u_2}^B)^{-1}(B_{u_2}) = (\Psi_{u_1}^B)^{-1}((\Psi_{u_2-u_1}^{B \circ \theta_{u_1}})^{-1}(B_{u_2} - B_{u_1}) + B_{u_1}).$$

It follows that

$$\begin{aligned} |\xi_{u_2} - \xi_{u_1}| &\leq (1 - 2 \sup_{y \in \mathbb{R}} \Lambda_{u_1}(y))^{-1} |(\Psi_{u_2-u_1}^{B \circ \theta_{u_1}})^{-1}(B_{u_2} - B_{u_1})| \\ &\leq (1 - 2 \sup_{y \in \mathbb{R}} \Lambda_{u_1}(y))^{-1} |B_{u_2} - B_{u_1}| \\ &\quad + (1 - 2 \sup_{y \in \mathbb{R}} \Lambda_{u_1}(y))^{-1} |(\Psi_{u_2-u_1}^{B \circ \theta_{u_1}})^{-1}(B_{u_2} - B_{u_1}) - (B_{u_2} - B_{u_1})| \\ &\leq (1 - 2 \sup_{y \in \mathbb{R}} \Lambda_{u_1}(y))^{-1} |B_{u_2} - B_{u_1}| + 2(1 - 2 \sup_{y \in \mathbb{R}} \Lambda_{u_1}(y))^{-1} (u_2 - u_1), \end{aligned}$$

which implies (1).

Let us show (2). Refining the above computation, one gets that

$$(2.5) \quad \begin{aligned} \xi_{u_{i,j}} - \xi_{u_{i,j-1}} &= (1 - 2\Lambda_{u_{i,j-1}}(\xi_{u_{i,j-1}}))^{-1} (B_{u_{i,j}} - B_{u_{i,j-1}}) \\ &\quad + o(|B_{u_{i,j}} - B_{u_{i,j-1}}| + 2(u_{i,j} - u_{i,j-1})) + O(u_{i,j} - u_{i,j-1}), \end{aligned}$$

where

$$\begin{aligned} |o(|B_{u_{i,j}} - B_{u_{i,j-1}}| + 2(u_{i,j} - u_{i,j-1}))| &\leq (|B_{u_{i,j}} - B_{u_{i,j-1}}| + 2(u_{i,j} - u_{i,j-1})) \\ &\quad \times \sup_{\substack{|y_1 - y_2| \leq |B_{u_{i,j}} - B_{u_{i,j-1}}| \\ + 2(u_{i,j} - u_{i,j-1})}} |(1 - 2\Lambda_{u_{i,j-1}}(y_2))^{-1} - (1 - 2\Lambda_{u_{i,j-1}}(y_1))^{-1}|, \end{aligned}$$

and

$$|O(u_{i,j} - u_{i,j-1})| \leq 2(1 - 2 \sup_{y \in \mathbb{R}} \Lambda_{u_1}(y))^{-1}(u_2 - u_1).$$

Thus, the sum in (2.4) behaves, as $i \rightarrow +\infty$, like

$$\sum_{j=1}^{N_i} (1 - 2\Lambda_{u_{i,j-1}}(\xi_{u_{i,j-1}}))^{-2} (B_{u_{i,j}} - B_{u_{i,j-1}})^2.$$

To conclude, we use that

$$\lim_{i \rightarrow +\infty} \sum_{j=1}^{N_i} |(B_{u_{i,j}} - B_{u_{i,j-1}})^2 - (u_{i,j} - u_{i,j-1})| = 0$$

in probability.

For (3), use (2.5) and write

$$\begin{aligned} \rho_{u_{i,j}} - \rho_{u_{i,j-1}} &= (1 - 2\Lambda_{u_{i,j-1}}(\xi_{u_{i,j-1}}))^{-1} (B_{u_{i,j}} - B_{u_{i,j-1}}) - \int_{u_{i,j-1}}^{u_{i,j}} (1 - 2\Lambda_v(\xi_v))^{-1} dB_v \\ &\quad + o(|B_{u_{i,j}} - B_{u_{i,j-1}}| + 2(u_{i,j} - u_{i,j-1})) + O(u_{i,j} - u_{i,j-1}). \quad \square \end{aligned}$$

Remark 2.3. The process $(\xi_u)_{u \geq 0}$ has a decomposition into a sum of a local martingale and a process with 0 quadratic variation, both adapted to the Brownian filtration $(\mathcal{F}_u^B)_{u \geq 0}$. Following Föllmer's terminology [Fö81], it is a Dirichlet process. However, it is believed not to be a semi-martingale [HW00], which would mean that $(\rho_u)_{u \geq 0}$ has an infinite total variation. The reason for that would be that the terms $o(|B_{u_{i,j}} - B_{u_{i,j-1}}| + 2(u_{i,j} - u_{i,j-1}))$ in (2.5) are not $O((B_{u_{i,j}} - B_{u_{i,j-1}})^2)$, since the flow $(\Psi_u^B)_{u \geq 2}$ is not \mathcal{C}^2 in space. One could push up to showing that $(\rho_u)_{u \geq 0}$ is locally $3/4 - \varepsilon$ Hölder continuous. We believe that this $3/4 - \varepsilon$ is optimal.

Next are some elementary properties of the LRM $(X_t)_{t \geq 0}$ (see Definition 1.1).

Proposition 2.4. *The following properties hold:*

- (1) Let $(\chi_\tau)_{\tau \geq 0}$ be the Linearly Reinforced Motion starting from 0, with initial occupation profile 1. Given $x_0 \in \mathbb{R}$ and another occupation profile L_0 , define the change of time

$$dt = L_0(S_0^{-1}(\chi_\tau))^3 d\tau,$$

and consider the change of scale S_0 given by (1.7). Then $X_t = S_0^{-1}(\chi_{\tau(t)})$ is a Linearly Reinforced Motion starting from x_0 , with initial occupation profile L_0 .

- (2) A.s., X_t is defined for all $t \geq 0$.
- (3) A.s., for any $\alpha \in (0, 1/2)$, the process $(X_t)_{t \geq 0}$ is locally α -Hölder continuous.
- (4) Let L_t be the occupation profile at time t , defined by (1.9). Then $(L_t(x) - L_0(x))_{x \in \mathbb{R}}$ is the occupation density of X on time-interval $[0, t]$, that is to say, for any $f : \mathbb{R} \rightarrow \mathbb{R}$ bounded,

$$\int_0^t f(X_s) ds = \int_{\mathbb{R}} f(x) (L_t(x) - L_0(x)) dx.$$

- (5) (Strong Markov property). Let T_0 be a stopping time for the natural filtration $(\mathcal{F}_t^X)_{t \geq 0}$ of $(X_t)_{t \geq 0}$. Then $(X_{T_0+t})_{t \geq 0}$ is distributed as a Linearly Reinforced Motion starting from X_{T_0} , with initial occupation profile L_{T_0} .
- (6) The process $(X_t)_{t \geq 0}$ is recurrent, that is to say, for all $t_0 \geq 0$, the process will visit a.s. all points after t_0 .
- (7) Let $x_1 < x_2 \in \mathbb{R}$. Then

$$(2.6) \quad \mathbb{P}(\text{After time } t_0, X_t \text{ hits } x_2 \text{ before } x_1 | \mathcal{F}_{t_0}^X, x_1 < X_{t_0} < x_2) \geq \frac{1}{2}$$

is equivalent to

$$(2.7) \quad \int_{X_{t_0}}^{x_2} L_{t_0}(x)^{-2} dx \leq \int_{x_1}^{X_{t_0}} L_{t_0}(x)^{-2} dx.$$

More precisely, let $y_1 < 0$ and $y_2 > 0$. Let $U_{y_1}^\downarrow$ be the first time the drifted Brownian motion $B_u - u$ hits y_1 and $U_{y_2}^\uparrow$ the first time $B_u + u$, hits y_2 , with $B_0 = 0$. Then

$$(2.8) \quad \mathbb{P}(\text{After time } t_0, X_t \text{ hits } x_2 \text{ before } x_1 | \mathcal{F}_{t_0}^X, x_1 < X_{t_0} < x_2) = \mathbb{P}(U_{y_2}^\uparrow < U_{y_1}^\downarrow),$$

where

$$y_1 = \int_{x_1}^{X_{t_0}} L_{t_0}(x)^{-2} dx, \quad y_2 = \int_{X_{t_0}}^{x_2} L_{t_0}(x)^{-2} dx.$$

(8) Let $t > 0$ and consider $(t_{i,j})_{0 \leq j \leq N_i, i \geq 0}$ a deterministic family such that

$$0 = t_{i,0} < t_{i,1} < \cdots < t_{i,N_i-1} < t_{N_i} = u$$

and

$$\lim_{i \rightarrow +\infty} \max_{1 \leq j \leq N_i} (t_{i,j} - t_{i,j-1}) = 0.$$

Then

$$\lim_{i \rightarrow +\infty} \sum_{j=1}^{N_i} (X_{t_{i,j}} - X_{t_{i,j-1}})^2 = \int_0^t L_s(X_s) ds$$

in probability. Let $(R_t)_{t \geq 0}$ be the process

$$R_t = X_t - \int_0^t L_s(X_s)^2 ds B_{u(s)},$$

where $u(\cdot)$ is the inverse time-change of (1.8). Then, for a family $(t_{i,j})$ as above,

$$\lim_{i \rightarrow +\infty} \sum_{j=1}^{N_i} (R_{t_{i,j}} - R_{t_{i,j-1}})^2 = 0$$

in probability.

Proof. (1): This is a straightforward consequence of Definition 1.1. One uses the same driving Brownian motion and reduced process.

(2): This is equivalent to

$$\int_0^{+\infty} L_0(X_u)^3 (1 - 2\Lambda_u(\xi_u))^{-\frac{3}{2}} du = +\infty \text{ a.s.}$$

Fix $y_1 < y_2 \in \mathbb{R}$. Since L_0 is positive bounded away from 0 on $[S_0^{-1}(y_2), S_0^{-1}(y_1)]$, it is enough to show that

$$\int_0^{+\infty} \mathbf{1}_{y_1 < \xi_u < y_2} (1 - 2\Lambda_u(\xi_u))^{-\frac{3}{2}} du = +\infty \text{ a.s.}$$

Using the elementary properties of occupation densities, one can show that the above integrals equals on $[S_0^{-1}(y_2), S_0^{-1}(y_1)]$, it is enough to show that

$$\int_{y_1 < y < y_2} \int_0^{+\infty} (1 - 2\Lambda_u(\xi_u))^{-\frac{3}{2}} d_u \Lambda_u(y) dy.$$

Applying the identity (2.3), get that it equals in turn

$$\int_{y_1 < y < y_2} \int_0^{+\infty} \exp(3\mathcal{L}_u(y)) d_u \mathcal{L}_u(y) dy = \frac{1}{3} \int_{y_1 < y < y_2} (\exp(3\mathcal{L}_{+\infty}(y)) - 1) dy.$$

Conclude using that a.s., $\forall y \in \mathbb{R}$, $\mathcal{L}_{+\infty}(y) = \lim_{u \rightarrow +\infty} \mathcal{L}_u(y) = +\infty$ [BB99, HW00].

(3): This follows from the local Hölder continuity of $(\xi_u)_{u \geq 0}$ and the fact that we perform \mathcal{C}^1 changes of scale and time.

(4): Use that

$$\begin{aligned}
\int_0^t f(X_s) ds &= \int_0^{u(t)} f(S_0^{-1}(\xi_v)) L_0(S_0^{-1}(\xi_v))^3 (1 - 2\Lambda_v(\xi_v))^{-\frac{3}{2}} dv \\
&= \int_{\mathbb{R}} \int_0^{u(t)} f(S_0^{-1}(y)) L_0(S_0^{-1}(y))^3 (1 - 2\Lambda_v(y))^{-\frac{3}{2}} d_v \Lambda_v(y) dy \\
&= \int_{\mathbb{R}} f(S_0^{-1}(y)) L_0(S_0^{-1}(y))^3 ((1 - 2\Lambda_{u(t)}(y))^{-\frac{1}{2}} - 1) dy \\
&= \int_{\mathbb{R}} f(x) L_0(x) ((1 - 2\Lambda_{u(t)}(S_0(x)))^{-\frac{1}{2}} - 1) dx.
\end{aligned}$$

(5): Let $U_0 = u(T_0)$. It is a stopping time for the driving Brownian motion $(B_u)_{u \geq 0}$. Let $\tilde{\xi}_u = (\Psi_u^{B \circ \theta_{U_0}})^{-1}((B \circ \theta_{U_0})_u)$. The process $(\tilde{\xi}_u)_{u \geq 0}$ has same law as $(\xi_u)_{u \geq 0}$. Moreover,

$$\xi_{U_0+u} = (\Psi_{U_0}^B)^{-1}(\tilde{\xi}_u + B_{U_0}).$$

Let

$$S_{T_0}(x) = \int_{X_{T_0}}^x L_{T_0}(r)^{-2} dr.$$

We have that

$$X_{T_0+t} = S_0^{-1}(\xi_{u(T_0+t)}) = (\Psi_{U_0}^B \circ S_0)^{-1}(\tilde{\xi}_{u(T_0+t)-U_0} + B_{U_0}).$$

Moreover,

$$(\Psi_{U_0}^B \circ S_0)^{-1}(0 + B_{U_0}) = X_{T_0}$$

and, following (2.2) and (2.3),

$$\frac{d}{dx} \Psi_{U_0}^B \circ S_0(x) = L_0(x)^{-2} (1 - 2\Lambda_{U_0}(S_0(x))) = L_{T_0}(x)^{-2}.$$

Thus,

$$(\Psi_{U_0}^B \circ S_0)^{-1}(y + B_{U_0}) = S_{T_0}^{-1}(y).$$

Finally,

$$u(T_0 + t) - U_0 = \int_{T_0}^{T_0+t} L_s(X_s)^{-3} ds.$$

So we get (5).

(6): This follows from the recurrence of $(\xi_u)_{u \geq 0}$.

(7): Since we have the Markov property, it is enough to show it for $t_0 = 0$. Then, if $X_0 \in (x_1, x_2)$,

$$\begin{aligned}
\mathbb{P}(X_t \text{ hits } x_2 \text{ before } x_1) &= \mathbb{P}(\xi_u \text{ hits } S_0(x_2) \text{ before } S_0(x_1)) \\
&= \mathbb{P}(B_u \text{ meets } \Psi_u^B \circ S_0(x_2) \text{ before } \Psi_u^B \circ S_0(x_1)),
\end{aligned}$$

which is exactly (2.8).

(8): The proof is similar to that of (2) and (3) in Proposition 2.2. One has to apply a time-change to go from $(\xi_u)_{u \geq 0}$ to $(X_t)_{t \geq 0}$, and thus, considers $(B_u)_{u \geq 0}$ at random stopping times rather than at fixed times. Note that, in the time change (1.8),

$$L_t(X_t) dt = \left(\frac{d}{dx} S_0(X_t) \right)^{-2} (1 - 2\Lambda_u(\xi_u))^{-2} du. \quad \square$$

Remark 2.5. The equivalence between (2.6) and (2.7) emphasizes the reinforcement property. Indeed, the motion tends to drift towards the places it has already visited a lot. Yet it is recurrent. Property (8) gives a decomposition of $(X_t)_{t \geq 0}$ as a local martingale plus an adapted process with zero quadratic variation. As for $(\xi_u)_{u \geq 0}$, we believe that the LRM $(X_t)_{t \geq 0}$ is not a semi-martingale.

3. THE VRJP-RELATED RANDOM ENVIRONMENT AND ITS CONVERGENCE

It was shown in [ST15], that a VRJP has same law as a time-change of a non-reinforced Markov jump process in a network with random conductances. In dimension one, the study of the mixing measure was already initiated in [DV02]. In our setting, one has the following:

Proposition 3.1 (Davis-Volkov [DV02], Sabot-Tarrès [ST15]). *Let $n \in \mathbb{N}$. Let $(V^{(n)-}(x))_{x \in 2^{-n}\mathbb{N}^*}$ and $(V^{(n)+}(x))_{x \in 2^{-n}\mathbb{N}^*}$ be two independent families of independent real random variables, where $V^{(n)\sigma}(x)$, $\sigma \in \{-1, +1\}$, is distributed according to*

$$2^{\frac{n-1}{2}} \pi^{-\frac{1}{2}} (L_0(\sigma x) L_0(\sigma(x - 2^{-n})))^{\frac{1}{2}} \exp(-2^n L_0(\sigma x) L_0(\sigma(x - 2^{-n})) \sinh(v/2)^2 + v/2) dv.$$

Define $(\mathcal{U}^{(n)-}(x))_{x \in 2^{-n}\mathbb{N}}$ and $(\mathcal{U}^{(n)+}(x))_{x \in 2^{-n}\mathbb{N}}$ by

$$\mathcal{U}^{(n)-}(0) = \mathcal{U}^{(n)+}(0) = 0, \quad \mathcal{U}^{(n)\sigma}(x) = \sum_{i=1}^{2^n x} V^{(n)\sigma}(2^{-n}i), \sigma \in \{-1, +1\}, x \in \mathbb{N}^*.$$

Set

$$(3.1) \quad \mathcal{U}^{(n)}(x) = \begin{cases} 0 & \text{if } x = 0, \\ \mathcal{U}^{(n)+}(x) & \text{if } x \in \mathbb{N}^*, \\ \mathcal{U}^{(n)-}(|x|) & \text{if } -x \in \mathbb{N}^*. \end{cases}$$

Let $(Z_q^{(n)})_{0 \leq q \leq q_{\max}^{(n)}}$ be the continuous-time process on $2^{-n}\mathbb{Z}$, which, conditional on the random environment $(\mathcal{U}^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}}$, is a (non-reinforced) Markov jump process, started from 0, with transition rate from $x \in 2^{-n}\mathbb{Z}$ to $x + \sigma 2^{-n}$, $\sigma \in \{-1, +1\}$, equal to

$$2^{2n-1} \frac{L_0(x + \sigma 2^{-n})}{L_0(x)} e^{-\mathcal{U}^{(n)}(x + \sigma 2^{-n}) + \mathcal{U}^{(n)}(x)}.$$

$q_{\max}^{(n)} \in (0, +\infty)$ is the time when the process explodes to infinity, whenever this happens. Otherwise $q_{\max}^{(n)} = +\infty$. Let $\lambda_q^{(n)}(x)$ be the local times of $Z_q^{(n)}$:

$$\lambda_q^{(n)}(x) = 2^n \int_0^q \mathbf{1}_{Z_r^{(n)}=x} dr.$$

Define the change of time

$$q^{(n)}(t) = \inf \left\{ q \geq 0 \mid 2^n \sum_{x \in 2^{-n}\mathbb{Z}} ((L_0(x)^2 + 2\lambda_q^{(n)}(x))^{\frac{1}{2}} - L_0(x)) \geq t \right\}.$$

Then the family of time changed processes

$$\left(Z_{q^{(n)}(t) \wedge q_{\max}^{(n)}}^{(n)}, (L_0(x)^2 + 2\lambda_{q^{(n)}(t) \wedge q_{\max}^{(n)}}^{(n)}(x))^{\frac{1}{2}} \right)_{x \in 2^{-n}\mathbb{Z}, t \geq 0}$$

has same distribution as the VRJP

$$(X_{t \wedge t_{\max}^{(n)}}^{(n)}, L_{t \wedge t_{\max}^{(n)}}^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}, t \geq 0}.$$

We will show that the random environment $(\mathcal{U}^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}}$ converges as $n \rightarrow +\infty$ to a process the process $(\mathcal{U}(x))_{x \in \mathbb{R}}$ introduced in (1.10). Out of this we deduce that the process $(Z_q^{(n)})_{0 \leq q \leq q_{\max}^{(n)}}$ has a limit in law $(Z_q)_{q \geq 0}$, which condition on the environment $(\mathcal{U}(x))_{x \in \mathbb{R}}$, is a Markov diffusion on \mathbb{R} . Then, we conclude that the VRJP $(X_t^{(n)})_{0 \leq t \leq t_{\max}^{(n)}}$ converges to a time-change of $(Z_q)_{q \geq 0}$.

Lemma 3.2. *The family of random processes $(\mathcal{U}^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}}$ defined in (3.1) converges in law, as $n \rightarrow +\infty$, to the process $(\mathcal{U}(x))_{x \in \mathbb{R}}$, defined by (1.10). $\mathcal{U}^{(n)}$ is considered to be interpolated linearly outside $2^{-n}\mathbb{Z}$. The space of continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ is considered to be endowed with the topology of uniform convergence on compact intervals.*

Proof. Notice that for n large, the law of $V^{(n)+}(x)$ is almost

$$\mathcal{N}(2^{-n}L_0(x)^{-1}L_0(x-2^{-n})^{-1}, 2^{-(n-1)}L_0(x)^{-1}L_0(x-2^{-n})^{-1}),$$

i.e. a normal with mean $2^{-n}L_0(x)^{-1}L_0(x-2^{-n})^{-1}$ and variance $2^{-(n-1)}L_0(x)^{-1}L_0(x-2^{-n})^{-1}$. More precisely, the total variation distance between the two laws is

$$O(2^{-2n}L_0(x)^{-2}L_0(x-2^{-n})^{-2}).$$

In other words, the total variation distance between the law of $V^{(n)+}(x)$ and that of $\mathcal{U}(x) - \mathcal{U}(x-2^{-n})$ is $O(2^{-2n})$, where O is uniform for x in a compact subset $\text{Fix } A > 0$. We get that the total variation between $(\mathcal{U}^{(n)}(x))_{x \in [0, A] \cap 2^{-n}\mathbb{Z}}$ and $(\mathcal{U}(x))_{x \in [0, A] \cap 2^{-n}\mathbb{Z}}$ is $O(2^{-n})$. \square

Recall that $(\lambda_q(x))_{x \in \mathbb{R}, q \geq 0}$ denotes the family of local times of $(Z_q)_{q \geq 0}$.

Proposition 3.3. *Consider the random environments $(\mathcal{U}^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}}$ and the random processes $(Z_q^{(n)})_{0 \leq q \leq q_{\max}^{(n)}}$, with local times $(\lambda_q^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}, 0 \leq q \leq q_{\max}^{(n)}}$, introduced in Proposition 3.1. As $n \rightarrow +\infty$, $q_{\max}^{(n)} \rightarrow +\infty$ in probability, and the process*

$$(\mathcal{U}^{(n)}(x), Z_{q \wedge q_{\max}^{(n)}}^{(n)}, \lambda_{q \wedge q_{\max}^{(n)}}^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}, q \geq 0}$$

converges in law to

$$(\mathcal{U}(x), Z_q(x), \lambda_q(x))_{x \in \mathbb{R}, q \geq 0}.$$

We interpolate $2^{-n}\mathbb{Z}$ -valued processes linearly, and use for $\lambda_{q \wedge q_{\max}^{(n)}}^{(n)}(x)$ and $\lambda_q(x)$ the topology of uniform convergence on compact subsets of $\mathbb{R} \times [0, +\infty)$.

Proof. The idea is to "embed" the processes $(Z_q^{(n)})_{0 \leq q \leq q_{\max}^{(n)}}$ for different values of n inside a Brownian motion, scale-changed. Let $(\beta_s)_{s \geq 0}$ be a standard Brownian motion started from 0, with a family of local times denoted $(\ell_s^\beta(x))_{x \in \mathbb{R}, s \geq 0}$. Take $(\mathcal{U}^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}}$ independent from $(\beta_s)_{s \geq 0}$. Define the change of scale $\mathcal{S}^{(n)} : 2^{-n}\mathbb{Z} \rightarrow \mathbb{R}$ by $\mathcal{S}^{(n)}(0) = 0$ and for $x \in 2^{-n}\mathbb{Z}$, $x \neq 0$,

$$\mathcal{S}^{(n)}(x) = 2^{-n} \sum_{i=1}^{2^n|x|} L_0(\text{sgn}(x)2^{-n}i)^{-1} L_0(\text{sgn}(x)2^{-n}(i-1))^{-1} e^{\mathcal{U}^{(n)}(\text{sgn}(x)2^{-n}i) + \mathcal{U}^{(n)}(\text{sgn}(x)2^{-n}(i-1))}.$$

Consider the time change

$$s^{(n)}(q) = \inf \left\{ s \geq 0 \mid 2^{-n} \sum_{x \in 2^{-n}\mathbb{Z}} L_0(x)^{-2} e^{2\mathcal{U}^{(n)}(x)} \ell_s^\beta(\mathcal{S}^{(n)}(x)) \geq q \right\}.$$

Then one can construct $Z_q^{(n)}$ and $\lambda_q^{(n)}(x)$ as

$$Z_q^{(n)} = (\mathcal{S}^{(n)})^{-1}(\beta_{s^{(n)}(q)}), \quad \lambda_q^{(n)}(x) = L_0(x)^{-2} e^{2\mathcal{U}^{(n)}(x)} \ell_{s^{(n)}(q)}^\beta(\mathcal{S}^{(n)}(x)).$$

Similarly, take $(\mathcal{U}(x))_{x \in \mathbb{R}}$ independent from $(\beta_s)_{s \geq 0}$. Consider the change of scale

$$\mathcal{S}(x) = \int_0^x L_0(r)^{-2} e^{2\mathcal{U}(r)} dr,$$

and the change of time

$$s(q) = \inf \left\{ s \geq 0 \mid \int_{\mathbb{R}} L_0(x)^{-2} e^{2\mathcal{U}(x)} \ell_s^\beta(\mathcal{S}(x)) dx \geq q \right\}.$$

One can construct Z_q and $\lambda_q(x)$ as

$$Z_q = \mathcal{S}^{-1}(\beta_{s(q)}), \quad \lambda_q(x) = L_0(x)^{-2} e^{2\mathcal{U}(x)} \ell_{s(q)}^\beta(\mathcal{S}(x)).$$

The convergence of $\mathcal{U}^{(n)}$ to \mathcal{U} (Lemma 3.2) implies then the other convergences. \square

Let be the time change

$$q(t) = \inf \left\{ q \geq 0 \mid \int_{x \in \mathbb{R}} ((L_0(x)^2 + 2\lambda_q(x))^{\frac{1}{2}} - L_0(x)) dx \geq t \right\}.$$

This is the same time-change as in (1.13). Set

$$X_t^* = Z_{q(t)}, \quad L_t^*(x) = (L_0(x)^2 + 2\lambda_{q(t)}(x))^{\frac{1}{2}}.$$

Lemma 3.4. *The function $t \mapsto q(t)$ is a.s. an increasing diffeomorphism of $[0, +\infty)$. The space-time process $(L_t^*(x) - L_0(x))_{x \in \mathbb{R}, t \geq 0}$ is the family of local times of $(X_t^*)_{t \geq 0}$, that is to say for any f bounded measurable function,*

$$\int_0^t f(X_s^*) ds = \int_{\mathbb{R}} f(x)(L_t^*(x) - L_0(x)) dx.$$

Proof. For the first point, one needs to check that

$$\lim_{q \rightarrow +\infty} \int_{x \in \mathbb{R}} ((L_0(x)^2 + 2\lambda_q(x))^{\frac{1}{2}} - L_0(x)) dx = +\infty.$$

But actually, a.s. for all $x \in \mathbb{R}$, $\lim_{q \rightarrow +\infty} \lambda_q(x) = +\infty$.

If we differentiate the time change $t \mapsto q(t)$, we get

$$dt = (L_0(Z_q)^2 + 2\lambda_q(Z_q))^{-\frac{1}{2}} dq.$$

Thus,

$$\begin{aligned} \int_0^t f(X_s^*) ds &= \int_0^{q(t)} f(Z_r)(L_0(x)^2 + 2\lambda_r(Z_r))^{-\frac{1}{2}} dr \\ &= \int_{\mathbb{R}} \int_0^{q(t)} f(x)(L_0(x)^2 + 2\lambda_r(x))^{-\frac{1}{2}} dr \lambda_r(x) dx = \int_{\mathbb{R}} f(x)(L_0(x)^2 + 2\lambda_r(x))^{\frac{1}{2}} dx \\ &= \int_{\mathbb{R}} f(x)(L_t^*(x) - L_0(x)) dx, \end{aligned}$$

which is our second point. \square

Combing Proposition 3.1 and Proposition 3.3, one immediately gets that the VRJP has a limit in law which is a time change of $(Z_q)_{q \geq 0}$:

Proposition 3.5. *As $n \rightarrow +\infty$, $t_{\max}^{(n)} \rightarrow +\infty$ in probability, and the VRJP*

$$(X_{t \wedge t_{\max}^{(n)}}^{(n)}, L_t^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}, t \geq 0}$$

converges in law to

$$(X_t^*, L_t^*(x))_{x \in \mathbb{R}, t \geq 0},$$

where we interpolate $L_t^{(n)}(x)$ linearly outside $x \in 2^{-n}\mathbb{Z}$.

Now let us recall how to obtain an ERRW as mixture of random walks.

Proposition 3.6 (Sabot-Tarrès [ST15]). *Let $(\gamma^{(n)}(x - 2^{-n}, x))_{x \in 2^{-n}\mathbb{Z}}$ be independent random variables where $\gamma^{(n)}(x - 2^{-n}, x)$ has the distribution $\Gamma(2^{n-1}L_0(x - 2^{-n})L_0(x), 1)$. Let $(\widehat{V}^{(n)-}(x))_{x \in 2^{-n}\mathbb{N}^*}$ and $(\widehat{V}^{(n)+}(x))_{x \in 2^{-n}\mathbb{N}^*}$ be conditionally two independent families of independent real random variables, where $\widehat{V}^{(n)\sigma}(x)$, $\sigma \in \{-1, +1\}$, has conditional distribution*

$$\pi^{-\frac{1}{2}} (\gamma^{(n)}(x - \sigma 2^{-n}, x))^{\frac{1}{2}} \exp \left(-2\gamma^{(n)}(x - \sigma 2^{-n}, x) \sinh(v/2)^2 + v/2 \right) dv.$$

Define $(\widehat{U}^{(n)-}(x))_{x \in 2^{-n}\mathbb{N}}$ and $(\widehat{U}^{(n)+}(x))_{x \in 2^{-n}\mathbb{N}}$ by

$$\widehat{U}^{(n)-}(0) = \widehat{U}^{(n)+}(0) = 0, \quad \widehat{U}^{(n)\sigma}(x) = \sum_{i=1}^{2^n x} \widehat{V}^{(n)\sigma}(2^{-n}i), \sigma \in \{-1, +1\}, x \in \mathbb{N}^*.$$

Set

$$\widehat{\mathcal{U}}^{(n)}(x) = \begin{cases} 0 & \text{if } x = 0, \\ \widehat{\mathcal{U}}^{(n)+}(x) & \text{if } x \in \mathbb{N}^*, \\ \widehat{\mathcal{U}}^{(n)-}(|x|) & \text{if } -x \in \mathbb{N}^*. \end{cases}$$

Consider the discrete time random walk on $2^{-n}\mathbb{Z}$, started from 0, in the random environment $(\gamma^{(n)}(x - 2^{-n}, x), \widehat{\mathcal{U}}^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}}$, with conditional transition probabilities from x to $x \pm 2^{-n}$ proportional to

$$\gamma^{(n)}(x, x \pm 2^{-n}) e^{-(\widehat{\mathcal{U}}^{(n)}(x) + \widehat{\mathcal{U}}^{(n)}(x \pm 2^{-n}))}.$$

Then, averaged by the environment, it has same distribution as the ERRW $(\widehat{Z}_k^{(n)})_{k \geq 0}$ of Proposition 1.8.

The following elementary convergence in probability holds:

Lemma 3.7. *Let $A > 0$. Let S_0 be the change of scale (1.7), with $x_0 = 0$. Then*

$$\sup_{x \in [-A, A] \cap 2^{-n}\mathbb{Z}} \left| \frac{1}{2} \sum_{i=1}^{2^n|x|} \gamma(2^{-n}i - \text{sgn}(x)2^{-n}, 2^{-n}i)^{-1} - S_0(x) \right|$$

converges in probability to 0 as $n \rightarrow +\infty$.

Proof. By the elementary properties of gamma distributions,

$$\mathbb{E}[\gamma^{(n)}(x - 2^{-n}, x)^{-1}] = \frac{\Gamma(\cdot - 1)}{\Gamma(\cdot)} (2^{n-1} L_0(x - 2^{-n}) L_0(x)) = 2^{-n+1} L_0(x - 2^{-n})^{-1} L_0(x)^{-1},$$

where Γ is Euler's Gamma function, and

$$\text{Var}(\gamma^{(n)}(x - 2^{-n}, x)^{-1}) = \left(\frac{\Gamma(\cdot - 2)}{\Gamma(\cdot)} - \frac{\Gamma(\cdot - 1)^2}{\Gamma(\cdot)^2} \right) (2^{n-1} L_0(x - 2^{-n}) L_0(x)) = O(2^{-3n}).$$

From Doob's maximal inequality follows that

$$\begin{aligned} \mathbb{E} \left[\sup_{x \in [0, A] \cap 2^{-n}\mathbb{Z}} \left| \frac{1}{2} \sum_{i=1}^{2^n x} \gamma(2^{-n}(i-1), 2^{-n}i)^{-1} - \mathbb{E}[\gamma(2^{-n}(i-1), 2^{-n}i)^{-1}] \right|^2 \right] \\ \leq 4 \sum_{i=1}^{\lfloor 2^n A \rfloor} \text{Var}(\gamma^{(n)}(2^{-n}(i-1), 2^{-n}i)^{-1}) = O(2^{-2n}). \end{aligned}$$

Moreover,

$$\lim_{n \rightarrow +\infty} \sup_{x \in [-A, A] \cap 2^{-n}\mathbb{Z}} \left| \frac{1}{2} \sum_{i=1}^{2^n|x|} \mathbb{E}[\gamma(2^{-n}i - \text{sgn}(x)2^{-n}, 2^{-n}i)^{-1}] - S_0(x) \right| = 0. \quad \square$$

Proof of Proposition 1.8. Lemma 3.7 implies that $(\widehat{\mathcal{U}}^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}}$ converges in law, for the topology of uniform convergence on compacts, to \mathcal{U} given by 1.10. This can be proved similarly to Lemma 3.2. Define the change of scale $\widehat{\mathcal{S}}^{(n)} : 2^{-n}\mathbb{Z} \rightarrow \mathbb{R}$ by $\widehat{\mathcal{S}}^{(n)}(0) = 0$, and for $x \in 2^{-n}\mathbb{Z}$, $x \neq 0$,

$$\widehat{\mathcal{S}}^{(n)}(x) = \frac{1}{2} \sum_{i=1}^{2^n|x|} \gamma(2^{-n}i - \text{sgn}(x)2^{-n}, 2^{-n}i)^{-1} e^{\widehat{\mathcal{U}}^{(n)}(2^{-n}i - \text{sgn}(x)2^{-n}) + \widehat{\mathcal{U}}^{(n)}(2^n i)}.$$

Under this change of scale, $(\widehat{\mathcal{S}}^{(n)}(\widehat{Z}_k^{(n)}))_{k \geq 0}$ conditional on the random environment $(\gamma^{(n)}(x - 2^{-n}, x), \widehat{\mathcal{U}}^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}}$, is a local martingale. Lemma 3.7 combined with the convergence of $\widehat{\mathcal{U}}^{(n)}$ to \mathcal{U} , implies in turn that $\widehat{\mathcal{S}}^{(n)}$ converges in law to \mathcal{S} given by (1.12). This

implies that $(\widehat{Z}_{\lfloor 4^n q \rfloor}^{(n)})_{q \geq 0}$ converges in law to $(Z_q)_{q \geq 0}$. Indeed, one can embed, as in the proof of Proposition 3.3, the process $(\widehat{Z}_{\lfloor 4^n q \rfloor}^{(n)})_{q \geq 0}$ inside a scale-changed (by $(\widehat{\mathcal{S}}^{(n)})^{-1}$) Brownian motion, and the change of scales converges. Compared to the proof of Proposition 3.3, here the jumps at exponential times are replaced by jumps at fixed times. Moreover, in this convergence, the edge-occupation times of $(\widehat{Z}_{\lfloor 4^n q \rfloor}^{(n)})_{q \geq 0}$ scaled by 2^{-n} converge to the local times of $(Z_q)_{q \geq 0}$. For the approximation of local times by the number of interval crossings, see [MP10], Section 6.2, and [Kni81], Section 5.1. \square

4. CONVERGENCE OF THE VRJP TO THE LINEARLY REINFORCED MOTION

In this section we prove that the Vertex Reinforced Jump Processes converges in law to a Linearly Reinforced Motion constructed using the Bass-Burdzy flow (Section 2). To this end, we will make appear something that looks like a Bass-Burdzy flow in discrete. We also use that we already have a limit obtained as a time-changed Markov diffusion in a random environment (Proposition 3.5).

Define the scale functions $x \mapsto S_t^{(n)}(x)$ by

$$S_0^{(n)}(x) = \begin{cases} 0 & \text{if } x = 0, \\ 2^{-n} \sum_{i=1}^{2^n x} L_0(2^{-n}i)^{-1} L_0(2^{-n}(i-1))^{-1} & \text{if } x \in 2^{-n}\mathbb{Z} \cap (0, +\infty), \\ 2^{-n} \sum_{i=1}^{2^n |x|} L_0(-2^{-n}i)^{-1} L_0(-2^{-n}(i-1))^{-1} & \text{if } x \in 2^{-n}\mathbb{Z} \cap (-\infty, 0), \\ (2^{-n} \lceil 2^n x \rceil - x) S_0^{(n)}(2^{-n} \lfloor 2^n x \rfloor) + (x - 2^{-n} \lfloor 2^n x \rfloor) S_0^{(n)}(2^{-n} \lceil 2^n x \rceil) & \text{if } x \notin 2^{-n}\mathbb{Z}, \end{cases}$$

and

$$\frac{\partial}{\partial t} S_t^{(n)}(x) = \begin{cases} 0 & \text{if } x = X_t^{(n)}, \\ -L_t^{(n)}(X_t^{(n)})^{-2} L_t^{(n)}(X_t^{(n)} + 2^{-n})^{-1} & \text{if } x \geq X_t^{(n)} + 2^{-n}, \\ +L_t^{(n)}(X_t^{(n)})^{-2} L_t^{(n)}(X_t^{(n)} - 2^{-n})^{-1} & \text{if } x \leq X_t^{(n)} - 2^{-n}, \\ -(x - 2^{-n} \lfloor 2^n x \rfloor) L_t^{(n)}(X_t^{(n)})^{-2} L_t^{(n)}(X_t^{(n)} + 2^{-n})^{-1} & \text{if } x \in (X_t^{(n)}, X_t^{(n)} + 2^{-n}), \\ +(2^{-n} \lceil 2^n x \rceil - x) L_t^{(n)}(X_t^{(n)})^{-2} L_t^{(n)}(X_t^{(n)} - 2^{-n})^{-1} & \text{if } x \in (X_t^{(n)} - 2^{-n}, X_t^{(n)}). \end{cases}$$

Remark 4.1. $x \mapsto S_t^{(n)}(x)$ is a strictly increasing function. $S_t^{(n)}$ has been constructed in a way so as to always have, for $x \in 2^{-n}\mathbb{Z}$,

$$S_t^{(n)}(x) - S_t^{(n)}(x - 2^{-n}) = 2^{-n} L_t^{(n)}(x)^{-1} L_t^{(n)}(x - 2^{-n})^{-1}.$$

In particular,

$$\begin{aligned} S_t^{(n)}(+\infty) - S_t^{(n)}(-\infty) &= 2^{-n} \sum_{i \in \mathbb{Z}} L_t^{(n)}(2^{-n}i)^{-1} L_t^{(n)}(2^{-n}(i-1))^{-1} \\ &\leq 2^{-n} \sum_{i \in \mathbb{Z}} L_0(2^{-n}i)^{-1} L_0(2^{-n}(i-1))^{-1}. \end{aligned}$$

Moreover,

$$(4.1) \quad \lim_{n \rightarrow +\infty} S_0^{(n)}(x) = \int_0^x L_0(r)^{-2} dr = S_0(x).$$

Condition (1.3) ensures that $S_0(+\infty) = +\infty$ and $S_0(-\infty) = -\infty$. However, for finite n , we do not necessarily have $S_0^{(n)}(+\infty) = +\infty$ and $S_0^{(n)}(-\infty) = -\infty$.

Consider the change of time

$$du^{(n)}(t) = \frac{L_t(X_t^{(n)} - 2^{-n}) + L_t(X_t^{(n)} + 2^{-n})}{2L_t(X_t^{(n)})^2 L_t(X_t^{(n)} - 2^{-n}) L_t(X_t^{(n)} + 2^{-n})} dt,$$

and the inverse time change $t^{(n)}(u)$, for $u \in (0, u_{\max}^{(n)}) = (0, u^{(n)}(t_{\max}^{(n)})) \subseteq (0, +\infty)$.

Lemma 4.2. *The process*

$$(4.2) \quad (M_u^{(n)})_{u \geq 0} := (S_{t^{(n)}(u) \wedge t_{\max}^{(n)}}(X_{t^{(n)}(u) \wedge t_{\max}^{(n)}}^{(n)}))_{u \geq 0}$$

is a martingale with respect to its natural filtration $(\mathcal{F}_u^{M^{(n)}})_{u \geq 0}$. It advances by jumps at discrete times. A.s., $u_{\max}^{(n)} = +\infty$. Moreover, for $u_1 > u_0 \geq 0$,

$$\mathbb{E}[(M_{u_1}^{(n)} - M_{u_0}^{(n)})^2 | \mathcal{F}_{u_0}^{M^{(n)}}] = u_1 - u_0.$$

Proof. Given $u \in [0, u_{\max}^{(n)})$, $M^{(n)}$ will make a jump on the infinitesimal time interval $(u, u + du)$ with infinitesimal probability

$$(4.3) \quad 2^{2n-1} (L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n}) + L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})) \frac{dt^{(n)}(u)}{du} du \\ = 4^n L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)})^2 L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n}) L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n}) du.$$

Conditional that the jump occurs, it will be of height

$$+2^{-n} L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)})^{-1} L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})^{-1}$$

with probability

$$\frac{L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})}{L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n}) + L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})},$$

and of height

$$-2^{-n} L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)})^{-1} L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n})^{-1}$$

with probability

$$\frac{L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n})}{L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n}) + L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})}.$$

So the expected height of the jump is 0, and the expected height squared is

$$4^{-n} L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)})^{-2} L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n})^{-1} L_{t^{(n)}(u)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})^{-1},$$

which is exactly the inverse of the jump rate (4.3).

Let $(U_N)_{N \geq 0}$ be the family of stopping times after performing N jumps. We get that $(M_{u_1 \wedge U_N}^{(n)} - M_{u_0 \wedge U_N}^{(n)})_{N \geq 0}$ is an \mathbb{L}^2 convergent martingale and at the limit,

$$\begin{aligned} \mathbb{E}[(M_{u_1}^{(n)} - M_{u_0}^{(n)})^2 | \mathcal{F}_{u_0}^{M^{(n)}}] &= \lim_{N \rightarrow +\infty} \mathbb{E}[(M_{u_1 \wedge U_N}^{(n)} - M_{u_0 \wedge U_N}^{(n)})^2 | \mathcal{F}_{u_0}^{M^{(n)}}] \\ &= \lim_{N \rightarrow +\infty} \mathbb{E}[u_1 \wedge U_N - u_0 \wedge U_N | \mathcal{F}_{u_0}^{M^{(n)}}] \\ &= \mathbb{E}[u_1 \wedge u_{\max}^{(n)} - u_0 \wedge u_{\max}^{(n)} | \mathcal{F}_{u_0}^{M^{(n)}}] \end{aligned}$$

Since on the event $u_{\max}^{(n)} \in (u_0, u_1)$ we would have $(M_{u_1}^{(n)} - M_{u_0}^{(n)})^2 = +\infty$, this in particular means that it has probability 0, and further that $u_{\max}^{(n)} = +\infty$ a.s. \square

We consider the process $((X_t^*)_{t \geq 0}, (L_t^*(x))_{x \in \mathbb{R}, t \geq 0})$ obtained as a limit in law of the VRJP $((X_t^{(n)})_{0 \leq t \leq t_{\max}^{(n)}}, (L_t^{(n)}(x))_{x \in 2^{-n}\mathbb{Z}, 0 \leq t \leq t_{\max}^{(n)}})$ in Theorem 3.5. We define

$$\tilde{S}_t^*(x) = \int_{X_t^*}^x L_t^*(r)^{-2} dr.$$

\tilde{S}_t^{*-1} is the inverse diffeomorphism of \tilde{S}_t^* on \mathbb{R} . We define the time change

$$du^*(t) = L_t^*(X_t^*)^{-3} dt,$$

and $t^*(u)$ the inverse time change.

Lemma 4.3. *A.s., $u^*(+\infty) = +\infty$.*

Proof.

$$u^*(+\infty) = \int_0^{+\infty} L_t^*(X_t^*)^{-3} dt = +\infty.$$

But the above integral equals

$$\int_{\mathbb{R}} \int_0^{+\infty} L_t^*(x)^{-3} d_t L_t^*(x) dx = \frac{1}{2} \int_{\mathbb{R}} L_0(x)^{-2} dx,$$

which is $+\infty$ by (1.3). □

In discrete, we define

$$\tilde{S}_t^{(n)}(x) = S_t^{(n)}(x) - S_t^{(n)}(X_t^{(n)}),$$

and $\tilde{S}_t^{(n)-1}$ the inverse function on $(S_t^{(n)}(-\infty) - S_t^{(n)}(X_t^{(n)}), S_t^{(n)}(+\infty) - S_t^{(n)}(X_t^{(n)}))$.

From Theorem 3.5 immediately follows the following convergence result:

Lemma 4.4. *We have a joint convergence in law of processes*

$$(X_t^{(n)}, L_t^{(n)}(x), u^{(n)}(t), t^{(n)}(u), \tilde{S}_t^{(n)}(x), \tilde{S}_t^{(n)-1}(y))$$

towards

$$(X_t^*, L_t^*(x), u^*(t), t^*(u), \tilde{S}_t^*(x), \tilde{S}_t^{*-1}(y)).$$

For $\tilde{S}_t^{(n)}(x)$ and $\tilde{S}_t^{(n)-1}(y)$ we use the topology of uniform convergence on compact subsets of $\mathbb{R} \times [0, +\infty)$. In particular, $t_{\max}^{(n)}$ converges in probability towards $+\infty$, and, for any $t_0 \geq 0$,

$$\left(\sup_{0 \leq t \leq t_0} \tilde{S}_t^{(n)}(-\infty), \inf_{0 \leq t \leq t_0} \tilde{S}_t^{(n)}(+\infty) \right)$$

converges in probability towards $(-\infty, +\infty)$.

Proposition 4.5. *The martingale $(M_u^{(n)})_{u \geq 0}$, introduced in (4.2), converges in law to a standard Brownian motion started at 0, $(B_u)_{u \geq 0}$, in the Skorokhod topology.*

Proof. For $A > 0$, let $T_A^{(n)}$ be the first time $X_t^{(n)}$ exits from the interval $[-A, A]$. Define $(M_u^{(n,A)})_{u \geq 0}$ to be the process that coincides with $(M_u^{(n)})_{u \geq 0}$ on the time-interval $[0, u^{(n)}(T_A^{(n)})]$, and after time $u^{(n)}(T_A^{(n)})$ behaves like conditional independent standard Brownian motion started from $M_{u^{(n)}(T_A^{(n)})}^{(n)}$. $(M_u^{(n,A)})_{u \geq 0}$ is constructed in a way such that it is a martingale started from 0 and moreover, $((M_u^{(n,A)})^2 - u)_{u \geq 0}$ is a martingale too. Furthermore, one has a uniform control on the size of the jump of $(M_u^{(n,A)})_{u \geq 0}$. All of them are smaller than or equal to

$$2^{-n} \left(\inf_{[-A-2^{-n}, A+2^{-n}]} L_0 \right)^{-2},$$

and, in particular,

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\sup_{u \geq 0} (M_u^{(n,A)} - M_{u^-}^{(n,A)})^2 \right] = 0.$$

According to Theorem 1.4, Section 7.1 in [EK86], $(M_u^{(n,A)})_{u \geq 0}$ converge in law as $n \rightarrow +\infty$ to a standard Brownian motion started from 0. Now, $T_A^{(n)}$ converges in law to T_A^* , the first time X_t^* exits $[-A, A]$, and $u^{(n)}(T_A^{(n)})$ converges to

$$\int_0^{T_A^*} L_t^*(X_t^*)^{-3} dt.$$

In particular,

$$\lim_{u \rightarrow +\infty} \sup_{n \in \mathbb{N}} \mathbb{P}(u^{(n)}(T_A^{(n)}) \leq u) = 0.$$

Thus, $(M_u^{(n)})_{u \geq 0}$ converges in law to a Brownian motion, too. \square

Proposition 4.6. *The limit process $((X_t^*)_{t \geq 0}, (L_t^*(x))_{x \in \mathbb{R}, t \geq 0})$ obtained in Proposition 3.5 has same law as a Linearly Reinforced Motion $((\tilde{X}_t)_{t \geq 0}, (L_t(x))_{x \in \mathbb{R}, t \geq 0})$ started from 0, with initial occupation profile L_0 . Consequently, one gets Theorem 1.3 and 1.5.*

Proof. From Lemma 4.4 and Proposition 4.5, the process

$$(X_t^{(n)}, L_t^{(n)}(x), u^{(n)}(t), t^{(n)}(u), \tilde{S}_t^{(n)}(x), \tilde{S}_t^{(n)-1}(y), M_u^{(n)})$$

is tight, and therefore has a subsequential limit in law

$$(4.4) \quad (X_t^*, L_t^*(x), u^*(t), t^*(u), \tilde{S}_t^*(x), \tilde{S}_t^{*-1}(y), B_u),$$

where $(B_u)_{u \geq 0}$ is a standard Brownian motion started from 0. Define

$$S_t^*(x) = \tilde{S}_t^*(x) + B_{u^*(t)},$$

and

$$\Psi_u^*(y) = S_{t^*(u)}^* \circ S_0^{*-1}(y) = \tilde{S}_{t^*(u)}^* \circ S_0^{-1}(y) + B_u,$$

where S_0 is given by (4.1). $\Psi_u^*(y)$ is the limit (along the subsequence we consider) of

$$\Psi_u^{(n)}(y) = S_{t^{(n)}(u)}^{(n)} \circ S_0^{(n)-1}(y).$$

We want to show that $(\Psi_u^*)_{u \geq 0}$ is the Bass-Burdzy flow associated to $(B_u)_{u \geq 0}$. We have, for $u < u_{\max}^{(n)}$ and $y \in (S_0^{(n)}(-\infty), S_0^{(n)}(+\infty))$, that

$$\frac{\partial}{\partial u} \Psi_u^{(n)}(y) = \begin{cases} 0 & \text{if } y = M_u^{(n)}, \\ -\frac{2L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n})}{L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n}) + L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})} & \text{if } (\Psi_u^{(n)})^{-1}(y) \geq X_{t^{(n)}(u)}^{(n)} + 2^{-n}, \\ +\frac{2L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})}{L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n}) + L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})} & \text{if } (\Psi_u^{(n)})^{-1}(y) \leq X_{t^{(n)}(u)}^{(n)} - 2^{-n}, \end{cases}$$

and in all other cases,

$$(4.5) \quad -\frac{2L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n})}{L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n}) + L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})} \leq \frac{\partial}{\partial u} \Psi_u^{(n)}(y) \\ \leq \frac{2L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})}{L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} - 2^{-n}) + L_{t^{(n)}(u)}^{(n)}(X_{t^{(n)}(u)}^{(n)} + 2^{-n})}.$$

Since $L_{t^{(n)}(u)}^{(n)}$ converges, for y away from B_u ,

$$\frac{\partial}{\partial u} \Psi_u^*(y) = -\mathbf{1}_{y > B_u} + \mathbf{1}_{y < B_u}.$$

(4.5) and the convergence of local times implies that $u \mapsto \Psi_u^*(y)$ is Lipschitz-continuous. Thus, according to Theorem 2.3 in [BB99], $(\Psi_u^*)_{u \geq 0}$ is the Bass-Burdzy flow associated to $(B_u)_{u \geq 0}$.

Let

$$\xi_u^* = (\Psi_u^*)^{-1}(B_u).$$

We have that

$$X_t^* = \tilde{S}_t^{*-1}(0) = S_0^{-1}(\xi_{u^*(t)}^*).$$

Thus, X_t^* follows the definition of a Linearly Reinforced Motion with driving Brownian motion $(B_u)_{u \geq 0}$ (Definition 1.1). It follows that the limit law for (4.4) is unique and we have the desired identity in law. \square

Proposition 4.7. *Let be a Linearly Reinforced Motion $((X_t)_{t \geq 0}, (L_t(x))_{x \in \mathbb{R}, t \geq 0})$ started from 0, with initial occupation profile L_0 . It is coupled with the random environment $(\mathcal{U}(x))_{x \geq 0}$ (see (1.10)). For any $x_1, x_2 \in \mathbb{R}$,*

$$\lim_{t \rightarrow +\infty} \frac{L_t(x_2)}{L_t(x_1)} = \frac{L_0(x_2)e^{-\mathcal{U}(x_2)}}{L_0(x_1)e^{-\mathcal{U}(x_1)}} \quad \text{a.s.}$$

Moreover, the convergence is a.s. uniform on compact subsets of \mathbb{R}^2 . In particular, the random environment $(\mathcal{U}(x))_{x \geq 0}$ is measurable with respect to $(X_t)_{t \geq 0}$.

Proof. The measure $L_0(x)^2 e^{-2\mathcal{U}(x)} dx$ is finite and invariant for $(Z_q)_{q \geq 0}$. According the ergodic theorem for one-dimensional diffusions (Section 6.8 in [IM74]),

$$\lim_{q \rightarrow +\infty} \frac{1}{q} \lambda_q(x) = c_1 L_0(x)^2 e^{-2\mathcal{U}(x)} \quad \text{a.s.},$$

where

$$(4.6) \quad c_1^{-1} = \int_{\mathbb{R}} L_0(r)^2 e^{-2\mathcal{U}(r)} dr = \int_{\mathbb{R}} e^{-2\sqrt{2}W(y) - 2|y|} dy.$$

For the uniform convergence, see [VZ03]. Then,

$$L_t(x) = (L_0(x)^2 + 2\lambda_{q(t)}(x))^{\frac{1}{2}} \sim \sqrt{2c_1 q(t)^{\frac{1}{2}}} L_0(x) e^{-\mathcal{U}(x)}. \quad \square$$

Remark 4.8. The measure $L_0(x)e^{-\mathcal{U}(x)}$ is not necessarily finite. We have that

$$\int_{\mathbb{R}} L_0(x) e^{-\mathcal{U}(x)} dx = \int_{\mathbb{R}} \left(\frac{d}{dy} S_0^{-1}(y) \right)^{-\frac{1}{2}} e^{-\sqrt{2}W(y) - |y|} dy,$$

where S_0^{-1} can be any increasing diffeomorphism from \mathbb{R} to \mathbb{R} . The integral above being finite is a 0-1 property, but there are examples where it is infinite. For that it is sufficient that

$$\int_{\mathbb{R}} \left(\frac{d}{dy} S_0^{-1}(y) \right)^{-\frac{1}{2}} e^{-(1+\varepsilon)|y|} dy = +\infty.$$

In the case when it is finite, the normalized occupation measure $\frac{1}{t}(L_t(x) - L_0(x))dx$ converges a.s., in the weak topology of measures, to

$$c_2 L_0(x) e^{-\mathcal{U}(x)} dx,$$

where c_2 is a normalization factor.

Next we give the large time behaviour of $(X_t)_{t \geq 0}$. Actually, the leading order is given by the deterministic drift part in the random potential $2\mathcal{U} - 2 \log(L_0)$.

Proposition 4.9. Consider a Linearly Reinforced Motion $((X_t)_{t \geq 0}, (L_t(x))_{x \in \mathbb{R}, t \geq 0})$ started from 0, with the initial occupation profile L_0 being equal to 1 everywhere, except possibly a compact interval. Then,

$$\limsup_{t \rightarrow +\infty} \frac{X_t}{\log(t)} = \frac{1}{3} \quad a.s., \quad \liminf_{t \rightarrow +\infty} \frac{X_t}{\log(t)} = -\frac{1}{3} \quad a.s.$$

The mixture of diffusions $(Z_q)_{q \geq 0}$ such that $X_t = Z_{q(t)}$, with

$$dt = (L_0(Z_q)^2 + 2\lambda_q(Z_q))^{-\frac{1}{2}} dq,$$

satisfies

$$\limsup_{q \rightarrow +\infty} \frac{Z_q}{\log(q)} = \frac{1}{6} \quad a.s., \quad \liminf_{q \rightarrow +\infty} \frac{Z_q}{\log(q)} = -\frac{1}{6} \quad a.s.$$

Proof. The measure $L_0(x)^2 e^{-2\mathcal{U}(x)} dx$ is a finite invariant measure for $(Z_q)_{q \geq 0}$. According to [VZ03],

$$\lim_{q \rightarrow +\infty} \sup_{x \in \mathbb{R}} |q^{-1} \lambda_q(x) - c_1 L_0(x)^2 e^{-2\mathcal{U}(x)}| = 0,$$

where c_1 is given by (4.6). Thus,

$$t = \int_{\mathbb{R}} ((L_0(x)^2 + 2\lambda_{q(t)}(x))^{\frac{1}{2}} - L_0(x)) dx \sim \sqrt{2c_1} \left(\int_{\mathbb{R}} L_0(x) e^{-\mathcal{U}(x)} dx \right) q(t)^{\frac{1}{2}},$$

and

$$(4.7) \quad \log(t) \sim \frac{1}{2} \log(q(t)).$$

So we are left to determine

$$\limsup_{q \rightarrow +\infty} \frac{Z_q}{\log(q)} \quad \text{and} \quad \liminf_{q \rightarrow +\infty} \frac{Z_q}{\log(q)}.$$

Consider the natural scale function \mathcal{S} of $(Z_q)_{q \geq 0}$, given by (1.12). We have that

$$(4.8) \quad \mathcal{S}^{-1}(\varsigma) \stackrel{+\infty}{\sim} \frac{1}{2} \log(\varsigma), \quad \mathcal{S}^{-1}(\varsigma) \stackrel{-\infty}{\sim} -\frac{1}{2} \log(|\varsigma|).$$

$(\mathcal{S}(Z_q))_{q \geq 0}$ is a Brownian motion $(\beta_s)_{s \geq 0}$ time-changed, with the time-change given by

$$ds = L_0(Z_q)^2 e^{-2\mathcal{U}(Z_q)} dq,$$

and the inverse time change

$$dq = L_0(\mathcal{S}^{-1}(\beta_s))^{-2} e^{2\mathcal{U}(\mathcal{S}^{-1}(\beta_s))} ds.$$

We have that for any $\alpha > 1$, a.s. there is $K_\alpha > 1$, such that

$$\forall \varsigma \in \mathbb{R}, K_\alpha^{-1} |\varsigma|^{\alpha-1} \leq L_0(\mathcal{S}^{-1}(\varsigma))^{-2} e^{2\mathcal{U}(\mathcal{S}^{-1}(\varsigma))} \leq K_\alpha |\varsigma|^\alpha.$$

Then, using the Brownian scaling, we get that for $\alpha > 1$ and some random $\tilde{K}_\alpha > 1$,

$$(4.9) \quad \tilde{K}_\alpha^{-1} s^{1+\frac{\alpha-1}{2}} \leq q(s) \leq \tilde{K}_\alpha s^{1+\frac{\alpha}{2}},$$

According the law of iterated logarithm,

$$\limsup_{s \rightarrow +\infty} \frac{\mathcal{S}(Z_{q(s)})}{(2s \log \log(s))^{\frac{1}{2}}} = 1, \quad \liminf_{s \rightarrow +\infty} \frac{\mathcal{S}(Z_{q(s)})}{(2s \log \log(s))^{\frac{1}{2}}} = 1.$$

Combining with (4.8) and (4.9), we get that

$$\limsup_{q \rightarrow +\infty} \frac{\log(Z_q)}{\log(q)} = \frac{1}{6}, \quad \liminf_{q \rightarrow +\infty} \frac{\log(Z_q)}{\log(q)} = -\frac{1}{6}.$$

Combining with (4.7), we get the result. □

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